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Cahn–Hilliard stochastic equation: existence of the solution and of its density

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We show the existence and uniqueness of a function-valued process solution to the stochastic Cahn–Hilliard equation driven by space-time white noise with a nonlinear diffusion coefficient. Then we show that the solution is locally differentiable in the sense of the Malliavin calculus, and, under some non-degeneracy condition on the diffusion coefficient, that the law of the solution is absolutely continuous with respect to Lebesgue measure.

Keywords: Cahn–Hilliard equation; Green function; Malliavin calculus; stochastic partial differential equations

1. Introduction

Set $D = [0, \pi]^d$, let W denote a one-dimensional $(d+1)$-parameter Wiener process, and consider the stochastic partial differential equation (SPDE)

$$\frac{\mathrm{d}u}{\mathrm{d}t} + (\Delta^2 u - \Delta f(u)) = \sigma(u)\dot{W}, \tag{1.1}$$

with initial condition $u(0, \cdot) = u_0$ and homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial n} = \frac{\partial \Delta u}{\partial n} = 0 \text{ on } \partial D. \tag{1.2}$$

This is a stochastic version of the Cahn–Hilliard equation ($\sigma = 0$). This equation describes the complicated phase separation and coarsening phenomena in a melted alloy that is quenched to a temperature at which only two different concentration phases can exist stably. u represents a scaled concentration, and $-\Delta u + f(u)$ represents the chemical potential. The Neumann boundary conditions reflect the conservation of mass and insulation from outside. The function f is the derivative of the homogeneous free energy F. In its original form, F contains a logarithmic term. In some cases, F can be approximated by an even-degree polynomial with positive dominant coefficient. For more physical background on this equation, see, for example, Cahn and Hilliard (1958) and Novick-Cohen and Segel (1984).

The existence and uniqueness of the solution to (1.1) have already been proved by Debussche and Dettori (1995) when f is the derivative of a logarithmic term in the case of logarithmic free energy and $\sigma = 0$, and by Da Prato and Debussche (1996) in the case of an additive noise ($\sigma = 1$), when f is a polynomial of odd degree in a set of distributions, and if

$d \geqslant 2$ in the set of $L^2(D)$ functions only for Gaussian perturbations more regular than the white noise W. Our study is restricted to dimension d, where $d \in \{1, 2, 3\}$, on the domain $D = [0, \pi]^d$. We assume that f is a polynomial of degree 3. To prove existence and uniqueness of an almost surely (a.s.) continuous solution, we need some information on the Green function G of the operator $\partial/\partial t + \Delta^2$. This operator has a well-defined expression as a Fourier series, but this is not easy to use. We need other estimates of G; for this, we use a result from Eidelman and Ivasisen (1970) on Green's functions on a smooth domain. We have extended these estimates from smooth domains to the domain D in the Appendix. In the first part of this paper we use the estimate of the Green function to prove first the existence and uniqueness of the solution for a similar equation with truncated coefficients. This yields the existence and uniqueness of the solution on the time interval $[0, \tau[$, where τ is a stopping time. To prove that $\tau = +\infty$, we need a priori estimates in the space $L^\infty([0, T]; L^q(D))$, $q \geqslant 4$. For this we prove two different upper estimates and use an interpolation method. In these computations we need the degree of f not to be larger than 3. The approach is similar to that used by Gyöngy (1998) and Da Prato and Gatarek (1995) for another nonlinear parabolic stochastic equation with correlated noise, the Burgers stochastic equation. We then study the regularity of u and prove that if u_0 is Hölder continuous, then u also has Hölder regular trajectories. More precisely, in the case $d = 1$, if u_0 is differentiable ($u_0 \in \mathscr{C}^2(D)$), the solution u is differentiable with respect to the space variable.

In the second part of this paper, we study the existence of the density of $u(t, x)$ for $t > 0$ and $x \in]0, \pi[^d$. For this we use the Malliavin calculus (see Nualart 1995) associated with the space-time white noise W. Similar results have been obtained by Morien (1999) in the case of the stochastic Burgers equation. Concerning the Burgers equation, let us mention the work of Lanjri Zaidi and Nualart (1999), which proves the existence of a density under weaker conditions. This refinement seems to be more difficult to obtain in our case, because we do not have good lower estimates of the Green function.

The paper is organized as follows. We first state the hypotheses and the main results of this paper; then we give some lemmas on the Green function G. In Section 2, we establish the existence of the solution to the SPDE (1.1) and study its regularity. In Section 3, we prove the absolute continuity of the solution $u(t, x)$ for $t > 0$ and $x \in]0, \pi[^d$. Finally, the basic pointwise upper estimates of the Green function and of its derivatives are shown in the Appendix. As usual, constants C and c may change from one line to the next; we denote by C_α a constant which depends on some parameter α. We denote the space variable by x and the space integral by $\int \ldots \mathrm{d}x$, even if the dimension d is not 1, and denote by $\|\cdot\|_q$ the L^q-norm with respect to $\mathrm{d}x$.

1.1. Hypotheses and results

We make the following assumptions:

Assumption 1. *f is a polynomial of degree 3 with positive dominant coefficient.*

Assumption 2. *$\sigma : \mathbb{R} \mapsto \mathbb{R}$ is a bounded and Lipschitz function.*

Assumption 3. *u_0 belongs to $L^q(D)$ for $q \geqslant 4$.*

Assumption 4. *u_0 is a continuous function on D.*

Assumption 4′. *u_0 is an α-Hölder continuous function on D, $\alpha \in]0, 1[$.*

Assumption 4″. *$d = 1$ and u_0 is an α-Hölder continuous function on D, $\alpha \geqslant 2$; moreover, $u_0'(0) = u_0'(\pi) = 0$.*

Assumption 5. *The function σ does not vanish ($\sigma \neq 0$).*

Remark 1.1. In Assumption 1, we have assumed that f is of degree 3; the case of degree 1 is easier and its proof is omitted.

We will use Assumptions 4, 4′ and 4″ only in Sections 2.3 and 3 to prove the regularity property of the solution, and Assumption 5 only in Section 3.2 to obtain the existence of a density.

We suppose that $W = \{W(t, x),\ t \in [0, T];\ x \in D\}$ is a one-dimensional, $(d+1)$-parameter Wiener process on the probability space $(\Omega, \mathscr{F}, P)$; as usual, we set $\mathscr{F}_t = \sigma(W(s, x);\ s \leqslant t, x \in D)$.

Let A denote the operator $-\Delta$ on the domain $\mathscr{D}(A) = \{u \in H^2(D);\ \partial u/\partial n = 0 \text{ on } \partial D\}$. The following family $(\varepsilon_k)_{k\in\mathbb{N}^d}$ is a basis of eigenfunctions of A in $L^2(D)$. If $d = 1$,

$$\begin{aligned} \varepsilon_k(x) &= \cos(kx)\sqrt{\frac{2}{\pi}} \text{ if } k \neq 0, \\ \varepsilon_0(x) &= \frac{1}{\sqrt{\pi}}; \end{aligned} \tag{1.3}$$

and for $d \in \{2, 3\}$,

$$\varepsilon_k(x) = \prod_{i=1}^{d} \varepsilon_{k_i}(x_i), \tag{1.4}$$

associated with the eigenvalues $\lambda_k = \sum_{i=1}^d k_i^2 = |k|^2$. By convention, denote by $\mathbb{N}^{d,*}$ the set $\mathbb{N}^d \backslash \{0\}$. The semigroup $S(t)$ generated by $-A^2$ is denoted by $S(t) = \mathrm{e}^{-tA^2}$, that is, for $z \in L^2(D)$,

$$S(t)z = \sum_{k\in\mathbb{N}^d} \mathrm{e}^{-\lambda_k^2 t}\langle z, \varepsilon_k\rangle \varepsilon_k,$$

where $\langle\cdot, \cdot\rangle$ denotes the usual scalar product in $L^2(D)$; this is a convolution semigroup with the Green function G defined by

$$G(t, x, y) = \sum_{k\in\mathbb{N}^d} \mathrm{e}^{-\lambda_k^2 t}\varepsilon_k(x)\varepsilon_k(y). \tag{1.5}$$

The following lemma is proved in the Appendix.

Lemma 1.2. *There exist $C > 0$ and $c > 0$ such that, for $t \in]0, T]$, $x, y \in D$, α a d-dimensional exponent satisfying $|\alpha| \leqslant 3$:*

$$|G(t, x, y)| \leqslant \frac{C}{t^{d/4}} \exp\left(-c\frac{|x-y|^{4/3}}{|t|^{1/3}}\right), \tag{1.6}$$

$$|\partial_x^\alpha G(t, x, y)| \leqslant \frac{C}{t^{(d+|\alpha|)/4}} \exp\left(-c\frac{|x-y|^{4/3}}{|t|^{1/3}}\right), \tag{1.7}$$

$$|\partial_t G(t, x, y)| \leqslant \frac{C}{t^{(d+4)/4}} \exp\left(-c\frac{|x-y^{4/3}}{|t|^{1/3}}\right). \tag{1.8}$$

We wish to prove the existence and uniqueness of the solution to the SPDE (1.1). Since the derivative of W is formal, this can be made rigorous only in a weak sense, as in Walsh (1986) for the stochastic heat equation. We say that u is a weak solution of (1.10) if, for each $\phi \in C^4(D)$ such that $\partial\phi/\partial n = \partial\Delta\phi/\partial n = 0$ on ∂D, u satisfies

$$\int_D (u(t, x) - u_0(x))\phi(x)\,\mathrm{d}x = -\int_0^t\int_D \Delta^2\phi(x)u(s, x)\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_D \Delta\phi(x)f(u(s, x))\,\mathrm{d}x\,\mathrm{d}s$$
$$+ \int_0^t\int_D \phi(s, x)\sigma(u(s, x))W(\mathrm{d}x, \mathrm{d}s). \tag{1.9}$$

As usual, u is a solution to (1.9) if and only if it solves the following evolution equation:

$$u(t, x) = \int_D G(t, x, y)u_0(y)\,\mathrm{d}y + \int_0^t\int_D \Delta G(t-s, x, y)f(u(s, y))\,\mathrm{d}y\,\mathrm{d}s$$
$$+ \int_0^t\int_D G(t-s, x, y)\sigma(u(s, y))W(\mathrm{d}y, \mathrm{d}s) \tag{1.10}$$

for $x \in D$, $t \in [0, T]$.

The main results of this paper are the following:

Theorem 1.3. *Suppose that Assumptions 1–3 hold; there exists a unique process $u = \{u(t, x),\ t \in [0, T],\ x \in D\}$ in $L^\infty([0, T], L^q(D))$, that is adapted, that is to say, $u(t, x)$ is $\mathscr{F}_t$-measurable for $(t, x) \in [0, T] \times D$, and satisfies the evolution equation (1.10).*

Theorem 1.4. *If Assumptions 1–4 are satisfied, then the solution to (1.10) has a.s. continuous trajectories.*

If the Assumptions 1–3 and 4′ are satisfied, then the trajectories of the solution to (1.10) *are a.s. β-Hölder continuous in t and β'-Hölder continuous in x, with $\beta \leqslant \alpha/4$, $\beta < \frac{1}{2}(1 - d/4)$ and $\beta' \leqslant \alpha$, $\beta' < (2 - d/2)$.*

If $d = 1$ and Assumptions 1–3 and 4 (or 4″) are satisfied, then the trajectories of the solution to (1.10) *are a.s.* β*-Hölder continuous in* t *and* β'*-Hölder continuous in* x *on* $[t_0, T] \times D$ *for* $0 < t_0 < T$ *(or* $[0, T] \times D$*), with* $\beta < \frac{3}{8}$ *and* $\beta' < \frac{3}{2}$.

Theorem 1.5. *Suppose that Assumptions* 1–4 *and* 5 *are satisfied, and let* u *denote the solution to* (1.10). *For* $(t, x) \in]0, T] \times]0, \pi[^d$*, the law of* $u(t, x)$ *is absolutely continuous with respect to the Lebesgue measure on* $\mathbb{R}$.

1.2. The Green function

The following result, similar to Gyöngy (1998, Lemma 3.1), gives precise estimates of the regularizing effect of convolution with G, ΔG and G^2.

Lemma 1.6. *Let* J *be defined, for all* $v \in L^1([0, T], L^\rho(D))$, $0 \leqslant t_0 \leqslant t \leqslant T$, *and* $x \in D$, *by*

$$J(v)(t_0, t, x) = \int_{t_0}^{t} \int_D H(t-s, x, y) v(s, y)\, \mathrm{d}y\, \mathrm{d}s.$$

Then for any $\rho \in [1, \infty]$, $q \in [\rho, +\infty]$, *and* $1/r = 1/q - 1/\rho + 1 \in [0, 1]$, J *is a bounded operator from* $L^1([0, T], L^\rho(D))$ *to* $L^\infty([0, T], L^q(D))$ *such that:*

1. *If* $H(s, x, y) = G(s, x, y)$*, there is a constant* $C > 0$ *such that*

$$\|J(v)(t_0, t, \cdot)\|_q \leqslant C \int_{t_0}^{t} (t-s)^{(d/4)(1/r-1)} \|v(s, \cdot)\|_\rho \, \mathrm{d}s. \tag{1.11}$$

2. *If* $H(s, x, y) = \Delta G(s, x, y)$ *(if* $d = 3$*, we also need* $r < 3$*; and if* $d = 2$*,* $r \neq \infty$*), there is a constant* $C > 0$ *such that*

$$\|J(v)(t_0, t, \cdot)\|_q \leqslant C \int_{t_0}^{t} (t-s)^{-(d+2)/4 + d/(4r)} \|v(s, \cdot)\|_\rho \, \mathrm{d}s. \tag{1.12}$$

3. *If* $H(s, x, y) = G^2(s, x, y)$ *(if* $d = 3$*, we also need* $r < \frac{3}{2}$*, and if* $d = 2$*,* $r \neq \infty$*), there is a constant* $C > 0$ *such that*

$$\|J(v)(t_0, t, \cdot)\|_q \leqslant C \int_{t_0}^{t} (t-s)^{-d/2 + d/(4r)} \|v(s, \cdot)\|_\rho \, \mathrm{d}s. \tag{1.13}$$

Proof. The proof is based on the inequalities of Minkowski and Young. We only prove (1.11); the proof of the two other inequalities is similar. Minkowski's inequality and Lemma 1.2 imply that

$$\begin{aligned}\|J(v)(t_0, t, \cdot)\|_q &\leqslant \int_{t_0}^{t} \left\| \int_D G(t-s, \cdot, y) v(s, y)\, \mathrm{d}y \right\|_q \mathrm{d}s \\ &\leqslant c \int_{t_0}^{t} |t-s|^{-d/4} \left\| \int_D \exp\left(-c \frac{|\cdot - y|^{4/3}}{|t-s|^{1/3}} \right) |v(s, y)|\, \mathrm{d}y \right\|_q \mathrm{d}s.\end{aligned}$$

We observe that for $t>0$,

$$\int_{\mathbb{R}^d} \exp\left(-\frac{c|x|^{4/3}}{t^{1/3}}\right) \mathrm{d}x = Ct^{d/4}. \tag{1.14}$$

Let $r \geqslant 1$ be such that $1/q+1 = 1/r+1/\rho$ (r exists because $1 \geqslant 1/q+1-1/\rho \geqslant 0$). Young's inequality and (1.14) imply that

$$\begin{aligned}\|J(v)(t_0, t, \cdot)\|_q &\leqslant \int_{t_0}^t |t-s|^{-d/4} \|v(s, \cdot)\|_\rho \left\| \exp\left(-c\frac{|\cdot|^{4/3}}{|t-s|^{1/3}}\right)\right\|_r \mathrm{d}s \\ &\leqslant C\int_{t_0}^t |t-s|^{(d/4)(1/r-1)} \|v(s, \cdot)\|_\rho \,\mathrm{d}s.\end{aligned}$$

Note that, again using Young's inequality, the left-hand sides of (1.11)–(1.13) converge with the conditions which are made on r when $d = 2, 3$. □

The proof of following lemma, which is similar to the preceding one, will be omitted.

Lemma 1.7. *Let J_α be defined, for all $v \in L^1([0, T], L^\rho(D))$, $0 \leqslant t_0 \leqslant t \leqslant T$, and $x \in D$, $\alpha \in]0, 1[$, by*

$$J_\alpha(v)(t_0, t, x) = \int_{t_0}^t \int_D (t-s)^{-\alpha} \Delta G(t-s, x, y) v(s, y) \,\mathrm{d}y \,\mathrm{d}s.$$

Then for any $\rho \in [1, \infty]$, $q \in [\rho, +\infty]$, and $1/r = 1/q - 1/\rho + 1 \in [0, 1]$, J is a bounded operator from $L^1([0, T], L^\rho(D))$ to $L^\infty([0, T], L^q(D))$ such that, for $d/(4r) > \alpha + d/4 - 1/2$,

$$\|J_\alpha(v)(t_0, t, \cdot)\|_q \leqslant C\int_{t_0}^t (t-s)^{-\alpha-(d+2)/4+d/(4r)} \|v(s, \cdot)\|_\rho \,\mathrm{d}s. \tag{1.15}$$

We also need upper estimates of increments of the Green kernel G.

Lemma 1.8. *For $\gamma < 4-d$ and $\gamma \leqslant 2$, $\gamma' < 1 - d/4$, there exists $C>0$ such that, for $t>s$, $x, z \in D$,*

$$\int_0^t \int_D |G(t-u, x, y) - G(t-u, z, y)|^2 \,\mathrm{d}y \,\mathrm{d}u \leqslant C|x-z|^\gamma, \tag{1.16}$$

$$\int_0^s \int_D |G(t-u, x, y) - G(s-u, x, y)|^2 \,\mathrm{d}y \,\mathrm{d}u \leqslant C|t-s|^{\gamma'}, \tag{1.17}$$

$$\int_s^t \int_D |G(t-u, x, y)|^2 \,\mathrm{d}y \,\mathrm{d}u \leqslant C|t-s|^{\gamma'}. \tag{1.18}$$

Proof. To prove these inequalities, we use the series decomposition (1.5) of G. Since the family $(\varepsilon_k)_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(D)$, we have

$$\int_0^t\int_D |G(t-u, x, y) - G(t-u, z, y)|^2\,\mathrm{d}y\,\mathrm{d}u$$

$$= \int_0^t\int_D \left| \sum_{k\in\mathbb{N}^d} \varepsilon_k(y)\exp(-\lambda_k^2(t-u))[\varepsilon_k(z)-\varepsilon_k(x)] \right|^2 \mathrm{d}y\,\mathrm{d}u$$

$$= \sum_{k\in\mathbb{N}^d} \int_0^t \exp(-2\lambda_k^2(t-u))|\varepsilon_k(z)-\varepsilon_k(x)|^2\,\mathrm{d}u$$

$$= \sum_{k\in\mathbb{N}^{d,*}} |\varepsilon_k(z)-\varepsilon_k(x)|^2[1-\exp(-2\lambda_k^2 t)]\lambda_k^{-2}2^{-1}.$$

Since $k \neq 0$, we have

$$|\varepsilon_k(z)-\varepsilon_k(x)| \leqslant C([\lambda_k^{1/2}|z-x|] \wedge 1),$$

and for $\alpha \in [0, 1[$,

$$\int_0^t\int_D |G(t-u, x, y) - G(t-u, z, y)|^2\,\mathrm{d}y\,\mathrm{d}u \leqslant C|z-x|^{2\alpha} \sum_{k\in\mathbb{N}^{d,*}} \lambda_k^{-2+\alpha}(1-\exp(-2\lambda_k^2 t))$$

$$\leqslant C|z-x|^{2\alpha} \sum_{k\in\mathbb{N}^{d,*}} \lambda_k^{-2+\alpha}.$$

Comparing the series $\sum_{k\in\mathbb{N}^{d,*}}\lambda_k^{-u}$ with a multiple integral, we deduce that it converges if $u > d/2$. So taking $0 < \alpha < (4-d)/2$ and $\alpha \leqslant 1$, we deduce (1.16).

We now prove inequality (1.17). Using (1.5), we obtain

$$\int_0^s\int_D |G(t-u, x, y) - G(s-u, x, y)|^2\,\mathrm{d}y\,\mathrm{d}u$$

$$= \sum_{k\in\mathbb{N}^d} |\varepsilon_k(x)|^2 \int_0^s [\exp(-\lambda_k^2(t-s)-1]^2 \exp(-2\lambda_k^2(s-u))\,\mathrm{d}u$$

$$\leqslant C \sum_{k\in\mathbb{N}^{d,*}} \lambda_k^{-2}[\exp(-\lambda_k^2(t-s))-1]^2[1-\exp(-2\lambda_k^2 s)].$$

Since $|1-\exp(-2\lambda_k^2 s| \leqslant 1 \wedge (2\lambda_k^2 s)$, taking $\alpha \in [0, 1]$, we have, for $s \leqslant T$,

$$\int_0^s\int_D |G(t-u, x, y) - G(s-u, x, y)|^2\,\mathrm{d}y\,\mathrm{d}u \leqslant C \sum_{k\in\mathbb{N}^{d,*}} |t-s|^\alpha \lambda_k^{2\alpha-2}.$$

The series converges if and only if $\alpha < 1 - d/4$, which yields (1.17).

The proof of (1.18), similar to the previous one, is omitted. □

2. Existence and uniqueness of the solution to the Cahn–Hilliard stochastic equation

Suppose that Assumptions 1–3 are satisfied; we first solve the problem when the function f is truncated to have globally Lipschitz coefficients.

2.1. The case of truncated f

Let $n > 0$ and denote by $K_n : [0, +\infty[\to [0, +\infty[$ a $\mathscr{C}^1$ function such that $|K_n| \leqslant 1$, $|K_n'| \leqslant 2$ and

$$K_n(x) = \begin{cases} 1 & \text{if } x < n, \\ 0 & \text{if } x \geqslant n+1. \end{cases} \tag{2.1}$$

Let $\beta \in [q, +\infty[$ if $d \in \{1, 2\}$; if $d = 3$, choose β such that $q \leqslant \beta < 6q/(6-q)^+$. We will prove the existence and uniqueness of the solution to the SPDE

$$\begin{aligned} u_n(t, x) = &\int_D G(t, x, y)u_0(y)\mathrm{d}y + \int_0^t\int_D \Delta G(t-s, x, y)K_n(\|u_n(s, \cdot)\|_q)f(u_n(s, y))\,\mathrm{d}y\,\mathrm{d}s \\ &+ \int_0^t\int_D G(t-s, x, y)\sigma(u_n(s, y))W(\mathrm{d}y, \mathrm{d}s) \end{aligned} \tag{2.2}$$

in the set $\mathscr{H}$ of $L^q(D)$-valued $\mathscr{F}_t$-adapted random processes $u(t, \cdot)$ such that the norm

$$\|u\|_{\mathscr{H}} = \sup_{0\leqslant t\leqslant T} \mathrm{E}(\|u(t, \cdot)\|_q^\beta)^{1/\beta} \tag{2.3}$$

is finite. Define two nonlinear operators on $\mathscr{H}$ by

$$H_n : \mathscr{H} \to \mathscr{H} \tag{2.4}$$

$$u \mapsto \int_0^t\int_D \Delta G(t-s, x, y)K_n(\|u(s, \cdot)\|_q)f(u(s, y))\,\mathrm{d}y\,\mathrm{d}s$$

and

$$L : \mathscr{H} \to \mathscr{H} \tag{2.5}$$

$$u \mapsto \int_0^t\int_D G(t-s, x, y)\sigma(u(s, y))W(\mathrm{d}y, \mathrm{d}s).$$

We prove that for T small enough, H_n and L are contractions.

We begin by studying H_n. Let $u \in \mathscr{H}$; (1.12) applied with $1 \leqslant \rho \leqslant q$, $1 \leqslant r_1 < \infty$ if $d = 2$ and $r_1 < 3$ if $d = 3$ such that $1/q = 1/r_1 + 1/\rho - 1$, implies that

$$\|H_n(u)(t, \cdot)\|_q \leqslant \int_0^t |t-s|^{-(d+2)/4 + d/(4r_1)}\|f(u(s, \cdot))K_n(\|u(s, \cdot)\|_q)\|_\rho\,\mathrm{d}s.$$

Taking $3\rho = q$, we have $1/r_1 = 1 - 2/q$, which satisfies $r_1 < +\infty$ if $d = 2$ and $r_1 < 3$ if $q = 3$. Also

$$\|H_n(u)(t, \cdot)\|_q \leqslant C(n+1)^3 T^{1+d/(4r_1)-(d+2)/4},$$

so that

$$\|H_n(u)\|_{\mathscr{H}} < \infty; \tag{2.6}$$

hence H_n is an operator mapping $\mathscr{H}$ into itself.

Let u and v belong to $\mathscr{H}$; notice that

$$\|K_n(\|u(s,\cdot)\|_q)f(u(s,\cdot)) - K_n(\|v(s,\cdot)\|_q)f(v(s,\cdot))\|_\rho \leqslant C_n\|u(s,\cdot) - v(s,\cdot)\|_q. \tag{2.7}$$

Indeed, without any loss of generality, we suppose that $\|u(s,\cdot)\|_q \leqslant \|v(s,\cdot)\|_q$; then

$$\begin{aligned}
&\|K_n(\|u(s,\cdot)\|_q)f(u(s,\cdot)) - K_n(\|v(s,\cdot)\|_q)f(v(s,\cdot))\|_\rho \\
&\quad\leqslant K_n(\|v(s,\cdot)\|_q)\|f(v(s,\cdot)) - f(u(s,\cdot))\|_\rho \\
&\qquad + \|[K_n(\|u(s,\cdot)\|_q) - K_n(\|v(s,\cdot)\|_q)]f(u(s,\cdot))\|_\rho.
\end{aligned}$$

Assumption 1 implies that

$$\begin{aligned}
&K_n(\|v(s,\cdot)\|_q)\|f(v(s,\cdot)) - f(u(s,\cdot))\|_\rho \\
&\quad\leqslant C[\|u(s,\cdot) - v(s,\cdot)\|_q(1 + \|v(s,\cdot)\|_q^2 + \|u(s,\cdot)\|_q^2)] \times K_n(\|v(s,\cdot)\|_q) \\
&\quad\leqslant C\|u(s,\cdot) - v(s,\cdot)\|_q \times [1 + 2(n+1)^2].
\end{aligned} \tag{2.8}$$

For the second term, notice that if $\|v(s,\cdot)\|_q \geqslant \|u(s,\cdot)\|_q > n+1$, then

$$\|[K_n(\|u(s,\cdot)\|_q) - K_n(\|v(s,\cdot)\|_q)]f(u(s,\cdot))\|_\rho = 0;$$

this yields

$$\begin{aligned}
&\|[K_n(\|u(s,\cdot)\|_q) - K_n(\|v(s,\cdot)\|_q)]f(u(s,\cdot))\|_\rho \leqslant C(1 + (n+1)^3)|\,\|u(s,\cdot)\|_q - \|v(s,\cdot)\|_q| \\
&\quad\leqslant C(1 + (n+1)^3)\|u(s,\cdot) - v(s,\cdot)\|_q.
\end{aligned} \tag{2.9}$$

Inequalities (2.8) and (2.9) imply (2.7). Using (2.7), (1.12) and Hölder's inequality with respect to $(t-s)^{-(d+2)/4+d/(4r_1)}\,\mathrm{d}s$, we conclude that

$$\begin{aligned}
\|H_n(u)(t,\cdot) - H_n(v)(t,\cdot)\|_{\mathscr{H}}^\beta &\leqslant C_n \sup_{t\in[0,T]}\Bigg[\left(\int_0^t (t-s)^{-(d+2)/4+d/(4r_1)}\,\mathrm{d}s\right)^{\beta-1} \\
&\qquad \times \int_0^t (t-s)^{-(d+2)/4+d/(4r_1)}\mathrm{E}(\|u(s,\cdot) - v(s,\cdot)\|_q^\beta)\,\mathrm{d}s\Bigg] \\
&\leqslant C_n T^{\beta([2-d]/4+d/(4r_1))}\|u - v\|_{\mathscr{H}}^\beta.
\end{aligned} \tag{2.10}$$

If T satisfies $C_n T^{(d/(4r_1)+[2-d]/4)\beta} < 1$, the map H_n is a contraction on $\mathscr{H}$.

We now turn to L. Let u belong to $\mathscr{H}$; Burkholder's inequality, (1.6) and Assumption 2 imply that, for an absolute constant C depending on G,

$$\sup_{(t,x)\in[0,T]\times D} \mathrm{E}\left(\left|\int_0^t\int_D G(t-s,\, x,\, y)\sigma(u(s,\, y))W(\mathrm{d}y,\, \mathrm{d}s)\right|^{\beta}\right)$$
$$\leqslant \sup_{(t,x)\in[0,T]\times D} \mathrm{E}\left(\left|\int_0^t\int_D G^2(t-s,\, x,\, y)\sigma^2(u(s,\, y))\,\mathrm{d}y\,\mathrm{d}s\right|^{\beta/2}\right)$$
$$\leqslant C\|\sigma\|_\infty^2 < +\infty; \tag{2.11}$$

this implies $\|L(u)\|_{\mathscr{H}} < \infty$, for every $u \in \mathscr{H}$. Let $u,\, v \in \mathscr{H}$; then, since $\beta \geqslant q$, we have

$$\mathrm{E}(\|L(u)(s,\,\cdot) - L(v)(s,\,\cdot)\|_q^\beta) \leqslant C\int_D \mathrm{E}(|L(u)(s,\, x) - L(v)(s,\, x)|^\beta)\,\mathrm{d}x.$$

Burkholder's inequality and (1.13) applied with $\beta/2$ instead of q and $q/2$ instead of ρ, $1/r_2 = 1 + 2/\beta - 2/q > 0$ (indeed, if $d = 3$, the condition $\beta < 6q/(6-q)^+$ yields $r_2 \leqslant \frac{3}{2}$), imply, for $t \leqslant T$,

$$\mathrm{E}(\|L(u)(t,\,\cdot) - L(v)(t,\,\cdot)\|_q^\beta)$$
$$\leqslant \int_D \mathrm{E}\left(\left|\int_0^t\int_D G^2(t-s,\, x,\, y)|\sigma(u(s,\, y)) - \sigma(v(s,\, y))|^2\,\mathrm{d}y\,\mathrm{d}s\right|^{\beta/2}\right)\mathrm{d}x$$
$$\leqslant C\mathrm{E}\left(\left|\int_0^t (t-s)^{-d/2+d/(4r_2)}\|u(s,\,\cdot) - v(s,\,\cdot)\|_q^2\,\mathrm{d}s\right|^{\beta/2}\right)$$
$$\leqslant CT^{\beta(1-d/2+d/(4r_2))} \sup_{s\in[0,T]} \mathrm{E}(\|u(s,\,\cdot) - v(s,\,\cdot)\|_q^\beta). \tag{2.12}$$

Again, for T small enough, L is a contraction on $\mathscr{H}$.

In conclusion, the operator $H_n + L$ working on $\mathscr{H}$ is a contraction for $T \leqslant T_0$. Hence, it admits a unique fixed point in the set

$$\{u \in \mathscr{H} \text{ such that } u(0,\,\cdot) = u_0\}.$$

By a classical argument, because T_0 does not depend on u_0, we can construct by concatenation on every interval $[0,\, T]$ a unique solution u_n to the SPDE (2.2).

2.2. Existence and uniqueness of the solution to (1.10)

Again let $\beta \in [q,\, +\infty[$ if $d = 1,\, 2,$ and $\beta \in [q,\, 6q/(6-q)^+[$ if $d = 3$; set

$$\tau_n = \inf\{t \geqslant 0 |\, \|u_n(t,\,\cdot)\|_q \geqslant n\}.$$

By uniqueness of the solution to (2.2), the local property of stochastic integrals yields, for $m > n$, $u_m(t, \cdot) = u_n(t, \cdot)$ if $t \leqslant \tau_n$, so that we can define a process u by $u(t, \cdot) = u_n(t, \cdot)$ on $t \leqslant \tau_n$. Set $\tau_\infty = \lim_n \tau_n$; clearly u is a solution to the SPDE (1.10) on the interval $[0, \tau_\infty)$, and is unique. We just need to prove that $\tau_\infty = +\infty$ a.s., and use an argument similar to that of Da Prato and Debussche (1996, Section 2.1).

Let L be defined by (2.5) and set $v_n = u_n - L(u_n)$; then, for every $T > 0$, v_n is the weak solution on $[0, T]$ to the SPDE

$$\begin{aligned} &\frac{\partial v_n}{\partial t}(t, x) + \Delta^2 v_n(t, x) - \Delta[K_n(\|v_n(t, \cdot) + L(u_n(t, \cdot)\|_q) f(v_n(t, x) + L(u_n)(t, x))] = 0, \\ &v_n(0, \cdot) = u_0(\cdot), \\ &\frac{\partial v_n}{\partial n} = \frac{\partial \Delta v_n}{\partial n} = 0 \text{ on } \partial D. \end{aligned} \tag{2.13}$$

Again (2.13) is formal and can be made rigorous as in (1.9) by requiring that, for any $\phi \in C^4(D)$ such that ϕ satisfies (1.2),

$$\begin{aligned} \int_D [v_n(t, x) - u_0(t, x)]\phi(x)\,\mathrm{d}x = &- \int_0^t \int_D \Delta^2 \phi(x) v_n(s, x)\,\mathrm{d}x\,\mathrm{d}s \\ &+ \int_0^t \int_D \Delta\phi(x) K_n(\|v_n(s, \cdot) + L(u_n)(s, \cdot)\|_q) f(v_n(s, x) \\ &+ L(u_n(s, x))\,\mathrm{d}x\,\mathrm{d}s. \end{aligned}$$

Since σ is bounded, (2.11) yields, for any $p, \delta \in]1, +\infty[$,

$$\sup_n \sup_{t \in [0,T]} \sup_{x \in D} \mathrm{E}(|L(u_n)(t, x)|^{2p\delta}) < +\infty. \tag{2.14}$$

Lemma 1.8 and Burkholder's inequality imply that, for every $\gamma < (4 - d)$, $\gamma \leqslant 2$, $\gamma' < 1 - d/4$, and $\alpha > 1$, $T > 0$, $(t, t', x, x') \in [0, T]^2 \times D^2$, for all $n \in \mathbb{N}$:

$$\mathrm{E}(|L(u_n)(t, x) - L(u_n)(t', x')|^{2\alpha}) \leqslant C(|t - t'|^{\gamma'} + |x - x'|^{\gamma})^{\alpha}. \tag{2.15}$$

Inequalities (2.14) and (2.15) and the Garsia–Rodemich–Ramsay lemma (see Garsia 1972) yield, if $\|L(u_n)\|_\infty = \sup_{t \in [0,T]} \sup_{x \in D} |L(u_n)(t, x)|$,

$$\sup_n \mathrm{E}(\|L(u_n)\|_\infty^{2p\delta}) < \infty. \tag{2.16}$$

On the other hand, since $G(t, x, y) = G(t, y, x)$ and $\int_D |G(t, x, y)|\,\mathrm{d}y < +\infty$, for every $q \in [1, +\infty[$,

$$\sup_{t \in [0,T]} \|G_t u_0\|_q \leqslant C\|u_0\|_q. \tag{2.17}$$

We just need to prove a uniform upper estimate for $H_n(u_n)$. Since the functional H_n has a regularizing effect, we first show that u_n belongs to the sets $L^a([0, T]; L^q(D))$ for $2 \leqslant a < +\infty$. As in Da Prato and Debussche (1996), we shall prove a priori estimates on v_n.

Recall that for $A = -\Delta$, $\alpha \neq 0$ and $u \in \mathrm{Dom}(A^\alpha)$,

$$A^\alpha u = \sum_{k \in \mathbb{N}^{d,*}} \lambda_k^\alpha \langle \varepsilon_k, u \rangle \varepsilon_k,$$

$A^\alpha u$ exists for every u such that $\sum_{k\in\mathbb{N}^{d,*}} \lambda_k^{2\alpha} \langle e_k, u\rangle^2 < \infty$. In what follows, for a function $u : [0, T] \times D \to \mathbb{R}$, we will set

$$m(u)(t) = \langle \varepsilon_0, u(t, \cdot)\rangle = \pi^{-d/2} \int_D u(t, x)\,\mathrm{d}x \quad \text{and} \quad \tilde{u}(t, y) = u(t, y) - m(u)(t).$$

Notice that $A^\alpha \tilde{u} = A^\alpha u$ for $\alpha \neq 0$, $u \in \mathrm{Dom}(A^\alpha)$. Apply A^{-1} to equation (2.24) and take its scalar product in $L^2(D)$ with $\tilde{v}_n(t, \cdot)$; this leads to

$$\begin{aligned} &\|A^{-1/2}\tilde{v}_n(t, \cdot)\|_2^2 - \|A^{-1/2}\tilde{v}_n(0, \cdot)\|_2^2 + \int_0^t \|A^{1/2} v_n(s, \cdot)\|_2^2\,\mathrm{d}s \\ &\quad + \int_0^t K_n(\|v_n(s, \cdot) + L(u_n)(s, \cdot)\|_q) \int_D f(v_n(s, x) + L(u_n)(s, x))\tilde{v}_n(s, x)\,\mathrm{d}x\,\mathrm{d}s = 0. \qquad (2.18) \end{aligned}$$

This equation is justified because v_n belongs to $L^\infty([0, T], L^q(D))$; since for $q \geqslant 2$, $L^q(D) \subset \mathrm{Dom}(A^{-1/2})$ and the first two terms are well defined so that $\int_0^t \|A^{1/2} v_n(s, \cdot)\|_2^2\,\mathrm{d}s$ converges too and $v_n(s, \cdot)$ belongs to $\mathrm{Dom}(A^{-1/2})$.

Let us deal with the last term of (2.18):

$$\begin{aligned} &\int_D f(v_n(t, x) + L(u_n)(t, x))\tilde{v}_n(t, x)\,\mathrm{d}x \qquad (2.19) \\ &= \int_D f(v_n(t, x) + L(u_n)(t, x))[v_n(t, x) + L(u_n)(t, x)]\,\mathrm{d}x \\ &\quad - \int_D f(v_n(t, x) + L(u_n)(t, x))[m(v_n)(t) + L(u_n)(t, x)]\,\mathrm{d}x. \end{aligned}$$

The polynomial f is of degree 3 with positive dominant coefficient; hence, $\lim_{|x|\to+\infty} xf(x) = +\infty$, and there exist a, C and $c > 0$ such that

$$xf(x) \geqslant \tfrac{7}{8}ax^4 - c, \qquad |f(x)| \leqslant \tfrac{5}{4}a|x|^3 + C.$$

The lower estimate of $xf(x)$ implies

$$\int_D f(v_n(t, x) + L(u_n(t, x)))(v_n(t, x) + L(u_n)(t, x))\,\mathrm{d}x \geqslant \tfrac{7}{8}a\|v_n(t, \cdot) + L(u_n)(t, \cdot)\|_4^4 - c. \qquad (2.20)$$

The upper estimate of $|f|$ and Hölder's inequality with conjugate exponents 4 and $\frac{4}{3}$ yield:

$$\int_D f(v_n(t, x) + L(u_n(t, x)))(m(v_n)(t) + L(u_n)(t, x))\,\mathrm{d}x$$

$$\leqslant \tfrac{5}{4}a\int_D |v_n(t, x) + L(u_n)(t, x)|^3(|m(v_n)(t)| + |L(u_n)(t, \cdot)|)\,\mathrm{d}x$$

$$+ C\int_D (|m(v_n)(t)| + |L(u_n)(t, \cdot)|)\,\mathrm{d}x$$

$$\leqslant \tfrac{5}{4}a[\|v_n(t, \cdot) + L(u_n)(t, \cdot)\|_4^3|m(v_n)(t)| + \|v_n(t, \cdot) + L(u_n)(t, \cdot)\|_4^3\|L(u_n)(t, \cdot)\|_4]$$

$$+ C(|m(v_n)(t)| + \|L(u_n)(t, \cdot)\|_4).$$

Notice that there exists $c > 0$ such that, for every x and y, $|x|^3|y| \leqslant \frac{1}{4}|x|^4 + c|y|^4$; this implies

$$\int_D f(v_n(t, x) + L(u_n(t, x)))(m(v_n)(t) + L(u_n)(t, x))\,\mathrm{d}x$$

$$\leqslant \frac{5a}{8}\|v_n(t, \cdot) + L(u_n)(t, \cdot)\|_4^4 + C[m(v_n(t))^4 + \|L(u_n)(t, \cdot)\|_4^4]. \tag{2.21}$$

K_n is a positive bounded function; using (2.19)–(2.21), (2.18) yields

$$\|A^{-1/2}\tilde{v}_n(t, \cdot)\|_2^2 - \|A^{-1/2}\tilde{v}_n(0, \cdot)\|_2^2 + \int_0^t \|A^{1/2}v_n(s, \cdot)\|_2^2\,\mathrm{d}s$$

$$+ \frac{a}{4}\int_0^t K_n(\|v_n(s, \cdot) + L(u_n)(s, \cdot)\|_q)\|v_n(s, \cdot) + L(u_n)(s, \cdot)\|_4^4\,\mathrm{d}s$$

$$\leqslant \int_0^t C(1 + m(v_n)(s)^4 + \|L(v_n)(s, \cdot)\|_4^4)\,\mathrm{d}s. \tag{2.22}$$

Taking the scalar product of the solution to (2.24) with the function ε_0, we obtain

$$\frac{\partial}{\partial t}\langle v_n(t, \cdot), \varepsilon_0\rangle = 0,$$

$$\langle v_n(0, \cdot), \varepsilon_0\rangle = \langle u_0(\cdot), \varepsilon_0\rangle;$$

hence for $m(u_0) = \langle \varepsilon_0, u_0\rangle$, $m(v_n)(t) = m(u_0)$. Since $A^{-1/2}\tilde{u}_0 = A^{-1/2}u_0$, equation (2.22) yields that, for every $T > 0$,

$$\|A^{-1/2}\tilde{v}_n(T, \cdot)\|_2^2 + \int_0^T \|A^{1/2}v_n(t, \cdot)\|_2^2\,\mathrm{d}t$$

$$+ \int_0^T \frac{a}{4}K_n(\|v_n(t, \cdot) + L(u_n)(t, \cdot)\|_q)\|v_n(t, \cdot) + L(u_n)(t, \cdot)\|_4^4\,\mathrm{d}t$$

$$\leqslant C\int_0^T (1 + m(u_0)^4 + \|L(u_n)(t, \cdot)\|_4^4)\mathrm{d}t + \|A^{-1/2}u_0(\cdot)\|_2^2.$$

This yields

$$\int_0^T K_n(\|v_n(t,\cdot)+L(u_n)(t,\cdot)\|_q)\|v_n(t,\cdot)+L(u_n)(t,\cdot)\|_4^4\,dt$$
$$\leqslant C_T(1+m(u_0)^4+\|L(u_n)\|_\infty^4)+\|A^{-1/2}u_0(\cdot)\|_2^2. \tag{2.23}$$

We need another estimate, for this we denote by v_n^m the Galerkin approximation of v_n. We define P_m to be the orthogonal projector on $\text{Span}\{\varepsilon_0, \ldots, \varepsilon_m\}$. For every ω, v_n^m is the 'strong' solution of the following PDE:

$$\begin{aligned}&\frac{\partial v_n^m}{\partial t}(t,x)+\Delta^2 v_n^m(t,x)\\&\quad-\Delta[K_n(\|v_n^m(t,\cdot)+L(u_n)(t,\cdot)\|_q)P_m(f(v_n^m(t,x)+L(u_n)(t,x)))]=0,\\&v_n^m(0,\cdot)=P_m(u_0(\cdot)),\\&\frac{\partial v_n^m}{\partial n}=\frac{\partial\Delta v_n^m}{\partial n}=0 \text{ on } \partial D.\end{aligned} \tag{2.24}$$

The proof of existence and uniqueness of the processes v_n^m is classical; we use deterministic methods and prove that v_n^m is unique on some time interval $[0, t_n^m[$. The following a priori estimates will prove that $t_n^m=+\infty$.

The boundary conditions satisfied by v_n^m and the Green formula yield

$$\int_D \Delta^2 v_n^m(t,x)\times v_n^m(t,x)\,dx=\|\Delta v_n^m(t,x)\|_2^2.$$

We now take the scalar product in $L^2(D)$ of (2.24) with v_n; using the Green formula once more, we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\|v_n^m(t,\cdot)\|_2^2+\int_D\Delta^2 v_n^m(t,x)\times v_n^m(t,x)\,dx$$
$$-K_n(\|v_n^m(t,\cdot)+L(u_n)(t,\cdot)\|_q)\int_D f(v_n^m(t,x)+L(u_n)(t,x))\Delta v_n^m(t,x)\,dx=0.$$

Thus

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial t}\|v_n^m(t,\cdot)\|_2^2+\|\Delta v_n^m(t,\cdot)\|_2^2\\&\quad=K_n(\|v_n^m(t,\cdot)+L(u_n)(t,\cdot)\|_q)\\&\qquad\times\Bigg[\int_D[f(v_n^m(t,x)+L(u_n)(t,x))-f(v_n^m(t,x))]\Delta(v_n^m(t,x))\,dx\\&\qquad\quad+\int_D f(v_n^m(t,x))\Delta v_n^m(t,x)\,dx\Bigg].\end{aligned} \tag{2.25}$$

f is a polynomial of degree 3; thus, for every $x, y\in\mathbb{R}$,

$$|f(x+y)-f(x)| \leqslant C|y|[1+x^2+y^2]. \tag{2.26}$$

Let $\epsilon > 0$; Schwarz's inequality, (2.26) and the inequality $xy \leqslant x^2/\epsilon + \epsilon y^2$ imply

$$\begin{aligned} &\left|\int_D [f(v_n^m(t,x)+L(u_n)(t,x)) - f(v_n^m(t,x))]\Delta v_n^m(t,x)\,\mathrm{d}x\right| \\ &\quad \leqslant \int_D C|L(u_n)(t,x)|[1+|L(u_n)(t,x)|^2+|v_n^m(t,x)|^2]|\Delta v_n^m(t,x)|\,\mathrm{d}x \\ &\quad \leqslant C\|L(u_n)\|_\infty[1+\|L(u_n)\|_\infty^2+\|v_n^m(t,\cdot)^2\|_2]\|\Delta v_n^m(t,\cdot)\|_2 \\ &\quad \leqslant C\epsilon\|\Delta v_n^m(t,\cdot)\|_2^2 + \frac{C}{\epsilon}[1+\|L(u_n)\|_\infty^4+\|v_n^m(t,\cdot)\|_4^4]\|L(u_n)\|_\infty^2. \end{aligned} \tag{2.27}$$

Let us now turn to the second term of the right-hand side of (2.25). First, let us prove that $\int_D (v_n^m(t,x)^3)\Delta v_n^m(t,x)\,\mathrm{d}x$ is negative. Integrating by parts and using the boundary conditions on v_n^m, we find if $d=2$ (the other cases are similar),

$$\begin{aligned} &\int_0^\pi\int_0^\pi \frac{\partial^2}{\partial x_1^2}(v_n^m(t,x_1,x_2)) \times v_n^m(t,x_1,x_2)^3\,\mathrm{d}x_1\,\mathrm{d}x_2 \\ &\quad = \int_0^\pi \left[\frac{\partial}{\partial x_1}(v_n^m(t,x_1,x_2)) \times v_n^m(t,x_1,x_2)^3\right]_{x_1=0}^{x_1=\pi}\mathrm{d}x_2 \\ &\qquad - 3\int_0^\pi\int_0^\pi \left[\frac{\partial}{\partial x_1}(v_n^m(t,x_1,x_2))\right]^2 v_n^m(t,x_1,x_2)^2\,\mathrm{d}x_1\,\mathrm{d}x_2 \\ &\quad = -3\int_0^\pi\int_0^\pi \left[\frac{\partial}{\partial x_1}(v_n^m(t,x_1,x_2))\right]^2 v_n^m(t,x_1,x_2)^2\,\mathrm{d}x_1\,\mathrm{d}x_2 \leqslant 0; \end{aligned}$$

this yields $\int_D v_n^m(t,x)^3\Delta v_n^m(t,x)\,\mathrm{d}x \leqslant 0$. Since the main coefficient of f is positive, this implies

$$\begin{aligned} \int_D f(v_n^m(t,x))\Delta v_n^m(t,x)\,\mathrm{d}x &\leqslant C\int_D (1+|v_n^m(t,x)|^2)|\Delta v_n^m(t,x)|\,\mathrm{d}x \\ &\leqslant C\epsilon\|\Delta v_n^m(t,\cdot)\|_2^2 + \frac{C}{\epsilon}[1+\|v_n^m(t,\cdot)\|_4^4]. \end{aligned} \tag{2.28}$$

Furthermore, $\|K_n\|\infty \leqslant 1$; using (2.27) and (2.28), equation (2.25) becomes

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t}\|v_n^m(t,\cdot)\|_2^2 + \|\Delta v_n^m(t,\cdot)\|_2^2 &\leqslant C_1\epsilon\|\Delta v_n^m(t,\cdot)\|_2^2 + \frac{C_2}{\epsilon}[1+\|L(u_n)\|_\infty^4+\|v_n^m(t,\cdot)\|_4^4] \\ &\quad \times (1+\|L(u_n)\|_\infty^2)K_n(\|v_n^m(t,\cdot)+L(u_n)(t,\cdot)\|_q). \end{aligned}$$

Choose $\epsilon = 1/(2C_1)$; then

$$\frac{1}{2}\frac{\partial}{\partial t}\|v_n^m(t,\cdot)\|_2^2+\frac{1}{2}\|\Delta v_n^m(t,\cdot)\|_2^2$$
$$\leqslant 2C_1C_2[1+\|L(u_n)\|_\infty^4+\|v_n^m(t,\cdot)\|_4^4](1+\|L(u_n)\|_\infty^2)K_n(\|v_n^m(t,\cdot)+L(u_n)(t,\cdot)\|_q).$$

Integrating on $[0, t]$, we obtain

$$\|v_n^m(t,\cdot)\|_2^2+\int_0^t\|\Delta v_n^m(s,\cdot)\|_2^2\,\mathrm{d}s\leqslant\|P_mu_0\|_2^2+CT(1+\|L(u_n)\|_\infty^6)$$
$$+C(1+\|L(u_n)\|_\infty^2)\int_0^t\|v_n^m(s,\cdot)\|_4^4K_n(\|v_n^m(s,\cdot)+L(u_n)(s,\cdot)\|_q)\,\mathrm{d}s.$$

Therefore the equality $m(v_n^m(t,\cdot))=m(v_n(t,\cdot))=m(u_0(\cdot))$ implies

$$\|v_n^m(t,\cdot)\|_2^2+\int_0^t[\|\Delta v_n^m(s,\cdot)\|_2^2+m(v_n^m(s,\cdot))^2]\,\mathrm{d}s\leqslant\|u_0\|_2^2+CT(1+\|L(u_n)\|_\infty^6+m(u_0))$$
$$+C(1+\|L(u_n)\|_\infty^2)\int_0^t\|v_n^m(s,\cdot)\|_4^4K_n(\|v_n^m(s,\cdot)+L(u_n)(s,\cdot)\|_q)\,\mathrm{d}s.$$

The norm $(\|\Delta\cdot\|_{L^2(D)}^2+m(\cdot)^2)^{1/2}$ is equivalent to the Sobolev norm of $W^{2,2}(D)$ (see, for example, Da Prato and Debussche 1996, p. 245). The sequence $(v_n^m)_{m\in\mathbb{N}}$ is bounded in $L^2([0,T],W^{2,2}(D))$. Thus, $t_n^m=\infty$ and this sequence converges as $m\to+\infty$ in the weak* topology of $L^2([0,T],W^{2,2}(D))$. Its weak limit is the weak solution to (2.13) and hence is equal to v_n. Therefore, v_n belongs to $L^2([0,T],W^{2,2}(D))$, and we can repeat the preceding computation with v_n instead of v_n^m, which yields

$$\|v_n(t,\cdot)\|_2^2+\int_0^t\Big[\|\Delta v_n(s,\cdot)\|_2^2+m(v_n(s,\cdot))^2\Big]\,\mathrm{d}s\leqslant\|u_0\|_2^2+CT(1+\|L(u_n)\|_\infty^6+m(u_0))$$
$$+C(1+\|L(u_n)\|_\infty^2)\int_0^t\|v_n(s,\cdot)\|_4^4K_n(\|v_n(s,\cdot)+L(u_n)(s,\cdot)\|_q)\,\mathrm{d}s.$$

Inequality (2.23) and Schwarz's inequality imply that

$$\|v_n(t,\cdot)\|_2^2+\int_0^t\Big[\|\Delta v_n(s,\cdot)\|_2^2+m(v_n(s,\cdot))^2\Big]\,\mathrm{d}s$$
$$\leqslant\|u_0\|_2^2+C_T(1+\|L(u_n)\|_\infty^6)+C_T(1+\|L(u_n)\|_\infty^2)[\|A^{-1/2}u_0\|_2^2+m(u_0)^4].$$

Inequality (2.16) yields that, for $\beta\in\,]1,+\infty[$,

$$\sup_n \mathrm{E}\left(\sup_{t\in[0,T]}\|v_n(t,\cdot)\|_2^{2\beta}\right)<\infty,\tag{2.29}$$

$$\sup_n \mathrm{E}\left(\left[\int_0^T\{\|\Delta v_n(t,\cdot)\|_2^2+m(v_n(t,.))^2\}\,\mathrm{d}t\right]^\beta\right)<\infty.\tag{2.30}$$

Furthermore, by Sobolev's embedding theorem (Adams 1975, Corollary 5.16) there exists $C>0$ such that for $r \geqslant 2$, $d<4$, if $u \in W^{2,2}(D)$,

$$\|u\|_{L^r(D)} \leqslant C\|u\|_{W^{2,2}(D)}.$$

Thus, (2.30) becomes for $2 \leqslant r < +\infty$,

$$\sup_n \mathrm{E}\left(\left[\int_0^T \|v_n(t,\cdot)\|_r^2 \,\mathrm{d}t\right]^\beta\right) < \infty. \tag{2.31}$$

Inequalities (2.16), (2.29) and (2.31) imply for $2 \leqslant r < +\infty$,

$$\sup_n \mathrm{E}\left(\sup_{t\in[0,T]} \|u_n(t,\cdot)\|_2^{2\beta}\right) < \infty, \tag{2.32}$$

$$\sup_n \mathrm{E}\left(\left[\int_0^T \|u_n(t,\cdot)\|_r^2 \,\mathrm{d}t\right]^\beta\right) < \infty. \tag{2.33}$$

Let us use the interpolation method to prove that u_n belongs a.s. to $L^a([0,T], \mathrm{L}^q(D))$, with $+\infty > r \geqslant q \geqslant 2$, $a \geqslant 1 \vee 2q/r$. Hölder's inequality implies, if $q=(1-\lambda)2+r\lambda$, for $\lambda \in [0,1]$, that

$$\int_0^T \|u_n(t,\cdot)\|_q^a \,\mathrm{d}t \leqslant \int_0^T \|u_n(t,\cdot)\|_2^{2a(1-\lambda)/q} \|u_n(t,\cdot)\|_r^{ar\lambda/q} \,\mathrm{d}t.$$

Taking $\lambda = 2q/(ar)$, we obtain

$$\int_0^T \|u_n(t,\cdot)\|_q^a \,\mathrm{d}t \leqslant \sup_{t\in[0,T]} \|u_n(t,\cdot)\|_2^{[2/q]a(1-\lambda)} \times \int_0^T \|u_n(t,\cdot)\|_r^2 \,\mathrm{d}t;$$

(2.32) and (2.33) imply that for $q \in [2,\infty[$ and $a \in [q,+\infty[$,

$$\sup_n \mathrm{E}\left(\left[\int_0^T \|u_n(t,\cdot)\|_q^a \,\mathrm{d}t\right]^\beta\right) < \infty. \tag{2.34}$$

Using (1.12) with $q \geqslant 4$ and $\rho = q/3$, so that $1/r = 1-2/q$, we obtain

$$\|H_n(u_n)(t,\cdot)\|_q \leqslant C\int_0^t (t-s)^{-(d+2)/4+d/(4r)}(\|u_n(s,\cdot)\|_q^3+1)\,\mathrm{d}s.$$

Let γ, $\gamma' \in]1,+\infty[$ be conjugate exponents, with γ close enough to unity to ensure $(-(d+2)/4+d/(4r))\gamma > -1$; then Hölder's inequality implies

$$\|H_n(u_n)(t,\cdot)\|_q \leqslant C\left[\int_0^t (t-s)^{(-[d+2]/4+[d/(4r)])\gamma}\,\mathrm{d}s\right]^{1/\gamma}\left[\int_0^t (\|u_n(s,\cdot)\|_q^3+1)^{\gamma'}\,\mathrm{d}s\right]^{1/\gamma'}.$$

Using (2.34), we obtain

$$\sup_n \mathrm{E}\left(\sup_{t\in[0,T]} H_n(u_n)(t,\cdot)\|_q^{\beta} \right) < \infty; \tag{2.35}$$

(2.16), (2.17) and (2.35) imply that, for $\beta \in [q, +\infty[$, and if $d = 3$ for $\beta \in]q, 6q/(6-q)^+[$ with $q \geqslant 4$,

$$\sup_n \mathrm{E}\left(\sup_{t\in[0,T]} \|u_n(t,\cdot)\|_q^{\beta} \right) < \infty.$$

We can now conclude that $\tau_\infty = +\infty$ a.s.; indeed, for every $T > 0$,

$$P(\tau_n \leqslant T) = P\left(\sup_{t\leqslant T} \|u_n(t,\cdot)\|_q \geqslant n \right) \leqslant \mathrm{E}\left(\sup_{t\leqslant T} \|u_n(t,\cdot)\|_q^{2\beta} \right) n^{-2\beta}, \tag{2.36}$$

so that $\lim_{n\to\infty} P(\tau_n \leqslant T) = 0$. Therefore, we can construct the solution to the SPDE (1.10) on every interval $[0, T]$; this completes the proof of Theorem 1.3. □

2.3. Path regularity of *u*

We prove Theorem 1.4, and study separately each term on the right-hand side of (1.10). Let us prove two lemmas on the regularity of Gu_0.

Lemma 2.1. *If u_0 is continuous, the function $G.u_0(*)$ is continuous.*

Proof. The function Gu_0 is continuous on $[a, T] \times D$, for $a > 0$, because the series which defines G is absolutely convergent on $[a, T] \times D$. We just need to prove continuity at $t = 0$. Let $x_0 \in D$ be fixed; at $\epsilon > 0$ given, there exists $\eta > 0$ such that $|x_0 - y| < \eta$ implies $|u_0(y) - u_0(x_0)| < \epsilon$. Using the fact that G is a semigroup and (1.6), we find

$$\begin{aligned} |G_t u_0(x_0) - u_0(x_0)| &= \left| \int_D G(t, x_0, y)(u_0(y) - u_0(x))\,\mathrm{d}y \right| \\ &\leqslant \epsilon \int_{|y-x_0|\leqslant\eta} |G(t, x_0, y)|\,\mathrm{d}y + M \int_{|y-x_0|\geqslant\eta} |G(t, x_0, y)|\,\mathrm{d}y \\ &\leqslant C\epsilon + C \int_{|z| > \eta t^{-1/4}} \exp(-z^{4/3})\,\mathrm{d}z. \end{aligned}$$

The last integral on the right-hand side converges to 0 as t tends to 0; this concludes the proof. □

Lemma 2.2. *If u_0 belongs to $\mathscr{C}^{\alpha}(D)$, for $0 < \alpha < 1$, the function $G.u_0(*)$ belongs to $\mathscr{C}^{\alpha/4,\alpha}([0, T], D)$.*

Proof. The proof is inspired by Bally *et al*. (1995, Lemma A.2), in the case of the heat kernel. Let us study the time increment. Because G is a semigroup, using (1.6) we find that

$$
\begin{aligned}
|G_t u_0(x) - G_s u_0(x)| &= \left| \int_D \int_D G(s, x, y) G(t-s, y, z) u_0(z)\, \mathrm{d}y\, \mathrm{d}z - \int_D G(s, x, y) u_0(y)\, \mathrm{d}y \right| \\
&= \left| \int_D G(s, x, y) \left(\int_D G(t-s, y, z)(u_0(z) - u_0(y))\, \mathrm{d}z \right) \mathrm{d}y \right| \\
&\leqslant C \int_D |G(s, x, y)| \int_D G(t-s, y, z) \| z - y|^\alpha\, \mathrm{d}z\, \mathrm{d}y \\
&\leqslant C \int_D |G(s, x, y)| |t-s|^{\alpha/4}\, \mathrm{d}y \\
&\leqslant C|t-s|^{\alpha/4}.
\end{aligned}
$$

For the space variable, we prove the regularity with respect to the first space coordinate; the proof is similar for the other ones. If $d \geqslant 2$, we set $x = (x_1, \tilde{x})$, where $\tilde{x} = (x_2, \ldots, x_d)$ and $D = [0, \pi] \times \tilde{D}$. With this notation, we find that

$$
G(t, x, y) = \sum_{k \in \mathbb{N}^d} \exp(-\lambda_k^2 t) \varepsilon_{k_1(x_1)} \varepsilon_{\tilde{k}}(\tilde{x}) \varepsilon_{k_1(y_1)} \varepsilon_{\tilde{k}}(\tilde{y}).
$$

Notice that

$$
\varepsilon_{k_1}(x_1) \varepsilon_{k_1}(y_1) = \tfrac{1}{2}[\varepsilon_{k_1}(x_1 + y_1) + \varepsilon_{k_1}(x_1 - y_1)],
$$

and hence

$$
G(t, x, y) = \phi_t(x_1 + y_1, \tilde{x}, \tilde{y}) + \phi_t(x_1 - y_1, \tilde{x}, \tilde{y}), \tag{2.37}
$$

with $\phi_t(x_1, \tilde{x}, \tilde{y}) = \frac{1}{2} G(t, (x_1, \tilde{x}), (0, \tilde{y}))$. Furthermore, $\phi_t(x_1 + 2\pi, \tilde{x}, \tilde{y}) = \phi_t(x_1, \tilde{x}, \tilde{y}) = \phi_t(-x_1, \tilde{x}, \tilde{y})$ and

$$
\sup_{t \in [0,T]} \sup_{(x_1, \tilde{x}) \in D} \int_D |\phi_T(x_1, \tilde{x}, \tilde{y})|\, \mathrm{d}y \leqslant C. \tag{2.38}
$$

Take $x = (x_1, \tilde{x})$ and $x' = (x_1', \tilde{x})$, with $x_1 > x_1'$; we have

$$
\begin{aligned}
Gu_0(t,x) - Gu_0(t,x') &= \int_0^\pi \int_{\tilde{D}} [\phi_t(x_1+y_1,\tilde{x},\tilde{y}) + \phi_t(x_1-y_1,\tilde{x},\tilde{y})] u_0(y)\,\mathrm{d}\tilde{y}\,\mathrm{d}y_1 \\
&\quad - \int_0^\pi \int_{\tilde{D}} [\phi_t(x'_1+y_1,\tilde{x},\tilde{y}) + \phi_t(x'_1-y_1,\tilde{x},\tilde{y})] u_0(y)\,\mathrm{d}\tilde{y}\,\mathrm{d}y_1 \\
&= \int_{x_1-x'_1)}^{\pi+(x_1-x'_1)} \int_{\tilde{D}} \phi_t(x'_1+y_1,\tilde{x},\tilde{y}) u_0(y_1-(x_1-x'_1),\tilde{y})\,\mathrm{d}\tilde{y}\,\mathrm{d}y_1 \\
&\quad - \int_0^\pi \int_{\tilde{D}} \phi_t(x'_1+y_1,\tilde{x},\tilde{y}) u_0(y)\,\mathrm{d}\tilde{y}\,\mathrm{d}y_1 \\
&\quad + \int_{-(x_1-x'_1)}^{\pi-(x_1-x'_1)} \int_{\tilde{D}} \phi_t(x'_1-y_1,\tilde{x},\tilde{y}) u_0(y_1+(x_1-x'_1),\tilde{y})\,\mathrm{d}\tilde{y}\,\mathrm{d}y_1 \\
&\quad - \int_0^\pi \int_{\tilde{D}} \phi_t(x'_1-y_1,\tilde{x},\tilde{y}) u_0(y)\,\mathrm{d}\tilde{y}\,\mathrm{d}y_1 \\
&= \sum_{i=1}^4 D_i(t,x,x'),
\end{aligned}
$$

with

$$
\begin{aligned}
D_1(t,x,x') &= \int_{x_1-x'_1}^{\pi} \int_{\tilde{D}} \phi_t(x'_1+y_1,\tilde{x},\tilde{y})[u_0(y_1-(x_1-x'_1),\tilde{y}) - u_0(y)]\,\mathrm{d}\tilde{y}\,\mathrm{d}y_1 \\
D_2(t,x,x') &= \int_0^{\pi-(x_1-x'_1)} \int_{\tilde{D}} \phi_t(x'_1-y_1,\tilde{x},\tilde{y})[u_0(y_1+(x_1-x'_1),\tilde{y}) - u_0(y)]\,\mathrm{d}\tilde{y}\,\mathrm{d}y_1 \\
D_3(t,x,x') &= -\int_0^{(x_1-x'_1)} \int_{\tilde{D}} \phi_t(x'_1+y_1,\tilde{x},\tilde{y}) u_0(y)\,\mathrm{d}\tilde{y}\,\mathrm{d}y_1 \\
&\quad + \int_{-(x_1-x'_1)}^{0} \int_{\tilde{D}} \phi_t(x'_1-y_1,\tilde{x},\tilde{y}) u_0(y_1+(x_1-x'_1),\tilde{y})\,\mathrm{d}\tilde{y}\,\mathrm{d}y_1 \\
D_4(t,x,x') &= \int_\pi^{\pi+(x_1-x'_1)} \int_{\tilde{D}} \phi_t(x'_1+y_1,\tilde{x},\tilde{y}) u_0(y_1-(x_1-x'_1),\tilde{y})\,\mathrm{d}\tilde{y}\,\mathrm{d}y_1 \\
&\quad - \int_{\pi-(x_1-x'_1)}^{\pi} \int_{\tilde{D}} \phi_t(x'_1-y_1,\tilde{x},\tilde{y}) u_0(y)\,\mathrm{d}\tilde{y}\,\mathrm{d}y_1.
\end{aligned}
$$

Using (2.38) and the fact that u_0 has α-Hölder continuous trajectories, we obtain

$$|D_1(t,x,x')| + |D_2(t,x,x')| \leqslant C|x_1 - x'_1|^\alpha.$$

For D_3, we notice first that

$$D_3(t, x, x') = \int_0^{(x_1-x_1')}\int_{\tilde{D}} \phi_t(x_1' + y_1, \tilde{x}, \tilde{y})[-u_0(y) + u_0(-y_1 + (x_1 - x_1'), \tilde{y})]\,\mathrm{d}\tilde{y}\,\mathrm{d}y_1;$$

the estimation is then similar to that of D_1 and D_2. For the last term, since ϕ_t is 2π-periodic, we have

$$D_4(t, x, x') = \int_0^{(x_1-x_1')}\int_{\tilde{D}} \phi_t(\pi + x_1' + y_1, \tilde{x}, \tilde{y})[u_0(\pi + y_1 - (x_1 - x_1'), \tilde{y}) - u_0(\pi - y_1, \tilde{y})]\,\mathrm{d}\tilde{y}\,\mathrm{d}y_1,$$

which immediately yields the same upper estimate as that of D_1 and D_2.

Using Lemma 1.8, the fact that σ is bounded and Kolmogorov's lemma, we see that the stochastic term is a.s. Hölder continuous; more precisely, it belongs to $\mathscr{C}^{\gamma,\gamma'}([0, T], D)$, with $\gamma < (1 - d/4)/2$, $\gamma' < 2 - d/2 \leqslant 1$.

We need to study the regularity of the drift term; for this, we use the factorization method (see, for example, Da Prato and Zabczyk 1992). We remark that

$$\Delta G(t, x, y) = \int_D G(t - s, x, z)\Delta G(s, z, y)\,\mathrm{d}z.$$

Fix $\alpha \in]0, 1[$ and set

$$\mathscr{J}(v)(t, x) = \int_0^t\int_D G(t - s, x, z)(t - s)^{-\alpha} v(s, z)\,\mathrm{d}s\,\mathrm{d}z, \tag{2.39}$$

$$\mathscr{K}(v)(s, z) = \int_0^s\int_D \Delta G(s - s', z, y)(s - s')^{\alpha-1}(f(v(s', y)))\,\mathrm{d}y\,\mathrm{d}s'; \tag{2.40}$$

then the drift term is $H(u)(t, x) = \pi^{-1}\sin(\pi\alpha)\mathscr{J}(\mathscr{K}(v))(t, x)$, for all $t, x \in [0, T] \times D$. Because u_0 is continuous, it belongs to all $L^q(D)$, $q \geqslant 4 > 3 \vee 3d/2$, and u belongs to $L^\infty([0, T], L^q(D))$ a.s. according to Theorem 1.3. We first show that $\mathscr{K}$ is an operator mapping $L^\infty([0, T], L^q(D))$ into itself. Using (1.15), with q and $q/3$, $1/r = 1 - 2/q$, we obtain

$$\|\mathscr{K}(v)(t, \cdot)\|_q \leqslant \int_0^t (t - s)^{-1+\alpha-(d+2)/4+(d/4)(1-2/q)}(1 + \|v(s, \cdot)\|_q^3)\,\mathrm{d}s.$$

We require $-1 + \alpha - 1/2 - d/(2q) > -1$, that is, $\alpha > 1/2 + d/(2q)$.

Then we study $\mathscr{J}$, and prove that $\mathscr{J}(v)$ is Hölder continuous if $v \in L^\infty([0, T], L^q(D))$. Let us first study the regularity with respect to the space variable. For $x, x' \in D$,

$$A(t, x, x') = \int_0^t\int_D 1_{\{|x-y|\leqslant|x-x'|\}}(t - s)^{-\alpha}(|G(t - s, x, y)| + |G(t - s, x', y|)|u(s, y)|\,\mathrm{d}y\,\mathrm{d}s,$$

$$B(t, x, x') = \int_0^t\int_D 1_{\{|x-y|>|x-x'|\}}(t - s)^{-\alpha}|G(t - s, x, y) - G(t - s, x', y)||u(s, y)|\,\mathrm{d}y\,\mathrm{d}s;$$

then

$$|\mathscr{J}(v)(t, x) - \mathscr{J}(v)(t, x')| \leqslant A(t, x, x') + B(t, x, x').$$

Using Hölder's inequality with q and its conjugate exponent, (1.6) and (1.14) we find, for $\beta \in \,]0, 4(1-\alpha-d/(4q))[$, that

$$A(t, x, x') \leqslant |x-x'|^{\beta} \int_0^t (t-s)^{-\alpha-\beta/4-d/(4q)} \|u(s, \cdot)\|_q \, \mathrm{d}s \leqslant C|x-x'|^{\beta}. \tag{2.41}$$

We notice that if $|x-y| \geqslant |x-x'|$, and $\tilde{x}$ denotes a point between x and x', then $|\tilde{x}-y| \geqslant 2^{-1/2}(|2x-x'-y| \wedge |x'-y|)$. Let $\lambda \in [0, 1]$; using Hölder's inequality, Taylor's formula, (1.7) and (1.14), we find that

$$\begin{aligned}
B(t, x, x') &\leqslant |x-x'|^{\lambda} \int_0^t \int_D (t-s)^{-\alpha} |G(t-s, x, y)| + |G(t-s, x', y)|^{1-\lambda} \\
&\quad \times 1_{\{|x-y| \geqslant |x-x'|\}} \left[\int_0^1 |\partial_x G(t-s, \theta x + (1-\theta)x', y)| \, \mathrm{d}\theta \right]^{\lambda} |u(s, y)| \, \mathrm{d}y \, \mathrm{d}s \\
&\leqslant C|x-x'|^{\lambda} \int_0^t \int_D (t-s)^{-\alpha-d(1-\lambda)/4-\lambda(d+1)/4} \\
&\quad \times \left[\exp\left(-\frac{c|2x-x'-y|^{4/3}}{|t-s|^{1/3}} \right) + \exp\left(-\frac{c|x'-y|^{4/3}}{|t-s|^{1/3}} \right) \right]^{\lambda} u(s, y) \, \mathrm{d}y \, \mathrm{d}s \\
&\leqslant C|x-x'|^{\lambda} \int_0^t (t-s)^{-\alpha-d/4-\lambda/4+(d/4)(1-1/q)} \|u(s, \cdot)\|_q \, \mathrm{d}s.
\end{aligned} \tag{2.42}$$

We need $-\alpha - d/(4q) - \lambda/4 > -1$, that is, $\lambda < 4(1-\alpha-d/(4q))$. Since q can be chosen as big as we want, $\mathscr{J}(v)$ is λ-Hölder continuous with respect to the space variable with $\lambda < 1$.

Let us study the time regularity; for $0 \leqslant t < t' \leqslant T$, we have

$$\begin{aligned}
&\mathscr{J}(v)(t', x) - \mathscr{J}(v)(t, x) \\
&= \int_t^{t'} \int_D G(t'-s, x, z)(t'-s)^{-\alpha} v(s, z) \, \mathrm{d}s \, \mathrm{d}z \\
&\quad + \int_0^t \int_D [G(t'-s, x, z)(t'-s)^{-\alpha} - G(t-s, x, z)(t-s)^{-\alpha}] v(s, z) \, \mathrm{d}s \, \mathrm{d}z.
\end{aligned}$$

Using (1.6) and Hölder's inequality with q and its conjugate exponent and (1.14), we have, for $\alpha + d/(4q) < 1$,

$$\begin{aligned}
\left| \int_t^{t'} \int_D G(t'-s, x, z)(t'-s)^{-\alpha} v(s, z) \, \mathrm{d}s \, \mathrm{d}z \right| &\leqslant \int_t^{t'} (t'-s)^{-\alpha-d/(4q)} \|v(s, \cdot)\|_q \, \mathrm{d}s \\
&\leqslant |t-t'|^{1-\alpha-d/(4q)} \sup_{s \in [0,T]} \|v(s, \cdot)\|_q.
\end{aligned}$$

For the second term, let $\mu \in [0, 1]$, $\epsilon > 0$; Taylor's formula, Hölder's inequality and the fact that if $\tilde{t} \in [t, t']$, then $|\tilde{t} - s| > |t-s|$ for $s \in [0, t]$, imply that

$$\left|\int_0^t\int_D [G(t'-s,x,y)(t'-s)^{-\alpha} - G(t-s,x,y)(t-s)^{-\alpha}]v(s,y)\,\mathrm{d}s\,\mathrm{d}y\right|$$

$$\leqslant C|t-t'|^{\mu}\int_0^t\int_D (t-s)^{-\alpha-[d(1-\mu)]/4}|v(s,y)|$$

$$\times\left[\int_0^1(\theta t+(1-\theta)t'-s)^{-(d+4)/4}\exp\left(-c\frac{|x-y|^{4/3}}{|\theta t+(1-\theta)t'-s|^{1/3}}\right)\mathrm{d}\theta\right]^{\mu}\mathrm{d}y\,\mathrm{d}s$$

$$\leqslant C|t-t'|^{\mu}\int_0^t\int_D (t-s)^{-\alpha-d/4-\mu+\epsilon}|x-y|^{-4\epsilon}|v(s,y)|\,\mathrm{d}y\,\mathrm{d}s$$

$$\leqslant C|t-t'|^{\mu}\int_0^t (t-s)^{-\alpha-d/4-\mu+\epsilon}\|(x-\cdot)^{-4\epsilon}\|_{q/q-1}\|v(s,\cdot)\|_q\,\mathrm{d}s \leqslant C|t-t'|^{\mu},$$

with $\mu<1-\alpha-d/4+\epsilon$ and $\epsilon<d(1-1/q)/4$. Since $\alpha>1/2+d/(2q)$, we obtain $\mu<1-\alpha-d/(4q)<1/2-3d/(4q)$. Since q can be chosen as big as we want, we conclude that the function $\mathscr{J}(v)$ is μ-Hölder continuous in time with $\mu<\frac{1}{2}$. This concludes the proof of Theorem 1.4. □

2.4. Improvement of the regularity of *u* when $d=1$

We suppose here that $d=1$ and that Assumption 4″ is satisfied. Let us again study the regularity of each term on the right-hand side of (1.10). As for the initial condition term, we prove the following result.

Lemma 2.3. *If u_0 belongs to $\mathscr{C}^{2+\epsilon}(D)$, for $0\leqslant\epsilon<1$, the function $G_{.}u_0(*)$ belongs to $\mathscr{C}^{(2+\epsilon)/4,2+\epsilon}([0,T],D)$.*

Proof. We first notice that $U(t,x)=u_0(x)-\int_0^t\int_D\Delta_x G(s,x,y)u_0''(y)\mathrm{d}y\,\mathrm{d}s$ satisfies the partial differential equation $\partial_t U+\Delta^2 U=0$ with the boundary conditions (1.2) and $U(0,\cdot)=u_0$; hence $U(t,x)=G_t u_0(x)$. Since $\int_D G(s,x,y)\,\mathrm{d}y=1$, $\int_D\Delta_x G(s,x,y)\,\mathrm{d}y=0$, and hence

$$G_t u_0(x)-u_0(x)=\int_0^t\int_D\Delta_x G(s,x,y)(u_0''(y)-u_0''(x))\,\mathrm{d}y\,\mathrm{d}s$$

Then using (1.7) and the hypothesis on u_0, we deduce that

$$|G_t u_0(x)-u_0(x)|\leqslant Ct^{(2+\epsilon)/4}.$$

Since $G_t u_0(x)-G_s u_0(x)=\int_D G(s,x,y)(G_{t-s}u_0(y)-u_0(y))\,\mathrm{d}y$, we deduce the time regularity of Gu_0.

For the space regularity, we use (2.37) and notice that

$$\Delta G_t u_0(x) = \int_D (\phi''(t, x+y) + \phi''(t, x-y)) u_0(y)\, \mathrm{d}y = \int_D (\phi(t, x+y) + \phi(t, x-y)) u_0''(y)\, \mathrm{d}y$$

for $t > 0$. Hence the proof of space regularity of Gu_0 is similar to the corresponding one in Lemma 2.2, using the same decomposition. □

Let us prove that the stochastic term $L(u)$ defined by (2.5) belongs to $\mathscr{C}^{\gamma,4\gamma}([0, T], D)$ with $\gamma < \frac{3}{8}$. This time regularity has already been established. It suffices to prove that $L(u)$ is differentiable with respect to the space variable, and that its derivative is λ-Hölder continuous with $\lambda < \frac{1}{2}$. We use the same methods as Kunita (1984, p. 219) and introduce $\eta(t, x, \xi) = (L(u)(t, x+\xi) - L(u)(t, x))\xi^{-1}$ for $\xi \in]0, \pi - x]$. We have to evaluate $\mathrm{E}(|\eta(t, x, \xi) - \eta(t, x', \xi')|^p)$. Using Burkholder's inequality and the fact that the sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(D)$ and that σ is bounded, we obtain

$$\begin{aligned}\mathrm{E}(|\eta(t, x, \xi) - \eta(t, x', \xi')|^p) &\leqslant C\Big[\int_0^t \int_D [(G(t-s, x+\xi, y) - G(t-s, x, y))\xi^{-1} \\ &\quad - (G(t-s, x'+\xi', y) - G(t-s, x', y))(\xi')^{-1}]^2\, \mathrm{d}y\, \mathrm{d}s\Big]^{p/2} \\ &\leqslant C\Big[\int_0^t \sum_{k\in\mathbb{N}^*} \exp(-2(t-s)k^4)[\{\cos(k(x+\xi)) - \cos(kx)\}\xi^{-1} \\ &\quad - \{\cos(k(x'+\xi')) - \cos(kx')\}(\xi')^{-1}]^2\, \mathrm{d}s\Big]^{p/2}.\end{aligned}$$

Using the identity $\cos(y + k\xi) - \cos(y) = -k\xi \int_0^1 \sin(y + k\xi u)\, \mathrm{d}u$, a simple computation shows that, for $0 \leqslant \lambda \leqslant 1$,

$$\mathrm{E}(|\eta(t, x, \xi) - \eta(t, x', \xi')|^p) \leqslant C\left[\int_0^t \sum_{k\in\mathbb{N}^*} \exp(-2(t-s)k^4) k^{2+2\lambda} (|x - x'| + |\xi - \xi'|)^{2\lambda}\, \mathrm{d}s\right]^{p/2}.$$

Thus, for $\lambda < \frac{1}{2}$, we finally obtain that

$$\mathrm{E}(|\eta(t, x, \xi) - \eta(t, x', \xi')|^p) \leqslant C(|x - x'| + |\xi - \xi'|)^{\lambda p}.$$

Therefore, $\eta(t, x, \cdot)$ can be extended as a continuous function on $[0, \pi]$, and $\eta(t, \cdot, 0)$ is the space derivative of $L(u)(t, x)$ which also has λ-Hölder continuous trajectories.

For the last term $H(u)$, we proceed in the same way as in the preceding subsection. We observe that $\partial_x^3 G(t, x, z) = \int_D \partial_x^3 G(t-s, x, y) G(s, y, z)\, \mathrm{d}y$, and replace in the function $\mathscr{K}$ defined by (2.40) the second-order derivative by the third-order one. Using the factorization method again, we prove that the space derivative of $H(u)$ is λ-Hölder with $\lambda < 1$, so that $H(u)$ belongs to $\mathscr{C}^{\mu/4,\mu}([0, T], D)$ with $\mu < \frac{3}{2}$.

3. Malliavin calculus for the Cahn–Hilliard SPDE

In this section we suppose that Assumption 4 holds, because we need u to be continuous. We first prove local regularity of u in the sense of the Malliavin calculus. Let us recall some classical notation of Malliavin calculus (see Nualart 1995). $\mathbb{D}^{1,2}$ is the set of random variables F such that the Malliavin derivative DF exists and satisfies

$$\|F\|_{1,2} = [\mathrm{E}(|F|^2) + \mathrm{E}(\|DF\|^2_{L^2([0,T]\times D)})]^{1/2} < \infty.$$

A process $X \in L^2(\Omega \times [0, T] \times D)$ belongs to $\mathbb{L}^{1,2}$ if, for each $(s, x) \in [0, T] \times D$, $X_{s,x}$ belongs to $\mathbb{D}^{1,2}$ and

$$\mathrm{E}\left(\int_0^T\int_D\int_0^T\int_D |D_{s,y}X(t, x)|^2 \,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t\right) < +\infty.$$

We define the 'local' versions $\mathbb{D}^{1,2}_{\mathrm{loc}}$ and $\mathbb{L}^{1,2}_{\mathrm{loc}}$ of these two spaces as follows. X belongs to $\mathbb{D}^{1,2}_{\mathrm{loc}}$ ($\mathbb{L}^{1,2}_{\mathrm{loc}}$) if there exists a sequence Ω_n such that, for every n, $X = X_n$ on Ω_n a.s., X_n belong to $\mathbb{D}^{1,2}_{\mathrm{loc}}$ ($\mathbb{L}^{1,2}_{\mathrm{loc}}$), and $\lim_{n\to\infty} P(\Omega_n) = 1$.

Lemma 3.1. *The solution u of the SPDE* (1.10) *belongs to* $\mathbb{L}^{1,2}_{\mathrm{loc}}$.

3.1. Approximation of u by a sequence of elements of $\mathbb{L}^{1,2}$

For every $n > 0$, let us denote by

$$\Omega_n = \left\{\omega \in \Omega | \sup_{0\leqslant t\leqslant T}\sup_{x\in D} |u(t, x, w)| \leqslant n\right\}. \tag{3.1}$$

Because the process u is a.s. continuous, $\lim_{n\to\infty} P(\Omega_n) = 1$. Let us construct an approximation $(u_n)_{n\in\mathbb{N}}$ of u, such that $u_n = u$ on Ω_n a.s. For this we still truncate the polynomial f, but in another way. Set $f_n(x) = K_n(|x|)f(x)$, where K_n is defined by (2.1); f_n is $\mathscr{C}^1$ function with bounded derivative. Let us denote by u_n the solution to the SPDE

$$u_n(t, x) = \int_D G(t, x, y)u_0(y)\,\mathrm{d}y + \int_0^t\int_D \Delta G(t - s, x, y) f_n(u(s, y))\,\mathrm{d}y\,\mathrm{d}s$$
$$+ \int_0^t\int_D G(t - s, x, y)\sigma(u_n(s, y))W(\mathrm{d}y, \mathrm{d}s). \tag{3.2}$$

Notice that since f_n is globally Lipschitz, a standard argument shows that (3.2) has a unique solution. The local property of stochastic integrals proves that $u = u_n$ on Ω_n a.s. In order to prove Lemma 3.1, according to Nualart (1995, p. 45), it suffices to check the following result.

Lemma 3.2. *The solution* u_n *to* (3.2) *exists and is unique; furthermore, it belongs to* $\mathbb{L}^{1,2}$ *and its Malliavin derivative satisfies the SPDE*

$$D_{s,y}u_n(t,x) = G(t-s,x,y)\sigma(u_n(s,y)) + \int_s^t\int_D \Delta G(t-\theta,x,\eta)f'_n(u_n(\theta,\eta))D_{s,y}u_n(\theta,\eta)\,\mathrm{d}\eta\,\mathrm{d}\theta$$

$$+ \int_s^t\int_D G(t-\theta,x,\eta)S_n(\theta,\eta)D_{s,y}u_n(\theta,\eta)W(\mathrm{d}\eta,\mathrm{d}\theta) \tag{3.3}$$

if $s \leqslant t$, and $D_{s,y}u_n(t,x) = 0$ if $s > t$, where $S_n(\theta,\eta)$ is $\mathscr{F}_\theta$-adapted, bounded and satisfies

$$D_{s,y}(\sigma(u_n(\theta,\eta))) = S_n(\theta,\eta)D_{s,y}u_n(\theta,\eta).$$

Proof. To prove the existence and the uniqueness of the solution to (3.3) we construct a Cauchy sequence $(u_{n,k})_{k\in\mathbb{N}}$ converging to u_n by the Picard iteration scheme, which means that

$$u_{n,0}(t,x) = G_t u_0(x)$$

and

$$u_{n,k+1}(t,x) = \int_D G(t,x,y)u_0(y)\,\mathrm{d}y + \int_0^t\int_D \Delta G(t-s,x,y)f_n(u_{n,k}(s,y))\,\mathrm{d}y\,\mathrm{d}s$$

$$+ \int_0^t\int_D G(t-s,x,y)\sigma(u_{n,k}(s,y))W(\mathrm{d}y,\mathrm{d}s), \tag{3.4}$$

for $k \geqslant 0$. Then for $k \geqslant 1$,

$$u_{n,k+1}(t,x) - u_{n,k}(t,x) = \int_0^t\int_D \Delta G(t-s,x,y)[f_n(u_{n,k}(s,y)) - f_n(u_{n,k-1}(s,y))]\,\mathrm{d}y\,\mathrm{d}s$$

$$+ \int_0^t\int_D G(t-s,x,y)[\sigma(u_{n,k}(s,y)) - \sigma(u_{n,k-1}(s,y))]W(\mathrm{d}y,\mathrm{d}s).$$

Burkholder's inequality and the fact that σ and f_n are Lipschitz functions imply, for $p \in [2, +\infty[$, that

$$\mathrm{E}(|u_{n,k+1}(t,x) - u_{n,k}(t,x)|^p)$$

$$\leqslant C_n\mathrm{E}\left(\left|\int_0^t\int_D |\Delta G(t-s,x,y)||u_{n,k}(s,y) - u_{n,k-1}(s,y)|\,\mathrm{d}y\,\mathrm{d}s\right|^p\right)$$

$$+ C\mathrm{E}\left(\left|\int_0^t\int_D |G^2(t-s,x,y)||u_{n,k}(s,y) - u_{n,k-1}(s,y)|^2\,\mathrm{d}y\,\mathrm{d}s\right|^{p/2}\right).$$

Lemma 1.6 applied twice with $q = \rho = \infty$ and $r = 1$, and Hölder's inequality, imply that

$$
\begin{aligned}
&\mathrm{E}\Big(\sup_{x\in D}|u_{n,k+1}(t, x) - u_{n,k}(t, x)|^p\Big) \\
&\quad\leqslant C_n \mathrm{E}\left(\left|\int_0^t (t-s)^{-(d+2)/4+d/4}\|u_{n,k}(s, \cdot) - u_{n,k-1}(s, \cdot)\|_\infty \,\mathrm{d}s\right|^p\right) \\
&\qquad + C\mathrm{E}\left(\left|\int_0^t (t-s)^{-d/2+d/4}\|u_{n,k}(s, \cdot) - u_{n,k-1}(s, \cdot)\|_\infty^2 \,\mathrm{d}s\right|^{p/2}\right) \\
&\quad\leqslant C_n \mathrm{E}\left(\int_0^t (t-s)^{-[(d+2)\vee(2d)]/4+d/4}\|u_{n,k}(t, \cdot) - u_{n,k-1}(t, \cdot)\|_\infty^p \,\mathrm{d}s\right).
\end{aligned}
$$

Let $b = \frac{1}{4}\{(d+2)\vee(2d)\}$; iterating this inequality and using Fubini's theorem as in Walsh (1986, Lemma 3.3), we obtain

$$
\begin{aligned}
&\mathrm{E}(\|u_{n,k+2}(t, \cdot) - u_{n,k+1}(t, \cdot)\|_\infty^p) \\
&\quad\leqslant C_n \int_0^t (t-s)^{-b+(d/4)}\,\mathrm{d}s \int_0^s (s-\tau)^{-b+d/4}\mathrm{E}(|u_{n,k}(\tau, \cdot) - u_{n,k-1}(\tau, \cdot)\|_\infty^p)\,\mathrm{d}\tau \\
&\quad\leqslant C_n \int_0^t \left(\int_\tau^t (t-s)^{-b+d/4}(s-\tau)^{-b+d/4}\,\mathrm{d}s\right)\mathrm{E}(\|u_{n,k}(\tau, \cdot) - u_{n,k-1}(\tau, \cdot)\|_\infty^p)\,\mathrm{d}\tau \\
&\quad\leqslant C_n \int_0^t \sup_{x\in D}\mathrm{E}(\|u_{n,k}(\tau, \cdot) - u_{n,k-1}(\tau, \cdot)\|_\infty^p)\,\mathrm{d}\tau.
\end{aligned}
$$

This implies, for $k \geqslant 0$, that

$$
\sup_{t\in[0,T]} \mathrm{E}(\|u_{n,2(k+1)}(t, \cdot) - u_{n,2k+1}(t, \cdot)\|_\infty^p) \leqslant \frac{(C_n)^k}{k!} \sup_{t\in[0,T]} \mathrm{E}(\|u_{n,2}(t, \cdot) - u_{n,1}(t, \cdot)\|_\infty^p)
$$

$$
\sup_{t\in[0,T]} \mathrm{E}(\|u_{n,2k+1}(t, \cdot) - u_{n,2k}(t, \cdot)\|_\infty^p) \leqslant \frac{(C_n)^k}{k!} \sup_{t\in[0,T]} \mathrm{E}(\|u_{n,1} - u_{n,0}\|_\infty^p).
$$

Hence

$$
\sum_{k>0} \sup_{t\in[0,T]} \mathrm{E}(\|u_{n,k}(t, \cdot) - u_{n,k-1}(t, \cdot)\|_\infty^p) < \infty,
$$

and the sequence $u_{n,k}(t, x)$ converges in $L^p(\Omega)$ as $k \to +\infty$, for $(t, x) \in [0, T] \times D$, to the solution to the SPDE (3.2) such that

$$
\sup_{t\in[0,T]} \mathrm{E}(\|u_n(t, \cdot)\|_\infty^p) < \infty,
$$

for $p \in [2, +\infty[$. Computations similar to preceding ones shows that if u_n and v_n are solutions to (3.2), for $p \in [2, +\infty[$,

$$\sup_{t\in[0,T]} \mathrm{E}(\|u_n(t,\cdot)-v_n(t,\cdot)\|_\infty^p) < C_n\int_0^t \mathrm{E}(\|u_n(s,\cdot)-v_n(s,\cdot)\|_\infty^p)\,\mathrm{d}s.$$

Gronwall's lemma implies $u_n(t, x) = v_n(t, x)$ a.s. Since the solution is a.s. continuous, the processes u_n and v_n are indistinguishable.

We now prove by induction that the sequence $u_{n,k}(t, x)$ belongs to $\mathbb{D}^{1,2}$. Since $u_{n,0}$ is deterministic, it belongs to $\mathbb{D}^{1,2}$ and $Du_{n,0} \equiv 0$. Suppose that, for $k \geqslant 0$ and for every $(t, x) \in [0, T] \times D$, $u_{n,k}(t, x)$ belongs to $\mathbb{D}^{1,2}$ and satisfies

$$\sup_{t\in[0,T]}\sup_{x\in D}\mathrm{E}\left(\int_0^T\int_D |D_{s,y}u_{n,k}(t,x)|^2\,\mathrm{d}y\,\mathrm{d}s\right) < \infty.$$

According to Nualart (1995, Proposition 1.2.3), since σ is Lipschitz, there exists $S_{n,k}(\theta, \eta)$ such that

$$D_{s,y}(\sigma(u_{n,k}(\theta,\eta))) = S_{n,k}(\theta,\eta)D_{s,y}u_{n,k}(\theta,\eta) \tag{3.5}$$

and

$$\sup_{k,n,\theta,\eta} |S_{n,k}(\theta,\eta)| = C_\sigma < \infty. \tag{3.6}$$

Let us take the Malliavin derivative of both sides of (3.4); then for $s \leqslant t$,

$$\begin{aligned} &D_{s,y}u_{n,k+1}(t,x) \\ &\quad = G(t-s,x,y)\sigma(u_{n,k}(s,y)) + \int_s^t\int_D \Delta G(t-\theta,x,\eta)f'_n(u_{n,k}(\theta,\eta))D_{s,y}u_{n,k}(\theta,\eta)\,\mathrm{d}\eta\,\mathrm{d}\theta \\ &\quad + \int_s^t\int_D G(t-\theta,x,\eta)S_{n,k}(\theta,\eta)D_{s,y}u_{n,k}(\theta,\eta)W(\mathrm{d}\eta,\mathrm{d}\theta), \end{aligned} \tag{3.7}$$

and for $s > t$ $D_{s,y}u_{n,k+1}(t, x) = 0$. We need to verify that

$$\sup_k \sup_{t\in[0,T]}\sup_{x\in D}\mathrm{E}\left(\int_0^T\int_D |D_{s,y}u_{n,k}(t,x)|^2\,\mathrm{d}y\,\mathrm{d}s\right) < \infty. \tag{3.8}$$

Clearly, $\mathrm{E}(\int_0^T\int_D |D_{s,n}u_{n,k}(t,x)|^2\,\mathrm{d}y\,\mathrm{d}s) \leqslant 3\sum_{i=1}^3 A_i(t,x)$, where

$$A_1(t,x) = \mathrm{E}\left(\int_0^T\int_D |G(t-s,x,y)\sigma(u_{n,k}(s,y))|^2\,\mathrm{d}y\,\mathrm{d}s\right),$$

$$A_2(t,x) = \mathrm{E}\left(\int_0^T\int_D \left|\int_s^t\int_D \Delta G(t-\theta,x,\eta)f'_n(u_{n,k}(\theta,\eta))D_{s,y}u_{n,k}(\theta,\eta)\,\mathrm{d}\eta\,\mathrm{d}\theta\right|^2\mathrm{d}y\,\mathrm{d}s\right),$$

$$A_3(t,x) = \mathrm{E}\left(\int_0^T\int_D \left|\int_s^t\int_D G(t-\theta,x,\eta)S_{n,k}(\theta,\eta)D_{s,y}u_{n,k}(\theta,\eta)W(\mathrm{d}\eta,\mathrm{d}\theta)\right|^2\mathrm{d}y\,\mathrm{d}s\right).$$

Since σ is bounded, (1.18) implies that

$$\sup_k \sup_{t\in[0,T]} \sup_{x\in D} A_1(t,x) \leqslant \|\sigma\|_\infty^2 \sup_{t\in[0,T]} \sup_{x\in D} \mathrm{E}\left(\int^T \int_D G^2(t-s,x,y)\,\mathrm{d}y\,\mathrm{d}s\right)$$
$$= C < \infty. \tag{3.9}$$

Since $|f'_n| \leqslant C_n$, Fubini's theorem and Hölder's inequality with respect to the measure $|\Delta G(t-\theta, x, \eta)|\,\mathrm{d}\eta\,\mathrm{d}\theta$ yield

$$A_2(t,x) \leqslant C_n \int_0^t \int_D |\Delta G(t-\theta,x,\eta)| \mathrm{E}\left(\int_0^\theta \int_D |D_{s,y} u_{n,k}(\theta,\eta)|^2\,\mathrm{d}y\,\mathrm{d}s\right)\mathrm{d}\eta\,\mathrm{d}\theta$$
$$\leqslant C_n \int_0^t \int_D |\Delta G(t-\theta,x,\eta)|\,\mathrm{d}\eta \sup_{\eta'} \mathrm{E}\left(\int_0^\theta \int_D |D_{s,y} u_{n,k}(\theta,\eta')|^2\,\mathrm{d}y\,\mathrm{d}s\right)\mathrm{d}\theta.$$

Thus (1.7) implies that

$$A_2(t,x) \leqslant C \int_0^t \frac{1}{\sqrt{t-\theta}} \sup_\eta \mathrm{E}\left(\int_0^\theta \int_D |D_{s,y} u_{n,k}(\theta,\eta)|^2\,\mathrm{d}y\,\mathrm{d}s\right)\mathrm{d}\theta. \tag{3.10}$$

Inequality (3.6), Burkholder's inequality, Fubini's theorem and (1.6) imply that

$$A_3(t,x) \leqslant C\mathrm{E}\left(\int_0^t \int_D \int_s^t \int_D |G(t-\theta,x,\eta)|^2 |D_{s,y} u_{n,k}(\theta,\eta)|^2\,\mathrm{d}\eta\,\mathrm{d}\theta\,\mathrm{d}y\,\mathrm{d}s\right)$$
$$\leqslant C \int_0^t (t-\theta)^{-d/4} \sup_\eta \mathrm{E}\left(\int_0^\theta \int_D |D_{s,y} u_{n,k}(\theta,\eta)|^2\,\mathrm{d}y\,\mathrm{d}s\right)\mathrm{d}\theta. \tag{3.11}$$

Therefore, (3.9)–(3.11) yield the existence of positive constants C and C_n such that, for every $t \in [0, T]$ and $k \geqslant 0$,

$$\sup_{x\in D} \mathrm{E}\left(\int_0^t \int_D |D_{s,y} u_{n,k+1}(t,x)|^2\,\mathrm{d}y\,\mathrm{d}s\right)$$
$$\leqslant C + C_n \int_0^t (t-\theta)^{-(1/2\vee d/4)} \sup_\eta \mathrm{E}\left(\int_0^\theta \int_D |D_{s,n} u_{n,k}(\theta,\eta)|^2\,\mathrm{d}y\,\mathrm{d}s\right)\mathrm{d}\theta;$$

therefore, iterating this inequality and using the convergence of the integral $\int_s^t (t-\theta)^{-(1/2\vee d/4)}(\theta-s)^{-(1/2\vee d/4)}\,\mathrm{d}\theta$, we deduce (3.8).

We have proved that $u_{n,k}(t,x)$ belongs to $\mathbb{D}^{1,2}$ for all $(t,x) \in [0,T] \times D$. Using Nualart (1995, Lemma 1.2.3), we deduce that the random variable $u_n(t,x)$ belongs to $\mathbb{D}^{1,2}$, and that the sequence $Du_{n,k}(t,x)$ converges to $Du_n(t,x)$ in the weak topology of $L^2([0,T] \times D \times \Omega)$. Let us define $S_n(\theta,\eta)$ as the weak limit of $(S_{n,k}(\theta,\eta))k \geqslant 0$ in $L^\infty(\Omega \times [0,T] \times D)$; then $D_{s,y}(\sigma(u_n(\theta,\eta))) = S_n(\theta,\eta) D_{s,y} u_n(\theta,\eta)$ and

$$\sup_{n,\theta,\eta} |S_n(\theta,\eta)| = C_\sigma < \infty. \tag{3.12}$$

Differentiating the SPDE (2.2), we obtain that for each (t, x), the process $D_{s,y}u_n(t, x)$ satisfies the following SPDE:

$$V^n_{s,y}(t, x) = G(t-s, x, y)\sigma(u_n(s, y)) + \int_s^t\int_D \Delta G(t-\theta, x, \eta) f'_n(u_n(\theta, \eta)) V^n_{s,y}(\theta, \eta)\,\mathrm{d}\eta\,\mathrm{d}\theta$$

$$+ \int_s^t\int_D G(t-\theta, x, \eta) S_n(\theta, \eta) V^n_{s,y}(\theta, \eta) W(\mathrm{d}\eta, \mathrm{d}\theta) \qquad (3.13)$$

for $s \leqslant t$ and $V^n_{s,y}(t, x) = 0$ if $s > t$. We need to prove the uniqueness of the solution to (3.13). Let V_n and U_n be two solutions of (3.13). Computations similar to those made to prove (3.9)–(3.11) imply that

$$\sup_{x\in D} \mathrm{E}\left(\int_0^t\int_D |V^n_{s,y}(t, x) - U^n_{s,y}(t, x)|^2\,\mathrm{d}y\,\mathrm{d}s\right)$$

$$\leqslant C_n\int_0^t (t-\theta)^{-(\frac{1}{2}\vee\frac{d}{4})} \times \sup_{x\in D} \mathrm{E}\left(\int_0^\theta\int_D |V^n_{s,y}(\theta, x) - U^n_{s,y}(\theta, x)|^2\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}\theta\right),$$

and Gronwall's generalized lemma yields

$$\sup_{t\in[0,T]}\sup_{x\in D} \mathrm{E}\left(\int_0^T\int_D |V^n_{s,y}(t, x) - U^n_{s,y}(t, x)|^2\,\mathrm{d}y\,\mathrm{d}s\right) = 0.$$

Hence (3.8) shows that the process u_n belongs to $\mathbb{L}^{1,2}$, which implies that

$$\sup_{t\in[0,T]}\sup_{x\in D} \mathrm{E}\left(\int_0^t\int_D |D_{s,y}u_n(t, x)|^2\,\mathrm{d}y\,\mathrm{d}s\right) = C_n < \infty; \qquad (3.14)$$

this concludes the proof of Lemma 3.2. □

3.2. Existence of a density for the random variable $u_n(t, x)$

Let $t > 0$ and $x \in\,]0, \pi[^d$; according to Nualart (1995, Theorem 2.1.3), we have to prove that

$$\int_0^t\int_D |D_{r,z}u_n(t, x)|^2\,\mathrm{d}r\,\mathrm{d}z > 0 \text{ a.s.} \qquad (3.15)$$

Let us prove the following technical result for time integrals over small time intervals:

Lemma 3.3. *There exists a constant C_n such that, for every $0 < \epsilon < t$,*

$$\sup_{s\in[t-\epsilon,t]}\sup_{y\in D} \mathrm{E}\left(\int_{t-\epsilon}^t\int_D |D_{r,z}u_n(s, y)|^2\,\mathrm{d}z\,\mathrm{d}r\right) \leqslant C_n\epsilon^{1-d/4}. \qquad (3.16)$$

Proof. For $t-\epsilon \leqslant s \leqslant t$, set $H^\epsilon_n(s, y) = \mathrm{E}(\int_{t-\epsilon}^s\int_D |D_{r,z}u_n(s, y)|^2\,\mathrm{d}z\,\mathrm{d}r)$; (3.14) shows that

$$\sup_{s\in[0,T]}\sup_{y\in D}H^{\epsilon}_n(s,\ y)=C_n<\infty. \tag{3.17}$$

According to (3.2),

$$H^{\epsilon}_n(s,\ y)\leqslant C\sum_{i=1}^{3}A_i(s,\ y,\ \epsilon),$$

with

$$A_1(s,\ y,\ \epsilon)=\mathrm{E}\left(\int_{t-\epsilon}^{s}\int_D|G(s-r,\ y,\ z)\sigma(u_n(r,\ z))|^2\,\mathrm{d}z\,\mathrm{d}r\right)$$

$$A_2(s,\ y,\ \epsilon)=\mathrm{E}\left(\int_{t-\epsilon}^{s}\int_D\left|\int_r^s\int_D\Delta G(s-\theta,\ x,\ \eta)f'_n(u_n(\theta,\ \eta))D_{r,z}u_n(\theta,\ \eta)\,\mathrm{d}\eta\,\mathrm{d}\theta\right|^2\mathrm{d}r\,\mathrm{d}z\right)$$

$$A_3(s,\ x,\ \epsilon)=\mathrm{E}\left(\int_{t-\epsilon}^{s}\int_D\left|\int_r^s\int_D G(s-\theta,\ x,\ \eta)S_n(\theta,\ \eta)D_{r,z}u_n(\theta,\ \eta)W(\mathrm{d}\eta,\ \mathrm{d}\theta)\right|^2\mathrm{d}r\,\mathrm{d}z\right)$$

Because σ is bounded, (1.6) implies that

$$A_1(s,\ y,\ \epsilon)\leqslant C\epsilon^{1-d/4}. \tag{3.18}$$

Burkholder's inequality, Fubini's theorem, inequalities (1.6), (3.12), (3.17) and equation (1.14) imply that

$$\begin{aligned}A_3(s,\ y,\ \epsilon)&\leqslant C\mathrm{E}\left(\int_{t-\epsilon}^{s}\int_D\int_r^s\int_D G^2(s-\theta,\ y,\ \eta)|D_{r,z}u_n(\theta,\ \eta)|^2\,\mathrm{d}\theta\,\mathrm{d}\eta\,\mathrm{d}z\,\mathrm{d}r\right)\\&\leqslant C\mathrm{E}\left(\int_{t-\epsilon}^{s}\int_D\int_{t-\epsilon}^{\theta}\int_D G^2(s-\theta,\ y,\ \eta)|D_{r,z}u_n(\theta,\ \eta)|^2\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\eta\,\mathrm{d}\theta\right)\\&\leqslant C\int_{t-\epsilon}^{s}\int_D G^2(s-\theta,\ y,\ \eta)H^{\epsilon}_n(\theta,\ \eta)\,\mathrm{d}\eta\,\mathrm{d}\theta\\&\leqslant\int_{t-\epsilon}^{s}(s-\theta)^{-d/4}\sup_{\eta\in D}H^{\epsilon}_n(\theta,\ \eta)\,\mathrm{d}\theta.\end{aligned} \tag{3.19}$$

For the other term, Schwarz's inequality with respect to the measure $|\Delta G(s-\theta,\ y,\ \eta)|\,\mathrm{d}\theta\,\mathrm{d}\eta$ implies that

$$A_2(s,\ x,\ \epsilon)\leqslant C_n\,\mathrm{E}\left(\int_{t-\epsilon}^{s}\int_D\int_r^s\int_D|\Delta G_{s-\theta}(y,\ \eta)||D_{r,z}u_n(\theta,\ \eta)|^2\,\mathrm{d}\theta\,\mathrm{d}\eta\,\mathrm{d}z\,\mathrm{d}r\right).$$

Using (1.7), by a computation similar to that of A_3, we find

$$A_2(s,\ y,\ \epsilon)\leqslant C_n\int_{t-\epsilon}^{s}(s-\theta)^{-1/2}\sup_{\eta\in D}H^{\epsilon}_n(\theta,\ \eta)\,\mathrm{d}\theta. \tag{3.20}$$

Inequalities (3.18)–(3.20) imply that

$$\sup_{y\in D} H^{\epsilon}_n(s, y) \leqslant C\epsilon^{1-d/4} + C_n \int_{t-\epsilon}^{s} (s-\theta)^{-(d/4\vee 1/2)} \sup_{\eta\in D} H^{\epsilon}_n(\theta, \eta)\,\mathrm{d}\theta;$$

Gronwall's generalized lemma concludes the proof of (3.16). □

Let us prove Theorem 1.5 when $\sigma = 1$.

Lemma 3.4. *If $\sigma = 1$, denote by u_n the solution to* (3.2). *For $t > 0$ and $x \in]0, \pi[^d$, the law of $u_n(t, x)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.*

Proof. It suffices to prove (3.15) in the particular case $\sigma = 1$. Denote by Q the process defined by

$$D_{r,z}u_n(t, x) = G(t-r, x, z) + Q_{r,z}(t, x), \tag{3.21}$$

and set

$$I_1(t, x, \epsilon) = \int_{t-\epsilon}^{t}\int_D G^2(t-r, x, z)\,\mathrm{d}z\,\mathrm{d}r,$$

$$I_2(t, x, \epsilon) = \int_{t-\epsilon}^{t}\int_D Q_{r,z}(t, x)^2\,\mathrm{d}z\,\mathrm{d}r.$$

According to (3.21),

$$\int_0^t\int_D |D_{r,z}u_n(t, x)|^2\,\mathrm{d}z\,\mathrm{d}r \geqslant \int_{t-\epsilon}^{t}\int_D |D_{r,z}u_n(t, x)|^2\,\mathrm{d}z\,\mathrm{d}r \geqslant \frac{1}{2} I_1(t, x, \epsilon) - I_2(t, x, \epsilon). \tag{3.22}$$

Let us find an upper estimate for $\mathrm{E}(I_2(t, x, \epsilon)) \leqslant 2(B_1(t, x, \epsilon) + B_2(t, x, \epsilon))$, where

$$B_1(t, x, \epsilon) = \int_{t-\epsilon}^{t}\int_D \mathrm{E}\left(\left|\int_r^t\int_D \Delta G(t-\theta, x, \eta) f'_n(u_n(\theta, \eta)) D_{r,z}u_n(\theta, \eta)\,\mathrm{d}\eta\,\mathrm{d}\theta\right|^2\right)\mathrm{d}z\,\mathrm{d}r,$$

$$B_2(t, x, \epsilon) = \int_{t-\epsilon}^{t}\int_D \mathrm{E}\left(\left|\int_r^t\int_D G(t-\theta, x, \eta) S_n(\theta, \eta) D_{r,z}u_n(\theta, \eta) W(\mathrm{d}\eta, \mathrm{d}\theta)\right|^2\right)\mathrm{d}z\,\mathrm{d}r.$$

Schwarz's inequality applied with respect to the measure $|\Delta G(t-\theta, x, \eta)|\,\mathrm{d}\eta\,\mathrm{d}\theta$ and Fubini's theorem imply that

$$\begin{aligned} B_1(t, x, \epsilon) &\leqslant C_n \int_{t-\epsilon}^{t}\int_D \mathrm{E}\left(\int_s^t\int_D |\Delta G(t-\theta, x, \eta)||D_{r,z}u_n(\theta, \eta)|^2\,\mathrm{d}\eta\,\mathrm{d}\theta\right)\mathrm{d}z\,\mathrm{d}r \\ &\leqslant C_n \int_{t-\epsilon}^{t}\int_D |\Delta G(t-\theta, x, \eta)|\,\mathrm{E}\left(\int_{t-\epsilon}^{\theta}\int_D |D_{r,z}u_n(\theta, \eta)|^2\,\mathrm{d}z\,\mathrm{d}r\right)\mathrm{d}\eta\,\mathrm{d}\theta. \end{aligned}$$

Using Lemma 3.3 with $t-\epsilon \leqslant \theta \leqslant t$ and (1.14), we deduce that

$$B_1(t, x, \epsilon) \leqslant C_n \epsilon^{1-d/4} \int_{t-\epsilon}^{t} \int_D |\Delta G(t-\theta, x, \eta)| \, d\eta \, d\theta \leqslant C_n \epsilon^{3/2-d/4}. \tag{3.23}$$

For the second term, Burkholder's inequality implies that

$$B_2(t, x, \epsilon) \leqslant C \int_{t-\epsilon}^{t} \int_D \mathrm{E}\left(\int_s^t \int_D |G(t-\theta, x, \eta)|^2 |D_{r,z} u_n(\theta, \eta)|^2 \, d\eta \, d\theta \right) dz \, dr.$$

Computations similar to that for B_1 imply that

$$B_2(t, x, \epsilon) \leqslant C\epsilon^{2-d/4},$$

so that

$$\mathrm{E}(I_2(t, x, \epsilon)) \leqslant C_n \epsilon^{(3/2-d/4)\wedge(2-d/4)}. \tag{3.24}$$

We now need to find lower estimates for I_1. For this, we use the exact expression of the Green function G given by (1.5), which yields

$$I_1(t, x, \epsilon) = \int_{t-\epsilon}^{t} \int_D \left[\sum_{k \in \mathbb{N}^d} \varepsilon_k(x) \varepsilon_k(y) \exp(-\lambda_k^2 (t-s)) \right]^2 dy \, ds.$$

Since the sequence $(\varepsilon_k)_{k \in \mathbb{N}^d}$ is an orthonormal basis for $L^2(D)$,

$$I_1(t, x, \epsilon) = \int_{t-\epsilon}^{t} \left[\sum_{k \in \mathbb{N}^d} \varepsilon_k^2(x) \exp(-2\lambda_k^2 (t-s)) \right] ds.$$

$$= \sum_{k \in \mathbb{N}^{d,*}} \varepsilon_k^2(x) \frac{1}{2\lambda_k^2} [1 - \exp(-2\lambda_k^2 \epsilon)] + C\epsilon.$$

The series is well defined because $\sum_{k \in \mathbb{N}^{d,*}} 1/\lambda_k^2 < \infty$. Let us choose $x_0 = (x_i)_{i \in [1,d]} \in \,]0, \pi[^d$. According to (1.4),

$$I_1(t, x, \epsilon) \geqslant C \sum_{k \in (\mathbb{N}^*)^d} \Pi_{1 \leqslant i \leqslant d} \cos^2(k_i x_i) \frac{1}{\left(\sum_{i=1}^{d} k_i^2 \right)^2} \left[1 - \exp\left(-2 \left(\sum_{i=1}^{d} k_i^2 \right)^2 \epsilon \right) \right].$$

Given α in $]0, x_1 \wedge \ldots \wedge x_d \wedge \pi/2[$, we observe the following:

Remark 3.5. If $k \in \mathbb{N}^*$, $x \in D$ are such that $kx \in \,]\pi/2 - \alpha/2, \pi/2 + \alpha/2[\pmod{\pi}$, then

$$(k+1)x \notin \,]\frac{\pi}{2} - \frac{\alpha}{2}, \frac{\pi}{2} + \frac{\alpha}{2}[\pmod{\pi}.$$

The intervals $]\pi/2 - \alpha/2, \pi/2 + \alpha/2[\pmod{\pi}$ correspond to small values of the cosine function; in fact there exists $\beta > 0$ such that $\cos^2(x) > \beta$ if $x \notin \,]\pi/2 - \alpha/2, \pi/2 + \alpha/2[\pmod{\pi}$. Using Remark 3.5, we obtain the following lower estimate, skipping every other term:

$$I_1(t, x_0, \epsilon) \geqslant C\beta^2 \sum_{k\in(\mathbb{N}^*)^d} \left[\sum_{i=1}^d (1+2k_i)^2\right]^{-2} \left[1-\exp\left(-2\epsilon\left[\sum_{i=1}^d (2k_i)^2\right]^2\right)\right]$$
$$\geqslant C\beta^2 \sum_{k\in(\mathbb{N}^*)^d} \left[\sum_{i=1}^d k_i^2\right]^{-2} \left(1-\exp\left(-8\epsilon\left[\sum_{i=1}^d k_i^2\right]^2\right)\right).$$

Take $M>0$ small enough ($M<2$) such that for $0<x<M$ we have $1-\exp(-x)\geqslant x/2$; then since the number of points with integer coordinate in a circle (or sphere) or radius r is dominated by Cr^d,

$$I_1(t, x_0, \epsilon) \geqslant C\beta^2 \sum_{k\in(\mathbb{N}^*)^d, 8\epsilon(\Sigma_{i=1}^d k_i^2)^2<M} \left[\sum_{i=1}^d k_i^2\right]^{-2} \left(1-\exp\left(-8\epsilon\left[\sum_{i=1}^d k_i^2\right]^2\right)\right)$$
$$\geqslant C\beta^2 \sum_{k\in(\mathbb{N}^*)^d, 8\epsilon(\Sigma_{i=1}^d k_i^2)^2<M} \epsilon$$
$$\geqslant C\beta^2\epsilon\left(\frac{M}{8\epsilon}\right)^{d/4} \geqslant C_1\beta^2\epsilon^{1-d/4}. \tag{3.25}$$

Inequalities (3.22), (3.24) and (3.25) yield

$$P\left(\int_0^T\int_D |D_{s,y}u_n(t, x)|^2\,\mathrm{d}y\,\mathrm{d}s>0\right) \geqslant \sup_{0<\epsilon\leqslant\epsilon_0} P\big(\tfrac{1}{2}I_1(t, x, \epsilon)-I_2(t, x, \epsilon)>0\big)$$
$$\geqslant \sup_{0<\epsilon\leqslant\epsilon_0} P(I_2(t, x, \epsilon)<C\tfrac{1}{2}\epsilon^{1-d/4})$$
$$\geqslant 1-\inf_{0<\epsilon\leqslant\epsilon_0}\left\{\mathrm{E}(I_2(t, x, \epsilon))\frac{2}{\epsilon^{1-d/4}C}\right\}$$
$$\geqslant 1-\inf_{0<\epsilon\leqslant\epsilon_0} C_n\epsilon^{1/2\wedge(1-d/4)}$$
$$\geqslant 1.$$

This concludes the proof of Lemma 3.4. □

3.3. Proof of Theorem 1.5

Let us now extend Lemma 3.4 to functions σ which do not vanish. Let us denote by $v_{s,y}^n$ the solution to the SPDE (3.3) with $\sigma=1$, that is, for $t\geqslant s$,

$$v^n_{s,y}(t,x) = G(t-s,x,y) + \int_s^t\int_D \Delta G(t-\theta,x,\eta) f'_n(u_n(\theta,\eta)) v^n_{s,y}(\theta,\eta)\,\mathrm{d}\eta\,\mathrm{d}\theta$$

$$+ \int_s^t\int_D G(t-\theta,x,\eta) S_n(\theta,\eta) v^n_{s,y}(\theta,\eta) W(\mathrm{d}\eta,\mathrm{d}\theta),$$

and $v^n_{s,y}(t,x) = 0$ if $t < s$. Because of the uniqueness of the solution to (3.3),

$$v^n_{s,y}(t,x)\sigma(u_n(s,y)) = D_{s,y}u_n(t,x).$$

Also, because σ does not vanish,

$$\begin{aligned} P\left(\int_0^t\int_D |D|v_{s,y}u_n(t,x)|^2\,\mathrm{d}y\,\mathrm{d}s > 0\right) &= P\left(\int_0^t\int_D |v^n_{s,y}(t,x)\sigma(u_n(s,y))|^2\,\mathrm{d}y\,\mathrm{d}s > 0\right) \\ &= P\left(\int_0^t\int_D 1_{\{|v^n_{s,y}(t,x)\sigma(u_n(s,y))|^2>0\}}\,\mathrm{d}y\,\mathrm{d}s > 0\right) \\ &= P\left(\int_0^t\int_D 1_{\{|v^n_{s,y}(t,x)|^2>0\}}\,\mathrm{d}y\,\mathrm{d}s > 0\right) \\ &= P\left(\int_0^t\int_D |v^n_{s,y}(t,x)|^2\,\mathrm{d}y\,\mathrm{d}s > 0\right) \\ &= 1. \end{aligned}$$

This proves (3.15), which implies that the law of $u_n(t,x)$ is absolutely continuous, and the localization $u_{|\Omega_n} = u_n$ yields Theorem 1.5.

Appendix: Proof of Lemma 1.2

We use a theorem from Eidelman and Ivasisen (1970), which gives a similar result for smooth domains. The difficulty is in the 'corners' of D, that is, the points x at least two of whose coordinates belong to $\{0, \pi\}$, and where the boundary of D is not smooth. Let us denote by $D^{\bullet}$ the parallelepiped D without its 'corners'.

Let D_n be an increasing sequence of smooth convex domains included in D such that the intervals $\{x_i \in [1/n, \pi - 1/n]\}$ of ∂D belongs to ∂D_n. Denote by G_n the Green function associated with the operator $\partial/\partial t + \Delta^2$ on D_n with the Neumann boundary conditions. According to Eidelman and Ivasisen (1970, Theorem 1.1), and the first chapter of Eidelman and Zhitarashu (1998), we conclude that this parabolic system in the sense of Petrovskii is well defined and that the following inequalities hold for $x, y \in D_n$, $\alpha \leqslant 1$, $|\beta| \leqslant 4$ with constants which do not depend on n:

$$|G_n(t-s,\, x,\, y)| \leqslant \frac{C}{|t-s|^{d/4}} \exp\left(-\frac{c|x-y|^{4/3}}{|t-s|^{1/3}}\right), \tag{A.1}$$

$$|D_t^\alpha D_x^\beta G_n(t-s,\, x,\, y)| \leqslant \frac{C}{|t-s|^{(d+4\alpha+|\beta|)/4}} \exp\left(-\frac{c|x-y|^{4/3}}{|t-s|^{1/3}}\right). \tag{A.2}$$

If $|\beta| \leqslant 3$, $D_x^\beta G_n$ is integrable; moreover, for $0 \leqslant t < t' \leqslant T$,

$$\int_t^{t'} \int_{\mathbb{R}^d} D_x^\beta G_n(t-s,\, x,\, y)\, \mathrm{d}y\, \mathrm{d}s \leqslant C|t-t'|^{1-d/4}. \tag{A.3}$$

Let ϕ be a C^∞ function on $[0,\, T] \times D$; for $(t,\, x) \in [0,\, T] \times D_n$, set

$$w_n(t,\, x) = \int_0^t \int_{D_n} G_n(t-s,\, x,\, y)\phi(s,\, y)\, \mathrm{d}y\, \mathrm{d}s.$$

The function w_n satisfies

$$\frac{\partial w_n}{\partial t}(t,\, x) + \Delta^2 w_n(t,\, x) = \phi(t,\, x) \tag{A.4}$$

on D_n with the homogeneous Neumann boundary conditions on ∂D_n. We first prove the convergence of (w_n) on $[0,\, T] \times D^\bullet$.

Lemma A.1. *For each $k > 0$, the sequence $(w_n)_{n \geqslant k}$ is relatively compact in $\mathscr{C}([0,\, T] \times D_k)$. We can extract a subsequence $\tilde{w}_n$ such that its time derivative converges uniformly on each interval $[\epsilon,\, T] \times D_k$, for $\epsilon > 0$, and such that its space derivative of order α, with $|\alpha| \leqslant 3$, converges uniformly on $[0,\, T] \times D_k$. Let us denote by $\overline{w}_k$ the limit of $\tilde{w}_n$ on $[0,\, T] \times D_k$; there exists a function w defined on $D^\bullet$ such that $w|_{D_k} = \overline{w}_n|_{D_k}$.*

Proof. According to the Arzelà–Ascoli theorem, we have to prove that the sequence is bounded and equicontinuous. Inequalities (A.1) and (1.14) imply the uniform boundedness of the sequence.

Equicontinuity in time and in space are proved by different arguments, since w_n is a convolution in time. Let $0 \leqslant t < t' \leqslant T$, $x \in D_k$; then

$$\begin{aligned} w_n(t',\, x) - w_n(t,\, x) = & \int_0^t \int_{D_n} G_n(s,\, x,\, y)[\phi(t'-s,\, y) - \phi(t-s,\, y)]\, \mathrm{d}y\, \mathrm{d}s \\ & + \int_t^{t'} \int_{D_n} G_n(t'-s,\, x,\, y)\phi(s,\, y)\, \mathrm{d}y\, \mathrm{d}s. \end{aligned} \tag{A.5}$$

The function ϕ is uniformly continuous on $[0,\, T] \times D_k$, and the upper estimate (A.1) of G_n is independent of n; this implies that the first term on the right-hand side of (A.5) can be made less than ϵ for $t' - t$ small enough uniformly in $x \in D_k$. Furthermore, using (A.1) and (1.14), we have

$$\left|\int_t^{t'}\int_{D_n} G_n(t'-s, x, y)\phi(s, y)\,\mathrm{d}y\,\mathrm{d}s\right| \leqslant C\|\phi\|_\infty |t-t'|;$$

this implies the equicontinuity of w_n in the time variable (uniformly in the space variable). To study the space increment, let $x, x' \in D_k$; then

$$w_n(t, x) - w_n(t, x') = \int_0^t\int_{D_n} [G_n(t-s, x, y) - G_n(t-s, x', y)]\phi(s, y)\,\mathrm{d}y\,\mathrm{d}s.$$

Using (A.2) for the first-order partial derivatives with respect to the space variable and the Taylor formula, we deduce, by a computation similar to that made to prove (2.41) and (2.42), that

$$|w_n(t, x) - w_n(t, x')| \leqslant C|x-x'|\int_0^t |t-s|^{-1/4}\|\phi\|_\infty\,\mathrm{d}s.$$

This yields the equicontinuity in the space variable in D_k, uniformly in the time variable. The sequence w_n admits a subsequence $(\tilde{w}_n)_{n\geqslant k}$ which converges to $\overline{w}_k$ in $C([0, T]\times D_k)$. For the time derivative, we proceed in a similar way. We observe that

$$\frac{\partial w_n}{\partial t}(t, x) = \int_0^t\int_{D_n} G_n(s, x, y)\phi'_t(t-s, y)\,\mathrm{d}y\,\mathrm{d}s + \int_{D_n} G_n(t, x, y)\phi(0, y)\,\mathrm{d}y.$$

We have to prove equicontinuity in space and time variables and the boundedness of the sequence. The first term can be studied as $w_n(t, x)$, ϕ being replaced by ϕ'_t. Because there is only a space integral, we prove equicontinuity and boundedness for the second term on $[\epsilon, T]\times D_k$. For the space increment, take $t, t' \in [\epsilon, T]$; then

$$\left|\int_{D_n} [G_n(t, x, y) - G_n(t', x, y)]\phi(0, y)\,\mathrm{d}y\right|$$

$$\leqslant |t-t'|\int_D\int_0^1 \frac{C}{(\theta t + (1-\theta)t')^{(d+4)/4}} \exp\left(\frac{-c|x-y|^{4/3}}{(\theta t + (1-\theta)t')^{1/3}}\right)\mathrm{d}\theta\,\mathrm{d}y,$$

Using (1.14), we conclude that the second term is equicontinuous in the time variable on $[\epsilon, T]$, uniformly in the space variable. We proceed in a similar way for the equicontinuity in the space variable. We can extract a further subsequence such that the time derivative converges too on $[\epsilon, T]\times D_k$ for $\epsilon > 0$.

For the space derivative, let $\beta \in \mathbb{N}^d$ be such that $|\beta| \leqslant 3$; then we have

$$D_x^\beta w_n(t, x) = \int_0^t\int_{D_n} D_x^\beta G_n(t-s, x, y)\phi(s, y)\,\mathrm{d}y\,\mathrm{d}s.$$

Uniform boundedness is a straightforward consequence of (A.2) and (1.14); the time increment of $D_x^\beta w_n$ can be studied like that of w_n. For the space increment, let $\lambda \in]0, 1[$, apply Taylor's formula, (A.2) and (1.14) and proceed as in the proof of (2.41) and (2.42); this yields, for $\lambda \in [0, 1]$,

$$|D_x^\beta w_n(t, x) - D_x^\beta w_n(t, x')| \leqslant |x - x'|^\lambda \int_0^t |t-s|^{-(\beta+\lambda)/4} \|\phi\|_\infty \, \mathrm{d}s.$$

We see that it suffices to take λ such that $\beta + \lambda < 4$. To obtain the equicontinuity of the derivatives of w_n of order less than 3 in the space variable, we can finally extract a further sequence such that all the partial derivatives of order less than 3 with respect to the space variable converge on $[0, T] \times D_k$. The fact that there exists a function w defined on $[0, T] \times D^\bullet$ comes from the fact that the values of the sequences do not depend on k. The function w also has a time derivative and space derivative of order up to 3, because the corresponding derivatives of (w_n) converge uniformly on $]\epsilon, T[\times D$. □

Since $\tilde{w}_n$ is a solution to (A.4), the sequence $\Delta^2 \tilde{w}_n$ restricted to D_k is uniformly convergent hence it also admits a limit point w which satisfies the PDE

$$\frac{\partial}{\partial t} w(t, x) = -\Delta^2 w(t, x) + \phi(t, x) \tag{A.6}$$

on $D^\bullet$. Furthermore, w satisfies the Neumann boundary conditions because the function w_n satisfies such conditions on the boundary of D_n, and we have constructed D_n so that $\partial D_n \backslash \partial D$ increases to $\partial D \backslash \{0, \pi\}^d$ as $n \to \infty$. The initial condition is again $w(0, \cdot) = 0$. Since (A.6) has a unique solution with homogeneous Neumann's condition, we deduce that

$$w(t, x) = \int_0^t \int_D G(t-s, x, y) f(s, y) \, \mathrm{d}y \, \mathrm{d}s,$$

where G is the Green kernel given by (1.5). The sequence $(G_n(\cdot, x, \cdot))_{n \geqslant k}$ also converges weakly on $[0, t] \times D_k$ for every $t \in]0, T]$ and every $x \in D_k$.

We need to extend upper estimates (A.1) and (A.2) from G_n to G. Let s_0, y_0, $x_0 \in]0, T[\times (]0, \pi[^d)^2$; there exist k such that y_0 and x_0 belong to $D_{k-1} \backslash \partial D_{k-1}$. Let χ be a positive C^∞ function on $\mathbb{R} \times \mathbb{R}^d$ with compact support included in $[-1, 1]^{d+1}$, such that $\int_{[-1,1]^{d+1}} \chi(x) \, \mathrm{d}x = 1$, and set

$$\chi_{s_0, y_0, \epsilon}(s, x) = \epsilon^{-(d+1)} \chi\left(\frac{s - s_0}{\epsilon}, \frac{x - y_0}{\epsilon}\right);$$

then the sequence $\chi_{s_0, y_0, \epsilon}$ converges weakly to δ_{s_0, y_0} in $\mathscr{D}([0, T] \times D_k)$. For ϵ small enough, the support of $\chi_{s_0, y_0, \epsilon}$ is included in $[0, t_0] \times D_k$; hence for fixed t_0, we have

$$\begin{aligned} \int_0^{t_0} \int_D G(t_0 - s, x_0, y) \chi_{s_0, y_0, \epsilon}(s, y) \, \mathrm{d}y \, \mathrm{d}s &= \int_0^{t_0} \int_{D_k} G(t_0 - s, x_0, y) \chi_{s_0, y_0, \epsilon}(s, y) \, \mathrm{d}y \, \mathrm{d}s \\ &= \lim_{n \to \infty} \int_0^{t_0} \int_{D_k} G_n(t_0 - s, x_0, y) \chi_{s_0, y_0, \epsilon}(s, y) \, \mathrm{d}y \, \mathrm{d}s. \end{aligned}$$

The function χ is positive, and (A.1) implies that

$$\left|\int_0^{t_0}\int_D G(t_0-s,\, x_0,\, y)\chi_{s_0,y_0,\epsilon}(s,\, y)\,\mathrm{d}y\,\mathrm{d}s\right|$$

$$\leqslant C\int_0^{t_0}\int_{D_k}\frac{1}{|t_0-s|^{d/4}}\exp\left(-\frac{c|x_0-y|^{4/3}}{|t_0-s|^{1/3}}\right)\chi_{s_0,y_0,\epsilon}(s,\, y)\,\mathrm{d}y\,\mathrm{d}s. \tag{A.7}$$

On the other hand, we know that the kernel G is defined by (1.5); hence G is a continuous function on $]0,\, T]\times D^2$, so that

$$\lim_{\epsilon\to 0}\int_0^{t_0}\int_D G(t_0-s,\, x_0,\, y)\chi_{s_0,y_0,\epsilon}(s,\, y)\,\mathrm{d}y\,\mathrm{d}s = G(t_0-s_0,\, x_0,\, y_0).$$

Therefore, as ϵ tends to 0, (A.7) yields

$$|G(t_0-s_0,\, x_0,\, y_0)|\leqslant\frac{C}{|t_0-s_0|^{d/4}}\exp\left(-\frac{c|x_0-y_0|^{4/3}}{|t_0-s_0|^{1/3}}\right).$$

We observe that for $|\alpha|\leqslant 3$,

$$\partial_x^\alpha w(t,\, x) = \int_0^t\int_D \partial_x^{|\alpha|}G(t-s,\, x,\, y)f(s,\, y)\,\mathrm{d}y\,\mathrm{d}s.$$

Because $\partial_x^\alpha\tilde{w}_n$ converges to $\partial_x^\alpha w$; $(\partial_x^\alpha G_n(\cdot,\, x,\, \cdot))_{n\geqslant k}$ converges weakly to $G(\cdot,\, x,\, \cdot)$ on $[0,\, T]\times D_k$. Arguments similar to the preceding ones imply that, for $|\alpha|\leqslant 3$,

$$|\partial_x^\alpha G(t_0-s_0,\, x_0,\, y_0)|\leqslant\frac{C}{|t_0-s_0|^{(d+|\alpha|)/4}}\exp\left(-\frac{c|x_0-y_0|^{4/3}}{|t_0-s_0|^{1/3}}\right)$$

and

$$|\Delta^2 G(t_0-s_0,\, x_0,\, y_0)|\leqslant\frac{C}{|t_0-s_0|^{(s+4)/4}}\exp\left(-\frac{c|x_0-y_0|^{4/3}}{|t_0-s_0|^{1/3}}\right). \tag{A.8}$$

Because G is the Green kernel associated with the operator $\partial/\partial t+\Delta^2$, G satisfies

$$\frac{\partial}{\partial t}G = -\Delta^2 G,$$

if $t>0$, $x,\, y\in D$, and the estimate (1.8) can be deduced from (A.8); this concludes the proof of Lemma 1.2.

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