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C_T for conformal higher spin fields from partition function on conically deformed sphere

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Abstract: We consider the one-parameter generalization S^4_q of 4-sphere with a conical singularity due to identification $\tau = \tau + 2\pi q$ in one isometric angle. We compute the value of the spectral zeta-function at zero $\widehat{\zeta}(q) = \zeta(0;q)$ that controls the coefficient of the logarithmic UV divergence of the one-loop partition function on S^4_q. While the value of the conformal anomaly a-coefficient is proportional to $\widehat{\zeta}(1)$, we argue that in general the second c $\sim C_T$ anomaly coefficient is related to a particular combination of the second and first derivatives of $\widehat{\zeta}(q)$ at $q = 1$. The universality of this relation for C_T is supported also by examples in 6 and 2 dimensions. We use it to compute the c-coefficient for conformal higher spins finding that it coincides with the "$r = -1$" value of the one-parameter Ansatz suggested in arXiv:1309.0785. Like the sums of a_s and c_s coefficients, the regularized sum of $\widehat{\zeta}_s(q)$ over the whole tower of conformal higher spins $s = 1, 2, \ldots$ is found to vanish, implying UV finiteness on S^4_q and thus also the vanishing of the associated Rényi entropy. Similar conclusions are found to apply to the standard 2-derivative massless higher spin tower. We also present an independent computation of the full set of conformal anomaly coefficients of the 6d Weyl graviton theory defined by a particular combination of the three 6d Weyl invariants that has a (2,0) supersymmetric extension.

Keywords: AdS-CFT Correspondence, Conformal Field Theory, Supergravity Models

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1 Introduction

In this paper we revisit the question about conformal anomaly c-coefficients for conformal higher spin (CHS) fields previously addressed in [1, 2]. Conformal higher spins in 4d [3–8] have higher derivative kinetic terms $h_s\partial^{2s}h_s$ (h_s are totally symmetric rank s tensors) and thus their generalization to a curved metric background is a non-trivial question (cf. [1, 2, 9, 10]). A curved space generalization is required in order to compute the corresponding conformal anomaly coefficients appearing in the (one-loop) effective action

$$\Gamma = -\log Z = -B_4 \log \Lambda_{\rm UV} \ + \ {\rm finite}\,, \tag{1.1}$$

$$B_4 = \int d^4x \sqrt{g}\; b_4(x), \qquad \bar{b}_4 = (4\pi)^2 b_4 = -{\rm a}\, R^*\, R^* + {\rm c}\, C^2\ . \tag{1.2}$$

Here $R^*\, R^*$ is $32\pi^2$ times the Euler density and C^2 is the square of the Weyl tensor. To compute the a_s-coefficient for spin s field it is enough to know the corresponding Weyl-covariant $\nabla^{2s} + \dots$ operator on a 4-sphere where it takes a simple factorized form of a product of s "partially-massless" 2nd order Laplacians [1, 9, 11, 12]. As a result, one finds [1, 13]

$$\mathrm{a}_s = \frac{1}{720}\,\nu\left(14\,\nu^2 + 3\nu\right), \qquad\qquad \nu \equiv s(s+1)\,, \tag{1.3}$$

where ν is the number of dynamical degrees of freedom of a spin s CHS field. Remarkably, the total a-anomaly defined as the finite part of the regularized sum $\sum_{s=1}^{\infty} e^{-\epsilon\, s}\mathrm{a}_s$ vanishes [1, 13].[1] The factorization of the Weyl-covariant CHS kinetic operators applies for any conformally flat background, e.g., on $S^1_q \times S^3$, where at large $q = 2\pi\beta$ one finds that the corresponding free energy proportional to the Casimir energy E_c on $\mathbb{R}\times S^3$ is given by [7][2]

$$-\log Z(S^1_q \times S^3)\Big|_{q\to\infty} = 2\pi q\, E_c + \dots\,, \qquad\qquad E_{c,s} = \frac{1}{720}\,\nu\left(18\nu^2 - 14\nu - 11\right)\,. \tag{1.4}$$

To determine the value of the c-coefficient in (1.2) (which is proportional to the coefficient C_T in the 2-point function of the flat-space stress tensor) one is to consider a more general non conformally flat background. Assuming that c_s has a similar cubic ν-polynomial structure as a_s in (1.3) and reproduce the known values for $s = 1$ [16] and $s = 2$ [3, 17], one is led to the following Ansatz [1]

$$\mathrm{c}_s = \frac{1}{1080}\nu\left[43\nu^2 - 59\nu + \ r\,(\nu-2)(\nu-6)\right]\,, \tag{1.5}$$

where r is a free parameter. If one further assumes that all $s > 2$ CHS kinetic operators factorize into 2nd-order Lichnerowitz-type operators also on a Ricci flat background (like it happens for the $s = 2$ Weyl graviton) one then finds the expression (1.5) with $r = \frac{1}{2}$ [1].

However, the assumption of factorization on a generic Ricci-flat background is expected to fail in general [9]. Moreover, when the Weyl tensor is non-zero, different spins appear to mix in the kinetic term [10] and the mixing terms lead to additional contributions to the total c-anomaly [2]. Also, the sum of c_s with $r = \frac{1}{2}$ in (1.5) regularized with $e^{-\epsilon\, s}$ does not vanish; it vanishes if instead one chooses $r = -1$ value [1].[3] The expression (1.5) with $r = -1$ was also shown to be selected by the consistency with the AdS/CFT-related correspondence between massless higher spin partition functions in (asymptotically) AdS_5 space and the conformal higher spin partition functions at the 4d boundary [18].

Below we will provide a strong independent evidence that the $r = -1$ value of (1.5)

$$\mathrm{c}_s = \frac{1}{360}\nu\left(14\nu^2 - 17\nu - 4\right)\,, \tag{1.6}$$

[1] This vanishing holds also in the more natural regularization $\sum_{s=1}^{\infty} e^{-\epsilon\,(s+\frac{1}{2})}\mathrm{a}_s$ consistent with AdS/CFT [14]. Besides, the regularized finite part of the total CHS partition function on S^4 is also trivial, i.e. [15], i.e. $\log Z_{\rm CHS} = \sum_{s=1}^{\infty}\log Z_s = 0$.

[2] One finds again that the finite part of $\sum_{s=1}^{\infty} e^{-\epsilon\,(s+\frac{1}{2})} E_{c,s}$ vanishes.

[3] In fact, the finite part of $\sum_{s=1}^{\infty} \mathrm{c}_s\, e^{-\epsilon\,(s+\frac{1}{2})}$ vanishes for any value of r. However, the special value $r = -1$ is still selected by demanding the consistency in the results for the total a and total c in the "minimal" case of a tower of even higher spins only where they should be opposite to the values for a complex 4d scalar for consistency with what happens for massless higher spins in AdS_5 (see footnote 9 in [14]).

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is indeed the correct value of the c-coefficient for the conformal higher spin fields. The main idea will be to extract the value of c_s from the CHS partition function on a 1-parameter deformation of the 4-sphere S^4_q which is an Einstein space with a conical singularity

$$ds_q^2 = d\theta^2 + \cos^2\theta\, d\tau^2 + \sin^2\theta\, d\Omega^2\,, \qquad d\Omega^2 = d\alpha_1^2 + \sin^2\alpha_1\, d\alpha_2^2\,, \tag{1.7}$$
$$\theta \in [0, \tfrac{\pi}{2}], \qquad \tau \in [0, 2\pi\, q], \qquad \alpha_1 \in [0, \pi],\ \ \alpha_2 \in [0, 2\pi]\ .$$

Here the deficit angle $2\pi(1-q)$ of the τ coordinate implies the presence of a conical singularity on a S^2 submanifold. For q=integer this is a multiple cover of a sphere while for $\gamma \equiv q^{-1}$=integer this may be interpreted as an orbifold $S^4/\mathbb{Z}_\gamma$.

The key observation is that since S^4_q is locally conformally flat (away from conical singularity), one may assume that the CHS kinetic operator defined on S^4_q still factorizes as it does on S^4, so that the expression for the partition function in terms of the contributions of determinants of 2nd order operators is then "inherited" from the S^4 case.[4] At the same time, having the Weyl tensor being non-zero at the singular subspace should allow one to extract the value of the c-coefficient from the q dependence of the B_4 coefficient of the UV divergence in (1.1).

Note that the expression for B_4 in (1.2) applies to regular geometries while in the presence of conical singularities there will be additional "surface" terms [19–21] with a non-trivial dependence on q entering effectively through boundary conditions. As a result, the coefficient B_4 in (1.2) will become a non-trivial function of q. The log UV divergent contribution of the determinant of a single 2nd order Laplace-type operator can be computed as the value of the corresponding spectral zeta function at zero, i.e.[5]

$$B_4(q) = \widehat{\zeta}(q)\,, \qquad \widehat{\zeta}(q) \equiv \zeta(0;q)\,, \qquad \zeta(z;q) = \sum_i \mathrm{d}_i \left[\lambda_i(q)\right]^{-z}\,, \tag{1.8}$$

where d_i are degeneracies of the eigenvalues λ_i. The general structure of $\widehat{\zeta}(q)$ for a bosonic field will be as follows

$$\widehat{\zeta}(q) = -\frac{\nu}{360\, q^3} + \frac{p_2}{q^2} + \frac{p_1}{q} + p_0 - 2E_c\, q\ , \tag{1.9}$$

where ν is the number of physical degrees of freedom (equal to 1 for a real ∂^2 scalar and $s(s+1)$ in the bosonic CHS field as in (1.3)) and E_c is the corresponding Casimir energy on S^3.

Assuming one is able to compute $\widehat{\zeta}(q)$ it remains to extract the conformal anomaly a- and c- coefficients from it. Since the $q=1$ case corresponds to the regular sphere S^4 when $B_4 = -2\mathrm{a}\chi(S^4) = -4\mathrm{a}$ we should have[6]

$$\mathrm{a} = -\frac{1}{4}\,\widehat{\zeta}(1)\ . \tag{1.10}$$

[4] Alternatively, one may define the corresponding heat kernel in terms of the one on S^4 using Sommerfeld-type "orbifold" or "sum over images" construction.

[5] Note that B_4 as coefficient of the logarithmic divergence receives contribution from all (zero and non-zero) modes on the Laplacian, so that $\widehat{\zeta}$ may need to be corrected by the contribution of the zero modes; here we formally assume that this correction is already taken into account.

[6] This relation is true for the final expression for $\widehat{\zeta}$ taking into account possible zero modes arising from decomposition of fields into transverse and longitudinal parts, see footnote 14.

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We shall propose that the expression for c in terms of $\widehat{\zeta}(q)$ should read

$$\mathrm{c} = -\frac{1}{4}\left[q\,\widehat{\zeta}(q)\right]''\Big|_{q=1} = -\frac{1}{4}\widehat{\zeta}''(1) - \frac{1}{2}\widehat{\zeta}'(1)\,, \tag{1.11}$$

where $\widehat{\zeta}'(q) \equiv \frac{d}{dq}\widehat{\zeta}(q)$. As we shall see, the same relation (k_d is dimension-dependent normalization factor)

$$C_T = k_d\left[q\,\widehat{\zeta}(q)\right]''\Big|_{q=1}\,, \tag{1.12}$$

is true also in $d = 2$ (C.4) and $d = 6$ (4.3) cases where c is replaced by the corresponding C_T coefficient (proportional to $\mathrm{a} = \frac{1}{3}c$ in 2d and c_3 in 6d). This suggests its universal validity.

The relation (1.11) can be directly verified in all low-spin ($s \leq 1$) cases. Since S^4_q is conformally related to $S^1_q \times \mathbb{H}^3$ space, in the low-spin cases where $\widehat{\zeta}'(1) = 0$, eq. (1.11) becomes equivalent the relation derived in [22]. As it turns out, for higher-spin $s \geq 2$ cases $\widehat{\zeta}'(1) \neq 0$ and thus it is the relation (1.11) that should be applied.[7]

Intuitively, the reason why c should be related to $\widehat{\zeta}''(1)$ can be understood from the fact that $\mathrm{c} \sim C_T$ should be proportional to the 2-point function of the stress tensor which itself should be given by the second variation of the effective action over the metric, i.e. the second term in the expansion in the small deformation of the metric (1.7) away from the sphere $q = 1$ case.

One may be tempted to represent $B_4(q)$ as in (1.2), i.e. as a curvature integral with the a and c as coefficients of different geometrical invariants. However, S^4_q has a singular curvature on S^2, and at best one may hope to get $B_4(q) = q\int_{S^4\setminus S^2} b_4^{\rm bulk} + \int_{S^2} b_4^{\rm surf.}(q)$, where $b_4^{\rm bulk}$ is as in (1.2) and is evaluated on a smooth metric, while $b_4^{\rm surf.}(q)$ non-trivially depends on q and invariants of S^2. Ref. [19] gave an explicit analysis of the conformal scalar operator on a singular manifold $\mathcal{M}_q = \mathscr{C}_q \times \Sigma$, where $\mathscr{C}_q$ is a flat cone with metric $ds^2 = dr^2 + r^2 d\varphi^2$, $0 \leq \varphi < 2\pi\, q$, while Σ does not depend on φ. The above splitting of $B_4(q)$ into "bulk" and "surface" parts can then be proved and also checked to be in agreement with the expression of $B_4(q)$ in terms of the spectral zeta-function [21] (see also appendix F below). It is important to stress that in general the surface term $b^{\rm surf.}(q)$ has coefficients depending on q in a non-universal way, i.e. its dependence on the spin of the field is only partly encoded in the values of the a and c coefficients. A major simplification occurs at first order in expansion in small $1-q$ and for low spin $s = 0, \frac{1}{2}, 1$. In these cases it is possible to use the integral density in (1.2) and take the singular manifold into account by a delta-function contribution to the curvature [20, 23] (see also [24–26]). However, this is not true at higher orders in $1-q$ (and even at leading order for bosons with spin $s \geq 2$ [27])

[7] One feature that distinguishes the cases with $s \leq 1$ from $s \geq 2$ conformal fields is that according to appendix D of [2] for $s \geq 2$ CHS field in flat space it is necessary to use the equations of motion to prove gauge invariance of the improved symmetric traceless stress tensor. This may be related to a non zero value of the one-point function $\langle T_{\mu\nu}\rangle$ on S^4_q or to $\widehat{\zeta}'(1) \neq 0$, suggesting a modification of the argument in [22] for $s \geq 2$.

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and this seems to prevent one from obtaining the general expression for the c-coefficient in this approach in a straightforward way.[8]

The rest of this paper is organized as follows. In section 2 we shall find $\widehat{\zeta}(q)$ (1.8) for the Laplace-type spin s operators on S_q^4 that enter the partition function of CHS fields. We shall first explicitly determine the eigenvalues and their degeneracies for $s = 0, 1, 2, 3, 4$ cases (with generalization to $s \geq 5$ discussed in appendix B). We shall then compute $\widehat{\zeta}(q)$ following the method discussed in appendix A. The total expressions for the $s = 1, 2, 3, 4, 5$ CHS fields are presented in (2.24)–(2.27), (B.6). In subsection 2.3 we shall discuss the general structure (1.9) of $\widehat{\zeta}(q)$ relating it to the free energy on $S_q^1 \times \mathbb{H}^3$.

In section 3 we shall use $\widehat{\zeta}(q)$ to determine the conformal anomaly coefficients corresponding to the CHS fields. We shall discuss the relations (1.10) and (1.11) and comment on similar relations following from Rényi entropy. We shall also determine the general expression for $\widehat{\zeta}_s(q)$ for any value of CHS spin s satisfying non-trivial consistency conditions. Like for the sums of a_s and c_s coefficients the regularized sum of $\widehat{\zeta}_s(q)$ over the whole tower of conformal higher spins is found to vanish, implying that the full CHS theory is one-loop UV finite on S_q^4 space and thus implying as well the vanishing of the total Rényi entropy. Similar conclusions apply to the standard 2-derivative massless higher spin tower (see appendix E).

In section 4 we shall discuss generalization to 6 dimensions. We shall compute the corresponding $\widehat{\zeta}(q)$ for low-spin $s = 0, 1, 2$ CHS fields on S_q^6 space and show that the expected relation (4.3) for the $\mathrm{c}_3 \sim C_T$ conformal anomaly coefficient is fully consistent not only with the previously known 2-derivative scalar and 4-derivative vector results but also with the new result for the 6d Weyl graviton conformal anomaly computed independently from the Seeley-DeWitt coefficient in appendix D.

The universality of the relation for C_T (1.12) is further supported by the discussion of the $d = 2$ case in appendix C. In appendix F we shall comment on the relation between the expressions for $\widehat{\zeta}(q)$ for spin $s = 0, 1, 2, 3$ Laplacians and some previous results [27] for the B_2 Seeley-DeWitt coefficient found in the "geometrical" approach.

2 Zeta-function of generalized spin s Laplacian on S_q^4

To compute the function $\widehat{\zeta}(q)$ in (1.8) for conformal higher spin fields on S_q^4 and thus the corresponding a and c anomaly coefficients using (1.10) and (1.11) , our starting point will be the CHS partition function on S^4 [1]. It is expressed in terms of the determinants of generalized Laplace (or Lichnerowitz-type) operators on unit-radius S^4 defined on a totally symmetric transverse traceless (TT) rank s tensor

$$\Delta_{s\,\perp}(M^2) \equiv (-D^2 + M^2)_{s\,\perp}\,, \tag{2.1}$$

where M^2 is a constant parameter that need not be positive (the scalar curvature is $R = 12$). For example, the one-loop S^4 partition functions of the standard 2-derivative massless

[8]For completeness, let us mention that if the expression of $\widehat{\zeta}(q)$ for a certain field were available as a function of the space dimension, then (1.11) could be cross checked against the representation of the surface contribution as a linear combination of specific conformal invariants on Σ, see for instance [28].

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conformally coupled scalar,[9] $s = 1$ Maxwell vector, $s = 2$ Weyl graviton and $s = 3$ and $s = 4$ CHS fields read

$$Z_0 = \left[\frac{1}{\det \Delta_0(2)}\right]^{1/2}, \qquad Z_1 = \left[\frac{\det \Delta_0(0)}{\det \Delta_{1\perp}(3)}\right]^{1/2}, \tag{2.2}$$

$$Z_2 = \left[\frac{\det \Delta_{1\perp}(-3)\ \det \Delta_0(-4)}{\det \Delta_{2\perp}(4)\ \det \Delta_{2\perp}(2)}\right]^{1/2}, \tag{2.3}$$

$$Z_3 = \left[\frac{\det \Delta_{2\perp}(-8) \det \Delta_{1\perp}(-9)\ \det \Delta_0(-10)}{\det \Delta_{3\perp}(5)\ \det \Delta_{3\perp}(3)\ \det \Delta_{3\perp}(-1)}\right]^{1/2}, \tag{2.4}$$

$$Z_4 = \left[\frac{\det \Delta_{3\perp}(-15) \det \Delta_{2\perp}(-16) \det \Delta_{1\perp}(-17)\ \det \Delta_0(-18)}{\det \Delta_{4\perp}(6)\ \det \Delta_{4\perp}(4)\ \det \Delta_{4\perp}(0)\ \det \Delta_{4\perp}(-6)}\right]^{1/2}. \tag{2.5}$$

We will assume that these partition functions extended to S^4_q have the same product structure with each operator now defined on S^4_q. Thus the problem of computing them reduces to finding the dependence of the spectrum of the operator (2.1) on the conical deformation parameter q.

We will closely follow the approach of [28] where the scalar and vector operators were discussed, generalizing it to the $s > 1$ case. We will first assume that $q = 1/\gamma \leq 1$ where for an integer γ the space S^4_q becomes the γ-quotient of S^4. For $q = \gamma = 1$ the spectrum should reduce to the regular S^4 one found in [29].

As explained later, for $\gamma > 1$ the spectrum will in general be different in the intervals $\gamma \in [n, n+1)$. Starting from a certain n, depending on the spin s, the structure of the spectrum will be independent of γ. The relevant range for us will be $\gamma \in [1, 2)$ since to find the conformal anomaly coefficients in (1.10), (1.11) we will interested in the expansion near $q \to 1$.[10]

The eigenvalues $\lambda_{n,m}$ of $\Delta_{s\perp}(M^2)$ in (2.1) will be parametrized by the two integers $n, m \geq 0$ as

$$\lambda_{n,m}(\gamma) = (n + \gamma\, m)(n + \gamma\, m + 3) - s + M^2\ . \tag{2.6}$$

The degeneracies $\mathrm{d}^{(s)}_{n,m}$ may be found using the correspondence between $\Delta_{s\perp}(M^2)$ and the Laplacian on the ambient flat space with coordinates $(x^1, x^2, x^3; x^4, x^5)$ and the constraint $|x|^2 \equiv x^a x^a = 1$ with the conical singularity implemented by the identification $x^a(\tau) = x^a(\tau + 2\pi q)$ where τ is the coordinate in (1.7) (see [28]).

The explicit spectrum for $\gamma \neq 1$ can be constructed by starting with a suitable Ansatz for the eigenstates consistent with periodicity on $S^4_{1/\gamma}$ generalizing to $s > 1$ the discussion of the scalar and vector cases in [28].

[9]The $s = 0$ member of CHS tower in $d = 4$ is non-dynamical, i.e. $Z_0 = 1$, but it is useful to consider also the 2-derivative conformally coupled scalar to be able to compare with previous results on S^4_q. In what follows the $s = 0$ case will stand for the ∂^2 scalar field.

[10]If $\gamma < 1$, there may be a finite number of normalizable eigenmodes that are, however, singular on some subspace. This has been noticed to happen already in the scalar case [28]. If only regular eigenmodes are considered, then our results extend to a neighbourhood of $\gamma = q^{-1} = 1$.

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2.1 Eigenvectors and degeneracies

In general, the eigenvectors of the Laplacian on the flat ambient space $(x^1, \ldots, x^5)$ will be tensors $(\Phi^{a_1\ldots a_s})_{n,m}$ corresponding to the eigenvalues (cf. (2.6))

$$\hat{\lambda}_{n,m}(\gamma) = (n+\gamma\, m)(n+\gamma\, m+3)\ . \tag{2.7}$$

They must be symmetric, traceless, and also tangential and transverse in the ambient space

$$x^a\, \Phi_a{}^{i_2\ldots i_s} = 0, \qquad\qquad \partial_{x^a}\, \Phi_a{}^{i_2\ldots i_s} = 0. \tag{2.8}$$

Here we split the coordinate indices as $a = (i, +, -)$ where $i = 1, 2, 3$ and $x^\pm = \frac{1}{\sqrt{2}}(x^5 \pm i\, x^4)$. The ansatz for the tensor components with all indices from the 3-subspace reads[11]

$$(\Phi^{i_1\ldots i_s})_{n,m} = |x|^{-n+\gamma\, m}\,(x^+)^{\gamma\, m} \sum_{p\geq 0} (B^{i_1\ldots i_s})^m_{i_1\ldots i_{s-2p}}\, x^{i_1}\cdots x^{i_{s-2p}}\,(x^+x^-)^p\ . \tag{2.9}$$

The sum over p involves a finite number of terms, i.e. monomials of total degree s with some explicit power of x^+x^-. In the case of one $\pm$ index we have instead

$$(\Phi^{i_1\ldots i_{s-1}\pm})_{n,m} = |x|^{-n+\gamma\, m}\,(x^+)^{\gamma\, m\pm 1} \sum_{p\geq 0} (B^{i_1\ldots i_{s-1}\pm})^m_{i_1\ldots i_{s-1+2p}}\, x^{i_1}\cdots x^{i_{s-1+2p}}\,(x^+x^-)^p, \tag{2.10}$$

and similar expression is assumed when there are more indices of the $+$ or $-$ type. The regularity of the eigentensor components with one or more "$-$" indices, i.e. the absence of negative powers of x^+, requires the sum over p to start at some positive value depending on the value of γm. This is the unique source of the γ-dependence of the spectrum. In practice, this is a feature that starts being relevant for $s \geq 2$.

All eigenvectors of the form (2.9), (2.10) or with more $\pm$ indices, appear together with a mirror copy where $x^+ \leftrightarrow x^-$ when $m > 0$. The solutions with $m = 0$ are automatically symmetric under this exchange. By an explicit enumeration, we can then determine the degeneracies $\mathrm{d}^{(s)}_{n,m}$. For the $s = 0$ and 1 cases we reproduce the results of [28] for S^d_q with any d:

scalar: $n + m \geq 0$

$$\begin{aligned}
\mathrm{d}^{(0)}_{n,m>0} &= 2\binom{n+d-2}{d-2} \overset{d=4}{\rightarrow} (n+1)(n+2),\\
\mathrm{d}^{(0)}_{n,0} &= \binom{n+d-2}{d-2} \overset{d=4}{\rightarrow} \frac{1}{2}\,(n+1)(n+2).
\end{aligned} \tag{2.11}$$

spin 1: $n + m \geq 1$

$$\begin{aligned}
\mathrm{d}^{(1)}_{n,m>0} &= 2\,(d-1)\binom{n+d-2}{d-2} \overset{d=4}{\rightarrow} 3\,(n+1)(n+2),\\
\mathrm{d}^{(1)}_{n,0} &= \frac{1}{n+1}\binom{n+d-3}{d-2}\left[d^2+(n-4)d+5-n\right] \overset{d=4}{\rightarrow} \frac{1}{2}\,n\,(3n+5)\ .
\end{aligned} \tag{2.12}$$

[11] The constraint $|x| \equiv \sqrt{x^a x^a} = 1$ is imposed after taking the derivatives when imposing the transversality condition and applying the Laplacian [28].

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For $s=2,3,4$ and $\gamma\in[1,2)$ we find the following results for the degeneracies in $d=4$:[12]

spin 2: $n+m\geq 2$

$$\mathrm{d}^{(2)}_{n,0}=\frac{1}{2}\,(n-1)\,(5n+8),\qquad \mathrm{d}^{(2)}_{n,1}=n(5n+11),\qquad \mathrm{d}^{(2)}_{n,m>1}=5\,(n+1)(n+2). \tag{2.13}$$

spin 3: $n+m\geq 3$

$$\begin{aligned}\mathrm{d}^{(3)}_{n,0}&=\frac{1}{2}\,(n-2)\,(7n+11), & \mathrm{d}^{(3)}_{n,1}&=(n-1)(7n+16),\\ \mathrm{d}^{(3)}_{n,2}&=n(7n+17), & \mathrm{d}^{(3)}_{n,m>2}&=7\,(n+1)(n+2).\end{aligned} \tag{2.14}$$

spin 4: $n+m\geq 4$

$$\begin{aligned}\mathrm{d}^{(4)}_{n,0}&=\frac{1}{2}\,(n-3)\,(9\,n+14), & \mathrm{d}^{(4)}_{n,1}&=3\,(n-2)\,(3\,n+7),\\ \mathrm{d}^{(4)}_{n,2}&=3\,(n-1)\,(3\,n+8), & \mathrm{d}^{(4)}_{n,3}&=n\,(9\,n+23),\\ \mathrm{d}^{(4)}_{n,m>3}&=9\,(n+1)\,(n+2).\end{aligned} \tag{2.15}$$

We suggest a generalization of these expressions for degeneracies to any integer $s>4$ in appendix B.

2.2 Computation of $\widehat{\zeta}(q)$

To find the spectral zeta-function and thus $\widehat{\zeta}(q)$ it remains to perform the sum in (1.8). Representing the eigenvalue (2.6) for particular s and M^2 as

$$\lambda_{n,m}=(n+\gamma\,m)(n+\gamma\,m+3)-s+M^2=(n+\gamma\,m+\mu)(n+\gamma\,m+\mu')\,, \tag{2.16}$$

we thus need to compute

$$\sum_{n\geq 0}\sum_{m\geq 0}\mathrm{d}^{(s)}_{n,m}\,\big[(n+\gamma\,m+\mu)(n+\gamma\,m+\mu')\big]^{-z}. \tag{2.17}$$

One possible approach is to follow [30] and expand $[\lambda_{n,m}]^{-z}$ in powers of the n,m independent term $-\frac{1}{4}(\mu-\mu')^2$.[13] Doing the sum over n, one can then reduce the expression for the spectral ζ function of $\Delta_{\perp\,s}$ on $S^4_{1/\gamma}$ to a sum of terms with coefficients being the Hurwitz zeta functions $\zeta_{\mathrm{R}}(a,b)$. Using the integral representation for $\zeta_{\mathrm{R}}(a,b)$

$$\zeta_{\mathrm{R}}(a,b)=\frac{1}{\Gamma(a)}\int_0^\infty dy\;\frac{y^{a-1}\,e^{-b\,y}}{1-e^{-y}}, \tag{2.18}$$

one can then do the sum over m in the integrand, expand in γ, integrate term by term in y, and finally send $z\to 0$ to obtain $\widehat{\zeta}(q)$ in (1.8) with $q=1/\gamma$. An alternative more

[12] It is useful to check the correspondence with the known results in the regular S^4 limit of $\gamma=1$. For general spin s and $\gamma=1$, setting $N=n+m$ we have in 4d: $\mathrm{d}^{(s)}_N=\frac{1}{6}\,(2s+1)\,(2N+3)(N+s+2)(N-s+1)$. We have checked that indeed in all cases $\sum_{s\leq n+m\leq N}\mathrm{d}^{(s)}_{n,m}=\mathrm{d}^{(s)}_N$.

[13] Only a finite number of terms will give a non-zero contribution in the limit $z\to 0$ so that the final result is stable for a sufficiently long expansion.

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straightforward approach (described in appendix A in the $s = 0$ case) is to use the heat kernel representation taking into account the factorized form of the eigenvalues in (2.17). Applying this procedure, the general expressions for $\widehat{\zeta}_{s\,\perp}(q; M^2)$ corresponding to the operator (2.1) defined on S^4_q with $s = 0, 1, 2, 3, 4$ are then found to be

$$
\begin{aligned}
\widehat{\zeta}_0(q;M^2) &= -\frac{1}{360\,q^3} + \frac{2-M^2}{12\,q} + \frac{1}{120}\,(19-30\,M^2+10\,M^4)\,q,\\
\widehat{\zeta}_{1\,\perp}(q;M^2) &= -\frac{1}{120\,q^3} + \frac{3-M^2}{4\,q} + M^2 - \frac{7}{3} + \frac{1}{40}\,(59-50\,M^2+10\,M^4)\,q,\\
\widehat{\zeta}_{2\,\perp}(q;M^2) &= -\frac{1}{72\,q^3} - \frac{2}{q^2} + \frac{68-5\,M^2}{12\,q} + 5\,M^2 - \frac{26}{3} + \frac{1}{24}\,(119-70\,M^2+10\,M^4)\,q,\\
\widehat{\zeta}_{3\,\perp}(q;M^2) &= -\frac{7}{360\,q^3} - \frac{14}{q^2} - \frac{7\,(M^2-53)}{12\,q} + 14\,M^2 - \frac{14}{3} + \frac{7}{120}(199-90\,M^2+10\,M^4)\,q,\\
\widehat{\zeta}_{4\,\perp}(q;M^2) &= -\frac{1}{40\,q^3} - \frac{54}{q^2} - \frac{3\,(M^2-150)}{4\,q} + 30\,M^2 + 56 + \frac{3}{40}\,(299-110\,M^2+10\,M^4)\,q.
\end{aligned}
\tag{2.19}
$$

To obtain the total values of $\widehat{\zeta}(q)$ for CHS fields it remains to sum up the contributions from different factors in the partition functions (2.2)–(2.5). When combining $\widehat{\zeta}(q)$ (2.19) for the operators $\Delta_{s\,\perp}$ one needs to account for the contribution of the number n_z of the artificial zero modes introduced by the splitting of the fields into transverse parts, i.e. the corrected expression is[14]

$$
\widehat{\zeta}(q) = \sum_{s'} \widehat{\zeta}_{s'\,\perp}(q) - n_{\rm z}\ . \tag{2.20}
$$

The number of zero modes associated to a TT spin s tensor is equal to the number of rank $s-1$ conformal Killing tensors on S^4 [33][15]

$$
{\rm k}_s = \frac{1}{12} s^2 (s+1)^2 (2s+1)\ . \tag{2.21}
$$

Explicitly, for the partition functions in (2.2)–(2.5) we get

$$
\begin{aligned}
&s=0: \quad n_{\rm z} = 0, && s=1: \quad n_{\rm z} = {\rm k}_1 = 1,\\
&s=2: \quad n_{\rm z} = 2{\rm k}_2 - {\rm k}_1 = 29, && s=3: \quad n_{\rm z} = 3{\rm k}_3 - {\rm k}_2 - {\rm k}_1 = 236,\\
&s=4: \quad n_{\rm z} = 4{\rm k}_4 - {\rm k}_3 - {\rm k}_2 - {\rm k}_1 = 1100.
\end{aligned}
\tag{2.22}
$$

[14] As the original action and thus the partition function is expressed in terms of unconstrained fields one has to remove spurious zero modes related to splitting the rank s tensor into its transverse plus longitudinal parts (see [31, 32] for the case of $s \le 2$ in $d = 4$). This splitting introduces additional n_z zero modes of the Jacobian of the change of variables. These modes were not present in the original unconstrained operator and their number must be subtracted from $\widehat{\zeta}_\perp$ leading to $B_4 = \widehat{\zeta} = \widehat{\zeta}_\perp - n_z$.

[15] In general d, this is the dimension of the $(s-1, s-1, 0, \ldots, 0)$ representation of SO$(d+1, 1)$.

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As a result, we find the following expressions for $\widehat{\zeta}(q)$ for the ∂^2 ($M^2 = 2$) and ∂^4 ($M^2 = 0, 2$) conformal scalars and $s = 1, 2, 3, 4$ CHS fields

$$\widehat{\zeta}_\varphi(q) = -\frac{1}{360\,q^3} - \frac{1}{120}q\,, \qquad \widehat{\zeta}_{\varphi^{(4)}}(q) = -\frac{1}{180\,q^3} + \frac{1}{6\,q} + \frac{3}{20}\,q, \tag{2.23}$$

$$\widehat{\zeta}_1(q) = -\frac{1}{180\,q^3} - \frac{1}{6\,q} - \frac{1}{3} - \frac{11}{60}\,q\,, \tag{2.24}$$

$$\widehat{\zeta}_2(q) = -\frac{1}{60\,q^3} - \frac{4}{q^2} + \frac{41}{6\,q} - 11 - \frac{553}{60}\,q\,, \tag{2.25}$$

$$\widehat{\zeta}_3(q) = -\frac{1}{30\,q^3} - \frac{40}{q^2} + \frac{227}{3\,q} - 92 - \frac{2413}{30}\,q\,, \tag{2.26}$$

$$\widehat{\zeta}_4(q) = -\frac{1}{18\,q^3} - \frac{200}{q^2} + \frac{1165}{3\,q} - \frac{1300}{3} - \frac{2303}{6}\,q\,. \tag{2.27}$$

The standard scalar and spin 1 cases were discussed in [28, 34]. The 4-derivative scalar expression was found in [35]. The vector expression in (2.24) agrees with the result of [36].

Similar analysis can be repeated in the case of the fermionic CHS fields with kinetic terms ∂^{2s} with $s = \frac{1}{2}, \frac{3}{2}, \ldots$ [1]. For $s = \frac{1}{2}$ fermion with the standard ∂-action or conformal ∂^3-action we find, using the results of [35][16]

$$\widehat{\zeta}_\psi(q) = -\frac{7}{1440\,q^3} - \frac{1}{48\,q} - \frac{17}{480}q, \qquad \widehat{\zeta}_{\psi^{(3)}}(q) = -\frac{7}{480\,q^3} + \frac{5}{48\,q} + \frac{29}{480}q\,. \tag{2.28}$$

While in this paper we are interested in fields defined on S^d_q with even d, let us note that in the case of odd d one may expect the coefficient of the log UV divergence in (1.1) to vanish (as the space has no boundary all log UV divergences should be bulk ones and thus should be built out of curvature invariants). Thus in odd d one may expect to find that $\widehat{\zeta}(q) = 0$. Indeed, one can check that this is the case for a scalar or spin 1 field using (2.11) and (2.12) (for any M^2 parameter in the operator and after subtracting as in (2.20) the constant $n_z = 1$ in the $s = 1$ case). However, in the $s = 2$, $d = 3$ case, i.e. for the operator $\Delta_{2\perp}(M^2)$ defined on S^3_q one finds $\widehat{\zeta}(q) = 4\,q^{-1} + 6$. Subtracting $n_z = 10$ gives $\widehat{\zeta}(q) = 4\,q^{-1} - 4$. This vanishes as expected for $q = 1$, i.e. for a round 3-sphere, but is non-zero in general. The same is then expected to happen also for $s > 2$ and requires an explanation.

2.3 General structure of $\widehat{\zeta}(q)$

The leading small q and large q asymptotics of the $\widehat{\zeta}(q)$-functions on S^4_q in (2.23)–(2.28) have the universal structure (1.9), i.e.

$$\widehat{\zeta}(q) = -\frac{\nu}{360\,q^3} + \ldots + (-2E_c)\,q\,, \tag{2.29}$$

where ν is the number of dynamical degrees of freedom in bosonic case (rescaled by $\frac{7}{8}$ in the fermionic case) and E_c is the Casimir energy on $\mathbb{R} \times S^3$. Indeed, the metric (1.7) effectively simplifies in these limits: for $q \to 0$ the τ-direction shrinks to zero (or the transverse 3-space

[16] One is to set $k = \frac{1}{2}$ and $\frac{3}{2}$ in eqs. (18), (19) in [35]. Note that q in [35] corresponds to our $\gamma = q^{-1}$.

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blows up) so we get effectively $S^1_q \times \mathbb{R}^3$, while for $q \to \infty$ the τ-direction decompactifies and the space becomes similar to $\mathbb{R} \times S^3$. This suggests that $\log Z$ should be related to the free energy on $S^1_q \times \mathbb{R}^3$ for $q \to 0$ and on $\mathbb{R} \times S^3$ for $q \to \infty$.[17]

To make this connection more explicit we may use that the metric of S^d_q is related by a conformal rescaling (by $\cos^2\theta$ in (1.7)) to the metric of $S^1_q \times \mathbb{H}^{d-1}$ where $\mathbb{H}^{d-1}$ is a real hyperbolic space of unit curvature radius.[18] The effective actions on S^d_q and on $S^1_q \times \mathbb{H}^{d-1}$ are then related by a finite integrated conformal anomaly term. This allows one to relate $\widehat{\zeta}(q)$ on S^d_q to the thermal free energy of a CFT on $S^1_q \times \mathbb{H}^{d-1}$ where the length of the thermal circle is $\beta = 2\pi q$.

In the case of the homogeneous space $S^1_q \times \mathbb{H}^3$ the free energy $F(q)$ is proportional to its volume $2\pi q \, \mathrm{Vol}(\mathbb{H}^3)$. Extracting the IR divergent factor in the volume, we may define the IR finite "free energy" $\mathcal{F}(q)$ by

$$F(q) \equiv \mathcal{F}(q) \, \log \Lambda_{\rm IR} \ . \tag{2.30}$$

Recalling that $\widehat{\zeta}(q)$ is the coefficient of the log of the UV cutoff (cf. (1.1), (1.8)), restoring the dependence on the curvature radius r and comparing the coefficients of $\log \mathrm{r}$ suggests a direct relation between $\widehat{\zeta}(q)$ and $\mathcal{F}(q)$, or explicitly $\widehat{\zeta}(q) = -\mathcal{F}(q)$. For $q \to \infty$ the free energy of $S^1_q \times \mathbb{H}^3$ should approach the one on $\mathbb{R} \times \mathbb{H}^3$.[19] Since $\mathbb{H}^3$ is related by the analytic continuation to S^3, that implies that $\mathcal{F}(q \gg 1) \to 2E_c q$ where E_c is the Casimir energy on S^3.[20]

In general, $F(q)$ computed on $S^1_q \times S^3$ or $S^1_q \times \mathbb{H}^3$ contains a non-universal (UV power-divergent) part proportional to the volume and thus linear in q and a universal finite part. One may define $F(q)$ in a particular scheme where all non-universal power UV divergences are subtracted out and the linear in q part is the Casimir energy, i.e.

$$\widehat{\zeta}(q) = -\mathcal{F}(q) = -\overline{\mathcal{F}}(q) - 2\, E_c \, q \,, \tag{2.31}$$

where $\overline{\mathcal{F}}(q)$ contains only non-positive powers of q. The function $\overline{\mathcal{F}}(q)$ was computed for free conformal fields with spins $s \leq 1$, including higher derivative cases, in [38, 39].[21]

In general, the definition of E_c on S^3 is scheme-dependent — it depends on the definition of the stress tensor or the coefficient g of the total derivative $D^2 R$ term in $\langle T^m_m \rangle$, i.e. $E_c = \frac{3}{4}\left(\mathrm{a} + \frac{1}{2} g\right)$ [40]. A natural scheme is the one when E_c is determined from the single particle partition function of the corresponding CFT using the standard zeta-function definition (see, e.g., [7, 18]). It is this E_c that appears as the q-coefficient in $\widehat{\zeta}(q)$ (1.9), (2.29) ($\widehat{\zeta}(q)$ itself is scheme-independent being the coefficient of log UV divergence on S^4_q).

---

[17] Let us mention in this connection a discussion [37] of an interesting duality between the $q \to 0$ and $q \to \infty$ limits of the partition function on $S^1_q \times S^3/\mathbb{Z}_n$.

[18] This conformally mapping has an important role in the discussions of Rényi entropy, see, e.g., [38].

[19] Note that in a conformal theory the partition function depends on the ratio of the scales of S^1 and $\mathbb{H}^3$.

[20] The proportionality coefficient can be understood as follows [35]: as $\mathrm{Vol}(\mathbb{H}^3) = -2\pi \log \Lambda_{\rm IR}$ and $\mathrm{Vol}(S^3) = 2\pi^2$, there is a relative $-\frac{1}{\pi}$ factor that free energy on $S^1_q \times S^3$ in (1.4) should be multiplied by.

[21] The relation (2.31) is valid also for a generic GJMS conformal higher derivative scalars [35]. Note that $\overline{\mathcal{F}}(q)$ was not so far computed directly on $S^1 \times \mathbb{H}^3$ for $s \geq 2$: it is non trivial to extend the analysis of [39] to spins higher than 1 due to several ambiguities discussed there.

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One can indeed check that the order q coefficients in (2.23)–(2.28) are the corresponding values of E_c for the conformal ∂^2 and ∂^4 scalars, ∂ and ∂^3 fermions and $s = 1, 2, 3, 4$ CHS fields summarized below (see also (1.4))

	φ	$\varphi^{(4)}$	ψ	$\psi^{(3)}$	CHS$_1$	CHS$_2$	CHS$_3$	CHS$_4$
E_c	$\frac{1}{240}$	$-\frac{3}{40}$	$\frac{17}{960}$	$-\frac{29}{960}$	$\frac{11}{120}$	$\frac{553}{120}$	$\frac{2413}{60}$	$\frac{2303}{12}$

(2.32)

In the opposite $q \to 0$ limit the free energy on $S^1_q \times \mathbb{H}^3$ should approach the one on $S^1_q \times \mathbb{R}^3$, i.e. should have the same $q \to 0$ asymptotics as the thermal free energy on $S^1_q \times S^3$ (see, e.g., [41]). Thus it should simply be proportional to the free energy of a single scalar or single fermion times the number of degrees of freedom. This pattern is indeed directly seen in (2.23)–(2.28). Note that this relation implies that in d dimensions the maximal power of q^{-1} in $\widehat{\zeta}(q)$ in (2.29) should be $d-1$.

3 Conformal anomaly coefficients from $\widehat{\zeta}(q)$

Having found $\widehat{\zeta}(q)$ for a CFT on S^4_q one should be able to extract the information about the corresponding conformal anomaly coefficients a and c in (1.2). The a-coefficient is the one appearing in the log divergent part of the partition function on S^4. It is thus simply proportional to $\widehat{\zeta}(1)$ as in (1.9),

$$\mathrm{a} = -\frac{1}{4}\,\widehat{\zeta}(1)\ . \tag{3.1}$$

Starting with (2.23)–(2.28) we indeed match the known values of the a-coefficient for the ∂^2 and ∂^4 conformal scalars, ∂ and ∂^3 fermions and spin $s = 1, 2, 3, 4$ CHS fields from (1.3)

	φ	$\varphi^{(4)}$	ψ	$\psi^{(3)}$	CHS$_1$	CHS$_2$	CHS$_3$	CHS$_4$
a	$\frac{1}{360}$	$-\frac{7}{90}$	$\frac{11}{720}$	$-\frac{3}{80}$	$\frac{31}{180}$	$\frac{87}{20}$	$\frac{171}{5}$	$\frac{1415}{9}$
c	$\frac{1}{120}$	$-\frac{1}{15}$	$\frac{1}{40}$	$-\frac{1}{120}$	$\frac{1}{10}$	$\frac{199}{30}$	$\frac{914}{15} + \frac{2\,r}{3}$	$\frac{890}{3} + \frac{14\,r}{3}$

(3.2)

Here we included also the known values of the c-coefficient for the same $s \le 2$ fields and also the $s = 3, 4$ values from (1.5) depending on the a priori unknown parameter r.

While the value of the function $\widehat{\zeta}(q)$ at $q = 1$ gives the a-coefficient, one observes from (2.23), (2.24), (2.28) that its first derivative vanishes at $q = 1$ for all low spin $s = 0, \frac{1}{2}, 1$ conformal fields ($\widehat{\zeta}'(q) = \frac{d}{dq}\widehat{\zeta}(q)$)

$$\widehat{\zeta}'_{\varphi}(1) = 0\,, \qquad \widehat{\zeta}'_{\varphi^{(4)}}(1) = 0\,, \qquad \widehat{\zeta}'_{\psi}(1) = 0\,, \qquad \widehat{\zeta}'_{\psi^{(3)}}(1) = 0\,, \qquad \widehat{\zeta}'_{1}(1) = 0\ . \tag{3.3}$$

Surprisingly, this is no longer true for CHS fields with $s \ge 2$, i.e. $\widehat{\zeta}'_s(1) \ne 0$.

The second derivative of $\widehat{\zeta}(q)$ at $q = 1$ is expected to be related to the conformal anomaly c-coefficient. We propose the following general expression for c (and also similar relation for C_T in other dimensions)

$$\mathrm{c} = -\frac{1}{4}\,\frac{d^2}{dq^2}\big[q\,\widehat{\zeta}(q)\big]\Big|_{q=1} = -\frac{1}{4}\,\widehat{\zeta}''(1) - \frac{1}{2}\widehat{\zeta}'(1)\ . \tag{3.4}$$

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In the low-spin cases when the first derivative vanishes (3.3), c is then given just by the second derivative term. Using the expressions for $\widehat{\zeta}(q)$ in (2.23)–(2.28) we indeed reproduce the known values of c for fields with $s \leq 1$ in (3.2).

For $s = 2$, i.e. the Weyl graviton, where $\widehat{\zeta}_2'(1) = -8$ is no longer zero, we get from (3.4) precisely the known value $\mathrm{c} = \frac{199}{30}$ [17]. The same agreement is found in the case of $d = 6$ Weyl graviton as will be discussed in section 4 and appendix D.

Note that as follows from (3.4) and the general form of $\widehat{\zeta}(q)$ in (1.9), (2.29) one has

$$\mathrm{c} = E_c + \frac{1}{240}\nu - \frac{1}{2}p_2 \ , \tag{3.5}$$

where p_2 is the coefficient of the $\frac{1}{q^2}$ term in $\widehat{\zeta}(q)$. Interestingly, $p_2 = 0$ for all lower-spin fields (see (2.23), (2.24), (2.28)) but is non-zero for higher spin CHS fields starting with Weyl graviton (cf. (2.25)–(2.27)).

In the case of CHS fields with $s = 3$ and 4 we find c_s in (3.2) corresponding to the value of the parameter r in (1.5) equal to -1, i.e.

$$r = -1 : \qquad \mathrm{c}_3 = \frac{904}{15}, \qquad \mathrm{c}_4 = 292 \ . \tag{3.6}$$

This provides a strong evidence that the correct value of the c-coefficient of the CHS fields is given by (1.6).

Let us now compare (3.1) and (3.4) with similar relations for a and c expected from the free energy (2.30), (2.31) on $S^1_q \times \mathbb{H}^3$. Let us first recall the expression for the Rényi entropy in terms of the free energy on $S^1_q \times \mathbb{H}^3$

$$\mathcal{S}(q) = \frac{\mathcal{F}(q) - q\,\mathcal{F}(1)}{q-1} \ . \tag{3.7}$$

Then the expected expressions for the a and c anomaly coefficients are [22]

$$\mathrm{a} = -\frac{1}{4}\,\mathcal{S}(1) \ = \frac{1}{4}\,\mathcal{F}(1) - \frac{1}{4}\,\mathcal{F}'(1)\,, \tag{3.8}$$

$$\mathrm{c} = \frac{1}{2}\,\mathcal{S}'(1) = \frac{1}{4}\,\mathcal{F}''(1) \ . \tag{3.9}$$

Using the relation (2.31) between $\widehat{\zeta}(q)$ for the conformal theory on S^4_q and $\mathcal{F}(q)$ on $S^1_q \times \mathbb{H}^3$ we conclude that in all low-spin cases when $\widehat{\zeta}'(1) = -\mathcal{F}'(1) = 0$ (3.3) the expressions (3.8) and (3.9) are indeed equivalent to (3.1) and (3.4).[22] In particular, for the $s = 1$ case the conformal anomaly coefficients are reproduced correctly in both S^4_q and $S^1 \times \mathbb{H}^3$ approaches (see also [39]).

[22] The expression for a-coefficient in terms of the entanglement entropy $\mathcal{S}(1)$ is assumed to incorporate the required edge mode contribution adding a constant term to $\mathcal{F}(q)$ (see [42–44]) which is effectively included in the systematic computation of $\widehat{\zeta}(q)$ on S^4_q. Recent work [36] extends this to the case of a conformally invariant p-form field in $d = 2p + 2$. As in the case of the Maxwell field in $d = 4$ the correct a-coefficient is found directly from the spectral computation on S^d_q, while a constant shift is needed in the computation on $S^1_q \times \mathbb{H}^{d-1}$. This shift is predicted [36] to be minus the entanglement entropy of a conformal $(p-1)$-form field, in agreement with [43].

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The first novel case is the $s = 2$ Weyl graviton when $\widehat{\zeta}'(1) = -\mathcal{F}'(1) \neq 0$ and the relation (3.9) is to be replaced by (3.4). The consistency of (3.4) for all three $s = 2, 3, 4$ CHS cases discussed explicitly above provides a strong evidence for its universal applicability. A similar expression is true also in $d = 6$ (where it leads to the correct $C_T \sim \mathrm{c}_3$ coefficient for the 6d conformal graviton, see section 4 and appendix D) and in $d = 2$ (see appendix C). It would be important to derive (3.4) in general using the approach analogous to the one in [22], taking fully into account the special features of stress tensor for higher spin fields.

Using the expected general structure (2.29) of $\widehat{\zeta}(q)$ with the expression (1.4) for E_c for spin s CHS field as well as the explicit results for $\widehat{\zeta}_s(q)$ with $s = 1, 2, 3$ in (2.24)–(2.26) it is possible determine the general form of $\widehat{\zeta}_s(q)$ for any value of s. Starting with an ansatz (with $\nu = s(s+1)$)

$$\widehat{\zeta}_s(q) = -\frac{\nu}{360\,q^3} + \frac{p_2(\nu)}{q^2} + \frac{p_1(\nu)}{q} + p_0(\nu) - \frac{\nu\,(18\,\nu^2 - 14\nu - 11)}{360}\,q, \tag{3.10}$$

where $p_i(\nu) = \nu(k_{i2}\nu^2 + k_{i1}\nu + k_{i0})$ are cubic polynomials in ν (so that $\widehat{\zeta}_s$ is at most cubic in ν and vanishes for $\nu = 0$ as required to match the structure of conformal anomaly coefficients) one is able to fix the 9 unknown coefficients k_{ij} by matching to the $s = 1, 2, 3$ expressions in (2.24)–(2.27). As a result,

$$\widehat{\zeta}_s(q) = -\frac{\nu}{360\,q^3} - \frac{\nu^2\,(\nu-2)}{36\,q^2} + \frac{\nu\,(2\,\nu^2 - 5\,\nu - 1)}{36\,q} - \frac{\nu^2(2\nu-1)}{36} - \frac{\nu\,(18\,\nu^2 - 14\nu - 11)}{360}\,q. \tag{3.11}$$

Then a highly non-trivial consistency check is that for $s = 4$ and 5 this expression reproduces also $\widehat{\zeta}_4(q)$ in (2.27) and $\widehat{\zeta}_5(q)$ in (B.6). Furthermore, applying (3.1) we then match the known a-coefficient in (1.3), while applying (3.4) we get the $r = -1$ expression for the c-coefficient in (1.6).[23] Note also that

$$\widehat{\zeta}_s'(1) = -\frac{1}{60}\nu(\nu-2)(3\nu+2)\,, \tag{3.12}$$

is a non-zero integer for all $s > 1$ CHS fields and thus contributes to c in (3.4).

We observe also that not only the regularized sums of a_s and c_s but also the sum of the full $\widehat{\zeta}_s(q)$ functions over all $s = 1, 2, \ldots$ vanishes, i.e.

$$\sum_{s=1}^{\infty} e^{-\epsilon(s+\frac{1}{2})}\,\widehat{\zeta}_s(q)\Big|_{\epsilon\to 0,\ \text{finite}} = 0\,, \tag{3.13}$$

so that the full CHS theory is one-loop UV finite on S^4_q space.

This implies also the vanishing of the total free energy on $S^1_q \times \mathbb{H}^3$ (2.31) and thus of the associated Rényi entropy (3.7). This vanishing appears to be consistent with the "topological" nature of the CHS theory [15]. Similar conclusions are reached for the massless higher spin tower in appendix E.

[23] The analog of (2.20) here contains $n_z = s\,\mathrm{k}_s - \sum_{s'=0}^{s-1}\mathrm{k}_{s'} = \frac{1}{36}\nu^2\,(5\,\nu - 1)$, generalizing the expressions in (2.22).

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Let us note also that as the (one-loop) logarithmic UV divergences cancel in the full CHS theory, the finite part of the corresponding partition function Z is scheme-independent. As was shown in [15], $Z = 1$ in flat space (assuming the same regularization as in (3.13), in which the total number of degrees of freedom vanishes) and also on S^4 (which could be expected given the cancellation of conformal a-anomalies). One may expect that since S^4_q has a non-zero Weyl tensor it is likely that $Z(S^4_q)$ is a non-trivial function of q. It would be interesting to compute it using the heat-kernel method in appendix A.

4 Generalization to six dimensions

Let us now demonstrate how similar computations of $\widehat{\zeta}(q)$ and related conformal anomaly coefficients can be performed in six dimensions. In 6d for a classically Weyl invariant theory one gets instead of (1.2)

$$B_6 = \tfrac{1}{(4\pi)^3} \int d^6x \sqrt{g}\, \bar{b}_6(x)\,, \qquad\qquad \bar{b}_6 = -\mathrm{a}\, E_6 + \mathrm{c}_1\, I_1 + \mathrm{c}_2\, I_2 + \mathrm{c}_3\, I_3\ , \tag{4.1}$$

where $E_6 = -\epsilon_6\epsilon_6 RRR$ is proportional to the 6d Euler density and the 3 independent Weyl invariants are $I_1 = C_{\alpha\mu\nu\beta}\, C^{\mu\rho\sigma\nu}\, C_\rho{}^{\alpha\beta}{}_\sigma$, $I_2 = C_{\alpha\beta}{}^{\mu\nu}\, C_{\mu\nu}{}^{\rho\sigma}\, C_{\rho\sigma}{}^{\alpha\beta}$ and $I_3 = C_{\mu\alpha\beta\gamma}\, D^2 C^{\mu\alpha\beta\gamma} + \ldots$ (see for details [45] and appendix D).

The aim will be to consider the conical deformation S^6_q of 6-sphere (with the metric as in (1.7) with S^2 singularity replaced by S^4 one), compute the spectral ζ-function at $z = 0$ or $\widehat{\zeta}(q) = B_6$ as in (1.8) and then extract the values of the conformal anomaly coefficients from it. As the log divergent part of the free energy on S^6 should be proportional to a, we should have again $\widehat{\zeta}(1) \sim \mathrm{a}$. The c_3 coefficient proportional to C_T in the 2-point function of stress tensors should be determined, as in 4d case, by the 2nd derivative of $\widehat{\zeta}(q)$ at $q = 1$. The coefficients c_3 and c_4 related to 3-point functions of stress tensor may be possible to extract from the 3rd (or higher) derivative of $\widehat{\zeta}(q)$ but we will not attempt this here.

Taking into account normalizations, the expected relations are then the direct analogs of (3.1) and (3.4) in 4d case:

$$\mathrm{a} = -\tfrac{1}{96}\, \widehat{\zeta}(1)\,, \tag{4.2}$$

$$\mathrm{c}_3 = \tfrac{1}{12} \tfrac{d^2}{dq^2} \big[q\, \widehat{\zeta}(q) \big]\Big|_{q=1} = \tfrac{1}{12} \widehat{\zeta}''(1) + \tfrac{1}{6} \widehat{\zeta}'(1)\ . \tag{4.3}$$

The bosonic totally symmetric rank s conformal higher spins in 6d have kinetic terms $h_s\, \Box^{s+\frac{d-4}{2}}\, h_s = h_s\, \Box^{s+1}\, h_s$. Below we shall consider only the lowest spin cases: $s = 0$ — the standard ∂^2 conformal scalar, $s = 1$ — the higher derivative ∂^4 vector [39, 46, 47] and $s = 2$ — the ∂^6 conformal graviton (see appendix D). The corresponding partition functions on S^6 are [48] (cf. (2.2), (2.3))

$$Z_0 = \left[\frac{1}{\det \Delta_0(6)} \right]^{1/2}, \qquad\qquad Z_1 = \left[\frac{\det \Delta_0(0)}{\det \Delta_{1\perp}(7)\ \det \Delta_{1\perp}(5)} \right]^{1/2}, \tag{4.4}$$

$$Z_2 = \left[\frac{\det \Delta_{1\perp}(-5)\ \det \Delta_0(-6)}{\det \Delta_{2\perp}(8)\ \det \Delta_{2\perp}(6)\ \det \Delta_{2\perp}(2)} \right]^{1/2}, \tag{4.5}$$

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where $\Delta_{s\perp}(M^2) = (-D^2 + M^2)_{s\perp}$ are defined on S^6. Assuming as in 4d case that the these partition functions have the same structure on S^6_q, their computation requires the knowledge of the spectrum of $\Delta_{s\perp}(M^2)$ on this space.

The analysis of the spectrum goes along the same lines as in section 2. The analogs of the eigenvalues in (2.6) and (2.7) are obtained after the replacement $n+\gamma m+3 \to n+\gamma m+5$. The degeneracies of the spectrum for the conformal scalar and the 4-derivative spin 1 field are found from (2.11) and (2.12) where now $d \to 6$. The spin 2 degeneracies turn out to be $(n+m \geq 2)$

$$
\mathrm{d}^{(2)}_{n,0} = \frac{1}{12}(n-1)(n+2)(n+3)(7n+22), \qquad \mathrm{d}^{(2)}_{n,1} = \frac{1}{6}n(3+n)(4+n)(7n+17),
$$
$$
\mathrm{d}^{(2)}_{n,m>1} = \frac{7}{6}(n+1)(n+2)(n+3)(n+4). \tag{4.6}
$$

As a result, $\widehat{\zeta}_{s\perp}(q)$ in (1.8) for the operators $\Delta_{s\perp}(M^2)$ on S^6_q are given by (cf. (2.19))

$$
\begin{aligned}
\widehat{\zeta}_0(q;M^2) &= \tfrac{1}{15120\,q^5} + \tfrac{6\,M^2-35}{4320\,q^3} + \tfrac{24-10\,M^2+M^4}{144\,q} + \tfrac{4315-3990\,M^2+1050\,M^4-84\,M^6}{30240}\,q, \\
\widehat{\zeta}_{1\perp}(q;M^2) &= \tfrac{1}{3024\,q^5} + \tfrac{6\,M^2-41}{864\,q^3} + \tfrac{5\,(35-12\,M^2+M^4)}{144\,q} \\
&\quad - \tfrac{553-210\,M^2+15\,M^4}{180} + \tfrac{9439-6342\,M^2+1302\,M^4-84\,M^6}{6048}\,q, \\
\widehat{\zeta}_{2\perp}(q;M^2) &= \tfrac{1}{1080\,q^5} - \tfrac{1}{6\,q^4} + \tfrac{1111+42\,M^2}{2160\,q^3} + \tfrac{6M^2-55}{6q^2} + \tfrac{1560-242\,M^2+7\,M^4}{72\,q} \\
&\quad - \tfrac{3166-1860M^2+105M^4}{180} + \tfrac{17167-9198\,M^2+1554\,M^4-84\,M^6}{2160}\,q.
\end{aligned} \tag{4.7}
$$

Forming the combinations of these functions corresponding to the partition functions (4.4), (4.5) taking into account as in (2.20) the zero mode contributions[24]

$$
s=0: \quad n_{\rm z} = 0; \qquad s=1: \quad n_{\rm z} = 2\,\mathrm{k}_1 = 2; \qquad s=2: \quad n_{\rm z} = 3\,\mathrm{k}_2 - \mathrm{k}_1 = 83\,, \tag{4.8}
$$

we find that the total coefficients $\widehat{\zeta}_s(q)$ of the log UV divergence of the CHS partition functions (4.4), (4.5) on S^6_q are given by (cf. (2.23)–(2.25))

$$
\widehat{\zeta}_0(q) = \frac{1}{15120\,q^5} + \frac{1}{4320\,q^3} + \frac{31}{30240}\,q, \tag{4.9}
$$
$$
\widehat{\zeta}_1(q) = \frac{1}{1680\,q^5} - \frac{1}{288\,q^3} - \frac{1}{6\,q} - \frac{14}{15} - \frac{39}{224}\,q \tag{4.10}
$$
$$
\widehat{\zeta}_2(q) = \frac{1}{420\,q^5} - \frac{1}{2\,q^4} + \frac{703}{360\,q^3} - \frac{23}{2\,q^2} + \frac{49}{3\,q} - \frac{181}{9} - \frac{4143}{280}\,q\,. \tag{4.11}
$$

As in the 4d case (2.29) the $q \to 0$ and $q \to \infty$ asymptotics of $\widehat{\zeta}(q)$ are controlled by free energies on $S^1_q \times \mathbb{R}^5$ and $\mathbb{R} \times S^5$ respectively, i.e.

$$
\widehat{\zeta}_s(q) = \frac{\nu}{15120}\,\frac{1}{q^5} + \ldots + (-2\,E_c)\,q\,, \tag{4.12}
$$

[24] The number of zero modes associated with a transverse traceless totally symmetric rank s field on S^6 is (see [48] and refs. there) $\mathrm{k}_s = \frac{1}{4320}(2s+3)s(s+1)^3(s+2)^3(s+3)$, i.e. $\mathrm{k}_0 = 0$, $\mathrm{k}_1 = 1$, $\mathrm{k}_2 = 28$, etc.

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where the number of dynamical degrees of freedom ν [48] and the Casimir energy on $\mathbb{R}\times S^5$ [7] for 6d CHS fields are given by

$$\nu = \frac{1}{4}(s+1)^2(s+2)^2, \qquad\qquad s = 0, 1, 2, \ldots \tag{4.13}$$

$$\mathrm{E}_{c,s} = \frac{1}{60480}\,\nu\,(96\,\nu^{3/2} - 232\nu - 12\nu^{1/2} + 117)\ . \tag{4.14}$$

The values of $\widehat{\zeta}_s$ at $q = 1$ reproduce (4.2) the known a-coefficients [48][25]

$$\mathrm{a}_s = \frac{1}{1814400}\,\nu\,(88\,\nu^{3/2} - 110\nu - 4\nu^{1/2} + 1)\ . \tag{4.15}$$

We also observe that as in the 4d case (3.3), (3.12) the first derivative $\widehat{\zeta}'(1)$ vanishes for $s = 0$ and $s = 1$ but not for $s = 2$:

$$\widehat{\zeta}_0'(1) = 0\,, \qquad\quad \widehat{\zeta}_1'(1) = 0\,, \qquad\quad \widehat{\zeta}_2'(1) = -12\ . \tag{4.16}$$

Using (4.3) we get the following values for the $\mathrm{c}_{3,s}$ coefficients

$$\mathrm{c}_{3,0} = \frac{1}{2520} \qquad\qquad \mathrm{c}_{3,1} = -\frac{5}{168} \qquad\qquad \mathrm{c}_{3,2} = -\frac{1639}{420}\ . \tag{4.17}$$

The $s = 0$ and $s = 1$ values match the known ones found earlier in [39, 45]. Remarkably, the $s = 2$ result for $\mathrm{c}_{3,2}$ agrees with the direct computation of the corresponding 6d Seeley-DeWitt coefficient for the 6d Weyl graviton that we present in appendix D where we also determine the values of the two other conformal anomaly coefficients c_1 and c_2 in (4.1) (see (D.10)). This provides a non-trivial check of the consistency of the relation (4.3) for the $C_T \sim \mathrm{c}_3$ coefficient in 6d.

For completeness, let us summarize the values of the conformal anomalies for the 6d $s = 0, 1, 2$ CHS fields below:

s	a	c_1	c_2	c_3
0	$-\frac{5}{72\times 7!}$	$-\frac{1}{540}$	$\frac{1}{3024}$	$\frac{1}{2520}$
1	$\frac{275}{8\times 7!}$	$\frac{97}{180}$	$\frac{911}{5040}$	$-\frac{5}{168}$
2	$\frac{3005}{2\times 7!}$	$\frac{1507}{45}$	$\frac{635}{126}$	$-\frac{1639}{420}$

(4.18)

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[25] With our normalization in the 6d conformal scalar case ($\nu = 1$) we get $\mathrm{a}_0 = -\frac{1}{72576}$.

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A Details of computation of $\widehat{\zeta}(q)$ for 4d massive scalar

Here we provide some details of the computation of $\widehat{\zeta}(q)$ in section 2 on the example of 4d scalar operator $\Delta_0(M^2) = -D^2 + M^2$. The case of the conformal coupling on unit-radius S^4 corresponds to $M^2 = 2$. Introducing the parameter μ related to M^2 by $M^2 = \frac{9-\mu^2}{4}$ and setting $\gamma = 1/q$ we then have for the corresponding spectral ζ-function in (1.8)

$$\zeta(z;\gamma) = \sum_{n,m=0}^{\infty} \mathrm{d}^{(0)}_{n,m} \left[\left(\tfrac{3}{2} + n + m\,\gamma\right)^2 - \tfrac{\mu^2}{4}\right]^{-z} . \tag{A.1}$$

We used (2.6) and (2.11), i.e. $\mathrm{d}^{(0)}_{n,0} = \frac{1}{2}\,(n+1)\,(n+2)$, $\mathrm{d}^{(0)}_{n,m>0} = (n+1)\,(n+2)$. The evaluation of (A.1) was first considered in [34] where it allowed to obtain the finite-temperature one-loop effective potential for a scalar field in de Sitter space-time. The result revealed an unexpected dependence of the logarithmic divergences on the temperature associated to the presence of a horizon which is directly related to q-dependence of $\widehat{\zeta}(q) = \zeta(0;q)$ we discussed in section 2.[26]

We will compute $\zeta(0;\gamma)$ in (A.1) by using a somewhat more direct method than employed in [34]. We first split the contribution from the $m = 0$ and $m > 0$ modes as

$$\zeta(z;q) = \zeta^{(a)}(z;q) + \zeta^{(b)}(z;q), \qquad \zeta^{(a)}(z;q) = \sum_{n=0}^{\infty} \tfrac{1}{2}(n+1)\,(n+2)\left[\,\left(\tfrac{3}{2}+n\right)^2 - \tfrac{\mu^2}{4}\right]^{-z},$$
$$\zeta^{(b)}(z;\gamma) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}(n+1)(n+2)\left[\left(n+m\,\gamma+\tfrac{3-\mu}{2}\right)\left(n+m\,\gamma+\tfrac{3+\mu}{2}\right)\right]^{-z}. \tag{A.2}$$

As we are interested only in the value at $z = 0$ each of the two terms $\zeta^{(a)}$ and $\zeta^{(b)}$ can be computed by expanding to quadratic order in μ^2 only (higher order terms in μ will give vanishing contributions in the $z \to 0$ limit, cf. also footnote 13). For the first term we get

$$\begin{aligned}\zeta^{(a)}(z;q) =&\tfrac{1}{8}\left[4\zeta_{\mathrm{R}}\left(2z-2,\tfrac{3}{2}\right) - \zeta_{\mathrm{R}}\left(2z,\tfrac{3}{2}\right)\right] + \tfrac{1}{32}\mu^2 z\left[4\zeta_{\mathrm{R}}\left(2z,\tfrac{3}{2}\right) - \zeta_{\mathrm{R}}\left(2z+2,\tfrac{3}{2}\right)\right]\\ &+\tfrac{1}{256}\mu^4 z(z+1)\left[4\zeta_{\mathrm{R}}\left(2z+2,\tfrac{3}{2}\right) - \zeta_{\mathrm{R}}\left(2z+4,\tfrac{3}{2}\right)\right] + \mathcal{O}(\mu^6)\,,\end{aligned} \tag{A.3}$$

where $\zeta_{\mathrm{R}}(a,b)$ is the Hurwitz zeta-function (2.18). Dropping the contributions that manifestly vanish at $z \to 0$ (due to explicit factors of z that multiply analytic terms) we find

$$\zeta^{(a)}(z;q) = \tfrac{1}{8}\left[4\,\zeta_{\mathrm{R}}\left(2z-2,\tfrac{3}{2}\right) - \zeta_{\mathrm{R}}\left(2z,\tfrac{3}{2}\right)\right] + \mathcal{O}(z)\ . \tag{A.4}$$

This vanishes at $z = 0$

$$\zeta^{(a)}(0;q) = \tfrac{1}{8}\left[4\times\left(-\tfrac{1}{4}\right) - (-1)\right] = 0. \tag{A.5}$$

Instead of following the same strategy in the case of $\zeta^{(b)}(0;q)$ we shall use a simpler approach by relating it to the t^0 coefficient in the expansion of the corresponding heat kernel replaced

[26] In the present context the parameter $\gamma = q^{-1}$ of the conical singularity corresponds to the ratio of the temperature and the Hawking temperature in [34].

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by the sum of the kernels corresponding to the "1-st order" factors in (A.2)[27]

$$\begin{aligned} K(t;\gamma) &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}(n+1)(n+2)\left[e^{-t\left(n+m\gamma+\frac{3-\mu}{2}\right)}+e^{-t\left(n+m\gamma+\frac{3+\mu}{2}\right)}\right] \\ &\overset{t\to 0}{\sim} \sum_{k=-4}^{\infty} h_k\, t^k\,, \qquad\qquad \zeta^{(b)}(0;q)=\tfrac{1}{2}h_0\ . \end{aligned} \tag{A.6}$$

Computing the two sums, we readily obtain

$$\begin{aligned} K(t;\gamma) =& \frac{e^{-\frac{1}{2}(\mu-3)t}\left(e^{\mu t}+1\right)}{\left(e^t-1\right)^3\left(e^{\gamma t}-1\right)} = \frac{2}{\gamma}\frac{1}{t^4}-\frac{1}{t^3}+\left(\frac{\mu^2-1}{4\gamma}+\frac{\gamma}{6}\right)\frac{1}{t^2}+\frac{1}{8}\left(1-\mu^2\right)\frac{1}{t} \\ & -\frac{\gamma^3}{360}+\frac{\mu^2-1}{48}\gamma+\frac{5\mu^4-30\mu^2+17}{960\gamma}+\mathcal{O}(t)\ . \end{aligned} \tag{A.7}$$

The t^0 term here gives (using (A.5) and $\gamma = 1/q$)

$$\zeta(0;q)=\zeta^{(a)}(0;q)+\zeta^{(b)}(0;q)=-\frac{1}{360\,q^3}+\frac{\mu^2-1}{48\,q}+\frac{5\,\mu^4-30\,\mu^2+17}{960}\,q\,, \tag{A.8}$$

which is equivalent to the expression in (2.19) after we recall that $\mu^2 = 9-4M^2$. For example, for the conformally coupled scalar with $\mu = 1$ we get, in agreement with (2.23),

$$\zeta(0;q)=-\frac{1}{360\,q^3}-\frac{q}{120}. \tag{A.9}$$

B Degeneracies of eigenvalues of bosonic spin s Laplacian on S^4_q

The degeneracies for $\gamma \in [1,2)$ for spin $s \le 4$ in (2.12)–(2.15) admit a natural generalization to all integer $s > 0$ (here $n+m \ge s$)

$$\begin{aligned} &\mathrm{d}^{(s)}_{n,0}=\tfrac{1}{2}\,(n-s+1)\,[(2s+1)\,n+K_0], &&\mathrm{d}^{(s)}_{n,1}=(n-s+2)\,[(2s+1)\,n+K_1], \\ &\mathrm{d}^{(s)}_{n,2}=(n-s+3)\,[(2s+1)\,n+K_2], &&\dots \\ &\mathrm{d}^{(s)}_{n,s-1}=n\,[(2s+1)\,n+K_{s-1}], &&\mathrm{d}^{(s)}_{n,m>s-1}=(2s+1)\,(n+1)(n+2)\,, \end{aligned} \tag{B.1}$$

where the integers K_p are

$$K_p = 2\,(1+2\,p)+(3+2\,p)\,(s-p), \qquad\qquad p=0,\dots,s-1. \tag{B.2}$$

One can check that the degeneracies in (B.1) are always non-negative and also even when $m > 0$ as follows from the expected symmetry of the spin s Laplacian eigenstates under the exchange $x^+ \leftrightarrow x^-$ in this case. The total degeneracy

$$\sum_{m=0}^{s-1}\mathrm{d}^{(s)}_{N-m,m}+\sum_{m=s}^{N}\mathrm{d}^{(s)}_{N-m,m}=\tfrac{1}{6}\,(2s+1)\,(2N+3)(N+s+2)(N-s+1) \tag{B.3}$$

[27] As the eigenvalues factorize, the same applies to the corresponding determinant, and thus the heat kernel can be replaced by a sum of heat kernels corresponding to the factors. The origin of the $\frac{1}{2}$ factor in the relation for $\zeta(0;q)$ may be understood by comparing dimensions of the proper-time cutoffs in the original heat kernel and its "factor" analogs, cf. also [30].

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is equal as it should to the degeneracy of the level N eigenvalue for the regular sphere S^4 (cf. footnote 12).

A further test of (B.1) is provided by the explicit calculation of the zeta function $\widehat{\zeta}_s(q)$ for the CHS field with spin s. For instance, for $s = 5$ (B.1) gives $(n+m \geq 5)$

$$\begin{aligned} &\mathrm{d}^{(5)}_{n,0} = \tfrac{1}{2}\,(n-4)\,(11\,n+17), && \mathrm{d}^{(5)}_{n,1} = (n-3)\,(11\,n+26), \\ &\mathrm{d}^{(5)}_{n,2} = (n-2)\,(11\,n+31), && \mathrm{d}^{(5)}_{n,3} = (n-1)\,(11\,n+32), \\ &\mathrm{d}^{(5)}_{n,4} = n\,(11\,n+29), && \mathrm{d}^{(5)}_{n,m>4} = 11\,(n+1)\,(n+2). \end{aligned} \tag{B.4}$$

Generalizing the spin 5 CHS partition function on S^4 [1]

$$Z_5 = \left[\frac{\det\Delta_{4\perp}(-24)\det\Delta_{3\perp}(-25)\det\Delta_{2\perp}(-26)\det\Delta_{1\perp}(-27)\det\Delta_{0}(-28)}{\det\Delta_{5\perp}(7)\det\Delta_{5\perp}(5)\det\Delta_{5\perp}(1)\det\Delta_{5\perp}(-5)\det\Delta_{5\perp}(-13)}\right]^{1/2}, \tag{B.5}$$

to S^4_q we find that the associated $\widehat{\zeta}_5(q)$ function determined using (B.4) is given by

$$\widehat{\zeta}_5(q) = -\frac{1}{12q^3} - \frac{700}{q^2} + \frac{8245}{6\,q} - 1475 - \frac{15769}{12}\,q\;. \tag{B.6}$$

This expression is in perfect agreement with our general proposal in (3.11) (here $\nu = 30$).

C Two-dimensional case

In two dimensions there is just the "a" coefficient of the Weyl anomaly that also has the interpretation of the coefficient C_T in the 2-point function of stress tensor, i.e. is the Virasoro central charge and thus is usually denoted as c. In standard normalization where a real ∂^2 scalar has $c = 1$ we have for the coefficient of the log UV divergence of the partition function (cf. (1.1))

$$B_2 = \frac{1}{4\pi}\int d^2x\sqrt{g}\;\bar{b}_2\,, \qquad \bar{b}_2 = \mathrm{a}\,R\;, \qquad \mathrm{a} \equiv \tfrac{1}{6}\,c\;. \tag{C.1}$$

On S^2 one thus finds $B_2 = \frac{1}{3}c$. For a conformal field defined on a conical deformation S^2_q of the 2-sphere we expect the corresponding spectral zeta-function at $z = 0$ to have a similar general form as in 4d (2.29) and in 6d (4.12), i.e.

$$\widehat{\zeta}(q) = \frac{\nu}{6\,q} + p_0 - 2\,E_c\,q, \tag{C.2}$$

where the first and the last terms are fixed by the asymptotics corresponding to $S^1_q \times \mathbb{R}$ $(q \to 0)$ and $\mathbb{R} \times S^1$ $(q \to \infty)$. Here ν is the number of effective degrees of freedom with $\nu = 1$ for a real ∂^2 scalar and $\nu = -2$ for the 2d conformal higher spin fields with kinetic terms $h_s\Box^{\frac{s+d-4}{2}}h_s = h_s\Box^{s-1}h_s$ with $s = 1, 2, \ldots$ [48]. The Casimir energy E_c on unit-radius S^1 should in general be related to the central charge c by [49, 50] $E_c = -\frac{1}{12}c$. Since $B_2(S^2) = \widehat{\zeta}(1)$ we then conclude that

$$c = 3\,\widehat{\zeta}(1) = -12E_c\;. \tag{C.3}$$

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As follows from (C.2) and $E_c = -\frac{1}{12}c$ we then have also the following representation for c

$$c = 3\,\tfrac{d^2}{dq^2}\Big[q\,\widehat{\zeta}(q)\Big]\Big|_{q=1} \,. \tag{C.4}$$

Remarkably, this relation for $c = C_T$ in 2d case is a direct counterpart of the similar 4d (3.4) and 6d (4.3) relations we proposed above. This supports their common origin and implies a universal applicability of $C_T \sim \widehat{\zeta}''(1) + 2\widehat{\zeta}'(1)$ relation in any dimension.

In the case of the standard ∂^2 scalar field (C.2), (C.3) imply that

$$\widehat{\zeta}(q) = \frac{1}{6\,q} + \frac{1}{6}\,q\,. \tag{C.5}$$

In the case of $d = 2$ CHS fields one finds [13, 14, 48]

$$c_s = -2\Big[1 + 6\,s(s-1)\Big], \qquad\qquad s \geq 2\,. \tag{C.6}$$

As a result, the function $\widehat{\zeta}(q)$ for the 2d CHS fields consistent with the above relations (C.3), (C.6) turns out to be

$$\widehat{\zeta}_s(q) = -\frac{1}{3\,q} - 2\,s\,(s-1) - \frac{1}{3}\Big[1 + 6\,s(s-1)\Big]\,q = -\frac{1}{3}\Big(\frac{1}{q} - 1\Big) + \frac{c_s}{6}\Big(q+1\Big)\,. \tag{C.7}$$

Note that while for a standard scalar (C.5) we find that $\widehat{\zeta}'(1) = 0$, for the CHS fields $\widehat{\zeta}'_s(1) = -2\,s\,(s-1)$ so that for $s \geq 2$ it is again non-zero as in the 4d and 6d cases.

D Conformal anomaly coefficients for the Weyl graviton in six dimensions

In 6d there are three dimension 6 non-trivial Weyl invariants I_1, I_2, I_3 that appear in (4.1)

$$\begin{aligned} &I_1 = C_{\alpha\mu\nu\beta}\,C^{\mu\rho\sigma\nu}\,C_{\rho}{}^{\alpha\beta}{}_{\sigma}\,, \qquad I_2 = C_{\alpha\beta}{}^{\mu\nu}\,C_{\mu\nu}{}^{\rho\sigma}\,C_{\rho\sigma}{}^{\alpha\beta}\,,\\ &I_3 = C_{\mu\alpha\beta\gamma}\,(D^2\,\delta^{\mu}_{\nu} + 4\,R^{\mu}_{\nu} - \tfrac{6}{5}\,R\,\delta^{\mu}_{\nu})\,C^{\nu\alpha\beta\gamma} + \text{total derivatives.} \end{aligned} \tag{D.1}$$

A candidate Weyl-invariant gravity action is then an integral of a linear combination of these 3 invariants.

There is a particular choice [51] $\mathcal{W}_6 = -I_3 + 3I_2 + 12I_1$ that has special properties: (i) it vanishes on a Ricci flat background, and (ii) it admits $(2,0)$ locally superconformal extension [46, 52, 53]. Related to (i) and (ii) is that $\mathcal{W}_6$ appears, respectively, as the coefficient of the logarithmic IR divergence of the Einstein action in AdS$_7$ evaluated on the solution of Dirichlet problem [54] and also as the log UV divergence of the $(2,0)$ tensor multiplet [45]. The resulting action may be written as

$$S = \int d^6x\,\sqrt{g}\,\Big[R^{\mu\nu}D^2R_{\mu\nu} - \tfrac{3}{10}\,R\,D^2\,R - 2\,R^{\mu\nu\rho\sigma}\,R_{\nu\rho}\,R_{\mu\sigma} - R^{\mu\nu}R_{\mu\nu}\,R + \tfrac{3}{25}\,R^3\Big]\,. \tag{D.2}$$

The fact that it is expressed in terms of the Ricci tensor and is at most linear in the Weyl tensor implies that it can be rewritten as a 2nd derivative action involving several tensors of rank ≤ 2 and it is uniquely selected by this requirement [55].

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The quadratic part of (D.2) expanded around a curved space is governed by the 6-order differential operator that factorizes, as it is easy to see, into the product of three 2nd order Lichnerowitz-type operators if the background is an Einstein one. Restricted to transverse traceless $h_{\mu\nu}$ the kinetic operator in (D.2) is (cf. (4.5)) $-D^6 + \ldots = \Delta_{2\perp}(8)\ \Delta_{2\perp}(6)\ \Delta_{2\perp}(2)$ with

$$\Delta_2(M^2)\, h_{\mu\nu} = \Big(- D^2 + \tfrac{1}{30} M^2 R \Big)\, h_{\mu\nu} - 2\, C_{\mu\rho\nu\sigma}\, h^{\rho\sigma}\,, \tag{D.3}$$

where R is the scalar curvature and C is the Weyl tensor. On a unit-radius S^6 where $R = 30$ and $C = 0$ we then get the one-loop partition function in (4.5).

This factorization implies that the one-loop conformal anomaly coefficients in (4.1) corresponding to (D.2) can be computed following [45] by using directly the general expression [56] for the b_6 Seeley-DeWitt coefficient of the corresponding 2nd order Laplace-type operator $\Delta = -D^2 + X$ defined on $k \leq 2$ tensors that enter the generalization of the S^6 partition function in (4.5).

To simplify the computation one may use a shortcut and consider several special backgrounds. Considering S^6 case one can easily determine the value of a-coefficient. For a symmetric-space Einstein background with a non-zero Weyl tensor (where I_i invariants satisfy one linear relation) one is able to find c_1 and c_3 in terms of c_2 [57]:

$$\mathrm{a} = \tfrac{3005}{2\times 7!}, \qquad \mathrm{c}_1 = \tfrac{5633}{105} - 4\,\mathrm{c}_2, \qquad \mathrm{c}_3 = -\tfrac{35543}{5040} + \tfrac{5}{8}\,\mathrm{c}_2. \tag{D.4}$$

The value of c_2 may be fixed by considering the case of a Ricci flat background where the partition function takes a simple form

$$Z_2 = \Big[\frac{(\det \Delta_1)^4}{(\det \Delta_2)^3} \Big]^{1/2}, \tag{D.5}$$

with Δ_1, Δ_2 being the standard Laplacians acting on unconstrained vector and traceless tensor with $R_{\mu\nu} = 0$,[28] $\Delta_1 h_\mu = -D^2 h_\mu\,, \quad \Delta_2\, h_{\mu\nu} = -D^2\, h_{\mu\nu} - 2\, C_{\mu\rho\nu\sigma}\, h^{\rho\sigma}$. To find the corresponding b_6 coefficient for the vector Laplacian Δ_1 from the general expressions in [45, 56] one is to use that here the covariant derivative contains an extra "internal" vector connection part with the curvature $(\mathcal{F}_{\mu\nu})_\alpha{}^\beta = C_{\mu\nu\alpha}{}^\beta$. Then one finds

$$\begin{aligned} 7!\, \bar b_6 \Big[\Delta_1\Big] &= \tfrac{80}{9} C_\alpha{}^\mu{}_\gamma{}^\nu C^{\alpha\beta\gamma\delta}\, C_{\beta\mu\delta\nu} - \tfrac{164}{3} C_{\alpha\beta}{}^{\mu\nu} C^{\alpha\beta\gamma\delta}\, C_{\gamma\delta\mu\nu} \\ &\quad - 96 C^{\alpha\beta\gamma\delta} D^2 C_{\alpha\beta\gamma\delta} - 58 (D_\mu C_{\alpha\beta\gamma\delta})^2. \end{aligned} \tag{D.6}$$

In the case of spin 2 operator $-D^2 + X$ one has $(\mathcal{F}_{\mu\nu})_{\alpha\beta,\rho\sigma} = \frac{1}{2} C_{\mu\nu\alpha\rho} g_{\beta\sigma} + \ldots$ (dots stand for 3 similar terms that symmetrize in $(\alpha\beta)$ and $(\rho\sigma)$) and $X_{\mu\nu,\rho\sigma} = -C_{\mu\rho\nu\sigma} - C_{\mu\sigma\nu\rho}$. A straightforward computation then gives

$$\begin{aligned} 7!\, b_6 \Big[\Delta_2\Big] &= \tfrac{49984}{9} C_\alpha{}^\mu{}_\gamma{}^\nu C^{\alpha\beta\gamma\delta}\, C_{\beta\mu\delta\nu} - \tfrac{1388}{9} C_{\alpha\beta}{}^{\mu\nu} C^{\alpha\beta\gamma\delta}\, C_{\gamma\delta\mu\nu} \\ &\quad + 1416\, C^{\alpha\beta\gamma\delta} D^2 C_{\alpha\beta\gamma\delta} + 544\, (D_\mu C_{\alpha\beta\gamma\delta})^2. \end{aligned} \tag{D.7}$$

[28] To compare, the usual Einstein theory partition function on a Ricci flat background is $Z_{2\,E} = \Big[\frac{\det \Delta_1}{\det \Delta_2\ \det \Delta_0} \Big]^{1/2}$.

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The full Seeley-DeWitt coefficient corresponding to (D.5) is given by the combination $3\,b_6[\Delta_2] - 4\,b_6[\Delta_1]$ and thus contains one of the total derivative terms discussed in [45]. Ignoring total derivative terms and using the relations between the invariants in (4.1) that exist in the Ricci flat case ($E_6 = 32(2I_1 + I_2)$, $I_3 = 4I_1 - I_2$) one has in general

$$\bar{b}_6\Big|_{R_{\mu\nu}=0} = -\Big[\mathrm{a} - \tfrac{1}{192}\,(\mathrm{c}_1 + 4\,\mathrm{c}_2)\Big]\,E_6 + (\mathrm{c}_1 - 2\mathrm{c}_2 + 6\mathrm{c}_3)\,I_1\ . \tag{D.8}$$

As a result, we find

$$\mathrm{a} - \tfrac{1}{192}\,(\mathrm{c}_1 + 4\,\mathrm{c}_2) = \tfrac{377}{20160}, \qquad \mathrm{c}_1 - 2\mathrm{c}_2 + 6\mathrm{c}_3 = -\tfrac{1}{210}. \tag{D.9}$$

Combining (D.9) with (D.4) we conclude that the first relation is satisfied identically while the second one determines c_2. Thus finally

$$\mathrm{c}_1 = \tfrac{1507}{45}, \qquad \mathrm{c}_2 = \tfrac{635}{126}, \qquad \mathrm{c}_3 = -\tfrac{1639}{420}. \tag{D.10}$$

E Massless higher spins on S^4_q

One may define free massless higher spin (MHS) fields on S^4 and consider, as in the case of AdS$_4$ [58], the corresponding partition function built out of the determinants of the operators $\Delta_{s\perp}(M^2)$ on spin s TT tensors (2.1)[29]

$$Z_0 = \left[\frac{1}{\det\Delta_0(2)}\right]^{1/2}, \qquad Z_s = \left[\frac{\det\Delta_{s-1\,\perp}(1-s^2)}{\det\Delta_{s\,\perp}(2+2s-s^2)}\right]^{1/2}. \tag{E.1}$$

Then extending these MHS partition functions to S^4_q we may use the results in (2.19) and appendix B to compute the corresponding total $\widehat{\zeta}(q)$ function which is the coefficient of the log UV divergence. The expressions for the scalar and spin 1 field are the same as in (2.23) and (2.24), while for $s = 2, 3, 4$ we obtain (cf. (2.24)–(2.27))

$$\begin{aligned}
\widehat{\zeta}_2(q) &= -\frac{1}{180\,q^3} - \frac{2}{q^2} + \frac{10}{3\,q} - \frac{22}{3} - \frac{401}{60}\,q\,,\\
\widehat{\zeta}_3(q) &= -\frac{1}{180\,q^3} - \frac{12}{q^2} + \frac{45}{2\,q} - 39 - \frac{2251}{60}\,q\,,\\
\widehat{\zeta}_4(q) &= -\frac{1}{180\,q^3} - \frac{40}{q^2} + \frac{232}{3\,q} - \frac{376}{3} - \frac{7361}{60}\,q\ .
\end{aligned} \tag{E.2}$$

Here we used (2.20) with the number of zero modes being (cf. (2.21))

$$n_{\mathrm{z},s} = \mathrm{k}_s - \mathrm{k}_{s-1} = \tfrac{1}{6}\,s^2\,(1 + 5\,s^2)\ . \tag{E.3}$$

A natural generalization to any $s > 0$ is then (cf. (3.11))

$$\widehat{\zeta}_s(q) = -\frac{1}{180\,q^3} + \frac{s^2(1-s^2)}{6\,q^2} - \frac{s^2(3-2s^2)}{6\,q} + \frac{s^2(1-3s^2)}{6} - \frac{1}{60}(1 - 20\,s^2 + 30\,s^4)\,q. \tag{E.4}$$

This has the expected general structure (2.29) with the number of degrees of freedom $\nu = 2$ and $E_{c,s}$ being the Casimir energy of the MHS field on $\mathbb{R}\times S^3$ (see eq. (5.7) in [59]).[30]

[29]The analytic continuation from AdS$_4$ to S^4 corresponds to changing the sign of the square of the curvature radius or $M^2 \to -M^2$. Note that the $s = 0$ case is a special case of Z_s assuming one drops the ghost contribution. Here we set the radius of S^4 to 1.

[30]One can obtain $E_{c,s}$ from the single particle partition function $\mathcal{Z}_s(x) = \frac{x^{s+1}}{(1-x)^3}\Big[2s + 1 - (2s-1)\,x\Big]$ where $x = e^{-\beta}$ using eq. (5.16) in [59].

The coefficient of the total UV log divergence of the tower of massless higher spin fields on S^4 is given by $\widehat{\zeta}_0(1) + \sum_{s=1}^{\infty} \widehat{\zeta}_s(1)$ and vanishes when regularized with an exponential cutoff or zeta-function [58, 60]. The same is true for the sum of the Casimir energies [59].

Similarly, on the conical S^4_q space we find that the total $\widehat{\zeta}(q)$ function also vanishes, i.e. the regularized sum[31]

$$\begin{aligned}
&\widehat{\zeta}_0(q) + \sum_{s=1}^{\infty} e^{-\epsilon\, s}\, \widehat{\zeta}_s(q) \\
&\qquad = \frac{-\frac{4}{q^2} - 12q + \frac{8}{q} + 8}{\epsilon^5} + \frac{\frac{1}{3q^2} + \frac{2q}{3} - \frac{1}{q} + \frac{2}{3}}{\epsilon^3} + \frac{-\frac{1}{180q^3} - \frac{q}{60}}{\epsilon} + 0 + \mathcal{O}(\epsilon),
\end{aligned} \tag{E.5}$$

has zero finite part. Then the sum of free energies on $S^1_q \times \mathbb{H}^3$ (2.31) and thus the Rényi entropies (3.7) also vanish. Such formally defined Rényi entropy may be associated to the tower of massless higher spins in flat space and thus its vanishing is consistent with "topological" nature of such higher spin theory [15].

F B_2 Seeley-DeWitt coefficient for 4d spins $s \leq 3$

It is of interest to compare consequences of our expressions for the zeta-functions in section 2 with some previous results in [27]. Given a spectral zeta-function $\zeta(z;q)$ for the operators $\Delta_{s\perp}(M^2)$ we can also extract the B_2 Seeley-DeWitt coefficient (of quadratic UV divergences) that appears in the $t \to 0$ expansion of the heat kernel in 4 dimensions, $K(t) = B_0 t^{-2} + B_2 t^{-1} + B_4 + O(t)$. After a convenient rescaling we have

$$\widetilde{B}_2 \equiv \frac{(4\pi)^2}{\mathrm{Vol}(S^4)}\, B_2 = 6\, B_2 = 6 \lim_{z\to 1} (z-1)\, \zeta(z;q)\ . \tag{F.1}$$

Reintroducing the factors of the scalar curvature (equal to 12 for a unit-radius S^4) we have from (2.19)

$$\begin{aligned}
&\widetilde{B}_2^{(0)}(q,M^2) = \frac{R}{24\, q} + q\left(\frac{R}{8} - M^2\right), \qquad \widetilde{B}_2^{(1\perp)}(q,M^2) = -\tfrac{R}{2} + \tfrac{R}{8\, q} + q\left(\tfrac{5R}{8} - 3M^2\right), \\
&\widetilde{B}_2^{(2\perp)}(q,M^2) = -\frac{5R}{2} + \frac{5R}{24\, q} + q\left(\frac{35R}{24} - 5M^2\right), \\
&\widetilde{B}_2^{(3\perp)}(q,M^2) = -7R + \frac{7R}{24\, q} + q\left(\frac{21R}{8} - 7M^2\right)\ .
\end{aligned} \tag{F.2}$$

[31] Note that for massless higher spins in d dimensions the regularization prescription is with cutoff factor $e^{-\epsilon(s+\frac{d-4}{2})}$ [14]. For conformal higher spins one has instead $e^{-\epsilon(s+\frac{d-3}{2})}$ as they are effectively associated with the boundary, i.e. one is to replace $d \to d-1$.

Using the relations in appendix A of [1], we then find for the coefficients in heat kernels of operators $\Delta_s(M^2)$ defined on fields without the transversality condition

$$\widetilde{B}_2^{(1)}(q,M^2) = \widetilde{B}_2^{(1\perp)}(q,M^2) + \widetilde{B}_2^{(0)}(q,M^2-3) = -\tfrac{R}{2} + \tfrac{R}{6\,q} + q\,\left(-4\,M^2 + R\right), \tag{F.3}$$

$$\begin{aligned}\widetilde{B}_2^{(2)}(q,M^2) &= \widetilde{B}_2^{(2\perp)}(q,M^2) + \widetilde{B}_2^{(1\perp)}(q,M^2-5) + \widetilde{B}_2^{(0)}(q,M^2-8)\\ &= -3\,R + \tfrac{3R}{8\,q} + q\,\left(-9\,M^2 + \tfrac{33R}{8}\right), \end{aligned}\tag{F.4}$$

$$\begin{aligned}\widetilde{B}_2^{(3)}(q,M^2) &= \widetilde{B}_2^{(3\perp)}(q,M^2) + \widetilde{B}_2^{(2\perp)}(q,M^2-7) + \widetilde{B}_2^{(1\perp)}(q,M^2-12) + \widetilde{B}_2^{(0)}(q,M^2-15)\\ &= -10\,R + \tfrac{2R}{3\,q} + q\,\left(-16\,M^2 + 12R\right). \end{aligned}\tag{F.5}$$

It is convenient to split the coefficient $\widetilde{B}_2$ into a regular "bulk" part and "surface" part coming from the conical singularity (cf. also discussion in Introduction)

$$\widetilde{B}_2(q,M^2) = \underbrace{q\,\widetilde{B}_2(1,M^2)}_{\text{“bulk”}} + \underbrace{\left[\widetilde{B}_2(q,M^2) - q\,\widetilde{B}_2(1,M^2)\right]}_{\text{“surface”}}. \tag{F.6}$$

The "bulk" part is given by the usual Seeley-DeWitt coefficient evaluated on S^4_q with the singular region excised: it is given by the standard S^4 (i.e. $q = 1$) expression

$$\widetilde{B}_2(1,M^2) = N_s\left(\tfrac{R}{6} - M^2\right), \qquad N_s = (s+1)^2\,, \tag{F.7}$$

times the q-factor which accounts for the volume of S^4_q. The "surface" part in (F.6) vanishes for $q = 1$ by construction.[32] The splitting (F.6) then takes the form

$$\begin{aligned}\widetilde{B}_2^{(0)}(q,M^2) &= q\left(\tfrac{R}{6} - M^2\right) + \tfrac{R}{24}\left(\tfrac{1}{q} - q\right),\\ \widetilde{B}_2^{(1)}(q,M^2) &= q\left(\tfrac{2R}{3} - 4M^2\right) + \left(-\tfrac{R}{2} + \tfrac{R}{6\,q} + \tfrac{R}{3}\,q\right),\\ \widetilde{B}_2^{(2)}(q,M^2) &= q\left(\tfrac{3R}{2} - 9M^2\right) + \left(-3R + \tfrac{3R}{8\,q} + \tfrac{21R}{8}\,q\right),\\ \widetilde{B}_2^{(3)}(q,M^2) &= q\left(\tfrac{8R}{3} - 16M^2\right) + \left(-10R + \tfrac{2R}{3\,q} + \tfrac{28R}{3}\,q\right). \end{aligned}\tag{F.8}$$

[32] In the approaches that represent the conical singularity in terms of a singular part in the curvature the "surface" term originates from an integral over S^2 as near the cone singularity $S^4_q \sim \mathscr{C}_q \times S^2$.

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The spin 0,1,2 "surface" terms in (F.8) may be compared with the results of [27] (see eqs. (2.8), (2.11), (2.13) there with $\beta = 2\pi q$)[33]

$$
\begin{aligned}
s=0: &\qquad \tfrac{\beta}{6}\Big[\Big(\tfrac{2\pi}{\beta}\Big)^2-1\Big]\,\tfrac{\mathrm{Vol}(S^2)}{\mathrm{Vol}(S^4)} = \frac{1}{2q}-\frac{q}{2},\\
s=1: &\qquad \Big[N_1\,\tfrac{\beta}{6}\Big[\Big(\tfrac{2\pi}{\beta}\Big)^2-1\Big]+2\,(\beta-2\pi)\Big]\,\tfrac{\mathrm{Vol}(S^2)}{\mathrm{Vol}(S^4)} = -6+\frac{2}{q}+4q,\\
s=2: &\qquad \Big[N_2\,\tfrac{\beta}{6}\Big[\Big(\tfrac{2\pi}{\beta}\Big)^2-1\Big]+12\,(\beta-2\pi)\Big]\,\tfrac{\mathrm{Vol}(S^2)}{\mathrm{Vol}(S^4)} = -36+\frac{9}{2q}+\frac{63}{2}q\,.
\end{aligned}
\tag{F.9}
$$

These match the "surface" terms in (F.8) after setting $R = 12$. The extension of the pattern in the l.h.s. of (F.9) to the spin 3 case that matches the $s = 3$ expression in (F.8) is

$$
s=3:\ \Big[N_3\,\tfrac{\beta}{6}\Big[(\tfrac{2\pi}{\beta})^2-1\Big]\ +40\,(\beta-2\pi)\Big]\,\tfrac{\mathrm{Vol}(S^2)}{\mathrm{Vol}(S^4)} = -120+\tfrac{8}{q}+112\,q. \tag{F.10}
$$

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