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Calculate the Moment Generating Function for Continuous Random Variable by Using Decomposition Method

Abdullah A. Ameen

Department of Mathematics, Faculty of Science
Basra University, Basra, Iraq
dr_abd64@yahoo.com

Abstract

In this paper, decomposition method is applied to develop a computational method for the moment generating function of continuous random variable. The proposed method is easy to implement from a computational viewpoint and can be employed for finding moment generating function of continuous random variable without solving any integral. Sometimes, this integral cannot be solved in general and in this case, the moment generating function remains in integral form. Some examples are illustrative for demonstrating the advantage of the proposed method.

Keywords: decomposition method, moment generating function

1. Introduction

Moment generating functions has been widely used by statisticians (in connection with various distribution problems) to derive moments of distributions, establish the distributions of sums and differences of independent random variables, and derive limiting distributions of sequences of random variables. Let X be a (one-dimensional) random variable and $F(x) = \Pr(X \le x)$ its distribution function [1,2]. The function $M_X(t) = E(e^{tX})$, for t in R, is called the moment generating function of X. For a continuous distribution with density function $f(x)$, the moment generating function exists if $M_X(t)=\int_{-\infty}^{\infty} e^{tx} f(x)dx$, is finite for any real number t in some open interval $-h<t<h$.

The classical method for finding moment generating function of continuous random variable requires solving the integral $M_X(t)=\int_{-\infty}^{\infty} e^{tx} f(x)\, dx$. Sometimes, this integral cannot be solved in general. For instance, the moment generating function of Pareto distribution is remained in integral form [3]

$$\begin{aligned} M_X(t) &= b(-at)^b \left(\int_{-at}^{\infty} e^{-s} s^{-b-1}\, ds \right) \\ &= b(-at)^b\, \Gamma(-b,-at), \quad t<0. \end{aligned} \tag{1}$$

In this paper, we propose a computational method for moment generating function of continuous random variable based on solving first order differential equation by DM.

Consider the following first order differential equation

$$\frac{dy(x)}{dx} + t\, y(x) = f(x) \tag{2}$$

where $y(0)=0$ and t is small parameter tend to zero.

The analytical solution of (1) is given by

$$e^{tx}\, y(x) = \int e^{tx} f(x)\, dx \tag{3}$$

2. Solution of (2) by DM

The DM is effective method to construct analytic approximate solutions for nonlinear problem. The method was firstly proposed by the American mathematician, G. Adomian (1923-1996) and it is one of a semi-analytical method for solving approximate solutions for large classes of nonlinear differential equations [4-7].

Let $y(x)=\sum_{k=0}^{\infty} y_k(x)$, so Eq. (1) can be written as

$$\sum_{k=0}^{\infty} \frac{dy_k(x)}{dx} + t \sum_{k=0}^{\infty} y_k(x) = f(x) \tag{4}$$

where the terms $y_0(x), y_1(x), y_2(x), \ldots$ are determined recursively as follows.

$$y_0(x) = \frac{f(x)}{t} \tag{5}$$

$$y_k(x) = -\frac{1}{t}\frac{dy_{k-1}(x)}{dx} \quad , \quad k = 1, 2, 3, \ldots \tag{6}$$

or

$$y_k(x) = \frac{(-1)^k}{t^{k+1}}\frac{d^k f(x)}{dx^k} \quad , \quad k = 1, 2, 3, \ldots \tag{7}$$

Finally, the solution of Eq.(2) in the series form (by using DM) is

$$y(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{t^{k+1}}\frac{d^k f(x)}{dx^k} \tag{8}$$

3. The new method

The aim of this section is to derive a new method for computing moment generating function $M_X(t)$ of continuous random variable X having the probability function $f(x)$

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx \tag{9}$$

By using Eq.(9) in Eq.(3), one can get

$$M_X(t) = \left.\left(e^{tx}\sum_{k=0}^{\infty}\frac{(-1)^k}{t^{k+1}}\frac{d^k f(x)}{dx^k}\right)\right|_{x=-\infty}^{x=\infty} \tag{10}$$

4. Test Examples

In this section, we adopt some examples to illustrate the advantage of Method for moment generating function $M_X(t)$ of continuous random variable X having probability function $f(x)$. The result the simplicity and reliability of the method.

Example 1 : Let X is uniformly distributed over $[a\ ,\ b]$,

$$f(x)=\frac{1}{b-a}\quad ,\ a\le x\le b\ ,\ -\infty<a<b<\infty$$

Clearly,

$$\frac{d^k f(x)}{dx^k}=0\quad ,\quad k=1,2,3,\ldots$$

So, one can have,

$$M_X(t)=\left(e^{tx}\sum_{k=0}^{\infty}\frac{(-1)^k}{t^{k+1}}\frac{d^k f(x)}{dx^k}\right)\Bigg|_{x=a}^{x=b}=e^{tx}\frac{f(x)}{t}\Bigg|_{x=a}^{x=b}=\frac{e^{tx}}{t(b-a)}\Bigg|_{x=a}^{x=b}=\frac{e^{bt}-e^{at}}{t(b-a)}$$

Example 2 : : Let X has an exponential distribution with parameter r,

$$f(x)=r\,e^{-rx}\quad ,\ x\ge 0\ ,\ r>0$$

$$\frac{df(x)}{dx}=-r^2\,e^{-rx}$$

$$\frac{d^2 f(x)}{dx^2}=r^3\,e^{-rx}$$

$\vdots$

$$\frac{d^k f(x)}{dx^k}=(-1)^k\,r^{k+1}\,e^{-rx}\quad ,\quad k=1,2,3,\ldots$$

$$M_X(t)=\left(e^{tx}\sum_{k=0}^{\infty}\frac{(-1)^k}{t^{k+1}}\frac{d^k f(x)}{dx^k}\right)\Bigg|_{x=0}^{x=\infty}=\sum_{k=0}^{\infty}\frac{r^{k+1}\,e^{-(r-t)x}}{t^{k+1}}\Bigg|_{x=0}^{x=\infty}=-\sum_{k=0}^{\infty}\frac{r^{k+1}}{t^{k+1}}\quad \text{for}\quad t<r$$

$$=-\frac{r}{t}\left(\frac{1}{1-r/t}\right)=\frac{r}{r-t}\quad ,\ t<r$$

Example 3 : : Let X has Pareto distribution with parameters $a\,,b$,

$$f(x)=\frac{b\,a^b}{x^{b+1}}\quad ,\ x\ge a\ ,\ a>0\ ,\ b>0$$

$$\frac{df(x)}{dx} = -b(b+1)a^b x^{-(b+2)}$$

$$\frac{d^2f(x)}{dx^2} = b(b+1)(b+2)a^b x^{-(b+3)}$$

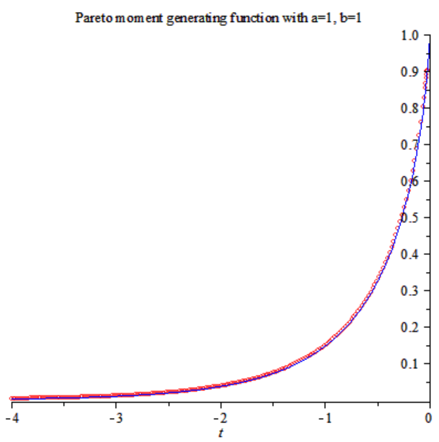
$$\vdots$$

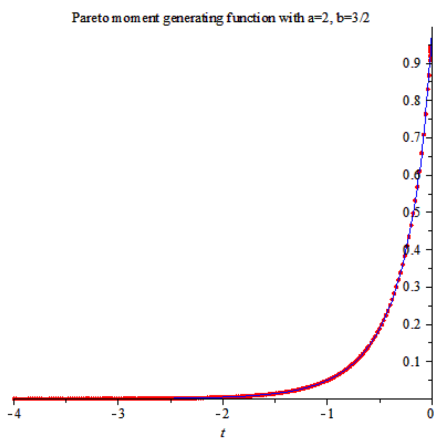
$$\frac{d^kf(x)}{dx^k} = (-1)^k \frac{a^b\ \Gamma(b+k+1)}{\Gamma(b)} x^{-(b+1+k)} \quad , \quad k = 1, 2, 3, \ldots$$

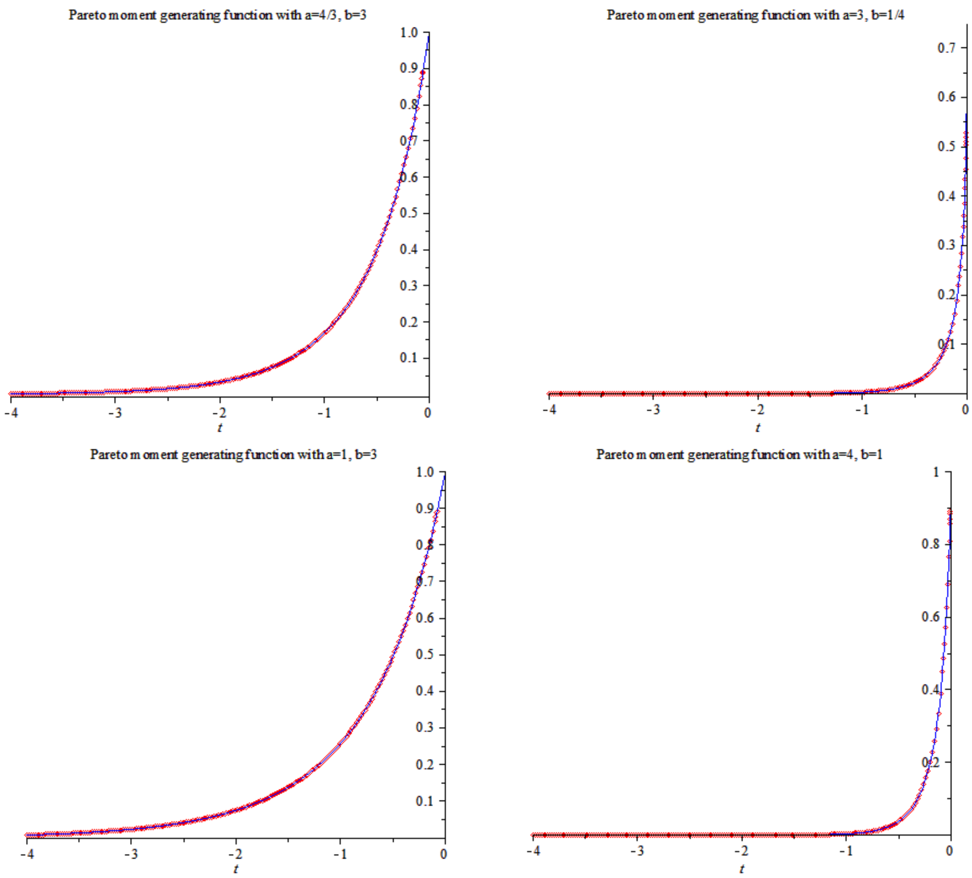
$$M_X(t) = \left(e^{tx} \sum_{k=0}^{\infty} \frac{(-1)^k}{t^{k+1}} \frac{d^kf(x)}{dx^k} \right) \Bigg|_{x=a}^{x=\infty}$$

$$= -e^{at} \left(\sum_{k=0}^{\infty} \frac{\Gamma(b+k+1)}{\Gamma(b)(at)^{k+1}} \right) \quad \text{for} \quad t < 0. \qquad (11)$$

It should be emphasis that the moment generating function of Pareto distribution is still in integral form (incomplete gamma function) as in Eq.(1). While in our method, we express to the moment generating function of Pareto distribution in the form Eq. (11) which is very simple and not contain any integral. The following Figs. show that Eq.(11) coincide with Eq. (1) for every parameters a and b .







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