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Other Asymptotic Methods for Deriving a Far Field Approximation

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Some asymptotic methods except for the reductive perturbation method are presented. The wave packet formalism is applied to the Vlasov-Poisson equations to derive a K-dV equation and also to the problem of the wave modulation by taking an example of the Bussinesque equation. The derivative expansion method and the extended Krylov-Bogoliubov-Mitropolsky method are also discussed.

§ 1. Introduction

Various methods have been used to treat the problems of nonlinear wave propagations in the dispersive or dissipative media.

Whitham[1] and Lighthill[2] start with the exact, uniform periodic wave train which can readily be obtained for typical nonlinear wave equations and, assuming the amplitude, wave number to be slowly varying function of space and time, derive differential equations for these quantities, averaging over the local oscillations in the medium. A subsequent method[3] allows these results to be obtained in a simpler manner by application of an averaging procedure to the Lagrangian of the original system. The desired equations then arise directly as the Euler equations of the averaged Lagrangian. The equations obtained in this way are hyperbolic and become inapplicable for rather large time inasmuch as the nonlinear increase in the steepness of the profile leads to the formation of shock waves. In addition to studying what happens physically when the predicted solutions become many-valued, Lighthill[2] discussed the general condition govering the dispersion equation is elliptic or hyperbolic, and obtained the result that the considered wave is unstable against changes in phase and amplitude for the hyperbolic dispersion. For their discussions of the stability of waves, only the knowledge of the nonlinear dispersion equation is needed. It should be noted that there are no restrictions on the order of magnitude of the amplitude in their treatment in which full nonlinearity is retained through wave trains as starting solutions. We may say that the averaging method is applicable to the nonlinear wave systems in which the nonlinearity exceeds the dispersion in magnitude. Therefore, the equations for phase and amplitude obtained by them can not be written in the form of a single equation for the complex amplitude, i.e., the nonlinear Schrödinger equation in contrast to, say, the reductive perturbation method.

On the contrary to the averaging method, these have been developed some methods devised to describe the nonstationary process in which the nonlinearity is comparable order with the dispersion. They are the wave packet formalism devised by Karpman-Krushkal[4] and some of the methods of multiple scales. In their treatments sinusoidal waves are taken as starting solutions in contrast to the averaging method.

Karpman-Krushkal[4] developed a simple, nonlinear generalization of the wave packet formalism. Expanding the dispersion law of the linear approximation around k_0 and ω_0, the wave number and the frequency of a carrier wave, and replacing $\omega-\omega_0$ and $k-k_0$ by $i\partial/\partial t$ and $-i\partial/\partial x$ respectively, they obtained the equation for phase with account of the nonlinear terms of the second order of amplitude and terms with derivatives up to the second order, giving a nonlinear Schrödinger equation with the aid of the energy equation of Whitham-Lighthill.[2] However, the Karpman-Krushkal approach does not lead to explicit expressions for the coefficients in the nonlinear Schrödinger equation, as does the reductive perturbation method, for example. Recently Kono-Sanuki[5] succeeded in the extention of this wave packet formalism so as to derive a far field approximation in the form of a single equation with the explicit coefficients within the framework. They discuss the reduction to the K-dV equation for the ion acoustic wave, from the Fourier transformed equations of the Vlasov-Poisson equations which have been widely used as the basis of the weak turbulent theory[6] in plasmas. This method suggests the possibility of the extention of the asymptotic methods to the systems with many modes. The similar method is applied to derive the nonlinear Schrödinger equation taking account of higher order effects.[7]

The reductive perturbation method, developed by Taniuti and his collaborators, belongs to the methods of multiple scales. Another form of the method of multiple scales is the derivative expansion method which has been devised by Sturrock[8] and developed by Sandri.[9] This method has also been used for various physical problems.[10],[11] Among them, Nayfeh[12] applied this method to the longitudinal nonlinear travelling waves in a hot electron plasma and obtained the result without the amplitude modulation. In the course of calculations, Nayfeh[12] omitted the procedure of moving his coordinate system with group velocity of the wave. This led him to the conclusion that nonlinearity affects only the phase and not the amplitude. Taking into account the effect of moving of the co-ordinate system with group velocity so as to be able to describe the neighbourhood of the center of the wave group, Kawahara[13] obtained an equation which describes the amplitude modulation of wave trains within the framework of the derivative expansion method. He also discussed the reduction to the K-dV equation for long waves based on the derivative expansion method.

Another useful method along the methods of multiple scales is the Krylov-Bogoliubov-Mitropolsky method[14),15)] (hereafter refered to KBM method) extended to the systems of nonlinear partial differential equations which was made by Montgomery and Tidman.[16)] Tidman and Stainer[17)] calculated the wave number (or frequency) shifts for nonlinear waves in plasmas. In the course of analysis, they made a simplifying (sufficient but not necessary) assumption, which led them to fail in taking account of full amplitude modulation. Kakutani and Sugimoto[18)] has succeeded in taking into account not only the wave number shifts but also long time slow modulation of the amplitude with the aid of extended KBM method.

In this article, we discuss three methods developed recently in our country, the wave packet formalism, the derivative expansion method and the extended KBM method. In the next section, we derive the K-dV equation for the ion acoustic wave to discuss the wave packet formalism according to Kono-Sanuki.[5)] In the third section, we apply the wave packet formalism to the problem of the nonlinear wave modulation. We also introduce the derivative expansion method due to Kawahara[13)] in the fourth section and the extended KBM method by Kakutani-Sugimoto[18)] in the last section.

§ 2. The wave packet formalism: Reduction to the K-dV equation

The basic equations are:

$$\frac{\partial}{\partial t}F_\alpha+v\frac{\partial}{\partial x}F_\alpha+\frac{e_\alpha}{m_\alpha}\frac{\partial}{\partial x}\phi\frac{\partial}{\partial v}F_\alpha=0, \tag{1}$$

$$\frac{\partial^2}{\partial x^2}\phi=-4\pi\sum_\alpha e_\alpha n_\alpha\int dvF_\alpha, \tag{2}$$

where α indicates the species of the plasma particle. F and ϕ denote the particle distribution function and the electric potential respectively. From the Fourier transform of Eq. (1), we get

$$\begin{aligned} f_\alpha(k,v,\omega)=&\frac{e_\alpha}{m_\alpha}\frac{1}{\omega-kv}k\phi(k,\omega)\frac{\partial}{\partial v}F_\alpha^{(0)}\\ &+\frac{e_\alpha}{m_\alpha}\frac{1}{\omega-kv}\sum_{k'}\int\frac{d\omega'}{2\pi}k'\phi(k',\omega')\frac{\partial}{\partial v}f_\alpha(k-k',v,\omega-\omega'), \end{aligned} \tag{3}$$

where $F_\alpha^{(0)}$ is the spatial homogeneous distribution function being independent of time, and $F_\alpha=F_\alpha^{(0)}+f$. Substitution of the iterated solution of Eq.(3) into Eq.(2) gives the following nonlinear equation for the potential $\phi(k,\omega)$ up to the second order with respect to ϕ inclusively,

$$\varepsilon(k,\omega)\phi(k,\omega)$$
$$=\sum_{k'}\int\frac{d\omega'}{2\pi}V(k',\omega';k-k',\omega-\omega')\phi(k',\omega')\phi(k-k',\omega-\omega'),\tag{4}$$

where

$$\varepsilon(k,\omega)=1-\frac{4\pi}{k}\sum_{\alpha}\frac{e_\alpha^2n_\alpha}{m_\alpha}\int dv\frac{1}{\omega-kv}\frac{\partial}{\partial v}F_\alpha^{(0)},\tag{5}$$

$$V(k',\omega';k-k',\omega-\omega')$$
$$=\frac{4\pi}{k^2}\sum_{\alpha}\frac{e_\alpha^3}{m_\alpha^2}n_\alpha k'(k-k')\int dv\frac{1}{\omega-kv}\frac{\partial}{\partial v}\frac{1}{\omega-\omega'-(k-k')v}\frac{\partial}{\partial v}F_\alpha^{(0)}.\tag{6}$$

Here the potential ϕ is assumed to be a well-behaved function and the quantity $1/(\omega-kv)$ in Eq.(6) means that $P/(\omega-kv)-\pi i\delta(\omega-kv)$ in view of the initial condition that initially there is no disturbance, the disturbance is put on after $t=0$, namely $f(k,v,t=0)=0$. Moreover the quantity $\varepsilon(k,\omega)$ means the linear dielectric permeability of the plasma. For the ion acoustic wave with small wave number, the dispersion relation is obtained by equating the real part of the dielectric function to zero:

$$\omega_k=\pm kc_s\left(1-\frac{1}{2}k^2\lambda_D^2\right)\quad\text{for}\quad k\lambda_D\ll1,\tag{7}$$

where $c_s=(Te/M)^{1/2}$ is the ion acoustic velocity. In the frame moving with velocity c_s, the oscillating frequency is $-1/2c_s\lambda_D^2k^3$. Here, noting the slow variation of ϕ and using the expansion of the real part of the dielectric permeability $\varepsilon_{\rm I}(k,\omega)\equiv{\rm Re}\,\varepsilon(k,\omega)$ around ω_k, we can rewrite Eq. (4) in the following form

$$\left.\frac{\partial\varepsilon_{\rm I}}{\partial\omega}\right|_{\omega=\omega_k}(\omega-\omega_k)\phi(k,\omega)+i\varepsilon_{\rm II}(k,\omega_k)\phi(k,\omega)$$
$$=\sum_{k'}\int\frac{d\omega'}{2\pi}V(k',\omega';k-k',\omega-\omega')\phi(k',\omega')\phi(k-k',\omega-\omega'),\tag{8}$$

where $\varepsilon_{\rm II}={\rm Im}\,\varepsilon$. Here introducing the slow variation $\Omega=\omega-c_sk$, and noting that the integral on the right hand side of Eq.(6) can be approximated in the following form, because of the relation $\Omega/kc_s\ll1$, provided that the resonance interaction is neglected,

$$\int\frac{1}{\omega-kv}\frac{\partial}{\partial v}\frac{1}{\omega-\omega'-(k-k')v}\frac{\partial}{\partial v}F_\alpha^{(0)}dv$$
$$\simeq\frac{1}{k(k-k')}\int dv\frac{P}{(v-c_s)^3}\frac{\partial}{\partial v}F_\alpha^{(0)},\tag{9}$$

we obtain the following equation of ϕ.

$$-i\Omega\phi(k,\Omega)+\frac{c_s}{2}\lambda_D^2(ik)^3\phi(k,\Omega)+A(c_s)\frac{k}{|k|}\phi(k,\Omega)$$
$$+B(c_s)\sum_{k'}\int\frac{d\omega'}{2\pi}ik'\phi(k',\Omega')\phi(k-k',\Omega-\Omega')=0, \tag{10}$$

where

$$A(c_s)=\sum_\alpha\omega_{p\alpha}^2\int dv\,\delta(c_s-v)\frac{\partial F_\alpha^{(0)}}{\partial v}\Big/\sum_\alpha\omega_{p\alpha}^2\int dv\frac{P}{(v-c_s)^2}\frac{\partial F_\alpha^{(0)}}{\partial v}, \tag{11}$$

$$B(c_s)=\sum_\alpha\frac{e_\alpha}{2m_\alpha}\omega_{p\alpha}^2\int dv\frac{P}{(v-c_s)^3}\frac{\partial F_\alpha^{(0)}}{\partial v}\Big/\sum_\alpha\omega_{p\alpha}^2\int dv\frac{P}{(v-c_s)^2}\frac{\partial F_\alpha^{(0)}}{\partial v}. \tag{12}$$

Then, introducing the slow time scale τ, and operating $\sum_k\int(d\Omega/2\pi)e^{ikx-i\Omega\tau}$ on Eq. (10), we have the nonlinear equation for electric potential.

$$\frac{\partial}{\partial\tau}\phi(x,\tau)+B(c_s)\frac{\partial}{\partial x}\phi^2(x,\tau)+\frac{1}{2}c_s^2\lambda_D\frac{\partial^3}{\partial x^3}\phi(x,\tau)$$
$$-A(c_s)\int\frac{P}{x-x'}\phi(x',\tau)dx'=0. \tag{13}$$

The last term in Eq. (13) represents the Landau damping effects. Similar equations are obtained by several authors.[19),20)] However, it should be noted that using the quasimonochromatic approximation and the time separation procedure, K-dV equation has been derived from Eq.(1) which is widely used as the basic equation for the turbulent theory.[6)] This methods suggests the possibility of the extention of the asymptotic methods to the systems with many modes.

§ 3. Application of the wave packet formalism to the problem of the wave modulation

The wave packet formalism can be applied to the problem of the wave modulation to derive the nonlinear Schrödinger equation. Ichikawa-Suzuki-Fried[7)] applied this method to the Vlasov-Poisson equations and discussed the effect of the nonlinear Landau damping[6)] on the wave modulation. In this section, we take the Bussinesque equation as a basic equation for simplicity.

$$L\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)u(x,t)=M\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)\{u(x,t)\}^2,$$
$$L\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)=\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}-\frac{\partial^4}{\partial t^2\partial x^2}, \tag{14}$$
$$M\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)=\frac{1}{2}\frac{\partial^2}{\partial x^2},$$

where $u(x,t)$ is a real function of the one-dimensional space co-ordinate x and time t, and represents a small but finite perturbation from a uniform state. Eq.(14) is an approximate equation for water waves, lattice waves and so forth.

Introducing the Fourier transformation with respect to x and t

$$u(x, t)=(2\pi)^{-2}\int d\omega \int dk\, u(k, \omega)\exp(-i\omega t+ikx)$$

Eq. (1) takes the following form in the Fourier representation:

$$D(y)u(y)=-\sum_{y_1} V(y, y_1)u(y_1)u(y-y_1), \tag{15}$$

$$D(y)=-\omega^2+k^2-k^2\omega^2, \tag{16}$$

$$V(y, y_1)=kk_1, \tag{17}$$

where y denotes the set (k, ω) and $\sum_y$ is an abbreviation symbol of integral $(2\pi)^{-2}\int d\omega \int dk$. Since we are concerned with the problem of the wave modulation, we may choose a monochromatic plane wave solution to the linealized equation of Eq. (14) as a starting solution, and study how modulations of amplitude and phase propagate due to the nonlinear mode couplings. For our purpose on the right hand side of Eq. (15) we consider only the mode coupling processes resulted in a fundamental mode. In the lowest order these couplings are those of a fundamental mode with a second harmonic mode and a "zero" mode.

The second harmonic mode and the "zero" mode can be represented simultaneously

$$u(y)=-\frac{P}{D(y)}\sum_{y_1} V(y, y_1)u(y_1)u(y-y_1)+\alpha\delta(y), \tag{18}$$

where P denotes a principal part of $1/D(y)$ and α is a constant to be determined by the appropriately specified initial and/or boundary conditions.

Substituting Eq.(18) into the right hand side of Eq.(15), we obtain the starting equation:

$$\begin{aligned} D(y)u(y)=&\sum_{y_1}\sum_{y_2}\frac{1}{D(y-y_1)}V(y, y)V(y-y_1, y_2)u(y_1)u(y_2)u(y-y_1-y_2) \\ &-\alpha V(y, y)u(y) \end{aligned} \tag{19}$$

with

$$u(x, t)=A(x, t)\exp[i(k_0x-\omega_0t)]+\text{complex conjugate}, \tag{20}$$

where $A(x,t)$ is the complex amplitude which varies slowly both in space and time, k_0 and ω_0 satisfy the dispersion relation. Equation (20) is rewritten in (k, ω) representation as follows:

$$u(y)=A(y-y_0)+A^*(-y-y_0). \tag{21}$$

Substituting Eq. (21) into Eq. (19), and transforming the variables from y to $Y=y-y_0$, $Y\equiv(K,\Omega)$ corresponding to the introduction of the slowly varying variables, there appear rapidly varying terms such as $A(Y-Y_1-Y_2-2y_0)$ with shift of integral multiple of y_0 in addition to the slowly varying terms such as $A(Y-Y_1)$. These rapidly varying terms can be dropped for the purpose of study of the slow variation of A. Then Eq. (19) reduces to the form:

$$D(y_0+Y)A(Y)=\sum_{Y_1}\sum_{Y_2}[M_0(Y,Y_1,Y_2)+M_2(Y,Y_1,Y_2)]$$
$$\times A(Y_1)A^*(-Y_2)A(Y-Y_1-Y_2)-\alpha V(y_0+Y,y_0+Y)A(Y), \quad (22)$$

$$M_0(Y,Y_1,Y_2)$$
$$=\frac{1}{D(Y-Y_1)}V(y_0+Y,y_0+Y)V(Y-Y_1,\ Y-Y_1), \quad (23)$$

$$M_2(Y,Y_1,Y_2)$$
$$=\frac{1}{D(Y-Y_2+2y_0)}V(y_0+Y,y_0+Y)V(2y_0+Y-Y_2,y_0+Y_1). \quad (24)$$

M_0 and M_2 represent the contributions of beat waves $l=0$ and $l=2$ respectively which are obtained in the reductive perturbation method.

Now the linear response function can be expanded as follows:

$$D(k_0+K,\omega_0+\Omega)=\frac{\partial D(k_0,\omega_0)}{\partial\omega_0}\left[\Omega-v_gK-\frac{1}{2}\frac{\partial v_g}{\partial k_0}K^2\right]$$
$$+\frac{1}{2}\frac{\partial^2 D(k_0,\omega_0)}{\partial\omega_0^2}[\Omega-v_gK]^2+\cdots, \quad (25)$$

where $v_g=d\omega_0/dk_0$. In our discussion of the slow modulation of the wave packet of the quasimonochromatic wave, it is natural to assume the following relations:

$$\frac{K}{k_0}\sim O(\varepsilon),\qquad \frac{\Omega}{\omega_0}\sim O(\varepsilon). \quad (26)$$

As $v_g\sim\Omega/K$, then $\Omega-v_gK$ is order ε^2. Therefore, in the expansion of Eq. (25), we can neglect the second term:

$$D(k_0+K,\omega_0+\Omega)\simeq\frac{\partial D(k_0,\omega_0)}{\partial\omega_0}\left[\Omega-v_gK-\frac{1}{2}\frac{\partial v_g}{\partial k_0}K^2\right]. \quad (27)$$

The matrix elements M_0 and M_2 are estimated from Eqs. (16) and (17) using the relations (26). First let us estimate M_2:

$$D(2y_0+Y-Y_2)\simeq D(2y_0)=-12k_0^2\omega_0^2,$$
$$V(y_0+Y,y_0+Y)\simeq V(y_0,y_0)=k_0^2,$$
$$V(2y_0+Y-Y_2,y_0+Y_1)\simeq V(2y_0,y_0)=2k_0^2.$$

From these,

$$M_2 \simeq -\frac{k_0^2}{6\omega_0^2}. \tag{28}$$

In the similar way as before M_0 can be estimated:

$$M_0(Y, Y_1, Y_2) = (k_0+K)^2 \Big/ \left\{1 - \left(\frac{\Omega-\Omega_1}{K-K_1}\right)^2 - (\Omega-\Omega_1)^2\right\} \simeq \frac{k_0^2}{1-v_g^2}, \tag{29}$$

where we have used the relation $(\Omega-\Omega')/(K-K') = v_g$ which is the case in the limit of the quasimonochromatic wave. v_g^2 is not unity for dispersive waves. From Eqs. (22), (27), (28) and (29), we can get the equation,

$$\begin{aligned}
&\left[\Omega - v_g K - \frac{1}{2}\frac{\partial v_g}{\partial k_0}K^2\right]A(K, \Omega) \\
&= -\frac{k_0^2}{\partial D(k_0, \omega_0)/\partial\omega_0}\left[\frac{1}{v_g^2-1} + \frac{1}{6\omega_0^2}\right]\frac{1}{(2\pi)^4} \\
&\qquad \int dK_1 dK_2 \int d\Omega_1 d\Omega_2 A(K_1, \Omega_1) A^*(-K_2, -\Omega_2) \\
&\times A(K-K_1-K_2, \Omega-\Omega_1-\Omega_2) - \frac{k_0^2}{\partial D(k_0, \omega_0)/\partial\omega_0}\alpha A(K, \Omega).
\end{aligned} \tag{30}$$

Here we change from the laboratory frame to the wave frame moving with a group velocity, i.e., $\Omega - v_g K \to \Omega'$. As was mentioned before the order of magnitude of Ω' is ε^2. From the expansion of Eq.(25), K and Ω' must be located inside the stripe with the width ε and ε^2 respectively along the dispersion curve in the (k, ω) space. Therefore, we can not straightforwadly perform the Fourier transformations of Eq.(30) with respect to K and Ω'. However, if we introduce the stretched variables $\tau = \varepsilon^2 t$ and $\xi = \varepsilon x$, we may consider that K/ε and Ω'/ε^2 can cover all the (k, ω) space. Thus, we obtain the nonlinear Schrödinger equation

$$\begin{aligned}
&i\frac{\partial}{\partial\tau}A(\xi, \tau) + \frac{1}{2}\frac{\partial v_g}{\partial k_0}\frac{\partial^2}{\partial\xi^2}A(\xi, \tau) \\
&= -\frac{k_0^2}{\partial D(k_0, \omega_0)/\partial\omega_0}\left[\frac{1}{v_g^2-1} + \frac{1}{6\omega_0^2}\right]|A(\xi, \tau)|^2 A(\xi, \tau) \\
&\qquad - \frac{\beta k_0^2}{\partial D(k_0, \omega_0)/\partial\omega_0}A(\xi, \tau),
\end{aligned} \tag{31}$$

where $\beta = \alpha/\varepsilon^2$. Here the linear interaction term is not so essential because it causes only a simple phase sift. In fact, $-\{\beta k_0^2/(\partial D/\partial\omega_0)\}A$ term can be removed by a simple substitution:

$$A \longrightarrow A \exp\left(i\frac{\beta k_0^2}{\partial D/\partial\omega_0}\tau\right).$$

In order to understand the scheme in this section intuitively, let us rewrite Eq. (30) in the following form:

$$\int dK' \int d\Omega' \tilde{D}(K,\Omega;K',\Omega')A(K',\Omega')=0,$$

$$\tilde{D}(K,\Omega;K',\Omega')=\left(\Omega'-v_gK'-\frac{1}{2}\frac{\partial v_g}{\partial k_0}K'^2\right)\delta(\Omega-\Omega')\delta(K-K')$$

$$+\frac{k_0^2}{\partial D(k_0,\omega_0)/\partial\omega_0}\left[\frac{1}{v_g^2-1}+\frac{1}{6\omega_0^2}\right]$$

$$\int d\Omega_1\int dK_1 A^*(-K_1,-\Omega_1)A(K-K'-K_1,\Omega-\Omega'-\Omega_1)$$

$$+\alpha\frac{k_0^2}{\partial D(k_0,\omega_0)/\partial\omega_0}\delta(K-K')\delta(\Omega-\Omega').$$

$\tilde{D}$ is the nonlinear response function taking account of an effect of finite amplitude. The second term of the above expression corresponds to "amplitude dispersion" called so by Lighthill.[2]

§ 4. Derivative expansion method

In this and following sections the basic equation we consider is also Eq. (14).

A derivative expansion method proceeds in the following manner. As in conventional perturbation analyses, we expand $u(x,t)$ in the small parameter ε according to

$$u=\varepsilon u^{(1)}+\varepsilon^2u^{(2)}+\cdots. \tag{32}$$

In addition, however, we make use of the fact that the characteristic scales for temporal and spatial variation of u due to the nonlinear coupling are much longer than the characteristic scales for the variation of the carrier. To incorporate the disparity between fast and slow scales in the expansion procedure, we arbitrarily extend the number of independent variables x and t to the sets of independent variables:

$$\begin{aligned}&x_0,\ x_1,\ x_2,\ \cdots,\ x_n,\\&t_0,\ t_1,\ t_2,\ \cdots,\ t_n,\end{aligned} \tag{33}$$

where $x_n=\varepsilon^n x$, $t_n=\varepsilon^n t$. Then Eq. (32) becomes

$$u=\sum_{m=1}^{N}\varepsilon^m u^{(m)}(x_0,x_1,\cdots,x_N,t_0,t_1,\cdots,t_N)+O(\varepsilon^{N+1}) \tag{34}$$

and the derivative operators $\partial/\partial x$ and $\partial/\partial t$ are expanded as

$$\frac{\partial}{\partial x}=\sum_{n=0}^{N}\varepsilon^n\frac{\partial}{\partial x_n},\qquad \frac{\partial}{\partial t}=\sum_{n=0}^{N}\varepsilon^n\frac{\partial}{\partial t_n},\tag{35}$$

respectively. According to the Eqs.(35), the linear operators in Eq.(14) can be expressed as

$$\begin{aligned}L\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)&=\sum_{n=0}^{N}\varepsilon^n L^{(n)}\left(\frac{\partial}{\partial x_0},\cdots,\frac{\partial}{\partial x_N},\frac{\partial}{\partial t_0},\cdots,\frac{\partial}{\partial t_N}\right)+O(\varepsilon^{N+1}),\\ M\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)&=\sum_{n=0}^{N}\varepsilon^n M^{(n)}\left(\frac{\partial}{\partial x_0},\cdots,\frac{\partial}{\partial x_N},\frac{\partial}{\partial t_0},\cdots,\frac{\partial}{\partial t_N}\right)+O(\varepsilon^{N+1})\end{aligned}\tag{36}$$

and the first few terms:

$$\left.\begin{aligned}L^{(0)}&=\frac{\partial^2}{\partial t_0^2}-\frac{\partial^2}{\partial x_0^2}-\frac{\partial^4}{\partial t_0^2\partial x_0^2},\\ L^{(1)}&=2\frac{\partial^2}{\partial t_0\partial t_1}-2\frac{\partial^2}{\partial x_0\partial x_1}-2\frac{\partial^4}{\partial t_0\partial t_1\partial x_0^2}-2\frac{\partial^4}{\partial t_0^2\partial t_1\partial x_1},\\ L^{(2)}&=\frac{\partial^2}{\partial t_1^2}+2\frac{\partial^2}{\partial t_0\partial t_2}-\left(\frac{\partial^2}{\partial x_1^2}+2\frac{\partial^2}{\partial x_0\partial x_2}\right)\\ &\quad-\left(\frac{\partial^4}{\partial t_0^2\partial x_1^2}+2\frac{\partial^4}{\partial t_0^2\partial x_0\partial x_2}+4\frac{\partial^4}{\partial t_0\partial t_1\partial x_0\partial x_1}+2\frac{\partial^4}{\partial t_0\partial t_2\partial x_0^2}\right),\end{aligned}\right\}\tag{37}$$

$$\left.\begin{aligned}M^{(0)}&=\frac{1}{2}\frac{\partial^2}{\partial x_0^2},\qquad M^{(1)}=\frac{\partial^2}{\partial x_0\partial x_1},\\ M^{(2)}&=\frac{1}{2}\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_0\partial x_2}.\end{aligned}\right\}\tag{38}$$

Substituting Eqs.(34) and (36) into Eq.(14) and equating to zero the coefficients of successive powers of ε, we find a set of equations to determine $u^{(m)}$ successively. These dependent quantities are to be determined so as to be nonsecular at each stage of perturbation.

Now, we shall proceed to find the far field approximation of Eq.(14). Using the operators given in Eqs.(37) and (38), we obtain the first four perturbation equations as follows:

$$L^{(0)}u^{(1)}=0,\tag{39 a}$$

$$L^{(0)}u^{(2)}+L^{(1)}u^{(1)}=M^{(0)}u^{(1)2},\tag{39 b}$$

$$L^{(0)}u^{(3)}+L^{(1)}u^{(2)}+L^{(2)}u^{(1)}=2M^{(0)}u^{(1)}u^{(2)}+M^{(1)}u^{(1)2},\tag{39 c}$$

$$\begin{aligned}&L^{(0)}u^{(4)}+L^{(1)}u^{(3)}+L^{(2)}u^{(2)}+L^{(3)}u^{(1)}\\ &\qquad=M^{(0)}[u^{(2)2}+2u^{(1)}u^{(3)}]+2M^{(1)}u^{(1)}u^{(2)}+M^{(2)}u^{(1)2}.\end{aligned}\tag{39 d}$$

Since we are considering the nonlinear modulation of the wave trains, we have the first-order solution from Eq. (29a)

$$u^{(1)}=A(x_1, \cdots, x_N, t_1, \cdots, t_N)e^{i(kx_0-\omega t_0)}+\text{c.c.}, \tag{40}$$

where A is a complex function of higher order scales, k and ω satisfy the dispersion relation

$$D(k, \omega)\equiv-\omega^2+k^2-k^2\omega^2, \tag{41}$$

c.c. stands for the complex conjugate of the preceding term. Introduction of this solution into the second-order equation (39b) gives

$$L^{(0)}u^{(2)}=\left[-i\left\{\frac{\partial D}{\partial\omega}\frac{\partial A}{\partial t_1}-\frac{\partial D}{\partial k}\frac{\partial A}{\partial x_1}\right\}e^{i\theta}+\text{c.c.}\right] \\ +P_2\{A^2e^{2i\theta}+\text{c.c.}\}, \tag{42}$$

where $\theta=kx_0-\omega_0 t$ and $P_m=-1/2mk^2$. If the condition for nonsecularity

$$\frac{\partial A}{\partial t_1}+v_g\frac{\partial A}{\partial x_1}=0, \qquad v_g=\frac{d\omega}{dk}, \tag{43}$$

and its complex conjugate relation are satisfied, Eq.(42) admits a uniformly valid solution,

$$u^{(2)}=\left\{\frac{P_2}{D_2}A^2e^{2i\theta}+\text{c.c.}\right\}+B(x_1, \cdots, x_N, t_1, \cdots, t_N) \\ +\{C(x_1, \cdots, x_N, t_1, \cdots, t_N)e^{i\theta}+\text{c.c.}\}, \tag{44}$$

where B(real) and C(complex) are functions of higher scales to be determined in higher order perturbations, and $D_m\equiv D(mk,m\omega)$.

Introdusing $u^{(1)}$ and $u^{(2)}$ into Eq.(39c), we find the second nonsecularity condition,

$$i\left\{\frac{\partial D}{\partial\omega}\frac{\partial A}{\partial t_2}-\frac{\partial D}{\partial k}\frac{\partial A}{\partial x_2}\right\}-\frac{1}{2}\left\{\frac{\partial^2 D}{\partial\omega^2}\frac{\partial^2 A}{\partial t_1^2}-2\frac{\partial^2 D}{\partial k\partial\omega}\frac{\partial^2 A}{\partial x_1\partial t_1}+\frac{\partial^2 D}{\partial k^2}\frac{\partial^2 A}{\partial x_1^2}\right\} \\ -2P_1\left\{\frac{P_2}{D_2}|A|^2+B\right\}A=-i\left\{\frac{\partial D}{\partial\omega}\frac{\partial C}{\partial t_1}-\frac{\partial D}{\partial k}\frac{\partial C}{\partial x_1}\right\}, \tag{45}$$

and its complex conjugate relation, where $|A|$ denotes the modulus of A. The relation which governs the variation of B can be derived from Eq.(39d) by the condition of nonsecularity for constant terms to give

$$\left(\frac{\partial^2}{\partial t_1^2}-\frac{\partial^2}{\partial x_1^2}\right)B=\frac{\partial^2}{\partial x_1^2}|A|^2. \tag{46}$$

If A and B depend on t_1 and x_1 through $\eta_1=x_1-v_g t_1$, i.e., if they are considered in the co-ordinate system moving with the group velocity, Eq.(46) yields

$$B=\frac{1}{v_g^2-1}|A|^2+\beta \tag{47}$$

provided that group velocity differs from the phase velocity. β is an absolute constant which is determined by boundary and/or initial conditions. In order to complete Eq.(45) for A, we must determine C. However as C is a coefficient of the secular producing term appeared in $u^{(2)}$, that is, a resonant term proportional to $e^{i\theta}$, C is related to the resonant terms of higher order $u^{(m)}$. In other words, the nonsecular conditions lead to the hierarchy of equations with respect to the coefficient of resonant terms. Therefore, we here make a plausible assumption. Noting the fact that Eq.(43) suggests physical quantities to be stationary in wave frame moving with the group velocity for the variation in the first order with respect to ε, we have already assumed that B depends on t_1 and x_1 through $\eta_1=x_1-v_gt_1$. Then it is natural to assume that C also obeys Eq.(43). Hence the right hand side of the Eq.(45) becomes zero. Using Eqs.(43), (46) and (47) into the second nonsecularity condition (45), we obtain a nonlinear equation for the complex amplitude A, that is

$$\begin{aligned} &i\left(\frac{\partial A}{\partial t_2}+v_g\frac{\partial A}{\partial x_2}\right)+\frac{1}{2}\frac{dv_g}{dk}\frac{\partial^2 A}{\partial x_1^2}\\ &=\frac{2P_1}{\partial D/\partial\omega}\left\{\frac{1}{v_g^2-1}+\frac{P_2}{D_2}\right\}|A|^2A+\frac{2P_1}{\partial D/\partial\omega}\beta A. \end{aligned} \tag{48}$$

It is easily seen that Eq.(48) can be transformed into the following nonlinear Schrödinger equation

$$i\frac{\partial A}{\partial\tau}+\frac{1}{2}\frac{d^2\omega}{dk^2}\frac{\partial^2 A}{d\xi^2}=-\frac{k^2}{\partial D/\partial\omega}\left\{\frac{1}{v_g^2-1}+\frac{1}{6\omega^2}\right\}|A|^2A-\frac{k^2}{\partial D/\partial\omega}\beta A \tag{49}$$

provided that we introduce the co-ordinate transformations defined as

$$\begin{aligned} &\xi=\frac{1}{\varepsilon}(x_2-v_gt_2)=x_1-v_gt_1=\varepsilon(x-v_gt),\\ &\tau=t_2=\varepsilon t_1=\varepsilon^2 t \end{aligned} \tag{50}$$

which are nothing but the co-ordinate transformations introduced in the reductive perturbation method.

In this analysis, if we assume that the dependent variable u has an asymptotic representation of the following form

$$u=\sum_{m=1}^{N}\delta^m u^{(m)}(x_0,\cdots,x_N,t_0,\cdots,t_N)+O(\delta^{N+1}) \tag{51}$$

instead of Eq.(34), where δ is another smallness parameter which measures the weakness of nonlinearity of the wave, there may occur several cases depending on the relative importance of the parameters δ and ε. We have already shown the case $\delta=\varepsilon$.

When $\delta=\varepsilon^2$, the system is dominated by linear envelopes. Proceeding in an analogous way as in the case $\delta=\varepsilon$, we obtain the same equation as Eq. (49) except that the nonlinear term is dropped.

In the case $\delta=\varepsilon^{1/2}$, nonlinear interaction is dominant and, therefore, the nonlinear term does appear in the first nonsecularity condition. The slow change part of $u^{(2)}$ can be determined by the same equation as Eq. (46) derived from higher order equations of the perturbation. Thus we have

$$i\left(\frac{\partial A}{\partial t_1}+v_g\frac{\partial A}{\partial x_1}\right)=-\frac{k^2}{\partial D/\partial\omega}\left\{\frac{1}{v_g^2-1}+\frac{1}{6\omega^2}\right\}|A|^2A-\frac{k^2}{\partial D/\partial\omega}\beta A, \tag{52}$$

where it should be noted that the independent variables of this equation are the first order slow scales.

§ 5. Krylov-Bogoliubov-Mitropolsky method

We choose a monochromatic plane wave solution $u^{(1)}$ to the linearized equation of Eq. (14) as a starting solution:

$$u^{(1)}=Ae^{i\theta}+A^*e^{-i\theta}, \tag{53}$$

where A is complex amplitude, θ is phase factor defined as $\theta=kx-\omega t$, k and ω being respectively wave number and frequency. In order that u_1 should not be trivial, the following linear dispersion relation should be satisfied:

$$D(k,\omega)=-\omega^2+k^2-\omega^2k^2=0. \tag{54}$$

We now seek KBM perturbation solution of the following form

$$u=\varepsilon u^{(1)}(A,A^*,\theta)+\varepsilon^2u^{(2)}(A,A^*,\theta)+\cdots. \tag{55}$$

In the above expression, $u^{(1)}$, $u^{(2)}$, $\cdots$ depend on x and t only through A, A^* and θ where the complex amplitude A is assumed to be a slowly varying function of x and t through the following relation:

$$\frac{\partial A}{\partial t}=\varepsilon a_1(A,A^*)+\varepsilon^2a_2(A,A^*)+\cdots, \tag{56}$$

$$\frac{\partial A}{\partial x}=\varepsilon b_1(A,A^*)+\varepsilon^2b_2(A,A^*)+\cdots \tag{57}$$

together with the complex conjugate relations to Eqs. (56) and (57), while the phase θ remains unchanged from the linearized limit, i.e., $\theta=kx-\omega t$, because nonlinear effects on the phase may be taken into account through the 'phase part' of the complex amplitude. The unknown function $a_1, b_1, a_2, b_2, \cdots$ should be determined so as to make the solution (55) free from secular terms. Substituting Eq.(55) into Eq.(14) with the relation (56) and (57), and equating coefficients of like powers of ε, we obtain a set of equations to determine $u^{(m)}$ successively.

From the coefficients of ε^2, we have the following ordinary differential equation for u_2 with respect to θ:

$$k^2\omega^2\left(\frac{\partial^4 u^{(2)}}{\partial\theta^4}+\frac{\partial^2 u^{(2)}}{\partial\theta^2}\right)$$
$$=i\left(\frac{\partial D}{\partial\omega}a_1-\frac{\partial D}{\partial k}b_1\right)e^{i\theta}+2k^2A^2e^{2i\theta}+\text{c.c.} \tag{58}$$

The solution $u^{(2)}$ would contain secular terms, i.e., θ proportional terms unless the coefficients of $e^{i\theta}$ and of $e^{-i\theta}$ in Eq. (58) vanish. If the condition for nonsecularity

$$a_1+v_g b_1=0, \qquad v_g=-\frac{\partial D}{\partial k}\Big/\frac{\partial D}{\partial\omega}=\frac{d\omega}{dk} \tag{59}$$

and its complex conjugate relations are statisfied, we can obtain the secular free solution $u^{(2)}$ as follows:

$$u_2=\frac{1}{6\omega^2}(A^2e^{2i\theta}+A^{*2}e^{-2i\theta})+B(A,A^*)e^{i\theta}$$
$$+B^*(A,A^*)e^{-i\theta}+C(A,A^*), \tag{60}$$

where $B(A,A^*)$, $B^*(A,A^*)$ (assumed to be complex) and $C(A,A^*)$ (assumed to be real) are constants with respect to θ and should be determined as functions of A and A^* from nonsecular conditions in higher orders.

Montgomery-Tidman[16] and Tidman-Stainer[17] concluded from Eq.(59) that $a_1=0$ and $b_1=0$, which are sufficient but not necessary conditions to avoid the secular terms. This is the origin why they could not take account of the full amplitude modulation. By virtue of the relations (56) and (57), it turns out that a_1 and b_1 may be regarded, respectively, as $\partial A/\partial t_1$ and $\partial A/\partial x_1$ to the lowest order in ϵ, where $t_1=\varepsilon t$ and $x_1=\varepsilon x$. Thus Eq.(59) may be interpreted as

$$\frac{\partial A}{\partial t_1}+v_g\frac{\partial A}{\partial x_1}\simeq 0 \tag{61}$$

which indicates that the amplitude A is constant in a frame of reference moving with the group velocity.

Let us further proceed to the third order solution $u^{(3)}$.

$$k^2\omega^2\left(\frac{\partial^4 u^{(3)}}{\partial\theta^4}+\frac{\partial^2 u^{(3)}}{\partial\theta^2}\right)=\left[i\left(\frac{\partial D}{\partial\omega}a_2-\frac{\partial D}{\partial k}b_2\right)-\frac{1}{2}\left\{\frac{\partial^2 D}{\partial\omega^2}\left(a_1\frac{\partial a_1}{\partial A}+a_1^*\frac{\partial a_1}{\partial A^*}\right)\right.\right.$$
$$\left.-2\frac{\partial^2 D}{\partial\omega\partial k}\left(a_1\frac{\partial b_1}{\partial A}+a_1^*\frac{\partial b_1}{\partial A^*}\right)+\frac{\partial^2 D}{\partial k^2}\left(b_1\frac{\partial b_1}{\partial A}+b_1^*\frac{\partial b_1}{\partial A^*}\right)\right\}$$
$$\left.+\frac{k^2}{6\omega^2}A^2A^*+Ck^2A\right]e^{i\theta}+\text{c.c.}+\text{terms in }(e^{\pm 2i\theta},\ e^{\pm 3i\theta}). \qquad (62)$$

For the solution $u^{(3)}$ to be secular-free, we must set the coefficients of $e^{\pm i\theta}$ in Eq. (62) equal to zero, giving rise to

$$i(a_2+v_g b_2)+\frac{1}{2}\frac{dv_g}{dk}\left(b_1\frac{\partial b_1}{\partial A}+b_1^*\frac{\partial b_1}{\partial A^*}\right)$$
$$=-\frac{k^2}{\partial D/\partial\omega}\left(\frac{1}{6\omega^2}|A|^2A+CA\right) \qquad (63)$$

together with its complex conjugate relation, where the first order relation (59) and the relation

$$\frac{dv_g}{dk}=-\left(\frac{\partial^2 D}{\partial k^2}+2v_g\frac{\partial^2 D}{\partial\omega\partial k}+v_g^2\frac{\partial^2 D}{\partial\omega^2}\right)\Big/\frac{\partial D}{\partial\omega}=-\frac{3\omega^5}{k^4} \qquad (64)$$

have been used.

It should be noted here that Eq.(63) does not contain the arbitrary constants B and B^* so that we need not determine them so far as Eq.(63) is concerned. But it does contain C. Therefore, in order to complete Eq.(63) for A, we must determine the functional form of $C(A,A^*)$ with respect to A and A^*. This can be done as follows. The secular producing terms so far appeared in obtaining $u^{(2)}$ and $u^{(3)}$ were only terms proportional to $e^{\pm i\theta}$, i.e., resonant terms. But it should be noted that, in general, constant terms, if they appear, with respect to θ are also secular producing and that secular terms resulting from constant terms grow faster than those due to resonant terms, because the former is proportional to θ^2. Therefore, in addition to resonant terms, we must also require non-secular condition for constant terms. Such a secular producing constant first appears in the fourth power in ε, therefore for $u^{(4)}$ to be secular-free, one must require at least

$$(v_g^2-1)\left\{b_1\frac{\partial}{\partial A}\left(b_1\frac{\partial C}{\partial A}\right)+b_1\frac{\partial}{\partial A}\left(b_1^*\frac{\partial C}{\partial A^*}\right)\right\}+\text{c.c.}$$
$$=\left(b_1\frac{\partial b_1}{\partial A}+b_1^*\frac{\partial b_1}{\partial A^*}\right)A^*+\text{c.c.}+2b_1b_1^*. \qquad (65)$$

Equation (65) can be satisfied if we choose the unknown constant C as follows:

$$C(A,\ A^*)=\frac{1}{v_g^2-1}AA^*+\beta, \qquad (66)$$

where β is an absolute constant not only with respect to θ but also to A and A^*.

Introducing Eq. (66) into Eq. (63), we have

$$i(a_2+v_g b_2)+\frac{1}{2}\frac{\partial v_g}{\partial k}\left(b_1\frac{\partial b_1}{\partial A}+b_1^*\frac{\partial b_1}{\partial A^*}\right)$$
$$=-\frac{k^2}{\partial D/\partial\omega}\left\{\left(\frac{1}{v_g^2-1}+\frac{1}{6\omega^2}\right)|A|^2A+\beta A\right\}. \qquad (67)$$

Since a_2,b_2 and $b_1(\partial b_1/\partial A)+b_1^*(\partial b_1/\partial A^*)$ can be interpreted, respectively, as $\partial A/\partial t-a_1/\varepsilon$, $\partial A/\partial x-b_1/\varepsilon$ and $\partial^2 A/\partial x^2$, where $t_2=\varepsilon^2 t$, $x_2=\varepsilon^2 x$ and $x_1=\varepsilon x$, and by virtue of Eq. (59), Eq. (67) implies that

$$i\left(\frac{\partial A}{\partial t_2}+v_g\frac{\partial A}{\partial x_2}\right)+\frac{1}{2}\frac{\partial v_g}{\partial k}\frac{\partial^2 A}{\partial x_1^2}=-\frac{k^2}{\partial D/\partial\omega}\left(\frac{1}{v_g^2-1}+\frac{1}{6\omega^2}\right)|A|^2A$$
$$-\frac{k^2}{\partial D/\partial\omega}\beta A. \qquad (68)$$

Equation (68) takes the same form as Eq. (48), and by introducing the wave frame moving with the group velocity, Eq. (68) is easily transformed into the nonlinear Schrödinger equation.

Here, we comment on the relation among the various methods of multiple scales from the procedural point of view. In the course of the reductive perturbation method, one must start by finding the Gardner-Morikawa transformation[21] which seems to be heuristic. On the contrary it seems that the derivative expansion method or the extended KBM method can suggest quite naturally coordinate transformations within the frameworks. However as is wellknown, the latters are straightfowardly applicable only to the problem of the wave propagations in the system in which the wave breaking does not occur. It should be emphasized that we must be awakened to the significance of the physical condsideration in the procedure of finding the Gardner-Morikawa transformation.

At the end we point out that Luke[22] developed an expansion procedure for slowly varying wavetrains which is a kind of the methods of the multiple scales, and showed that the first order results agree with those obtained by averaging technique. It is a distinctive feature of his method that full nonlinearity is retained and the scale of the nonuniformity introduce a small parameter.

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