CERN TH.6058/91

CALABI–YAU MODULI SPACE, SPECIAL GEOMETRY AND MIRROR SYMMETRY

S. Ferrara*)
CERN, Geneva, Switzerland

ABSTRACT

We review some aspects of the geometry of the moduli space of superstring vacua with (2,2) superconformal symmetry, its connection with the deformation theory of holomorphic three forms and its relation to space–time supersymmetry.

To appear in Brief Revs. for Mod. Phys. Lett. A

---

*) Work supported in part by the Department of Energy of the U.S.A. under contract No. DOE–AT03–88ER40384, Task E.

CERN TH.6058/91
March 1991

Calabi–Yau threefolds [1], i.e. three-dimensional Kähler manifolds with vanishing first Chern class (Ricci flatness) are both perturbative [2] and non-perturbative [3] solutions of string vacua with $N = 1$ and $N = 2$ four-dimensional space–time supersymmetry for heterotic [4] and type-II [5] strings, respectively.

In the language of σ-model perturbation theory, perturbative vacua are those obtained in a coupling-constant expansion, namely the moduli parameters of the Calabi–Yau (C–Y) space, small coupling being in correspondence with large values of the moduli fields.

Non-perturbative string vacua are those which include all non-perturbative effects due to the world-sheet topology, i.e. arbitrary instanton configurations. Non-perturbative solutions correspond to exact superconformal field theories on the world-sheet [3], [6].

The $N = 2$ superconformal invariant σ-model describes deformations of the Kähler class which correspond to the H^2 cohomology.

However, in string theory, quantum symmetries exist, which exchange a large radius for a small one, i.e. small for large σ-model coupling constants [7]. To incorporate these symmetries one needs to solve the two-dimensional σ-model exactly. For complex-structure deformations, the other deformation parameters of C–Y manifolds, tree-level string results, actually give exact results [8], [9], since the correlators of complex structure moduli are independent of the σ-model coupling constants, which are (1,1) moduli, and the two sets of deformation parameters form a product space [10]–[13].

Another important discovery is the mirror symmetry [14] among pairs of C–Y threefolds C, C', in which the even and odd cohomology classes are exchanged:

$$H^3(C) \simeq \sum_{i=0}^{3} H^{2i}(C') \qquad (C \to C'); \tag{1}$$

this implies an intriguing symmetry between Kähler class and complex structure deformations. However, the very same symmetry reduces the problem of computing non-perturbative instanton corrections to a 'classical' problem in the mirror image.

Mirror symmetry implies that solving exactly a (2,2) superconformal field theory reduces to a problem of algebraic geometry [15], that of studying the deformation theory of three-form cohomology.

The metric in the moduli space of C–Y manifolds is locally a product metric for the two types of moduli. The two spaces are (locally) special Kähler, i.e. they are Hodge manifolds with Riemann tensors given by

$$R_{i\bar{j}\ell\bar{m}} = g_{i\bar{j}}g_{\ell\bar{m}} + g_{i\bar{m}}g_{\ell\bar{j}} - \frac{1}{Y^2}W_{i\ell p}\overline{W}_{\bar{j}\bar{m}\bar{p}}g^{p\bar{p}}\,, \tag{2}$$

where $W_{i\ell p}$ is a totally symmetric (tensor) section on a U(1) bundle whose first Chern class is the Kähler class. Equation (2) was derived in three ways: from space–time supersymmetry arguments [10], [12], as a Ward identity among string amplitudes in four-dimensional compactifications on (2,2) vacua [13], and from the moduli geometry of Kähler class and complex structure deformations of C–Y threefolds [11], [12].

The Kähler potential is $-\log Y$. Equation (2) actually implies the existence of $h + 1$ holomorphic sections [16]–[18] $L^I(Z)$ with the Kähler potential satisfying the following condition [19]

$$\frac{\partial L^I}{\partial Z_i}\frac{\partial L^J}{\partial Z_j}\frac{\partial L^K}{\partial Z_k}\frac{\partial}{\partial L_I}\frac{\partial}{\partial L_J}\frac{\partial}{\partial L_K}Y = W_{ijk}\,. \tag{3}$$

Equation (3) is solved by setting

$$Y = L^A N_{AB} \overline{L}^B \qquad N_{AB} = F_{,AB} + \overline{F}_{,AB}$$

where $F(L^I)$ is a holomorphic section homogeneous of degree 2:

$$L^J F_I = 2F\,.$$

In special, projective coordinates $L^I/L^0 = Z_i$, Eq. (3) becomes

$$\partial_i \partial_j \partial_{\overline{k}} Y = W_{ijk} = \partial_i \partial_j \partial_k \mathcal{F}\,, \qquad \mathcal{F} = (L^0)^2 F \tag{4}$$

Equation (4) is actually a relation within different correlators of the underlying (2,2) conformal field theory, since it relates the holomorphic three-point function which appears in O.P.E. coefficients of the chiral ring [20] with the (non-holomorphic) correlation of the top chiral primary field [21] (with its Hermitian conjugate)

$$Y = \langle \Omega(Z) \overline{\Omega}(\overline{Z}) \rangle \tag{5}$$

The function F (up to terms for which $\partial^3 F = 0$) can be identified [19], [22] with a holomorphic generating function (zero-point function) for all (holomorphic) h-point ($h \geq 3$) functions of the corresponding $N = 2$ (twisted) TQFT [23].

A special property of the moduli geometry is that, using the W coefficients, one can construct two (closed) forms in terms of the Kähler and Ricci form

$$\|C\|^2 = (h+1)J - R\,, \tag{6}$$

in such a way that (in components)

$$\|C\|^2_{i\overline{j}} = \frac{1}{Y^2} W_{i\ell p} \overline{W}_{\overline{j}\overline{m}\overline{p}} g^{p\overline{p}} g^{\ell \overline{m}} = -\partial_i \partial_{\overline{j}} \log(-N_{IJ})\,. \tag{7}$$

The factorN_{IJ} is actually the real part of the period matrix F_{IJ}. Equations (6) and (7) are proven by contracting Eq. (2) with the inverse metric and by noticing that

$$N_{IJ} = -i \int \frac{\partial \Omega}{\partial Z^I} \wedge \frac{\partial \overline{\Omega}}{\partial Z^J}\,, \quad F_{IJ} = -i \int \frac{\partial \Omega}{\partial Z^I} \wedge \alpha_J\,, \quad \Omega = L^I \alpha_I - i F_I \beta^I\,, \tag{8}$$

where α, β is an integral cohomology basis in H^3. Note that under a $\mathrm{Sp}(2(1+h);\mathbf{Z})$ transformation, the F matrix undergoes a projective transformation (F_{IJ} is actually a global section of a U(1) line bundle since it is of degree zero):

$$iF_{IJ}(Z_\Gamma) \to (iAF + B)(iCF + D)^{-1}\,, \tag{9}$$

where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2h+2;\mathbf{Z})\,,$$

i.e. AB^T, CD^T are symmetric and $AD^T - BC^T = \mathbb{1}$; Z_Γ is the action of the modular group on the moduli space so that, by definition (g is a holomorphic function of the moduli):

$$\begin{pmatrix} iF_I(Z_\Gamma) \\ L^I(Z_\Gamma) \end{pmatrix} \to e^g \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} iF_I(Z) \\ L^I(Z) \end{pmatrix}\,. \tag{10}$$

In terms of the matrix F_{IJ}, it is then possible to introduce C–Y modular forms in two ways [24]. One is to take matrix-valued modular functions $\theta(F)$ transforming covariantly under Eq. (9). The other possibility is to introduce a sort of generalized Dedekind function defined through

$$f = -\sum_{M,N(\Gamma)} \log \frac{|M^A X_A - iN_A F^A|^2}{X_A \bar{F}_A + \bar{X}^A F_A} \tag{11}$$

(the sum is meant to be performed over an 'orbit' of Γ).

For the particular case

$$F = i\frac{L^1 L^2 L^3}{L^0}, \qquad t_i = -i\frac{L^i}{L^0},$$

after regularization, f reduces to

$$\sum_{i=1}^{3} \log |\eta(t_i)|^4 (t_i + \bar{t}_i). \tag{12}$$

This case corresponds to the geometry of the moduli space of the Z_3/Z_3 orbifold [25]. In this case F is the free-energy for the three-torus, an orbifold version of a C–Y space.

Observe that Eq. (11) defines (after regularization) a C–Y superpotential [24] for general functions F:

$$-\log W_{\text{regul}} = \sum_{M,N(\Gamma)} \log \left(M^A X_A - iN_A F^A\right), \tag{13}$$

which is indeed a holomorphic section of a U(1) bundle.

The $(h+1)$ holomorphic sections have a more direct meaning in type II superstrings [5] where in fact they are in correspondence with $(h+1)$ (Ramond–Ramond vectors) vector fields, the $N=2$ superpartners of the moduli scalars, and the graviphoton [26]–[28].

The self-dual 10-dimensional five-form $\mathcal{F}$ of type IIB supergravity can be projected on $M_4 \times K_6$, where K_6 is C–Y; it becomes a two-form over M_4 and a three-form over the C–Y manifold. This five-form is conserved over space–time:

$$\partial_\mu \mathcal{F}_{\mu\nu} = 0.$$

Projecting this equation over the homology cycles one obtains the Bianchi identities and the equations of motion for the vector fields coupled to the moduli fields, exactly as dictated by supergravity.

On the other hand, if one expands F on the physical basis of three-forms, elements of the $H^{3,0}$, $H^{2,1}$ Delbault cohomology, one obtains moduli-dependent combinations of the 4-D field strengths F^{-I}, T^-; these are exactly the ones that appear in the transformation laws of the gaugino and gravitino fields, respectively [27]. In mirror manifolds, because of the dual rate of even and odd forms, type IIA and type IIB theories, in the massless sector, coincide with the chirality-reversed theories in the mirror image.

Given a particular C–Y manifold, it is in principle possible to compute the prepotential function $F(Z)$. Then all other relevant geometrical quantities can be computed. Because of its relation with $N=2$ superconformal theories, this exercise amounts to exactly solving a two-dimensional superconformal field theory with methods of algebraic geometry.

Equations (11)–(13) can be computed by the knowledge of the periods X_A, F^A [12]. Recently a non-trivial example of an exactly solvable model was provided [29]. This was the mirror manifold of $P_4(5)$ for which $h_{21} = 1$. The periods are found to be solutions of a differential equation with hypergeometric functions as solutions. The modular group is, in this case, a discrete subgroup of SL(2,R), different from SL(2,$\mathbb{Z}$), which acts on the periods X^A, F_A, with a linear action of Sp(4;$\mathbb{Z}$) matrices. Inserting the period functions into Eqs. (11)–(13), one may obtain the generalized free-energy and 'Dedekind function' for this case. In a more general setting, the (Picard–Fuchs) differential equations satisfied by the periods [15], [30] should have [19] the same content as certain differential equations satisfied by the generating function F [23] in the corresponding $N = 2$ superconformal field theory.

REFERENCES

[1] E. Calabi, *in* Algebraic geometry and topology (Univ. Press, Princeton, 1957), p. 78; S.T. Yau, Proc. Nat. Acad. Sci. **74** (1977) 1798.

[2] P. Candelas, G. Horowitz, A. Strominger and E. Witten, Nucl. Phys. **B258** (1985) 46.

[3] D. Gepner, Nucl. Phys. **B296** (1988) 757 and Phys. Lett. **B299** (1987) 380.

[4] D.J. Gross, J. Harvey, E. Martinec and R. Rohm, Phys. Rev. Lett. **54** (1985) 1502.

[5] M.B. Green and J.H. Schwarz, Phys. Lett. **B122** (1983) 143;
J.H. Schwarz, Nucl. Phys. **B226** (1983) 269.

[6] M. Ademollo, L. Brink, A. D'Adda, R. D'Auria, E. Napolitano, S. Sciuto, E. del Giudice, P. Di Vecchia, S. Ferrara, F. Gliozzi, R. Musto and R. Pettorino, Phys. Lett. **B62** (1976) 105;
W. Boucher, D. Friedan and A. Kent, Phys. Lett. **B172** (1986) 316;
A. Sen, Nucl. Phys. **B278** (1986) 289 and **B284** (1987) 1423;
L. Dixon, D. Friedan, E. Martinec and S.H. Schenker, Nucl. Phys. **B282** (1987) 13;
T. Banks and L. Dixon, Nucl. Phys. **B307** (1988) 93;
W. Lerchen, D. Lüst and A.N. Schellekens, Nucl. Phys. **B287** (1987) 477.

[7] V.P. Nair, A. Shapere, A. Strominger and F. Wilczek, Nucl. Phys. **B287** (1987) 402;
A. Shapere and F. Wilczek, Nucl. Phys. **B320** (1989) 669;
A. Giveon, E. Rabinovici and G. Veneziano, Nucl. Phys. **B322** (1989) 363;
S. Ferrara, R. Lüst, A. Shapere and S. Theisen, Phys. Lett. **B225** (1989) 363;
A. Giveon and M. Porrati, Phys. Lett. **B246** (1990) 54.

[8] J. Distler and B. Greene, Nucl. Phys. **B304** (1989) 1 and **B309** (1988) 295.

[9] M. Dine, P. Huet and N. Seiberg, Nucl. Phys. **B322** (1989) 301.

[10] S. Cecotti, S. Ferrara and L. Girardello, Int. J. Mod. Phys. **4** (1989) 24; Phys. Lett. **B213** (1989) 443.

[11] P. Candelas, P. Green and T. Hubsch, *in* String '88 (eds. J. Gates, L. Preitschof and W. Siegel)(World Scientific, Singapore, 1989);
P. Candelas and X. de la Ossa, Nucl. Phys. **B355** (1991) 455.

[12] S. Ferrara and A. Strominger, *in* String '89 (eds. Arnowitt, Bryan, Duff, Nanopoulos and Pope)(World Scientific, Singapore, 1989), p. 245.

[13] L. Dixon, V.S. Kaplunovski and J. Louis, Nucl. Phys. **B239** (1990) 27.

[14] P. Candelas, M. Lynker and R. Schimmrigk, Nucl. Phys. **B341** (1990) 383.
B.R. Green and R. Plesser, Harvard preprint HUTP–89/B241 and Nucl. Phys. **B338** (1990) 15.
P. Aspinwall, C. Lütken and G. Ross, Phys. Lett. **B241** (1990) 373.
P. Aspinwall and C. Lütken, Nucl. Phys. **B353** (1991) 427 and **B355** (1991) 482.
M. Cvetic, J. Molera and B. Ovrut, Phys. Lett. **B248** (1990) 83;
L. Castellani, M. Monteiro, P. Fré and F. Gliozzi, Turin preprint DFTT/7/90 (1990).
[15] R. Briant and P. Griffiths, Progress in mathematics (Birkhäuser, Boston, 1983), Vol. 36, p. 77.
G. Tian, *in* Mathematical aspects of string theory (ed. S.T. Yau) (World Scientific, Singapore, 1987), p. 629.
[16] E. Cremmer, C. Kounnas, A. van Proeyen, J.P. Derendinger, S. Ferrara, B. de Wit and L. Girardello, Nucl. Phys. **B250** (1985) 385.
[17] A. Strominger, Commun. Math. Phys. **133** (1990) 163.
[18] L. Castellani, R. D'Auria and S. Ferrara, Phys. Lett. **B241** (1990) 57; Classical Quantum Gravity **7** (1990) 1767.
[19] A. Ceresole, S. Ferrara and W. Lerche (unpublished).
[20] W. Lerche, C. Vafa and N.P. Warner, Nucl. Phys. **B324** (1989) 427.
[21] V. Perival and A. Strominger, Phys. Lett. **B235** (1990) 261.
[22] A. Giveon and D.-J. Smit, Lawrence Berkeley Laboratory preprint LBL–30342, UCB–PTH–91/9 and LBL–30388, UCB–PTH–91/10.
[23] R. Dijkgraaf, F. Verlinde and F. Verlinde, Nucl. Phys. **348** (1991) 435 and **B352** (1991) 59.
[24] S. Ferrara, C. Kounnas, D. Lüst and F. Zwirner, CERN TH.6090/91.
[25] D. Schevitz, Nucl. Phys. **B338** (90) 283.
[26] S. Cecotti, Commun. Math. Phys. **131** (1990) 517.
[27] S. Ferrara, M. Bodner and C. Cadavid, Phys. Lett. **B247** (1990) 25 and Classical Quantum Gravity **8** (1991) 789.
[28] R. D'Auria, S. Ferrara and P. Fré, CERN–TH.5954/90, DFPD/90/TH/35–Ref. SISSA 192/90/EP, to appear in Nucl. Phys. B.
[29] P. Candelas, X. de la Ossa, P. Green and L. Parkes, Austin preprint UTTG-25-1990 and Phys. Lett. **B258** (1991) 118.
[30] P. Griffith and W. Schmid, *in* Discrete subgroups of Lie groups and application to moduli (The University Press, Oxford, 1973).