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CALCULATING COHOMOLOGY GROUPS OF MODULI SPACES OF CURVES VIA ALGEBRAIC GEOMETRY

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In this paper we compute the first, second, third, and fifth rational cohomology groups of $\overline{\mathcal{M}}_{g,n}$, the moduli space of stable n-pointed genus g curves. It turns out that $\mathrm{H}^1(\overline{\mathcal{M}}_{g,n}, \mathbf{Q})$, $\mathrm{H}^3(\overline{\mathcal{M}}_{g,n}, \mathbf{Q})$, and $\mathrm{H}^5(\overline{\mathcal{M}}_{g,n}, \mathbf{Q})$ are zero for all values of g and n, while $\mathrm{H}^2(\overline{\mathcal{M}}_{g,n}, \mathbf{Q})$ is generated by tautological classes, modulo relations that can be written down explicitly; the precise statements are given by Theorems (2.1) and (2.2). We are convinced that the computation of the fourth cohomology of all moduli spaces $\overline{\mathcal{M}}_{g,n}$ should also be accessible to our methods.

It must be observed that some of these results are not new. In fact, it is known that $\overline{\mathcal{M}}_{g,n}$ is simply connected (cf. [2], for instance), while Harer has determined $\mathrm{H}^2(\mathcal{M}_{g,n}, \mathbf{Q})$ [8]; once this is known, it is not hard to compute the corresponding group for $\overline{\mathcal{M}}_{g,n}$. Harer [10] has also shown that $\mathrm{H}^3(\mathcal{M}_{g,n}, \mathbf{Q})$ vanishes, at least for large enough genus. What is really new here is the method of proof, which is based on standard algebro-geometric techniques, rather than geometric topology. Especially for odd cohomology, this provides proofs that are quite short and, we hope, rather transparent. It should also be noticed that the odd cohomology of $\overline{\mathcal{M}}_{g,n}$, at least in the range we can deal with, seems to be somewhat better behaved than the one of $\mathcal{M}_{g,n}$, for it is certainly not the case that the first and third cohomology groups of $\mathcal{M}_{g,n}$ are always zero.

Roughly speaking, the idea of the proof is as follows. If one could apply the Lefschetz hyperplane theorem, one might reduce the computation of $\mathrm{H}^k(\overline{\mathcal{M}}_{g,n}, \mathbf{Q})$ to the one of $\mathrm{H}^k(\partial\mathcal{M}_{g,n}, \mathbf{Q})$, for low k. Although the standard Lefschetz theorem cannot be used, since $\mathcal{M}_{g,n}$ is almost never affine, a foundational result of Harer, which is a direct consequence of the construction of a cellular decomposition of $\mathcal{M}_{g,n}$ by means of Strebel differentials, provides a suitable substitute. A little Hodge theory then shows that, always for low enough k, $\mathrm{H}^k(\overline{\mathcal{M}}_{g,n}, \mathbf{Q})$ injects not only in $\mathrm{H}^k(\partial\mathcal{M}_{g,n}, \mathbf{Q})$, but also in the k-th cohomology group of the normalization N of $\partial\mathcal{M}_{g,n}$. Put otherwise, the combinatorics of the boundary does not contribute new classes to $\mathrm{H}^k(\overline{\mathcal{M}}_{g,n}, \mathbf{Q})$. Since the components of N are essentially products of moduli spaces $\overline{\mathcal{M}}_{\gamma,\nu}$ such that either $\gamma < g$ or $\gamma = g$ and $\nu < n$, one may try to compute $\mathrm{H}^k(\overline{\mathcal{M}}_{g,n}, \mathbf{Q})$ by double induction on g and n, starting from a few seed cases to be handled directly. This turns out to be possible, and reduces to elementary linear algebra. An interesting, and somewhat

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unexpected, byproduct of the proof is that, for any k, the k-th cohomology group of $\overline{\mathcal{M}}_{g,n}$ injects into the k-th cohomology of the normalization of the component of the boundary parametrizing irreducible singular curves, provided g is large enough.

There are other cases, in addition to the ones mentioned above, in which the cohomology of moduli spaces of curves has been computed. First of all, Harer [11] has computed the fourth cohomology of $\mathcal{M}_g$ for large enough g. In a different direction, the entire cohomology ring of $\overline{\mathcal{M}}_{0,n}$ has been described for any n by Keel [13] in terms of generators and relations, while Getzler [6] has shown how to recursively compute the cohomology of $\overline{\mathcal{M}}_{1,n}$, for any n. Mumford [18] and Faber [5] have computed the cohomology of $\overline{\mathcal{M}}_2$ and $\overline{\mathcal{M}}_{2,1}$, and Getzler [7] the one of $\overline{\mathcal{M}}_{2,2}$ and $\overline{\mathcal{M}}_{2,3}$. Finally, Looijenga [14] has computed the cohomology of $\mathcal{M}_3$ and $\mathcal{M}_{3,1}$.

We will assume Keel's result, which is derived entirely via algebraic geometry, although the part of it that we use could be proved without much effort by our methods. Some of Getzler's and Looijenga's results will be needed, while computing H^5, to deal with some of the initial cases of the induction; except for this, our treatment will be self-contained and for the most part quite elementary.

We are grateful to Eduard Looijenga for indicating to us that Proposition (2.8) could be proved via Hodge theory; our original proof was based on a fairly involved combinatorial argument and worked with certainty only for $k \leqslant 2$.

1. Notation

All the varieties we shall consider in this paper will be over $\mathbf{C}$. Only rational cohomology will be used; when we omit mention of the coefficient group, we always implicitly assume rational coefficients.

Let g and n be non-negative integers such that $2g - 2 + n > 0$. We denote by $\overline{\mathcal{M}}_{g,n}$ the moduli space of stable n-pointed genus g curves and by $\mathcal{M}_{g,n}$ its subspace parametrizing smooth curves. More generally, if P is a set with n elements, it will be technically convenient to consider also stable P-pointed curves. These are simply stable curves whose marked points are indexed by P, and not by $\{1, \ldots, n\}$. We shall denote by $\overline{\mathcal{M}}_{g,\mathrm{P}}$ and $\mathcal{M}_{g,\mathrm{P}}$ the corresponding moduli spaces. The *boundary* of $\mathcal{M}_{g,\mathrm{P}}$ is $\partial\mathcal{M}_{g,\mathrm{P}} = \overline{\mathcal{M}}_{g,\mathrm{P}} \setminus \mathcal{M}_{g,\mathrm{P}}$.

By a *graph* we shall mean the datum G of:

- a non-empty finite set $\mathrm{V} = \mathrm{V(G)}$ (the *vertices* of G),
- a non-negative integer g_v for every $v \in \mathrm{V}$,
- a finite set $\mathrm{L} = \mathrm{L(G)}$ (the *half-edges* of G),
- a partition $\mathcal{P}$ of L in subsets with one or two elements,
- a partition $\{\mathrm{L}_v\}_{v \in \mathrm{V}}$ of L.

Whe shall call the elements of $\mathcal{P}$ with one element *legs* of G and those with two elements *edges*; the set of all the latter will be denoted $\mathrm{E} = \mathrm{E(G)}$. We also set

$l_v = |L_v|$. In what follows we shall implicitly consider only graphs which are *connected*, in an obvious sense. If P is a finite set, by a P-*labelled graph* we shall mean the datum of a graph G plus a bijection between the set of its legs and P.

To every stable P-pointed genus g curve $(C; \{q_p\}_{p\in P})$ we may associate a P-labelled graph G as follows. Let $\pi : N \to C$ be the normalization of C. We let V(G) be the set of all components of N, and L(G) the set of all points of N which map either to nodes or to marked points of C; two of these constitute an edge if they map to the same point (a node) of C, while the remaining ones are legs. The indexing of the legs by P is the obvious one. We also set g_v = genus of v, and let L_v be the set of all elements of L belonging to v. Notice that

$$g = \sum_{v\in V(G)} g_v + 1 - |V(G)| + |E(G)|\,.$$

Conversely, this formula can be used to define the *genus* of any connected P-labelled graph. The graph associated to a stable P-pointed genus g curve is *stable* in the sense that $2g_v - 2 + l_v > 0$ for every vertex v; more exactly, to say that a curve is stable is equivalent to saying that its graph is.

Let G be a connected stable P-labelled graph of genus g. We denote by $\mathcal{M}(G)$ the moduli space of all P-pointed genus g stable curves whose associated graph is G; it is a locally closed subspace of $\overline{\mathcal{M}}_{g,P}$ of codimension $|E(G)|$. We also denote by δ_G the orbifold fundamental class of $\overline{\mathcal{M}(G)}$, that is, the crude fundamental class divided by the order of the automorphism group of a general element of $\mathcal{M}(G)$. The degree two classes correspond to graphs with one edge. These come in two kinds; there is the graph G_{irr}, with one vertex of genus $g-1$, and there are the graphs $G_{a,A}$, which have two vertices, one of genus a, with attached the legs indexed by A, and one of genus $g-a$, with attached the legs indexed by $A^c = P\setminus A$. We shall write δ_{irr} and $\delta_{a,A}$ instead of $\delta_{G_{irr}}$ and $\delta_{G_{a,A}}$; these classes are the fundamental classes of orbifold Cartier divisors Δ_{irr} and $\Delta_{a,A}$ supported on $\overline{\mathcal{M}(G_{irr})}$ and $\overline{\mathcal{M}(G_{a,A})}$, respectively. It is clear that $\delta_{a,A} = \delta_{g-a,A^c}$, and also that $\delta_{a,A}$ does not make sense as defined unless $G_{a,A}$ is a stable graph, i.e., unless $2a-2+|A| \geqslant 0$ and $2(g-a)-2+|A^c| \geqslant 0$. In practice, this means that $\delta_{a,A}$ is still undefined if $a=0$ and $|A|<2$ or $a=g$ and $|A|>|P|-2$. We will find it convenient to set $\delta_{a,A}$ to zero if $a<0$, $a>g$, $2a-2+|A|<0$, or $2(g-a)-2+|A^c|<0$. The class δ_{irr}, as defined above, also does not make sense in genus zero; we will set it to zero in this case. Finally, for any integer a, we write δ_a to denote the sum of all classes $\delta_{a,A}$; notice that, in case $g=2a$, the summand $\delta_{a,A} = \delta_{a,A^c}$ occurs only once, and not twice, in this sum.

The basic maps between moduli spaces we shall consider are

$$\pi : \overline{\mathcal{M}}_{g,\mathrm{P}\cup\{q\}} \to \overline{\mathcal{M}}_{g,\mathrm{P}} ,$$
$$\xi : \overline{\mathcal{M}}_{g-1,\mathrm{P}\cup\{q,r\}} \to \overline{\mathcal{M}}_{g,\mathrm{P}} ,$$
$$\eta : \overline{\mathcal{M}}_{a,\mathrm{A}\cup\{q\}} \times \overline{\mathcal{M}}_{g-a,(\mathrm{P}\setminus\mathrm{A})\cup\{r\}} \to \overline{\mathcal{M}}_{g,\mathrm{P}} ,$$

which are defined as follows. The image under π of a $\mathrm{P}\cup\{q\}$-pointed curve is obtained by "forgetting" the point labelled q and passing to the stable model. The image under ξ of a $\mathrm{P}\cup\{q, r\}$-pointed genus $g-1$ curve is obtained by identifying the points labelled q and r; likewise, the image under η of a pair consisting of an $\mathrm{A}\cup\{q\}$-pointed genus a curve and a $(\mathrm{P}\setminus\mathrm{A})\cup\{r\}$-pointed genus $g-a$ curve is the P-pointed curve of genus g obtained by identifying the points labelled q and r.

The map $\pi : \overline{\mathcal{M}}_{g,\mathrm{P}\cup\{q\}} \to \overline{\mathcal{M}}_{g,\mathrm{P}}$ is also called the *universal curve* over $\overline{\mathcal{M}}_{g,\mathrm{P}}$. It has $|\mathrm{P}|$ sections, indexed by P; the section σ_p attaches to any P-pointed curve $(C; \{x_i\}_{i\in\mathrm{P}})$ the $\mathrm{P}\cup\{q\}$-pointed curve obtained by attaching to C a copy of $\mathbf{P}^1$ by identifying x_p and $0 \in \mathbf{P}^1$, and labelling the points 1 and ∞ by p and q. One may use the universal curve to define further cohomology classes on $\overline{\mathcal{M}}_{g,\mathrm{P}}$ as follows. We denote by ω_π the relative dualizing sheaf and by D_p the image of σ_p. One then sets

$$\psi_p = \sigma_p^*(c_1(\omega_\pi)); \qquad p \in \mathrm{P} ,$$
$$\kappa_i = \pi_*(c_1(\omega_\pi(\textstyle\sum \mathrm{D}_p))^{i+1}) , \qquad i \geqslant 0 .$$

The classes ψ_p have degree 2, while κ_i has degree $2i$. In the rest of this paper, whenever we speak of *tautological* or *natural* classes (of degree 2) on $\overline{\mathcal{M}}_{g,\mathrm{P}}$, we refer to κ_1, the ψ_p, δ_{irr}, and the $\delta_{a,\mathrm{A}}$. The classes δ_{irr} and $\delta_{a,\mathrm{A}}$ will be called *boundary* classes. We set $\psi = \Sigma\psi_p$.

2. The main results

The first of our main results describes the first, third, and fifth cohomology groups of $\overline{\mathcal{M}}_{g,n}$.

Theorem **2.1.** — *We have* $\mathrm{H}^k(\overline{\mathcal{M}}_{g,n}) = 0$ *for* $k = 1, 3, 5$ *and all* g *and* n *such that* $2g-2+n > 0$.

The next result describes the second cohomology group of $\overline{\mathcal{M}}_{g,n}$ in terms of generators and relations; it turns out that this group is always generated by the natural classes.

Theorem **2.2.** — *For any* g *and* n *such that* $2g-2+n > 0$, $\mathrm{H}^2(\overline{\mathcal{M}}_{g,n})$ *is generated by* κ_1, *the classes* ψ_i, δ_{irr}, *and the classes* $\delta_{a,\mathrm{A}}$ *such that* $0 \leqslant a \leqslant g$, $2a-2+|\mathrm{A}| \geqslant 0$ *and* $2(g-a)-2+|\mathrm{A}^c| \geqslant 0$. *The relations among these classes are as follows.*

a) *If* $g > 2$ *all relations are generated by those of the form*

$$\delta_{a,A} = \delta_{g-a,A^c}\,. \tag{2.3}$$

b) *If* $g = 2$ *all relations are generated by the (2.3) plus the following one*

$$5\kappa_1 = 5\psi + \delta_{\mathrm{irr}} - 5\delta_0 + 7\delta_1\,.$$

c) *If* $g = 1$ *all relations are generated by the (2.3) plus the following ones*

$$\kappa_1 = \psi - \delta_0\,,$$
$$12\psi_p = \delta_{\mathrm{irr}} + 12\sum_{\substack{S\ni p\\ |S|\geqslant 2}} \delta_{0,S}\,, \qquad p \in \{1, \dots, n\}.$$

d) *If* $g = 0$ *all relations are generated by the (2.3) plus the following ones*

$$\kappa_1 = \sum_{A\not\ni x,y} (|A| - 1)\delta_{0,A}\,, \qquad x, y \in \{1, \dots, n\},\ x \neq y\,,$$
$$\psi_z = \sum_{\substack{A\ni z\\ A\not\ni x,y}} \delta_{0,A}\,, \qquad x, y, z \in \{1, \dots, n\},\ x, y, z \text{ distinct}\,,$$
$$\delta_{\mathrm{irr}} = 0\,.$$

Observe, first of all, that the moduli spaces $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$, although in general not smooth, are orbifolds; in particular, Poincaré duality holds for them in rational cohomology.

The proof of (2.1) and (2.2) begins with a simple remark. Look first at $\mathcal{M}_{0,n}$; it can be viewed as the space of all n-tuples $(0, 1, \infty, z_4, \dots, z_n)$ of distinct points of $\mathbf{P}^1$ or, which is the same, as the space of all $(n-3)$-tuples $(z_4, \dots, z_n)$ in $\mathbf{C}^{n-3}$ such that $z_i \neq 0, 1$ for all i and $z_i \neq z_j$ for all $i \neq j$. In other words, $\mathcal{M}_{0,n}$ is nothing but $\mathbf{C}^{n-3}$ minus a bunch of hyperplanes, so in particular it is an $(n-3)$-dimensional affine variety. It follows that $H_k(\mathcal{M}_{0,n}) = 0$ for $k > n - 3$. Things are similar in higher genus. In fact, when $g > 0$, $n > 0$, Harer [9] (see also [16]) constructs a $(4g - 4 + n)$-dimensional spine for $\mathcal{M}_{g,n}$; thus $H_k(\mathcal{M}_{g,n})$ vanishes for $k > 4g - 4 + n$. The spine in question is constructed starting from the cellular decomposition of $\mathcal{M}_{g,n}$ defined in terms of Strebel differentials. Harer [9] also shows, by a spectral sequence argument, that $H_k(\mathcal{M}_g) = 0$ for $k > 4g - 5$. By Poincaré duality this implies that the cohomology with compact support $H^k_c(\mathcal{M}_{g,n})$ vanishes for $k \leqslant d(g, n)$, where

$$d(g, n) = \begin{cases} n - 4 & \text{if } g = 0\,, \\ 2g - 2 & \text{if } n = 0\,, \\ 2g - 3 + n & \text{if } g > 0,\ n > 0. \end{cases} \tag{2.4}$$

Looking at the exact sequence of cohomology with compact supports

$$\cdots \to H^k_c(\mathcal{M}_{g,n}) \to H^k(\overline{\mathcal{M}}_{g,n}) \to H^k(\partial\mathcal{M}_{g,n}) \to H^{k+1}_c(\mathcal{M}_{g,n}) \to \cdots$$

then proves

Lemma **2.5.** — *The homomorphism* $H^k(\overline{\mathcal{M}}_{g,n}) \to H^k(\partial\mathcal{M}_{g,n})$ *is an isomorphism for* $k < d(g, n)$ *and is injective for* $k = d(g, n)$.

Let g and n be non-negative integers such that $2g-2+n>0$, and let P be a set with n elements. Denote by D_1, D_2, ... the different irreducible components of $\partial\mathcal{M}_{g,P}$. Each of these is the image of a map $\mu_i : X_i \to \overline{\mathcal{M}}_{g,P}$, where X_i can be of two different kinds. Either $X_i = \overline{\mathcal{M}}_{g-1,P\cup\{q,r\}}$, or else $X_i = \overline{\mathcal{M}}_{a,A\cup\{q\}} \times \overline{\mathcal{M}}_{b,B\cup\{r\}}$, where $a+b=g$, $A \coprod B = P$, and both $2a-2+|A|$ and $2b-2+|B|$ are non-negative. In any case q and r are distinct points not belonging to P, and the map μ_i is gotten by identifying q and r.

Lemma **2.6.** — *The map* $H^k(\overline{\mathcal{M}}_{g,P}) \to \bigoplus_i H^k(X_i)$ *is injective whenever* $H^k(\overline{\mathcal{M}}_{g,P}) \to H^k(\partial\mathcal{M}_{g,P})$ *is.*

The proof uses a bit of Hodge theory, in the form of the following result of Deligne.

Proposition **2.7.** ([4], *Proposition* (8.2.5)) — *Let* Y *be proper. If* $u : X \to Y$ *is a proper surjective morphism, and* X *is smooth, then the weight* k *quotient of* $H^k(Y, \mathbf{Q})$ *is the image of* $H^k(Y, \mathbf{Q})$ *in* $H^k(X, \mathbf{Q})$.

In our application, Y is $\partial\mathcal{M}_{g,P}$, and X is the disjoint union of the X_i. Of course, Deligne's result is stated for varieties and not for orbifolds, and the X_i are smooth as orbifolds, but usually not as varieties. There are at least two ways out. One is to convince oneself that Deligne's proof works also in the orbifold context. The other is to appeal to the results of Looijenga [15] and Boggi-Pikaart [2] which imply that each of the X_i is the quotient of a smooth variety Z_i by the action of a finite group; one may then take as X the disjoint union of the Z_i and prove injectivity of the map $H^k(\overline{\mathcal{M}}_{g,P}) \to H^k(X) = \bigoplus_i H^k(Z_i)$, which obviously implies the injectivity of $H^k(\overline{\mathcal{M}}_{g,P}) \to \bigoplus_i H^k(X_i)$.

Whatever road we choose, the proof proceeds as follows. The homomorphism

$$\rho : H^k(\overline{\mathcal{M}}_{g,P}) \to H^k(\partial\mathcal{M}_{g,P})$$

is a morphism of mixed Hodge structures, and hence is strictly compatible with the filtrations. Thus

$$\rho(H^k(\overline{\mathcal{M}}_{g,P})) \cap W_{k-1}(H^k(\partial\mathcal{M}_{g,P})) = \rho(W_{k-1}(H^k(\overline{\mathcal{M}}_{g,P}))) = \rho(\{0\}) = \{0\},$$

since $H^k(\overline{\mathcal{M}}_{g,P})$ is pure of weight k. As we are assuming that ρ is injective, this shows that $H^k(\overline{\mathcal{M}}_{g,P})$ injects into $H^k(\partial\mathcal{M}_{g,P})/W_{k-1}(H^k(\partial\mathcal{M}_{g,P}))$. On the other hand (2.7) says that $H^k(\partial\mathcal{M}_{g,P})/W_{k-1}(H^k(\partial\mathcal{M}_{g,P}))$ injects into $H^k(X)$.

In view of (2.5), an immediate corollary of Lemma (2.6) is the following result.

Proposition **2.8.** — *The map* $H^k(\overline{\mathcal{M}}_{g,P}) \to \bigoplus_i H^k(X_i)$ *is injective when* $k \leqslant d(g, |P|)$.

Proposition (2.8) makes it possible to give a quick inductive proof of (2.1), based on the following intermediate result.

Lemma **2.9.** — *Let k be an odd integer, h a non-negative integer, and suppose* $H^q(\overline{\mathcal{M}}_{g,n}) = 0$ *for all odd $q \leqslant k$, all $g \leqslant h$, and all n such that $q > d(g, n)$. Then* $H^q(\overline{\mathcal{M}}_{g,n}) = 0$ *for all odd $q \leqslant k$, all $g \leqslant h$, and all n. If* $H^q(\overline{\mathcal{M}}_{g,n}) = 0$ *for all odd $q \leqslant k$ and all g and n such that $q > d(g, n)$, then* $H^q(\overline{\mathcal{M}}_{g,n}) = 0$ *for all odd $q \leqslant k$ and all g and n.*

Clearly, it suffices to prove the first assertion. We argue by induction on k. We may assume, inductively, that $H^q(\overline{\mathcal{M}}_{g,n}) = 0$ for all odd $q < k$, all $g \leqslant h$ and all n. If $g \leqslant h$ and $k \leqslant d(g, n)$, Proposition (2.8) says that $H^k(\overline{\mathcal{M}}_{g,n})$ injects into a direct sum of vector spaces $H^k(X_i)$, where X_i is $\overline{\mathcal{M}}_{g-1,\, n+2}$ or a product of two moduli spaces $\overline{\mathcal{M}}_{a,\alpha}$ and $\overline{\mathcal{M}}_{b,\beta}$ such that $a + b = g$, $\alpha + \beta = n + 2$; in the latter case either $a < g$ or $a = g$ and $\alpha < n$, and similarly for b and β. By the Kunneth formula, $H^k(\overline{\mathcal{M}}_{g,n})$ injects into the direct sum of $H^k(\overline{\mathcal{M}}_{g-1,\, n+2})$ and of all the tensor products $H^l(\overline{\mathcal{M}}_{a,\alpha}) \otimes H^m(\overline{\mathcal{M}}_{b,\beta})$ with $l + m = k$. Since either l or m must be odd, the induction hypothesis guarantees that all these tensor products vanish, except possibly those for which l or m is zero. This means that $H^k(\overline{\mathcal{M}}_{g,n})$ injects into a direct sum of vector spaces $H^k(\overline{\mathcal{M}}_{\gamma,\nu})$ such that $\gamma < g$ or $\gamma = g$ and $\nu < n$. But then the result follows by double induction on g and n.

Lemma (2.9) reduces the proof of the vanishing of odd cohomology (so long as it does vanish!) to checking it explicitly for finitely many values of g and n in each odd degree k, that is, those for which $k > d(g, n)$. When $k = 1$ this means $g = 0$, $n \leqslant 4$ or $g = n = 1$. Now, $\overline{\mathcal{M}}_{0,3}$ is a point, while $\overline{\mathcal{M}}_{0,4}$ and $\overline{\mathcal{M}}_{1,1}$ are both isomorphic to the projective line, so the first cohomology groups of all three are zero. This concludes the proof of (2.1) in case $k = 1$. For $k = 3, 5$ the initial cases of the induction are slightly more complicated and will be dealt with in section 5. Lemma (2.9) cannot be applied as such if $k \geqslant 11$, since it is known that $H^{11}(\overline{\mathcal{M}}_{1,11})$ is not zero. So far as we know, the cases $k = 7, 9$ are still open.

The proof of (2.2) will be carried out in the next two sections and will make essential use of the following result; actually, all we will need is the somewhat weaker version described in Remark (2.11) below.

Theorem **2.10.** — *Let $g \geqslant 1$ be an integer, let P be a finite set such that $2g-2+|\mathrm{P}|>0$, and let q, r be distinct and not belonging to P. Then, if $\xi : \overline{\mathcal{M}}_{g-1,\mathrm{P}\cup\{q,r\}} \to \overline{\mathcal{M}}_{g,\mathrm{P}}$ is the morphism obtained by identifying the points labelled q and r, the pullback map $\xi^* : \mathrm{H}^k(\overline{\mathcal{M}}_{g,\mathrm{P}}) \to \mathrm{H}^k(\overline{\mathcal{M}}_{g-1,\mathrm{P}\cup\{q,r\}})$ is injective for any $k \leqslant 2g-2$ if $g \leqslant 7$, and for any $k \leqslant g+5$ if $g \geqslant 7$.*

This we will prove by triple induction on k, g and $n = |\mathrm{P}|$. The statement is true when $k=0$, and also when $k=1$, since $\mathrm{H}^1(\overline{\mathcal{M}}_{g,n}) = 0$ for any g and n. Suppose then that $g \leqslant 7$, that $k \leqslant 2g-2$, and that the result is known to hold for all triples (k', g', n') such that either $k' < k$, or $k' = k$ and $g' < g$, or $k' = k$, $g' = g$, and $n' < n$. In view of (2.8), what we have to show is that, if x is any element of $\mathrm{H}^k(\overline{\mathcal{M}}_{g,\mathrm{P}})$ such that $\xi^*(x) = 0$, then x pulls back to zero under any one of the maps

$$\overline{\mathcal{M}}_{a,\mathrm{A}\cup\{s\}} \times \overline{\mathcal{M}}_{b,\mathrm{B}\cup\{t\}} \to \overline{\mathcal{M}}_{g,\mathrm{P}},$$

where $g = a+b$ and $\mathrm{P} = \mathrm{A} \coprod \mathrm{B}$. By the Künneth formula, $\mathrm{H}^k(\overline{\mathcal{M}}_{a,\mathrm{A}\cup\{s\}} \times \overline{\mathcal{M}}_{b,\mathrm{B}\cup\{t\}})$ breaks up into a direct sum of summands $\mathrm{H}^l(\overline{\mathcal{M}}_{a,\mathrm{A}\cup\{s\}}) \otimes \mathrm{H}^m(\overline{\mathcal{M}}_{b,\mathrm{B}\cup\{t\}})$, where $k = l+m$. Thus we have to show that x goes to zero under any one of the maps

$$\rho : \mathrm{H}^k(\overline{\mathcal{M}}_{g,\mathrm{P}}) \to \mathrm{H}^l(\overline{\mathcal{M}}_{a,\mathrm{A}\cup\{s\}}) \otimes \mathrm{H}^m(\overline{\mathcal{M}}_{b,\mathrm{B}\cup\{t\}}).$$

Suppose $l \geqslant 2a-1$ and $m \geqslant 2b-1$; then $k = l+m \geqslant 2(a+b)-2 = 2g-2$. The only possibility is that $l = 2a-1$, $m = 2b-1$; in particular, l and m are both odd. Since they add to $k \leqslant 2g-2 \leqslant 12$, one of them must equal 5 or less. In view of (2.1), this implies that $\mathrm{H}^l(\overline{\mathcal{M}}_{a,\mathrm{A}\cup\{s\}}) \otimes \mathrm{H}^m(\overline{\mathcal{M}}_{b,\mathrm{B}\cup\{t\}}) = 0$, so we are done in this case. We may then suppose that either $l \leqslant 2a-2$ or $m \leqslant 2b-2$. Say $l \leqslant 2a-2$; this implies, in particular, that $a > 0$. Then ρ fits into a commutative diagram

$$\begin{array}{ccc}
\mathrm{H}^k(\overline{\mathcal{M}}_{g,\mathrm{P}}) & \xrightarrow{\rho} & \mathrm{H}^l(\overline{\mathcal{M}}_{a,\mathrm{A}\cup\{s\}}) \otimes \mathrm{H}^m(\overline{\mathcal{M}}_{b,\mathrm{B}\cup\{t\}}) \\
{\scriptstyle\xi^*}\downarrow & & {\scriptstyle\lambda}\downarrow \\
\mathrm{H}^k(\overline{\mathcal{M}}_{g-1,\mathrm{P}\cup\{q,r\}}) & \longrightarrow & \mathrm{H}^l(\overline{\mathcal{M}}_{a-1,\mathrm{A}\cup\{s,q,r\}}) \otimes \mathrm{H}^m(\overline{\mathcal{M}}_{b,\mathrm{B}\cup\{t\}})
\end{array}$$

If $a = g$, that is, if $b = 0$, then $|\mathrm{B}| > 1$, and hence $|\mathrm{A} \cup \{s\}| < n$. Thus λ is always injective, by induction hypothesis. Since $\xi^*(x) = 0$, and so $\lambda\rho(x) = 0$, this implies that $\rho(x) = 0$, as desired.

When $g > 7$ and $k \leqslant g+5$, the argument is similar, but simpler. Set $f(n) = 2n-2$ if $n \leqslant 7$ and $f(n) = n+5$ if $n \geqslant 7$. To show that $\rho(x) = 0$ we may argue as in the previous case, provided we can show that either $a > 0$ and $l \leqslant f(a)$ or $b > 0$ and $m \leqslant f(b)$. If $a = 0$, then $m \leqslant k \leqslant f(g) = f(b)$, and similarly if $b = 0$; we may therefore assume that $a > 0$ and $b > 0$. Three cases are possible. Suppose first that $a \leqslant 7$ and $b \leqslant 7$. If $l > f(a) = 2a-2$, $m > f(b) = 2b-2$, then $k \geqslant 2g-2 > g+5$, against the assumptions. Suppose next that $a \leqslant 7$ and $b > 7$. If $l > f(a) = 2a-2$, $m > f(b) = b+5$, then $k \geqslant a + g + 5 > g + 5$, contrary to what we have assumed. Suppose finally that $a \geqslant 7$ and $b \geqslant 7$. If $l > f(a) = a + 5$, $m > f(b) = b + 5$, then $k > g + 10 > g + 5$, again against the assumptions.

Remark **2.11.** — The above argument actually shows that, if v is an odd integer and we know that the odd cohomology of all the $\overline{\mathcal{M}}_{g,n}$ vanishes in degree not exceeding v, then $H^k(\overline{\mathcal{M}}_{g,P}) \to H^k(\overline{\mathcal{M}}_{g-1,P\cup\{q,r\}})$ is injective for $k \leqslant 2g-2$ if $g \leqslant v+2$, and for $k \leqslant g + v$ if $g \geqslant v + 2$. Thus we could improve slightly on (2.10) if we could prove the vanishing of the odd cohomology in degree greater than 5. In a different direction, just knowing that the first cohomology vanishes, which is all that has been fully proved up to now, suffices to show that $H^2(\overline{\mathcal{M}}_{g,P})$ injects into $H^2(\overline{\mathcal{M}}_{g-1,P\cup\{q,r\}})$ as soon as $g \geqslant 2$. This is the only consequence of (2.10) that we will need in the proof of (2.2).

3. Relations among degree two tautological classes

Our main goal in this section is to find all the relations satisfied by tautological classes in $H^2(\overline{\mathcal{M}}_{g,P})$, for all g and P. We begin by describing how the natural classes pull back under the basic maps π, ξ, and η defined in section 1.

Lemma **3.1.** — (i) $\pi^*(\kappa_1) = \kappa_1 - \psi_q$;
(ii) $\pi^*(\psi_p) = \psi_p - \delta_{0,\{p,q\}}$ *for any* $p \in P$;
(iii) $\pi^*(\delta_{\mathrm{irr}}) = \delta_{\mathrm{irr}}$;
(iv) $\pi^*(\delta_{a,A}) = \delta_{a,A} + \delta_{a,A\cup\{q\}}$.

Part *i)* of the lemma is proved in [1], while *iii)* and *iv)* are clear. To prove *ii)* we reason as follows. Consider the diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,P\cup\{q,r\}} & \xrightarrow{\mu} & \overline{\mathcal{M}}_{g,P\cup\{r\}} \\ {\scriptstyle\varphi'}\downarrow & & {\scriptstyle\varphi}\downarrow \\ \overline{\mathcal{M}}_{g,P\cup\{q\}} & \xrightarrow{\pi} & \overline{\mathcal{M}}_{g,P} \end{array}$$

where φ' and φ are defined as "forgetting the point labelled r" and μ as "forgetting the point labelled q". It is known (cf. [1], for instance) that

$$\mu^*(\omega_\varphi) = \omega_{\varphi'}(-\sum_{x\in P} \Delta_{0,\{x,q,r\}}).$$

Thus, if τ_x, $x \in P$ (resp., τ'_x, $x \in P \cup \{q\}$), are the canonical sections of φ (resp., φ'), then

$$\pi^*(\tau_p^*(\omega_\varphi)) = \tau_p'^*(\mu^*(\omega_\varphi)) = \tau_p'^*(\omega_{\varphi'}(-\sum_{x\in P}\Delta_{0,\{x,q,r\}}))$$

for any $p \in P$. This translates into *ii)*, finishing the proof of the lemma.

Lemma **3.2.** — i) $\xi^*(\kappa_1) = \kappa_1$;
(ii) $\xi^*(\psi_p) = \psi_p$ *for any* $p \in \mathrm{P}$;
(iii) $\xi^*(\delta_{\mathrm{irr}}) = \delta_{\mathrm{irr}} - \psi_q - \psi_r + \sum_{q\in \mathrm{B}, r\notin \mathrm{B}} \delta_{b,\mathrm{B}}$;
(iv) $\xi^*(\delta_{a,\mathrm{A}}) = \begin{cases} \delta_{a,\mathrm{A}} & \textit{if } g = 2a,\ \mathrm{A} = \mathrm{P} = \emptyset, \\ \delta_{a,\mathrm{A}} + \delta_{a-1,\mathrm{A}\cup\{q,r\}} & \textit{otherwise.} \end{cases}$

Part *i)* is proved in [1], while the other parts of the lemma are straightforward. We now turn to η. We shall not actually compute the pullbacks of the natural classes under η, but only under the map

$$\vartheta : \overline{\mathcal{M}}_{a,\mathrm{A}\cup\{q\}} \to \overline{\mathcal{M}}_{g,\mathrm{P}}$$

which associates to any $\mathrm{A}\cup\{q\}$-pointed genus a curve the P-pointed genus g curve obtained by glueing to it a *fixed* $\mathrm{A}^c \cup \{r\}$-pointed genus $g-a$ curve C via identification of q and r. On the other hand, we know that the first cohomology of $\overline{\mathcal{M}}_{\gamma,\mathrm{V}}$ always vanishes, so, by the Kunneth formula, the second cohomology of $\overline{\mathcal{M}}_{a,\mathrm{A}} \times \overline{\mathcal{M}}_{g-a,\mathrm{A}^c}$ is the direct sum of $\mathrm{H}^2(\overline{\mathcal{M}}_{a,\mathrm{A}})$ and $\mathrm{H}^2(\overline{\mathcal{M}}_{g-a,\mathrm{A}^c})$. Thus knowing how the natural classes pull back under ϑ actually tells us how they pull back under η. It is important to stress that, although of course ϑ depends on the choice of C, any two choices give rise to homotopic maps so that, in cohomology, the pullback map ϑ^* is independent of the choice of C.

Lemma **3.3.** — i) $\vartheta^*(\kappa_1) = \kappa_1$;
(ii) $\vartheta^*(\psi_p) = \begin{cases} \psi_p & \textit{if } p \in \mathrm{A}, \\ 0 & \textit{if } p \in \mathrm{A}^c; \end{cases}$
(iii) $\vartheta^*(\delta_{\mathrm{irr}}) = \delta_{\mathrm{irr}}$.

Suppose $\mathrm{A} = \mathrm{P}$. *Then*
(iv) $\vartheta^*(\delta_{b,\mathrm{B}}) = \begin{cases} \delta_{2a-g,\mathrm{P}\cup\{q\}} - \psi_q & \textit{if } (b,\mathrm{B}) = (a,\mathrm{P}) \textit{ or } (b,\mathrm{B}) = (g-a,\emptyset), \\ \delta_{b,\mathrm{B}} + \delta_{b+a-g,\mathrm{B}\cup\{q\}} & \textit{otherwise.} \end{cases}$

Suppose $\mathrm{A} \neq \mathrm{P}$. *Then*
(iv') $\vartheta^*(\delta_{b,\mathrm{B}}) = \begin{cases} -\psi_q & \textit{if } (b,\mathrm{B}) = (a,\mathrm{A}) \textit{ or } (b,\mathrm{B}) = (g-a,\mathbf{A}^c), \\ \delta_{b,\mathrm{B}} & \textit{if } \mathrm{B} \subset \mathrm{A} \textit{ and } (b,\mathrm{B}) \neq (a,\mathrm{A}), \\ \delta_{b+a-g,(\mathrm{B}\setminus\mathrm{A}^c)\cup\{q\}} & \textit{if } \mathrm{B} \supset \mathrm{A}^c \textit{ and } (b,\mathrm{B}) \neq (g-a,\mathrm{A}^c), \\ 0 & \textit{otherwise.} \end{cases}$

Again, the only part that needs justification is i), which is proved in [1].

We may now determine all relations among tautological classes in degree two. We have already observed that there are trivial relations

(3.4) $$\delta_{a,\mathrm{A}} = \delta_{g-a,\mathrm{A}^c}$$

for any $a \leqslant g$ and any $A \subset P$. What the remaining relations are depends on the genus. We begin with genus zero. In this case it has been observed by Keel [13] that, for any four distinct elements p, q, r, s of P, the following relations hold among the classes $\delta_{0,A}$ such that $|A| \geqslant 2$ and $|A^c| \geqslant 2$:

$$\sum_{\substack{A \ni p,q \\ A \not\ni r,s}} \delta_{0,A} = \sum_{\substack{A \ni p,r \\ A \not\ni q,s}} \delta_{0,A} = \sum_{\substack{A \ni p,s \\ A \not\ni q,r}} \delta_{0,A} . \tag{3.5}$$

What is more important, Keel proves that $H^2(\overline{\mathcal{M}}_{0,P})$ is the quotient of the vector space generated by the $\delta_{0,A}$ such that $|A| \geqslant 2$ and $|A^c| \geqslant 2$ modulo the trivial relations (3.4) and the relations (3.5) for all possible choices of p, q, r, s.

Proposition **3.6.** — *For any choice of distinct elements $x, y, z \in P$, the following relations hold in $H^2(\overline{\mathcal{M}}_{0,P})$:*

$$\psi_z = \sum_{\substack{A \ni z \\ A \not\ni x,y}} \delta_{0,A} , \tag{3.7}$$

$$\kappa_1 = \sum_{A \not\ni x,y} (|A| - 1)\, \delta_{0,A} . \tag{3.8}$$

These, together with the relation $\delta_{irr} = 0$ and relations (3.4) and (3.5), generate all relations in $H^2(\overline{\mathcal{M}}_{0,P})$ among the natural classes κ_1, ψ_i, δ_{irr}, and $\delta_{0,A}$ with $|A| \geqslant 2$ and $|A^c| \geqslant 2$.

In view of Keel's result, all that needs to be shown is that (3.7) and (3.8) hold. The proof is by induction on $|P|$, starting from the obvious remark that $0 = \kappa_1 = \psi_1 = \psi_2 = \psi_3$ on $\overline{\mathcal{M}}_{0,P}$ when $|P| = 3$. The induction step is based on Lemma (3.1). Suppose that (3.7) and (3.8) hold in $\overline{\mathcal{M}}_{0,P}$. By symmetry, it suffices to prove their analogues in $\overline{\mathcal{M}}_{0,P\cup\{q\}}$ for $x, y, z \in P$. Pulling back (3.7) via π gives that

$$\psi_z - \delta_{0,\{z,q\}} = \sum_{\substack{z \in A \subset P \\ x,y \notin A}} (\delta_{0,A} + \delta_{0,A\cup\{q\}}) ,$$

which is nothing but the analogue of (3.7) for $\overline{\mathcal{M}}_{0,P\cup\{q\}}$. Similarly, pulling back (3.8) yields

$$\kappa_1 - \psi_q = \sum_{x,y \notin A \subset P} (|A| - 1)(\delta_{0,A} + \delta_{0,A\cup\{q\}}) ,$$

that is, using (3.7) to express ψ_q in terms of boundary classes,

$$\kappa_1 = \sum_{x,y \notin A \subset P} \delta_{0,A\cup\{q\}} + \sum_{x,y \notin A \subset P} (|A| - 1)(\delta_{0,A} + \delta_{0,A\cup\{q\}}) ,$$

which is exactly what had to be shown.

We now prove a useful technical consequence of Keel's results, which will be used time and again in the sequel.

Lemma **3.9.** — *Let Q be a finite set with at least four elements, and let* $x, y, z \in Q$ *be distinct. Then* $\delta_{0,\{y,z\}}$ *and all the classes* $\delta_{0,S}$ *such that* $x \in S$ *and* $2 \leqslant |S| \leqslant |Q| - 3$ *constitute a basis of* $H^2(\overline{\mathcal{M}}_{0,Q})$.

As Keel observes, his results imply that the dimension of $H^2(\overline{\mathcal{M}}_{0,n})$ is $2^{n-1} - \binom{n}{2} - 1$. Now denote by V the subspace spanned by the classes listed in the statement of the lemma. Since there are $2^{|Q|-1} - \binom{|Q|}{2} - 1$ of these, it suffices to show that all classes $\delta_{0,T}$ belong to V. The only classes that are not already present in the list are those of the form $\delta_{0,\{a,b\}}$, where $x \notin \{a, b\}$ and $\{a, b\} \neq \{y, z\}$. Let p, q, r be elements of Q, all different from x and such that $p \neq r \neq q$. We claim that

$$\delta_{0,\{q,r\}} \equiv \delta_{0,\{p,r\}} \bmod V: \tag{3.10}$$

If $p = q$ there is nothing to prove. If $p \neq q$ then, in view of (3.4), one of Keel's relations (3.5) is

$$\delta_{0,\{x,p\}} + \sum_{\substack{x,p \in S \not\ni q,r \\ 3 \leqslant |S| \leqslant |Q|-3}} \delta_{0,S} + \delta_{0,\{q,r\}} = \delta_{0,\{x,\, q\}} + \sum_{\substack{x,q \in S \not\ni p,r \\ 3 \leqslant |S| \leqslant |Q|-3}} \delta_{0,S} + \delta_{0,\{p,r\}} ,$$

which implies (3.10). One among y and z, say z, is different from both a and b. Two applications of (3.10) then give

$$\delta_{0,\{a,b\}} \equiv \delta_{0,\{a,z\}} \equiv \delta_{0,\{y,z\}} \equiv 0 \bmod V,$$

proving the lemma.

We will call a basis such as the one constructed in Lemma 3.9) a *standard basis* (with respect to x, y and z). We next state and prove the analogues of (3.6) in higher genus.

Proposition **3.11.** — (i) *The following relations hold in* $H^2(\overline{\mathcal{M}}_{1,P})$, *for any* $p \in P$:

$$\kappa_1 = \psi - \delta_0 , \tag{3.12}$$

$$12\psi_p = \delta_{\mathrm{irr}} + 12 \sum_{\substack{S \ni p \\ |S| \geqslant 2}} \delta_{0,S} . \tag{3.13}$$

These, together with the (3.4), generate all relations in $H^2(\overline{\mathcal{M}}_{1,P})$ *among the natural classes* κ_1, ψ_i, δ_{irr}, *and* $\delta_{a,A}$ *with* $0 \leqslant a \leqslant 1$ *and* $2 \leqslant |A| \leqslant |P| - 2$ *if* $a = 0$.

(ii) *The following relation holds in* $H^2(\overline{\mathcal{M}}_{2,P})$:

$$5\kappa_1 = 5\psi + \delta_{irr} - 5\delta_0 + 7\delta_1 \tag{3.14}$$

This relation and the (3.4) generate all relations in $H^2(\overline{\mathcal{M}}_{2,P})$ *among the natural classes* κ_1, ψ_i, δ_{irr}, *and* $\delta_{a,A}$ *with* $0 \leqslant a \leqslant 2$ *and* $2 \leqslant |A| \leqslant |P| - 2$ *if* $a = 0$.

(iii) *If* $g \geqslant 3$, *the (3.4) generate all relations in* $H^2(\overline{\mathcal{M}}_{g,P})$ *among the classes* κ_1, ψ_i, δ_{irr}, *and* $\delta_{a,A}$ *with* $0 \leqslant a \leqslant g$ *and* $2 \leqslant |A| \leqslant |P| - 2$ *if* $a = 0$.

The proofs of (3.12), (3.13), and (3.14) are the exact analogues of those of (3.8) and (3.7). The initial cases of the induction are as follows. First of all, for any g and any P, one has Mumford's relation [17] [3]

$$\kappa_1 = 12\lambda - \delta + \psi, \tag{3.15}$$

where δ stands for the sum of δ_{irr} and all the δ_a with $2a \leqslant g$, and $\lambda = c_1(\pi_*(\omega_\pi))$ is the Hodge class.

For $g = |P| = 1$, one knows that $\psi = \lambda$, and that $12\lambda - \delta = 0$ [12], so that $\kappa_1 = \psi$ and $\delta = 12\psi$. Since in this case $\delta = \delta_{irr}$ and $\delta_0 = 0$, these identities are just the relations (3.12) and (3.13). For $g = 2$, $P = \emptyset$, Mumford [18] has shown that $10\lambda = \delta_{irr} + 2\delta_1$. Coupled with (3.15), this says that

$$\begin{aligned} 5\kappa_1 &= 60\lambda - 5\delta + 5\psi = 6\delta_{irr} + 12\delta_1 - 5\delta_{irr} - 5\delta_0 - 5\delta_1 + 5\psi \\ &= 5\psi + \delta_{irr} - 5\delta_0 + 7\delta_1\,, \end{aligned}$$

as desired.

What remains to be shown is that there are no relations in addition to the ones listed above. We begin with the case of genus 1. We need the following simple remark.

Lemma **3.16.** — *The homomorphism* $\xi^* : H^2(\overline{\mathcal{M}}_{1,P}) \to H^2(\overline{\mathcal{M}}_{0,P\cup\{q,r\}})$ *maps* δ_{irr} *to zero.*

Let p be an element of P, and let $\rho : \overline{\mathcal{M}}_{1,P} \to \overline{\mathcal{M}}_{1,\{p\}}$ be the morphism defined by forgetting the points labelled by elements of P other than p. Repeated applications of Lemma (3.1) show that $\delta_{irr} = \delta^*(\delta_{irr})$. On the other hand, the composition of ξ and ρ maps $\overline{\mathcal{M}}_{0,P\cup\{q,r\}}$ to a single point. It follows that $\xi^*(\delta_{irr}) = \xi^*(\rho^*(\delta_{irr})) = 0$, as desired.

What we need to do to finish the genus 1 case is to show that δ_{irr} and the classes $\delta_{1,S}$ are independent. First consider the inclusion $\vartheta : \overline{\mathcal{M}}_{1,1} \hookrightarrow \overline{\mathcal{M}}_{1,P}$ obtained by sending any 1-pointed genus 1 curve C to the union of C with a fixed $P \cup \{q\}$-pointed smooth rational curve E, with the marked point of C identified with the point of E labelled q. Notice that, by (3.3), $\vartheta^*(\delta_{irr}) = \delta_{irr}$, so $\delta_{irr} \in H^2(\overline{\mathcal{M}}_{1,P})$ is not zero since its pullback to $H^2(\overline{\mathcal{M}}_{1,1})$ does not vanish.

Now look at the pullbacks of the boundary classes via the morphism $\xi\colon \overline{\mathcal{M}}_{0,P\cup\{q,r\}} \to \overline{\mathcal{M}}_{1,P}$. We have seen that $\delta_{\rm irr}$ pulls back to zero. On the other hand it is clear that $\xi^*(\delta_{1,S}) = \delta_{0,S\cup\{q,r\}}$. The independence of the classes $\delta_{\rm irr}$ and $\delta_{1,S}$ will then follow from the remark that $\delta_{\rm irr} \neq 0$ and from the following result.

Lemma **3.17.** — *The classes* $\delta_{0,S\cup\{q,r\}}$, *where* S *runs through all subsets of* P *with at most* $|P|-2$ *elements, are independent in* $H^2(\overline{\mathcal{M}}_{0,P\cup\{q,r\}})$.

Let x, y be any two distinct points of P, and let $\vartheta : \overline{\mathcal{M}}_{0,\{x,y,q,t\}} \to \overline{\mathcal{M}}_{0,P\cup\{q,r\}}$ be the morphism consisting in attaching a fixed tail at the point labelled t. Lemma (3.3) shows that the only class of the form $\delta_{0,S\cup\{q,r\}}$ whose pullback under ϑ does not vanish is $\delta_{0,\{x,y\}} = \delta_{0,P\setminus\{x,y\}\cup\{q,r\}}$. This shows that, if a linear combination of the $\delta_{0,S\cup\{q,r\}}$ vanishes, it must involve only those classes such that $|S| \leqslant |P|-3$. These, however, are independent by Lemma (3.9), as they belong to a standard basis. This finishes the proof of part *i)* of (3.11).

To complete the proof of part *ii)* it remains to show that the boundary classes and the classes ψ_p, $p\in P$, are independent in $H^2(\overline{\mathcal{M}}_{2,P})$ modulo the trivial relations (3.4). Consider the morphism $\xi : \overline{\mathcal{M}}_{1,P\cup\{q,r\}} \to \overline{\mathcal{M}}_{2,P}$. Using (3.2) and (3.13) we find that

$$\begin{aligned}
\xi^*(\psi_p) &= \frac{1}{12}\delta_{\rm irr} + \sum_{p\in S}\delta_{0,S}\,,\\
\xi^*(\delta_{\rm irr}) &= \delta_{\rm irr} - \psi_q - \psi_r + \sum_{q\in S, r\notin S}\delta_{0,S} + \sum_{r\in S, q\notin S}\delta_{0,S}\\
&= \delta_{\rm irr} - \frac{1}{12}\delta_{\rm irr} - \sum_{q\in S}\delta_{0,S} - \frac{1}{12}\delta_{\rm irr} - \sum_{r\in S}\delta_{0,S} + \sum_{q\in S, r\notin S}\delta_{0,S} + \sum_{r\in S, q\notin S}\delta_{0,S}\\
&= \frac{5}{6}\delta_{\rm irr} - 2\sum_{q,r\in S}\delta_{0,S}\,,\\
\xi^*(\delta_{0,A}) &= \delta_{0,A}\,.
\end{aligned}$$

Moreover, if $P \neq \emptyset$ then

$$\xi^*(\delta_{1,A}) = \delta_{0,A\cup\{q,r\}} + \delta_{0,A^c\cup\{q,r\}}\,,$$

while if $P = \emptyset$ then $\xi^*(\delta_1) = \delta_{0,\{q,r\}}$. Now suppose a linear combination of the ψ_p, $\delta_{\rm irr}$, the $\delta_{0,A}$ and the $\delta_{1,A}$ vanishes. This linear combination cannot involve the ψ_p, since the class $\delta_{0,\{p,q\}}$ appears in the expression for $\xi^*(\psi_p)$, but not in the ones for the pullbacks under ξ of the remaining classes, and part *i)* says that the boundary classes are independent in $H^2(\overline{\mathcal{M}}_{1,P\cup\{q,r\}})$. On the other hand the above formulas, together with part *i)*, clearly show that the pullbacks of the boundary classes under ξ are independent.

To do part *iii)* we proceed by induction on g. When $P \neq \emptyset$, fix an element $p \in P$; we have to show that κ_1, the ψ_i, δ_{irr} and the $\delta_{a,A}$ such that $p \in A$ are independent. When $P = \emptyset$, instead, we have to show independence of κ_1, the ψ_i, δ_{irr} and the δ_a with $2a \leqslant g$. For $g > 3$, the formulas in Lemma (3.2), plus the induction hypothesis, show directly that the pullbacks of these classes via $\xi : \overline{\mathcal{M}}_{g-1,P\cup\{q,r\}} \to \overline{\mathcal{M}}_{g,P}$ are already independent. For $g = 3$, we argue as follows. Suppose there is a relation

$$0 = a\kappa_1 + \sum b_i \psi_i + c\,\delta_{irr} + \cdots$$

among them. Pulling back via ξ, and using (3.2) and (3.14), we find a relation on $\overline{\mathcal{M}}_{2,P\cup\{q,r\}}$ of the form

$$0 = (a - c)\psi_q + \cdots + (c + a/5)\,\delta_{irr} + \cdots$$

By *ii)*, all coefficients in this relation must vanish, so $a = 0$. At this point we may proceed as for $g > 3$. The proof of (3.11) is now complete.

4. Inductive computation of $H^2(\overline{\mathcal{M}}_{g,n})$

We now turn to the proof of (2.2). Clearly, all that has to be shown is that $H^2(\overline{\mathcal{M}}_{g,n})$ is always generated by tautological classes, the relations among these having been determined in the preceding section. As was the case for (2.1), the proof is by double induction on g and n. Here we shall describe the induction step. The initial cases of the induction, that is, those for which $2 > d(g, n)$, will be dealt with in section 5.

Our strategy is quite simple. Suppose we want to show that $H^2(\overline{\mathcal{M}}_{g,n})$ is generated by tautological classes, assuming the same is known to be true in genus less than g, or in genus g but with fewer than n marked points. Proposition (2.8) shows that $H^2(\overline{\mathcal{M}}_{g,n})$ injects into the direct sum of the second cohomology groups of the X_i. By induction hypothesis, these are generated by tautological classes, all relations among which are known. By (3.2) and (3.3), we have complete control on the effect of each map $H^2(\overline{\mathcal{M}}_{g,n}) \to H^2(X_i)$ on tautological classes, so that, at least in principle, we can decide which classes in $\bigoplus H^2(X_i)$ come from tautological classes on $H^2(\overline{\mathcal{M}}_{g,n})$. On the other hand, given any class in $H^2(\overline{\mathcal{M}}_{g,n})$, its pullbacks to the X_i satisfy obvious compatibility relations on the "intersections" of the X_i. The subspace of $\bigoplus H^2(X_i)$ defined by these compatibility relations can be completely described using (3.2) and (3.3), at least in principle, because the spaces $H^2(X_i)$ are generated by tautological classes. What we will show, in essence, is that it coincides with the one generated by the images of the tautological classes of $H^2(\overline{\mathcal{M}}_{g,n})$. By the injectivity of $H^2(\overline{\mathcal{M}}_{g,n}) \to \bigoplus H^2(X_i)$, this will conclude the proof. We shall first describe the inductive step in genus 3 or more. The cases of lower genus are a bit more involved, and will be treated later in this section.

Let then $g \geqslant 3$ be an integer, and let P be a finite set. If P is not empty, let p be a fixed element of P. Let x, y be distinct and not belonging to P. Let $\xi : \overline{\mathcal{M}}_{g-1,P\cup\{x,y\}} \to \overline{\mathcal{M}}_{g,P}$ be the map that is obtained by identifying the points labelled x and y. We wish to show that $H^2(\overline{\mathcal{M}}_{g,P})$ is generated by tautological classes, assuming the analogous statement is known to hold for $\overline{\mathcal{M}}_{\gamma,\nu}$ whenever $\gamma < g$ or $\gamma = g$ and $\nu < |P|$. We will do this only for $P \neq \emptyset$, the argument for $P = \emptyset$ being entirely similar. Let z be any element of $H^2(\overline{\mathcal{M}}_{g,P})$. The pullback $\xi^*(z)$ is invariant under the operation of interchanging x and y. Therefore, by the induction assumptions, it is a linear combination of κ_1, the ψ_i, $i \in P$, $\psi_x + \psi_y$, δ_{irr}, and the classes $\delta_{u,U}$, $\delta_{u,U\cup\{x,y\}}$ and $\delta_{u,U\cup\{x\}} + \delta_{u,U\cup\{y\}}$, where u is any integer between 0 and g and U runs through all subsets of P containing p; when $g = 3$, we can even do without κ_1. Formulas (3.2) tell us that there is a linear combination v of tautological classes such that the pullback of $\alpha = z - v$ is of the form

$$\textbf{(4.1)} \qquad \xi^*(\alpha) = f(\psi_x + \psi_y) + \sum_{\substack{p\in U\subset P\\ 0\leqslant u\leqslant g-2}} g_{u,U}\,\delta_{u,U\cup\{x,y\}} + \sum_{\substack{p\in U\subset P\\ 0\leqslant u\leqslant g-1}} h_{u,U}(\delta_{u,U\cup\{x\}} + \delta_{u,U\cup\{y\}})$$

for suitable coefficients f, $g_{u,U}$, $h_{u,U}$. In case $g = 3$, we may even assume, using (3.14), that $f = 0$. We will show that, in fact, $\xi^*(\alpha) = 0$; (2.10) and (2.11) will then tell us that α itself vanishes, proving that z is a linear combination of tautological classes, as desired.

Suppose $s \notin P \cup \{x,y\}$, and let $\vartheta : \overline{\mathcal{M}}_{g-1,P\cup\{s\}} \to \overline{\mathcal{M}}_{g,P}$ be the map that is obtained by attaching a fixed elliptic tail at the point labelled s. Look at the diagram

$$\textbf{(4.2)} \qquad \begin{array}{ccc} \overline{\mathcal{M}}_{g-2,P\cup\{x,y,s\}} & \xrightarrow{\ \beta\ } & \overline{\mathcal{M}}_{g-1,P\cup\{x,y\}} \\ \gamma\big\downarrow & {}^{\varphi}\nearrow & \xi\big\downarrow \\ \overline{\mathcal{M}}_{g-1,P\cup\{s\}} & \xrightarrow{\ \vartheta\ } & \overline{\mathcal{M}}_{g,P} \end{array}$$

where φ attaches the point labelled t of a $\{t, x, y\}$-pointed projective line to the point labelled s of a variable curve in $\overline{\mathcal{M}}_{g-1,P\cup\{s\}}$, and γ and β are the analogues of ξ and ϑ, respectively. In this diagram, the outer rectangle and the lower triangle are commutative up to homotopy. The identity $\varphi^*\xi^*(\alpha) = \vartheta^*(\alpha)$, together with formulas (4.1) and (3.3), applied to φ, implies that

$$\textbf{(4.3)} \qquad \vartheta^*(\alpha) = \sum_{\substack{p\in U\\ 0\leqslant u\leqslant g-2}} g_{u,U}\,\delta_{u,U\cup\{s\}}\,.$$

On the other hand, if we write down explicitly the identity $\gamma^*\vartheta^*(\alpha) = \beta^*\xi^*(\alpha)$ using formulas (4.1), (4.3), (3.2), and (3.3), we get a relation

$$\begin{aligned}(4.4)\qquad &\sum_{\substack{p\in \mathrm{U}\\ 0\leqslant u\leqslant g-2}} g_{u,\mathrm{U}}(\delta_{u,\mathrm{U}\cup\{s\}} + \delta_{u-1,\mathrm{U}\cup\{s,x,y\}})\\ &= f(\psi_x+\psi_y) + \sum_{\substack{p\in \mathrm{U}\\ 0\leqslant u\leqslant g-2}} g_{u,\mathrm{U}}(\delta_{u,\mathrm{U}\cup\{x,y\}} + \delta_{u-1,U\cup\{s,x,y\}})\\ &+ \sum_{\substack{p\in \mathrm{U}\\ 0\leqslant u\leqslant g-1}} h_{u,\mathrm{U}}(\delta_{u,\mathrm{U}\cup\{x\}} + \delta_{u,\mathrm{U}\cup\{y\}} + \delta_{u-1,\mathrm{U}\cup\{s,x\}} + \delta_{u-1,\mathrm{U}\cup\{s,y\}})\end{aligned}$$

in $\mathrm{H}^2(\overline{\mathcal{M}}_{g-2,\mathrm{P}\cup\{x,y,s\}})$. If $g \geqslant 4$, all the tautological classes appearing in (4.4) are independent, so $f = g_{u,\mathrm{U}} = h_{u,\mathrm{U}} = 0$ for all u and U. When $g = 3$, we already know that $f = 0$; since the boundary classes are independent in genus 1, we conclude that $g_{u,\mathrm{U}} = h_{u,\mathrm{U}} = 0$ for all u and U in this case as well. This shows that $\xi^*(\alpha) = 0$, as desired.

To complete the proof of (2.2) it remains to deal with the genus 1 and genus 2 cases.

Genus 1. We begin by improving on Lemma (3.16). Let P be a finite set, set $n = |\mathrm{P}|$ and let x and y be distinct and not belonging to P. As usual, we let $\xi : \overline{\mathcal{M}}_{0,\mathrm{P}\cup\{x,y\}} \to \overline{\mathcal{M}}_{1,\mathrm{P}}$ be the map gotten by identifying the points labelled x and y.

Lemma **4.5.** — *The kernel of* $\xi^* : \mathrm{H}^2(\overline{\mathcal{M}}_{1,\mathrm{P}}) \to \mathrm{H}^2(\overline{\mathcal{M}}_{0,\mathrm{P}\cup\{x,y\}})$ *is one-dimensional and is generated by* δ_{irr}.

Lemma (3.16) and part *i)* of Proposition (3.11) say in particular that δ_{irr} is not zero and belongs to the kernel of ξ^*. It remains to show that any other element of $\ker(\xi^*)$ is a multiple of δ_{irr}. The case $|\mathrm{P}| = 1$ is trivial. In the next section we shall see that, when $|\mathrm{P}| = 2$, $\mathrm{H}^2(\overline{\mathcal{M}}_{1,\mathrm{P}})$ has dimension two. On the other hand, the class $\delta_{1,\emptyset} \in \mathrm{H}^2(\overline{\mathcal{M}}_{1,\mathrm{P}})$ maps to $\delta_{0,\{x,y\}} \in \mathrm{H}^2(\overline{\mathcal{M}}_{0,\mathrm{P}\cup\{x,y\}})$, which is not zero. This takes care of the case when $|\mathrm{P}| = 2$. We then proceed by induction on $n = |\mathrm{P}|$. Let α be an element of the kernel of ξ^*. Let s, t be distinct and not belonging to $\mathrm{P}\cup\{x,y\}$. For any subset S of P with at most $n-2$ elements consider the diagram

$$\begin{array}{ccc}\overline{\mathcal{M}}_{0,\mathrm{S}\cup\{x,y,s\}} \times \overline{\mathcal{M}}_{0,\mathrm{S}^c\cup\{t\}} & \longrightarrow & \overline{\mathcal{M}}_{0,\mathrm{P}\cup\{x,y\}}\\ \eta\big\downarrow & & \xi\big\downarrow\\ \overline{\mathcal{M}}_{1,\mathrm{S}\cup\{s\}} \times \overline{\mathcal{M}}_{0,\mathrm{S}^c\cup\{t\}} & \xrightarrow{\nu_{\mathrm{S}}} & \overline{\mathcal{M}}_{1,\mathrm{P}}\end{array}$$

where the vertical arrows are obtained by identifying the points labelled x and y, and the horizontal ones by identifying the points labelled s and t. Using the Künneth formula and the vanishing of H^1 we may write $\nu_S^*(\alpha) = (\beta, \gamma)$, where $\beta \in H^2(\overline{\mathcal{M}}_{1,S\cup\{s\}})$ and $\gamma \in H^2(\overline{\mathcal{M}}_{0,S^c\cup\{t\}})$. From $\xi^*(\alpha) = 0$ we deduce that $\eta^*(\beta, \gamma) = 0$. On the other hand $\eta^*(\beta, \gamma) = (\beta', \gamma)$, where β' is the pullback of β to $\overline{\mathcal{M}}_{0,S\cup\{x,y,s\}}$. It follows that $\gamma = 0$ and, by induction hypothesis, that $\beta = a_S\,\delta_{\mathrm{irr}}$ for a suitable constant a_S. In other words, $\nu_S^*(\alpha) = (a_S\,\delta_{\mathrm{irr}}, 0)$. We now wish to show that, actually, $a = a_S$ does not depend on S. This will conclude the proof, since then the difference $\alpha - a\,\delta_{\mathrm{irr}}$ will restrict to zero on all components of $\partial\overline{\mathcal{M}}_{1,P}$ and hence will be zero by (2.8). To show that a_S is independent of S we proceed as follows. If $S \neq \emptyset$ write $S = T \cup \{w\}$, where $w \notin T$, and consider the diagram

$$\begin{array}{ccc}
\overline{\mathcal{M}}_{1,T\cup\{w\}} \times \overline{\mathcal{M}}_{0,\{w,y,s\}} \times \overline{\mathcal{M}}_{0,S^c\cup\{t\}} & \xrightarrow{\tau} & \overline{\mathcal{M}}_{1,T\cup\{x\}} \times \overline{\mathcal{M}}_{0,S^c\cup\{w,r\}} \\
\sigma\big\downarrow & & \nu_T\big\downarrow \\
\overline{\mathcal{M}}_{1,S\cup\{s\}} \times \overline{\mathcal{M}}_{0,S^c\cup\{t\}} & \xrightarrow{\nu_S} & \overline{\mathcal{M}}_{1,P}
\end{array}$$

where the vertical arrows are obtained by identifying the points labelled x and y, and the horizontal ones by identifying the points labelled s and t. We find that

$$(a_T\,\delta_{\mathrm{irr}},0,0) = \tau^*(a_T\,\delta_{\mathrm{irr}}, 0) = \tau^*(\nu_T^*(\alpha)) = \sigma^*(\nu_S^*(\alpha)) = \sigma^*(a_S\,\delta_{\mathrm{irr}}, 0) = (a_S\,\delta_{\mathrm{irr}}, 0, 0)\,,$$

and hence that $a_S = a_T$. Repeated applications of this argument show that $a_S = a_\emptyset$ for any S. This proves the lemma.

Because of the relations between tautological classes given in Proposition (3.11), to prove (2.2) in genus one it suffices to show that $H^2(\overline{\mathcal{M}}_{1,P})$ is generated by boundary classes. We prove this claim by induction on n. The result is obvious for $n = 1$, while the case $n = 2$ will be settled in section 5. Denote by $V = V_{1,P}$ the subspace of $H^2(\overline{\mathcal{M}}_{1,P})$ generated by the elements $\delta_{1,S}$, where S runs through all subset of P with at most $n-2$ elements. To prove our claim it suffices, in view of Lemma (3.16), to show that the morphism ξ^* vanishes modulo V.

To simplify notation, from now on in the genus zero case we shall write δ_S instead of $\delta_{0,S}$. Let $Q = P \cup \{x,y\}$. Let $\mathcal{B}$ be the standard basis of $H^2(\overline{\mathcal{M}}_{0,Q})$ relative to z, x, y. Let $\alpha \in H^2(\overline{\mathcal{M}}_{1,P})$. Using the fact that ξ is invariant under the involution exchanging x and y, we can write $\xi^*\alpha$ in terms of $\mathcal{B}$:

$$\begin{aligned}
\xi^*\alpha = {} & a_{\{x,y\}}\delta_{\{x,y\}} + \sum_{S\subset X, z\in S, |S|\geq 2, |X\setminus S|\geq 1} a_S\,\delta_S \\
& + \sum_{S\subset X, z\in S, |X\setminus S|\geq 3} b_S\,\delta_{S\cup\{x,y\}} + \sum_{S\subset X, z\in S, |X\setminus S|\geq 2} c_S(\delta_{S\cup\{x\}} + \delta_{S\cup\{y\}})\,.
\end{aligned} \tag{4.6}$$

Consider a subset R of P such that $|P \setminus R| \geqslant 2$, and look at the morphism

$$\vartheta_R : \overline{\mathcal{M}}_{1,R\cup\{u\}} \to \overline{\mathcal{M}}_{1,P}$$

defined by taking a varying stable, genus 1, $R\cup\{u\}$-pointed curve, a fixed stable, genus zero, $(P\setminus R)\cup\{v\}$-pointed curve C_0, and identifying the points labelled u and v. As we already noticed before stating Lemma (3.3), the homomorphism ϑ_R^* does not depend on the choice of C_0. By induction hypothesis, we have

$$\vartheta_R^*(\alpha) = \sum_{S\subset R,\, |R\setminus S|\geqslant 1} f_S^R\, \delta_{1,S} + \sum_{S\subset R,\, |R\setminus S|\geqslant 2} g_S^R\, \delta_{1,S\cup\{u\}} + f^R\, \delta_{\mathrm{irr}}\,. \tag{4.7}$$

Consider the elements $\delta_{1,S} \in H^2(\overline{\mathcal{M}}_{1,P})$ with $|P\setminus S| \geqslant 2$. Recalling the convention about the symbols $\delta_{a,A}$, Lemma (3.2) says that

$$\begin{aligned} &\xi^*\,\delta_{1,X\setminus\{p,q\}} = \delta_{\{p,q\}} \notin \mathscr{B}, && \text{if } p \neq z,\, q \neq z, \\ &\xi^*\,\delta_{1,S} = \begin{cases} \delta_{S\cup\{x,y\}} \in \mathscr{B}, \\ \delta_{P\setminus S} \in \mathscr{B}, \end{cases} && \begin{array}{l} \text{if } z\in S,\, S \neq P\setminus\{p,q\},\, p\neq z,\, q\neq z, \\ \text{if } z\notin S. \end{array} \end{aligned} \tag{4.8}$$

So all the classes $\delta_{1,S}$ restrict to elements of $\mathscr{B}$ except when $S = P\setminus\{p,q\}$. We also have, by lemma (3.3), that

$$\vartheta_R^*\delta_{1,S} = \begin{cases} \delta_{1,S} & \text{if } S\subset R,\, S\neq R, \\ -\psi_u = -\displaystyle\sum_{S'\subset R,|R\setminus S'|\geqslant 1} \delta_{1,S'} - \frac{1}{12}\delta_{\mathrm{irr}} & \text{if } S = R, \\ \delta_{1,R\setminus(P\setminus S)\cup\{u\}} & \text{if } S\supset P\setminus R, \\ 0 & \text{otherwise.} \end{cases} \tag{4.9}$$

Remark **4.10.** — *The only class $\delta_{1,S}$ with $S\subset P$ and $|P\setminus S| \geqslant 2$ such that, in the expression of $\vartheta_R^*\delta_{1,S}$, the class $\delta_{1,(R\setminus\{p,q\})\cup\{u\}}$ appears with non zero coefficient, is $\delta_{1,P\setminus\{p,q\}}$ and in this case $\vartheta_R^*\delta_{1,P\setminus\{p,q\}} = \delta_{1,R\setminus\{p,q\}\cup\{u\}}$*

Let us go back to the expression (4.7). Let $\{r,s\}\subset R'\subset R$. We claim that

$$g^R_{R\setminus\{r,s\}} = g^{R'}_{R'\setminus\{r,s\}}\,. \tag{4.11}$$

Clearly, it suffices to prove the claim in case $R' = R\setminus\{q\}$. Look at the diagram

$$\begin{tikzcd} \overline{\mathcal{M}}_{1,(R\setminus\{q\})\cup\{w\}} \arrow[r, "\vartheta_{R\setminus\{q\}}"] \arrow[d, "\zeta"'] & \overline{\mathcal{M}}_{1,P} \\ \overline{\mathcal{M}}_{1,R\cup\{u\}} \arrow[ur, "\vartheta_R"'] & \end{tikzcd}$$

where the maps are defined in the obvious way. To prove the claim just use the preceding remark with $P = R\cup\{u\}$, together with the fact that $\zeta^*\vartheta_R^*\alpha = \vartheta_{R\setminus\{q\}}^*\alpha$.

After this preparation we are ready to modify α. Suppose first that $P = \{z, p, q\}$. The first move consists in adding to α a suitable multiple of $\delta_{1,\emptyset}$ so as to make $a_{\{x,y\}} = 0$. The second move consists in adding to α a suitable multiple of $\delta_{1,\{z\}}$ so as to make $f_\emptyset^{\{z\}} = 0$. The third move consists in adding to α a suitable linear combination of $\delta_{1,\{p\}}$ and $\delta_{1,\{q\}}$ so as to make $a_{\{z,p\}} = a_{\{z,q\}} = 0$. The three moves, taken in that order, do not interfere with each other. As a result

$$\xi^*\alpha = c_{\{z\}}(\delta_{\{z,x\}} + \delta_{\{z,y\}}), \quad \vartheta^*_{\{z\}}\alpha = f^{\{z\}}\delta_{\text{irr}}. \tag{4.12}$$

Assume now that $|P| \geqslant 4$. Let $p \neq z$ and $q \neq x$. Observe that, by (4.11), whenever $|P \setminus R| \geqslant 2$, $|P \setminus S| \geqslant 2$ and $\{p, q\} \subset R \cap S$, we have

$$g^{R}_{R\setminus\{p,q\}} = g^{R\cap S}_{R\cap S\setminus\{p,q\}} = g^{S}_{S\setminus\{p,q\}},$$

so that $g^{R}_{R\setminus\{p,q\}} = \gamma_{p,q}$ does not depend on R. Therefore subtracting from α the class $\sum_{X\supset\{p,q\}} \gamma_{p,q}\delta_{1,X\setminus\{p,q\}}$, we get that $g^{R}_{R\setminus\{p,q\}} = 0$, for all p, q and R such that $p \neq z$, $q \neq z$, $|P \setminus R| \geqslant 2$ and $\{p, q\} \subset R$.

The second move consists in adding to α a linear combination of elements of type $\delta_{1,S}$, with $S \neq P \setminus \{p, q\}$, $p \neq z$, $q \neq z$, in such a way that

$$\xi^*\alpha = \sum_{z\in S\subset X, |X\setminus S|\geqslant 2} c_S(\delta_{S\cup\{x\}} + \delta_{S\cup\{y\}}). \tag{4.13}$$

By the above remark, the second move does not alter what has been accomplished by the preceding one. For convenience we shall set $c_T = 0$ when $z \notin T$.

To prove our initial claim we must prove that all the c_S are equal to 0. Consider the square

$$\begin{array}{ccc}
\overline{\mathcal{M}}_{0,R\cup\{x,y,u\}} & \xrightarrow{\eta_R} & \overline{\mathcal{M}}_{0,P\cup\{x,y\}} \\
\eta\big\downarrow & & \xi\big\downarrow \\
\overline{\mathcal{M}}_{1,R\cup\{u\}} & \xrightarrow{\vartheta_R} & \overline{\mathcal{M}}_{1,P}
\end{array}$$

where η is the morphism obtained by identifying the points labelled x and y, while η_R is obtained by identifying the point labelled u on the varying curve in $\overline{\mathcal{M}}_{0,R\cup\{x,y,u\}}$ with the point labelled v on the fixed curve C_0. The equality $\eta_R^*\xi^*\alpha = \eta^*\vartheta_R^*\alpha$, where $R \subset P$ and $|P \setminus R| \geqslant 2$, implies that

$$\begin{aligned}
&\sum_{S\subset R, |R\setminus S|\geqslant 1} f^{R}_{S}\,\delta_{S\cup\{x,y\}} + \sum_{S\subset R, |R\setminus S|\geqslant 2} g^{R}_{S}\,\delta_{S\cup\{x,y,u\}} = \\
&\sum_{S\subset R, |P\setminus S|\geqslant 2} c_S(\delta_{S\cup\{x\}} + \delta_{S\cup\{y\}}) + \sum_{P\setminus S\subset R, |P\setminus S|\geqslant 2} c_S(\delta_{(P\setminus S)\cup\{y\}} + \delta_{(P\setminus S)\cup\{x\}}).
\end{aligned} \tag{4.14}$$

We are going to prove that $c_T = 0$ for every T by descending induction on $|T|$. The first non-trivial case occurs when $|T| = n - 2$.

If $P = \{z, p, q\}$ this amounts to showing that $c_{\{z\}} = 0$. This follows immediately from (4.12) and (4.14) for $R = \{z\}$.

Suppose $|P| \geqslant 4$, and set $P \setminus \{p, q\} = T$. We can assume that $p \neq z$ and $q \neq z$, otherwise $c_T = 0$. For $R = \{p, q\}$ the equality (4.14) now reads

$$f_{\emptyset}^{\{p,q\}}\delta_{\{x,y\}} + f_{\{p\}}^{\{p,q\}}\delta_{\{p,x,y\}} + f_{\{q\}}^{\{p,q\}}\delta_{\{q,x,y\}} = c_{P\setminus\{p,q\}}(\delta_{\{p,q,y\}} + \delta_{\{p,q,x\}}),$$

where on the left-hand side we used the fact that, by our first move, $g_{\emptyset}^{\{p,q\}} = 0$ while, to simplify the right-hand side, we used again the fact that $c_S = 0$ if $z \notin S$. The above is a linear relation among elements of a standard basis of $H^2(\overline{\mathcal{M}}_{0,\{p,q,x,y,u\}})$, hence

$$f_{\emptyset}^{\{p,q\}} = f_{\{p\}}^{\{p,q\}} = f_{\{q\}}^{\{p,q\}} = c_{P\setminus\{p,q\}} = 0. \tag{4.15}$$

The first step in the induction is completed.

Now let $r \geqslant 3$, assume that $c_S = 0$ if $|P \setminus S| < r$, let T be such that $|P \setminus T| = r$, and set $R = P \setminus T$. As usual we can assume that $z \notin R$. Let us first assume that $|T| = |P \setminus R| \geqslant 2$. Relation (4.14) reads

$$\sum_{S\subset R, |R\setminus S|\geqslant 1} f_S^R\, \delta_{(R\setminus S)\cup\{u\}} + \sum_{S\subset R, |R\setminus S|\geqslant 2} g_S^R\, \delta_{S\cup\{x,y,u\}} = c_T(\delta_{\{x,u\}} + \delta_{\{y,u\}}),$$

where, to simplify the right-hand side, we used the inductive hypothesis together with the fact that $c_S = 0$ if $z \notin S$. This is an equality in $H^2(\overline{\mathcal{M}}_{0,R\cup\{x,y,u\}})$. As a basis for $H^2(\overline{\mathcal{M}}_{0,R\cup\{x,y,u\}})$ we take a standard basis where now the role of Q is played by $R \cup \{x, y, u\}$ and the role of z is played by u. Since $z \notin R$, and $g^R_{R\setminus\{p,q\}} = 0$, the classes appearing in the above equality belong to the standard basis; as they are all distinct, we are done.

Let us finally assume that $T = \{z\}$. We start with a general remark. Pulling back

$$\vartheta^*_{\{p,q\}}\alpha = f_{\emptyset}^{\{p,q\}}\delta_{1,\emptyset} + f_{\{p\}}^{\{p,q\}}\delta_{1,\{p\}} + f_{\{q\}}^{\{p,q\}}\delta_{1,\{q\}} + g_{\emptyset}^{\{p,q\}}\delta_{1,\{u\}} + f^{\{p,q\}}\delta_{\mathrm{irr}}$$

from $H^2(\overline{\mathcal{M}}_{1,\{p,q,u\}})$ to $H^2(\overline{\mathcal{M}}_{1,\{q,w\}})$, comparing it with $\vartheta^*_{\{q\}}\alpha = f_{\emptyset}^{\{q\}}\delta_{1,\emptyset} + f^{\{q\}}\delta_{\mathrm{irr}}$, and looking at the coefficient of $\delta_{1,\emptyset}$, we get

$$f_{\emptyset}^{\{p,q\}} - f_{\{q\}}^{\{p,q\}} = f_{\emptyset}^{\{q\}} \qquad \forall\ p, q. \tag{4.16}$$

But if $|T| = 1$ (and $|P| \geqslant 4$), then relations (4.15) have already been proved so that, in particular, $f_{\emptyset}^{\{q\}} = 0$ for $q \neq z$. Using (4.16) again we get

$$f_{\emptyset}^{\{z,q\}} = f_{\{q\}}^{\{z,q\}}. \tag{4.17}$$

Now look at relation (4.14) for $R = \{z, q\}$. Using the induction hypothesis to simplify the right-hand side one then gets

$$f_{\emptyset}^{\{z,q\}}\delta_{\{x,y\}} + f_{\{z\}}^{\{z,q\}}\delta_{\{z,x,y\}} + f_{\{q\}}^{\{z,q\}}\delta_{\{q,x,y\}} + g_{\emptyset}^{\{z,q\}}\delta_{\{z,q\}} = c_{\{z\}}(\delta_{\{z,y\}} + \delta_{\{z,x\}}),$$

which, by Keel's relations, can be written as

$$\begin{aligned} f_{\emptyset}^{\{z,q\}}\, \delta_{\{x,y\}} + f_{\{z\}}^{\{z,q\}}\, (\delta_{\{z,q\}} + \delta_{\{z,u\}} - \delta_{\{z,x\}} - \delta_{\{z,y\}} + \delta_{\{x,y\}}) \\ + f_{\{q\}}^{\{z,q\}}\delta_{\{z,u\}} + g_{\emptyset}^{\{z,q\}}\delta_{\{z,q\}} = c_{\{z\}}(\delta_{\{z,y\}} + \delta_{\{z,x\}}), \end{aligned}$$

which in turn is a linear relation among elements of a standard basis for $H^2(\overline{\mathcal{M}}_{0,\{z,q,x,y,u\}})$. In particular it follows that $f_{\emptyset}^{\{z,q\}} = -f_{\{q\}}^{\{z,q\}}$ which, together with (4.17), implies that all coefficients in the above identity must vanish, concluding the proof of the lemma.

Genus 2. In order to prove (2.2) in genus two we must show that, for any finite set P, the space $H^2(\overline{\mathcal{M}}_{2,P})$ is generated by the classes ψ_q, with $q \in P$, and by the boundary classes δ_{irr}, $\delta_{1,A}$, $\delta_{2,B}$, where A and B run through all subset of P such that $|B^c| \geqslant 2$, and such that, if $P \neq \emptyset$, then A contains a preassigned point $p \in P$. We set $n = |P|$.

We first consider the cases $n = 0$ and $n = 1$. By Theorem (2.10), $H^2(\overline{\mathcal{M}}_2)$ injects in $H^2(\overline{\mathcal{M}}_{1,2})$ via ξ^*. On the other hand, $H^2(\overline{\mathcal{M}}_{1,2})$ is two-dimensional and the classes δ_{irr} and $\delta_{1,\emptyset}$ are independent in $H^2(\overline{\mathcal{M}}_2)$, so that ξ^* is an isomorphism, and the result follows in this case.

We next consider the case $n = 1$. Again $H^2(\overline{\mathcal{M}}_{2,\{p\}})$ injects in $H^2(\overline{\mathcal{M}}_{1,\{p,x,y\}})$ via ξ^*. Look at diagram (4.2) for $g = 2$. Using Lemma (3.2), Lemma (3.3) and Proposition (3.11), we find that

$$\xi^*(\psi_p) = \psi_p = \frac{1}{12}\delta_{\mathrm{irr}} + \delta_{1,\{x\}} + \delta_{1,\{y\}} + \delta_{1,\emptyset}, \quad \xi^*(\delta_{1,\emptyset}) = \delta_{1,\emptyset} + \delta_{1,\{p\}},$$

$$\xi^*(\delta_{\mathrm{irr}}) = \delta_{\mathrm{irr}} - \psi_x - \psi_y + \delta_{1,\{x\}} + \delta_{1,\{y\}} = \frac{5}{6}\delta_{\mathrm{irr}} - 2\delta_{1,\{p\}} - 2\delta_{1,\emptyset},$$

$$\vartheta^*(\psi_p) = \psi_p = \frac{1}{12}\delta_{\mathrm{irr}} + \delta_{1,\emptyset}, \quad \vartheta^*(\delta_{1,\emptyset}) = \delta_{1,\emptyset} - \psi_s = -\frac{1}{12}\delta_{\mathrm{irr}}, \quad \vartheta^*(\delta_{\mathrm{irr}}) = \delta_{\mathrm{irr}}.$$

A priori, given a class α in $H^2(\overline{\mathcal{M}}_{2,\{p\}})$, we have

$$\xi^*(\alpha) = a\,\delta_{\mathrm{irr}} + b(\delta_{1,\{x\}} + \delta_{1,\{y\}}) + c\,\delta_{1,\{p\}} + d\,\delta_{1,\emptyset}, \quad \vartheta^*(\alpha) = t\,\delta_{1,\emptyset} + r\,\delta_{\mathrm{irr}},$$

so that, by adding to α a suitable linear combination of ψ_p, δ_{irr}, and $\delta_{1,\{p\}}$, one can assume that $r = b = d = 0$. Then the equality $\gamma^*\vartheta^*(\alpha) = \beta^*\xi^*(\alpha)$ gives $c = t$, while the equality $\varphi^*\xi^*(\alpha) = \vartheta^*(\alpha)$ gives

$$a\,\delta_{\mathrm{irr}} - t\,\psi_s = a\,\delta_{\mathrm{irr}} - t(\frac{1}{12}\,\delta_{\mathrm{irr}} + \delta_{1,\emptyset}) = t\,\delta_{1,\emptyset},$$

so that $a = t = c = 0$, and the result is proved also in this case.

We now turn to the case when $|P| = n \geqslant 2$, and proceed by induction on n. Fix, once and for all, a point p in P. Consider subsets $R \subset P$ such that $R^c = P \setminus R$ contains two or more points and $p \in R^c$. In addition to ξ and ϑ we also consider the map

$$\vartheta_R : \overline{\mathcal{M}}_{2,R\cup\{z\}} \to \overline{\mathcal{M}}_{2,P} ,$$

defined as identifying the point labelled w of a fixed $R^c \cup \{w\}$-pointed projective line with the point labelled z of a variable curve in $\overline{\mathcal{M}}_{2,R\cup\{z\}}$. Given a class $\alpha \in H^2(\overline{\mathcal{M}}_{2,P})$, a priori one has

$$\xi^*(\alpha) = a\,\delta_{\rm irr} + \sum_{S\subset P} a_S\,\delta_{1,S} + \sum_{S\subset P,|S^c|\geqslant 2} b_S\,\delta_{1,S\cup\{x,y\}} + \sum_{S\subset P,|S^c|\geqslant 1} c_S(\delta_{1,S\cup\{x\}} + \delta_{1,S\cup\{y\}}) ,$$

$$\vartheta_R^*(\alpha) = a^R\,\delta_{\rm irr} + \sum_{r\in R} b_r^R\psi_r + b_z^R\psi_z + \sum_{S\subset R,|S|\geqslant 2} c_S^R\,\delta_{0,S} + \sum_{S\subset R,|S|\geqslant 1} d_S^R\,\delta_{0,S\cup\{z\}} + \sum_{S\subset R} h_S^R\,\delta_{1,S} .$$

Let us show that, by adding to α a suitable linear combination of $\delta_{\rm irr}$ and of the ψ_i with $i \neq p$, one can assume that $a^R = b_r^R = 0$ for all $R \subset P$ such that $|R^c| \geqslant 2$ and for all $r \in R$. For each proper subset R' of R there is an obvious diagram

$$\textbf{(4.18)}\qquad \begin{array}{ccc} \overline{\mathcal{M}}_{2,R'\cup\{z'\}} & \xrightarrow{\vartheta_{R'}} & \overline{\mathcal{M}}_{2,P} \\ \zeta\big\downarrow & \nearrow \vartheta_R & \\ \overline{\mathcal{M}}_{2,R\cup\{z\}} & & \end{array}$$

which is commutative up to homotopy, and it is evident that $a^R = a^{R'}$ and $b_r^R = b_r^{R'}$ whenever $r \in R'$. But then it suffices to annihilate $a^\emptyset$ and $b_r^{\{r\}}$, for every $r \in P \setminus \{p\}$, which can be achieved by adding to α a suitable linear combination of $\delta_{\rm irr}$ and ψ_r for $r \in P \setminus \{p\}$. Next observe that, in view of (3.2) and (3.13), by adding to α a suitable multiple of ψ_p we can assume that $a = 0$. Finally, since

$$\xi^*(\delta_{2,S}) = \delta_{1,S\cup\{x,y\}} , \quad \xi^*(\delta_{1,S}) = \delta_{1,S} + \delta_{0,S\cup\{x,y\}} = \delta_{1,S} + \delta_{1,S^c} ,$$

we can assume that $b_S = 0$ and that $a_S = 0$ if $p \notin S$. In conclusion

$$\textbf{(4.19)}\qquad \xi^*(\alpha) = \sum_{p\in S\subset P} a_S\,\delta_{1,S} + \sum_{S\subset P,|S^c|\geqslant 1} c_S(\delta_{1,S\cup\{x\}} + \delta_{1,S\cup\{y\}}) ,$$

$$\textbf{(4.20)}\qquad \vartheta_R^*(\alpha) = b_z^R\psi_z + \sum_{S\subset R,|S|\geqslant 2} c_S^R\,\delta_{0,S} + \sum_{S\subset R,|S|\geqslant 1} d_S^R\,\delta_{0,S\cup\{z\}} + \sum_{S\subset R} h_S^R\,\delta_{1,S}$$

Now set $R = P \setminus \{p, q\}$, for some $q \in P$ with $q \neq p$, and look at the diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}_{1,R\cup\{x,y,z\}} & \xrightarrow{\nu} & \overline{\mathcal{M}}_{1,P\cup\{x,y\}} \\ \mu\big\downarrow & & \xi\big\downarrow \\ \overline{\mathcal{M}}_{2,R\cup\{z\}} & \xrightarrow{\vartheta_R} & \overline{\mathcal{M}}_{2,P}. \end{array}$$

Using Lemma (3.3), we get

$$\begin{aligned} \nu^*\xi^*(\alpha) = & \sum_{\{p,q\}\subset S\subset P} a_S\, \delta_{1,(S\setminus\{p,q\})\cup\{z\}} + \sum_{S\subset R} c_S(\delta_{1,S\cup\{x\}} + \delta_{1,S\cup\{y\}}) \\ & + \sum_{\{p,q\}\subset S\subset P,\,|S^c|\geq 1} c_S(\delta_{1,(S\setminus\{p,q\})\cup\{x,z\}} + \delta_{1,(S\setminus\{p,q\})\cup\{y,z\}}), \end{aligned} \tag{4.21}$$

$$\begin{aligned} \mu^*\vartheta_R^*(\alpha) = & \, b_z^R\psi_z + \sum_{S\subset R,\,|S|\geq 2} c_S^R\, \delta_{1,(R\setminus S)\cup\{x,y,z\}} \\ & + \sum_{S\subset R,\,|S|\geq 1} d_S^R\, \delta_{1,(R\setminus S)\cup\{x,y\}} + \sum_{S\subset R} h_S^R(\delta_{1,S} + \delta_{1,(R\setminus S)\cup\{z\}}). \end{aligned} \tag{4.22}$$

The equality $\nu^*\xi^*(\alpha) = \mu^*\vartheta_R^*(\alpha)$ provides a relation of linear dependence between classes of the following types

$$\begin{gathered} \delta_{1,T\cup\{z\}},\ \ \delta_{1,T\cup\{x\}},\ \ \delta_{1,T\cup\{y\}},\ \ \delta_{1,T\cup\{x,z\}},\ \ \delta_{1,T\cup\{y,z\}}, \\ \psi_z,\ \ \delta_{1,T\cup\{x,y,z\}},\ \ \delta_{1,T\cup\{x,y\}},\ \ \delta_{1,T} + \delta_{1,R\setminus T\cup\{z\}}, \end{gathered}$$

where $T \subset R$. By the results on $H^2(\overline{\mathcal{M}}_{1,\nu})$, and since $\psi_z = (1/12)\delta_{irr} + \cdots$, the above classes are linearly independent, so that all the coefficient in (4.21) and (4.22) are zero. As q is any element in P different from p, this means that (4.19) becomes

$$\begin{aligned} \xi^*(\alpha) = & \, a_{\{p\}}\delta_{1,\{p\}} + c_{\{p\}}(\delta_{1,\{p,x\}} + \delta_{1,\{p,y\}}) \\ & + c_{(P\setminus\{p\})}(\delta_{1,(P\setminus\{p\})\cup\{x\}} + \delta_{1,(P\setminus\{p\})\cup\{y\}}), \end{aligned} \tag{4.23}$$

while $\vartheta_R^*(\alpha) = 0$. We now look again at diagram (4.2). The identity $\varphi^*\xi^*(\alpha) = \vartheta^*(\alpha)$, together with (3.3), applied to φ, gives

$$\vartheta^*(\alpha) = a_{\{p\}}\delta_{1,\{p\}}, \tag{4.24}$$

while the identity $\gamma^*\vartheta^*(\alpha) = \beta^*\xi^*(\alpha)$ gives

$$\begin{aligned} a_{\{p\}}\delta_{\{p,x,y\}} = & \, a_{\{p\}}\delta_{\{p,s\}} + c_{\{p\}}(\delta_{\{p,x,s\}} + \delta_{\{p,y,s\}}) \\ & + c_{(P\setminus\{p\})}(\delta_{(P\setminus\{p\})\cup\{x,s\}} + \delta_{(P\setminus\{p\})\cup\{y,s\}}). \end{aligned}$$

As long as $|P| \geq 3$, the boundary classes appearing in the above relation belong to the standard basis of $H^2(\overline{\mathcal{M}}_{0,P\cup\{x,y,s\}})$ relative to p, x, y, so that all coefficients must vanish and we are done. If $P = \{p, q\}$, we further simplify notation and rewrite the above relation as

$$a\,\delta_{\{q,s\}} = a\,\delta_{\{p,s\}} + c(\delta_{\{q,y\}} + \delta_{\{q,x\}}) + d(\delta_{\{p,y\}} + \delta_{\{p,x\}}).$$

We choose $\delta_{\{q,s\}}$, $\delta_{\{q,x\}}$, $\delta_{\{q,y\}}$, $\delta_{\{q,p\}}$ and $\delta_{\{p,s\}}$ as a basis for $\overline{\mathcal{M}}_{0,5}$ and, using the relations

$$\delta_{\{p,y\}} = \delta_{\{p,s\}} + \delta_{\{q,y\}} - \delta_{\{q,s\}}\,, \quad \delta_{\{p,x\}} = \delta_{\{p,s\}} + \delta_{\{q,x\}} - \delta_{\{q,s\}}\,,$$

we get $a = -2d$ and $c = -d$. Thus

$$\xi^*(\alpha) = c(2\delta_{1,\{p\}} + \delta_{1,\{p,x\}} + \delta_{1,\{p,y\}} - \delta_{1,\{q,x\}} - \delta_{1,\{q,y\}}).$$

On the other hand, using (3.2) and (3.13), one finds that

$$2\delta_{1,\{p\}} + \delta_{1,\{p,x\}} + \delta_{1,\{p,y\}} - \delta_{1,\{q,x\}} - \delta_{1,\{q,y\}} = \xi^*(\delta_{1,\{p\}} + \psi_q - \psi_p)\,,$$

so (2.10) implies that $\alpha = c(\delta_{1,\{p\}} + \psi_q - \psi_p)$. The proof of (2.2) is now complete.

5. The initial cases of the induction

In this section we calculate those cohomology groups which are needed to start the inductive proofs of (2.1) and (2.2). More exactly, we shall compute the k-th cohomology group of $\overline{\mathcal{M}}_{g,n}$ for all k, g and n such that $k \leqslant 3$ or $k = 5$ and $d(g, n) < k$. Our treatment will be elementary and self-contained for $k \leqslant 3$, while for $k = 5$ we shall use, directly or indirectly, some of the results of [6], [7], and [14].

We have already settled the case $k = 1$ in the body of the proof of (2.1). For $k = 2$, the values of g and n involved are $g = 0$ and $n \leqslant 5$, and $g = 1$ and $n \leqslant 2$. For $k = 3$ they are $g = 0$ and $n \leqslant 6$, $g = 1$ and $n \leqslant 3$, and $g = 2$ and $n \leqslant 1$, while for $k = 5$ they are $g = 0$ and $n \leqslant 8$, $g = 1$ and $n \leqslant 5$, $g = 2$ and $n \leqslant 3$, and $g = 3$ and $n \leqslant 1$.

As we said, in genus zero we rely on Keel's results [13], although the computations could be easily done directly. What Keel shows, among other things, is that $H^k(\overline{\mathcal{M}}_{0,n})$ vanishes for all odd k, and that $H^2(\overline{\mathcal{M}}_{0,n})$ is generated by tautological classes, modulo the relations described in (3.6), and has dimension $2^{n-1} - \binom{n}{2} - 1$.

We now turn to higher genus. Since $\overline{\mathcal{M}}_{1,1}$ is isomorphic to $\mathbf{P}^1$, its second cohomology is one-dimensional (and generated by δ_{irr}). We next show that there are surjective morphisms $\alpha : \overline{\mathcal{M}}_{0,6} \to \overline{\mathcal{M}}_{2,0}$ and $\beta : \overline{\mathcal{M}}_{0,7} \to \overline{\mathcal{M}}_{2,1}$. This implies that the third cohomology groups of $\overline{\mathcal{M}}_{2,0}$ and $\overline{\mathcal{M}}_{2,1}$ vanish. Let $(C; p_1, \ldots, p_6)$ be a 6-pointed stable genus zero curve. The morphism α associates to it the stable model of the double admissible covering of C branched at the p_i. As for β, the image under it of a 7-pointed stable genus zero curve $(C; p_1, \ldots, p_7)$ is defined to be the stable model of $(D; q)$, where D is the double admissible covering of C branched at $p_1, \ldots, p_6$, and

q is one of the points lying above p_7. Notice that it is immaterial which of the two possible choices for q we make, since they yield isomorphic 1-pointed curves.

To complete our analysis for $k = 2, 3$ it remains to compute the second cohomology of $\overline{\mathcal{M}}_{1,2}$ and the third cohomology of $\overline{\mathcal{M}}_{1,3}$. It will suffice to prove the following.

Lemma **5.1.** — *We have* $h^2(\overline{\mathcal{M}}_{1,2}) = 2$, $h^3(\overline{\mathcal{M}}_{1,3}) = 0$.

In fact, there are exactly two boundary classes in $H^2(\overline{\mathcal{M}}_{1,2})$, namely δ_{irr} and $\delta_{1,\emptyset}$, which are independent by part *i)* of (3.11). Thus the first part of (5.1) implies that δ_{irr} and $\delta_{1,\emptyset}$ generate $H^2(\overline{\mathcal{M}}_{1,2})$.

The proof of the second part of (5.1) relies on Theorem (2.2), and hence also on the first part, in that we will use the fact that $H^2(\overline{\mathcal{M}}_{1,3})$ is freely generated by boundary classes. Since the boundary classes are δ_{irr}, $\delta_{1,\emptyset}$, $\delta_{1,\{1\}}$, $\delta_{1,\{2\}}$, and $\delta_{1,\{3\}}$, this shows in particular that $h^2(\overline{\mathcal{M}}_{1,3}) = 5$. As we know that $h^1(\overline{\mathcal{M}}_{1,2}) = h^1(\overline{\mathcal{M}}_{1,3}) = 0$, Poincaré duality implies that $\chi(\overline{\mathcal{M}}_{1,2}) = 2 + h^2(\overline{\mathcal{M}}_{1,2})$ and $\chi(\overline{\mathcal{M}}_{1,3}) = 12 - h^3(\overline{\mathcal{M}}_{1,3})$. Lemma (5.1) is then a consequence of the following result.

Lemma **5.2.** — *We have* $\chi(\overline{\mathcal{M}}_{1,2}) = 4$, $\chi(\overline{\mathcal{M}}_{1,3}) = 12$.

In [6], Getzler computes the Euler characteristic of $\overline{\mathcal{M}}_{1,n}$ for any n, and (5.2) is a very special case of his result. Here, however, we shall give an elementary and self-contained proof, which uses the following simple remark. Suppose X is a quasi-projective algebraic variety, and let

$$X = \overline{X}_d \supset \overline{X}_{d-1} \supset \ldots \supset \overline{X}_1 \supset \overline{X}_0$$

be a filtration of X by closed subvarieties. Suppose that $X_i = \overline{X}_i \setminus \overline{X}_{i-1}$ is of pure complex dimension i (or is empty) for every i. Then the exact sequence of cohomology with compact supports shows that the Euler characteristic with compact supports of X is the sum of those of $\overline{X}_{d-1}$ and X_d. Repeating this argument for $\overline{X}_{d-1}$, then for $\overline{X}_{d-2}$, and so on, shows that $\chi_c(X) = \sum \chi_c(X_i)$. If Poincaré duality holds for the X_i, for instance if they are orbifolds, then $\chi_c(X_i) = \chi(X_i)$ for any i. If in addition X is compact, or an orbifold, we conclude that $\chi(X) = \sum \chi(X_i)$. For us, the main consequence of this is that, if one stratifies $\overline{\mathcal{M}}_{g,n}$ according to graph type, then the Euler characteristic of $\overline{\mathcal{M}}_{g,n}$ is the sum of the Euler characteristics of the open strata. Before we can use this to prove Lemma (5.2), we need to compute some auxiliary Euler characteristics. First recall that

$$\chi(\mathcal{M}_{0,n}) = (-1)^{n-3}(n-3)!\,.$$

In fact, this can be proved inductively, starting from the case when $n = 3$, by observing that omitting the last marked point yields a fibration $\mathcal{M}_{0,n} \to \mathcal{M}_{0,n-1}$ whose fibers are projective lines minus $n-1$ points, so that $\chi(\mathcal{M}_{0,n}) = (3-n)\chi(\mathcal{M}_{0,n-1})$.

We next compute the Euler characteristics of the quotients of $\mathcal{M}_{0,4}$ and $\mathcal{M}_{0,5}$ modulo the operation of interchanging the labelling of two of the marked points, which we will denote by $\mathcal{M}'_{0,4}$ and $\mathcal{M}'_{0,5}$, respectively. We claim that

$$\text{(5.3)} \qquad \chi(\mathcal{M}'_{0,4}) = 0\,; \quad \chi(\mathcal{M}'_{0,5}) = 1\,.$$

The map $\mathcal{M}_{0,4} \to \mathcal{M}'_{0,4}$ has degree 2. We claim that there is a unique fiber consisting of only one point, so that

$$-1 = \chi(\mathcal{M}_{0,4}) = 2\chi(\mathcal{M}'_{0,4}) - 1\,,$$

proving the first identity in (5.3). Suppose in fact that there is an isomorphism between the two 4-pointed curves $(\mathbf{P}^1; 0, \infty, 1, x)$ and $(\mathbf{P}^1; 0, \infty, x, 1)$. This means that there is an automorphism α of $\mathbf{P}^1$ such that $\alpha(0) = 0$, $\alpha(\infty) = \infty$, $\alpha(1) = x$, and $\alpha(x) = 1$. The first two conditions imply that α is of the form $\alpha(z) = az$, for some nonzero complex number a. The last two conditions say that $x = a$ and $a^2 = 1$. Thus the curve in question, up to isomorphism, is $(\mathbf{P}^1; 0, \infty, 1, -1)$. As for the second identity in (5.3), the morphism $\mathcal{M}_{0,5} \to \mathcal{M}'_{0,5}$ is clearly unramified and of degree 2, so that $\chi(\mathcal{M}'_{0,5})$ is one half of $\chi(\mathcal{M}_{0,5})$. This completes the proof of (5.3).

We next observe that

$$\text{(5.4)} \qquad \chi(\mathcal{M}_{1,2}) = 1\,; \quad \chi(\mathcal{M}_{1,3}) = 0\,.$$

The Euler characteristic of $\mathcal{M}_{1,n}$ can be easily calculated inductively by examining the morphism $\mathcal{M}_{1,n} \to \mathcal{M}_{1,n-1}$, beginning from the observation that $\mathcal{M}_{1,1}$ is the affine line, and hence its Euler characteristic is 1. First of all, let $(C; p)$ be a smooth 1-pointed genus 1 curve, and τ its -1 involution. Let φ be a non-trivial automorphism of $(C; p)$; we wish to describe the fixed point set of φ. When $\varphi = \tau$, this clearly consists of the 2-torsion points. In general, notice that φ commutes with τ, and hence descends to an automorphism $\overline{\varphi}$ of $\mathbf{P}^1 = C/\tau$ fixing the image of p. On the other hand, φ may differ from τ only in two cases. The first is when C is the double covering of $\mathbf{P}^1$ branched at $0, 1, -1, \infty$ (and p maps to ∞, for instance); $\overline{\varphi}$ must leave 0 fixed, and interchange 1 and -1, so it is just multiplication by -1. In particular, its fixed points are just 0 and ∞, so φ has two fixed points, that is, p and the one lying above 0. The other case is when C is the double covering of $\mathbf{P}^1$ branched at ∞ and at the cubic roots of 1, with p mapping to ∞. Since $\overline{\varphi}$ must fix 0, it is just multiplication by a cubic root of unity. Two cases are possible. If φ interchanges the two points of C lying above 0, it has order 6 and its only fixed point is p; otherwise, φ has order 3 and 3 fixed points, namely p and those lying above 0.

Now look at $\pi : \mathcal{M}_{1,2} \to \mathcal{M}_{1,1}$. Its fiber over $(C;p)$ is just the quotient of C by the automorphism group of $(C;p)$, minus the image of p; in particular, it is dominated by $(C \setminus \{p\})/\tau$, where τ is the -1 involution, which is just the affine line. Hence all the fibers of π are affine lines, so $\chi(\mathcal{M}_{1,2}) = \chi(\mathbf{C})\chi(\mathcal{M}_{1,1}) = 1$.

We can do something similar with $\pi : \mathcal{M}_{1,3} \to \mathcal{M}_{1,2}$. Denote by X the locus in $\mathcal{M}_{1,2}$ of all curves $(C;p_1,p_2)$ such that p_2 is a 2-torsion point with respect to the group law with origin in p_1. Clearly, X is isomorphic to $\mathcal{M}'_{0,4}$, so $\chi(X) = 0$. We also denote by x the point of $\mathcal{M}_{1,2}$ corresponding to the covering C of $\mathbf{P}^1$ ramified at ∞ and at the third roots of unity, whith p_1 lying above ∞ and p_2 above 0, and denote by U the complement of $X \cup \{x\}$. The fiber of π above any point of U is an elliptic curve minus two points, while the fiber above any point of $X \cup \{x\}$ is a projective line minus two points. Thus

$$\begin{aligned}\chi(U) &= \chi(\mathcal{M}_{1,2}) - \chi(X) - 1 = 0\,,\\ \chi(\mathcal{M}_{1,3}) &= \chi(\pi^{-1}(U)) + \chi(\pi^{-1}(X)) + \chi(\pi^{-1}(x))\\ &= -2\chi(U) + \chi(\mathbf{C} \setminus \{0\})\chi(X) + \chi(\mathbf{C} \setminus \{0\}) = 0\,.\end{aligned}$$

We may now compute the Euler characteristics of $\overline{\mathcal{M}}_{1,2}$ and $\overline{\mathcal{M}}_{1,3}$ using the stratification by graph type. The open strata of $\overline{\mathcal{M}}_{1,2}$, other than $\mathcal{M}_{1,2}$, are indexed by the graphs in Figure 1. Here, and elsewhere, we adopt the convention that a solid dot stands for a component of genus zero, and a hollow one for a component of genus one.

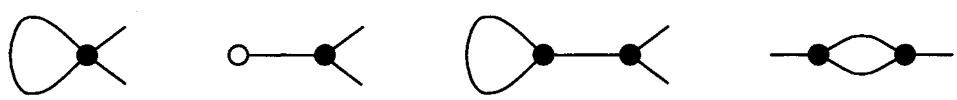

FIG. 1.

There are two one-dimensional strata V_1 and V_2, corresponding to the first two graphs, while the zero-dimensional strata are the two points corresponding to the last two graphs. It is evident that V_1 is isomorphic to $\mathcal{M}'_{0,4}$, and V_2 to $\mathcal{M}_{1,1}$. Thus

$$\chi(\overline{\mathcal{M}}_{1,2}) = \chi(\mathcal{M}_{1,2}) + \chi(\mathcal{M}'_{0,4}) + \chi(\mathcal{M}_{1,1}) + 2 = \chi(\mathcal{M}_{1,2}) + 3 = 4\,.$$

This proves the first statement in (5.2), and consequently also the first one in (5.1). Theorem (2.2) is now fully proved.

We finally turn to $\overline{\mathcal{M}}_{1,3}$. Its open strata other than $\mathcal{M}_{1,3}$ correspond to the graphs in Figure 2 (plus a labelling of the legs by 1, 2, 3).

There are: one stratum corresponding to graph A, isomorphic to $\mathcal{M}'_{0,5}$, one corresponding to graph B, isomorphic to $\mathcal{M}_{1,1} \times \mathcal{M}_{0,4}$, three strata corresponding

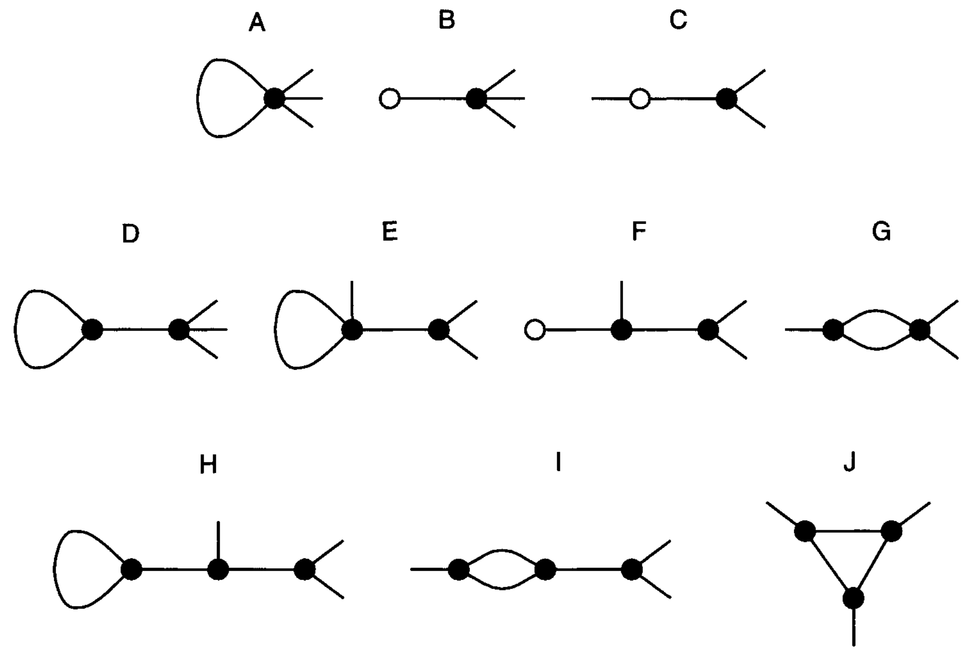

FIG. 2.

to graph C and isomorphic to $\mathcal{M}_{1,2}$, one stratum corresponding to graph D, isomorphic to $\mathcal{M}_{0,4}$, three corresponding to graph E and isomorphic to $\mathcal{M}'_{0,4}$, three corresponding to graph F and isomorphic to $\mathcal{M}_{1,1}$, three corresponding to graph G and isomorphic to $\mathcal{M}'_{0,4}$, and seven zero-dimensional strata, all points, three corresponding to graph H, three to graph I, and one to graph J. Putting everything together we conclude that

$$\begin{aligned}\chi(\overline{\mathcal{M}}_{1,3}) &= \chi(\mathcal{M}_{1,3}) + \chi(\mathcal{M}'_{0,5}) + \chi(\mathcal{M}_{1,1})\chi(\mathcal{M}_{0,4}) + 3\chi(\mathcal{M}_{1,2}) \\ &\quad + \chi(\mathcal{M}_{0,4}) + 6\chi(\mathcal{M}'_{0,4}) + 3\chi(\mathcal{M}_{1,1}) + 7 = 12\,,\end{aligned}$$

as desired.

Theorem (2.1) is now completely proved for $k = 1, 3$. It remains to examine the initial cases of the induction for $k = 5$, in positive genus. These are: $g = 1$ and $n \leqslant 5$, $g = 2$ and $n \leqslant 3$, $g = 3$ and $n \leqslant 1$. The group $H^5(\overline{\mathcal{M}}_{1,n})$ vanishes when $n \leqslant 2$ for dimension reasons, and when $n \leqslant 4$ by Poincaré duality; in fact, $H^5(\overline{\mathcal{M}}_{1,3})$ and $H^5(\overline{\mathcal{M}}_{1,4})$ are Poincaré dual to $H^1(\overline{\mathcal{M}}_{1,3})$ and $H^3(\overline{\mathcal{M}}_{1,4})$. Likewise, $H^5(\overline{\mathcal{M}}_{2})$ and $H^5(\overline{\mathcal{M}}_{2,1})$ are Poincaré dual to $H^1(\overline{\mathcal{M}}_{2})$ and $H^3(\overline{\mathcal{M}}_{2,1})$, which are both zero. On the other hand, Getzler has shown in [6] that $H^5(\overline{\mathcal{M}}_{1,5})$ vanishes, while in [7] he has proved that $H^5(\overline{\mathcal{M}}_{2,2})$ and $H^5(\overline{\mathcal{M}}_{2,3})$ are zero. At this point 2.9) implies that $H^5(\overline{\mathcal{M}}_{g,n})$ vanishes for $g \leqslant 2$ and all n. In genus 3 we may argue as follows.

Looijenga [14] proves that $H^7(\mathcal{M}_3)$ and $H^9(\mathcal{M}_{3,1})$ are zero. By Poincaré duality, this is the same as saying that $H^5_c(\mathcal{M}_3)$ and $H^5_c(\mathcal{M}_{3,1})$ vanish, so the exact sequence of cohomology with compact supports shows that $H^5(\overline{\mathcal{M}}_3)$ and $H^5(\overline{\mathcal{M}}_{3,1})$ inject into $H^5(\partial\mathcal{M}_3)$ and $H^5(\partial\mathcal{M}_{3,1})$, respectively. Lemma (2.6) then says that both $H^5(\overline{\mathcal{M}}_3)$ and $H^5(\overline{\mathcal{M}}_{3,1})$ inject into sums $\bigoplus H^5(X_i)$, where the X_i are products of moduli spaces $\overline{\mathcal{M}}_{g,n}$ with $g < 3$. By what has already been proved, $H^5(X_i) = 0$ for all i, hence $H^5(\overline{\mathcal{M}}_3) = H^5(\overline{\mathcal{M}}_{3,1}) = 0$. This concludes the proof of (2.1).

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