CALCULATION BY A MOMENT TECHNIQUE OF THE PERTURBATION OF THE GEOMAGNETIC FIELD BY THE SOLAR WIND

Thesis by

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In Partial Fulfillment of the Requirements

For the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1963

ACKNOWLEDGMENTS:

Both the initial problem and the idea of solving it using moments of the surface currents were suggested to me by Dr. Leverett Davis, Jr. I am deeply grateful to him, not only for this but also for his constant interest in the problem and numerous valuable suggestions in overcoming the obstacles which arose during its solution.

I am also grateful to the National Science Foundation for Fellowships during part of the time I worked on this research and to the National Aeronautics and Space Administration who have supported the completion of the research, under Grant NsG-151-61.

ABSTRACT:

An iterative method is developed by which one can calculate approximately the boundary of a magnetic field confined by a plasma. This method consists essentially of varying an assumed surface until the magnetic multipole moments of the currents, which would flow on that surface to balance the plasma pressure, cancel the corresponding moments of the magnetic sources within the surface. The method is applied to two problems.

For a dipole source of moment $\underset{\sim}{M}$ emu in a plasma of uniform pressure p dynes/cm^2 that does not penetrate the magnetic field, the approximate equation of the surface is $r=0.82615\ M^{1/3}\ p^{-1/6}(1-0.120039\alpha^2-.004180\alpha^4-.001085\alpha^6+.000200\alpha^8-.000597\alpha^{10}+.000326\alpha^{12}-.000094\alpha^{14})$ cm, where α is the latitude in radians from the plane normal to $\underset{\sim}{M}$.

The surface formed by a cold plasma of density N_o and pair mass M_t moving past a dipole of moment $M\underset{\sim}{e}_y$ with a velocity $-U_o\underset{\sim}{e}_z$ extends to infinity downwind. In a coordinate system (x, y, z) centered at the dipole, neutral points, where the surface is parallel to the wind direction, occur at the points $(0, \pm R_n, .27R_n)$, and other points on the surface are $(0, 0, 1.02R_n)$, $(0, \pm 2R_n, -\infty)$ and $(\pm 1.97R_n, 0, -\infty)$. $R_n = 1.0035\ (M/(M_t N_o U_o^2)^{\frac{1}{2}})^{1/3}$ is about 9 earth radii for the solar wind case.

TABLE OF CONTENTS:

LIST OF ILLUSTRATIONS

1. Introduction.

It has long been believed that there exists a flow of plasma from the sun which, because of its high conductivity, compresses the earth's magnetic field, confining it to a tear-drop shaped cavity, such as illustrated in Figure 1.

Solar plasma bursts were first suggested by Chapman and Ferraro (1) as an explanation for magnetic storms--the sudden arrival of the plasma stream giving rise to the sudden commencement of the storm. Later Biermann's (2) observations of comet tails supported the existance of a solar plasma flux and indicated that it was probably a continual phenomenon. Following Unsold and Chapman (3) he estimated its velocity at 1000 Km/sec and its particle density at anywhere from 100 particles/cc in quiet times to 10^5 particles/cc in active times. He assumed the stream to have a temperature of 10^4 °K.

Parker (4) developed a hydrodynamic theory of the solar corona which included heating out to about eight sun radii by hydromagnetic waves. His theory indicated that the corona should be in a state of constant expansion giving rise to a "solar wind" with a velocity of 300 Km/sec and density of 30 protons/cc at the radius of the earth's orbit. Chamberlain (5) objected that a hydrodynamic approach was not appropriate and that the loss of matter from the corona was limited by evaporation of particles from the tail of the Maxwellian distribution. His theory also indicates a

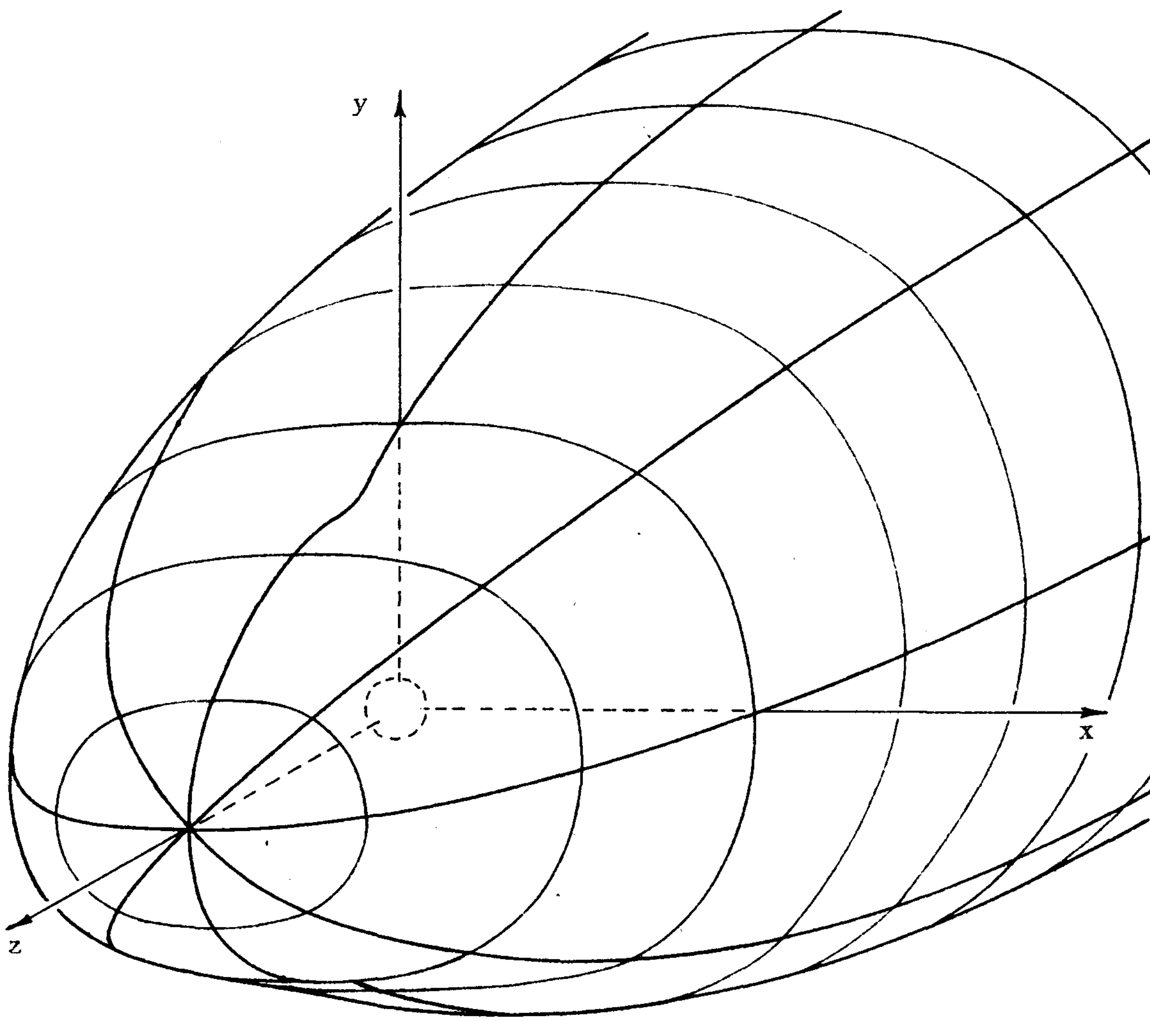


Figure 1 Exterior view of the bounding surface of the earth's dipole field (oriented in the y direction) for a plasma wind in the -z direction.

density of about 30 protons/cc but predicts the velocity at the radius of the earth to be only about 20 Km/sec. The recent results from the Mariner II plasma detector (6) indicate that the stream probably has a mean velocity of about 500 Km/sec, a density between 2.5 and 5 ions/cc and a temperature in excess of 10^5 °K. This of course favors Parker's theory over Chamberlain's.

The qualitative aspects of the transient phenomena involved when a plasma burst impinges on the earth's magnetic field have been studied by consideration of several idealized problems. Chapman and Ferraro (7) first considered the two dimensional axially symmetric problem of plasma injected radially into a magnetic field which fell off radially as r^{-3}. They deduced that a thin sheath, which would screen the plasma from the field, would form and move inward until the pressure of the field just inside it was sufficient to balance the plasma pressure. Later Ferraro (8) solved the idealized one dimensional problem, where the field falls off as x^{-3}, in considerable detail and came to the same general conclusion concerning the formation of a current sheath and its deceleration to rest.

The question of the transient disturbances involved when the solar wind changes its intensity is not, however, within the scope of this paper. Certainly before any quantitative work could be done on that for the real three dimensional problem,one must be able to solve quantitatively the simpler problem of the steady state interaction of the earth's field with a constant intensity plasma stream.

Dungey (9) seems to have been the first one to realize that the cavity must certainly close on the night side due to the finite plasma pressure, and that therefore the earth's field must be entirely confined by the solar wind.

The topological description of the field within the cavity is due to Johnson (10) who introduced the idea that within the cavity those field lines that lie near the poles do not rotate rigidly with the earth as do the field lines at lower latitudes but instead remain in the tail of the cavity and counter-rotate as described in section 8.

Zhigulev and Romishevskii (11) seem to have been the first to have suggested that the wind is supersonic and that therefore a detached bow shock should be formed upstream from the cavity. The plasma itself is essentially collisionless, so, in order to have such a shock, it is necessary to have magnetic fields in the plasma which can serve to randomize the particle motions, and the necessary condition for a shock is that the flow velocity exceed the Alfven velocity. Lees (12) has shown that if there is a radial (from the sun) magnetic field giving rise to such a bow shock that the plasma which has become subalvénic on passing through the shock will accelerate again to superalvénic velocities on flowing around the cavity, and that this converging plasma will therefore form a conical "wake shock" at the tail of the cavity.

Once the general principles governing the confinement of the earth's field were well understood, numerous investigators set to work to try to obtain a more quantitative picture of the resulting cavity. It turns out that the related two dimensional problem of plasma flow past a line dipole can be done analytically by the technique of a conformal transformation. This was done for the stream normal to the dipole axis by Dungey, whose earlier solution was not published until 1961 (13), and for arbitrary orientation by Zhigulev and Romishevskii (11). Later Hurley (14) solved the same problem but by a slightly different method.

Beard (15) was the first one to attempt a solution of the three dimensional problem. He simplified the problem by assuming that at any point just inside the surface the field is just twice the tangential component of the undisturbed dipole field. He justifies this by pointing out that it would be exact if the surface were an infinite plane, which of course it is far from being. However, this simplification enabled him to write down a partial differential equation for the surface, and the solution of this equation seemed to give a reasonable shape for the surface. Beard only applied his method to the non-polar regions on the sunlit side of the earth for normal incidence of the stream; however, soon papers began to appear applying this approximate boundary condition to the solution of more and more complex problems. For instance Spreiter and Briggs (16) extended the solution to the night side and considered various orientations of the dipole relative to the stream, but solved only for the trace of the surface in the meridian plane containing the earth-sun line. Beard (17) attempted to improve his approximation by including as part of his "source field" the field of a current system on the sunlit portion of his surface. When he carried this out, it changed his results very little. Spreiter and Alksne (18) recalculated the meridian and equatorial cross sections for the case when there is a westward flowing ring current of about five million amperes at a distance of about ten earth radii.

In the meantime others who were unsatisfied with Beard's approximation have attempted to obtain solutions by more rigorous methods, two of which have been proposed. Both of these methods essentially involve setting up a trial surface, testing to see if the surface satisfies the complete boundary conditions, modifying the surface in such a way as to improve the agreement, and iterating the process of testing and modification until the result converges to the correct answer. Slutz (19) proposed to solve for the scalar potential of the field inside the surface, treating it as a cavity in a diamagnetic medium, and then test the surface by seeing whether the field had the correct value at each point just inside the surface. Leverett Davis, Jr. and the author (20) proposed to solve for the currents (proportional to the field just inside) which must flow on the surface in order to balance the pressure and then test the accuracy of the surface by computing the moments of the field outside. The two papers just cited apply these methods to the simple three dimensional problem of a dipole field in a uniform pressure plasma, which served primarlly to test the convergence of the methods. In what follows, the moment technique will be extended to the wind problem.

2. Model for the Calculations.

Despite the long history of the problem and the large amount of effort that has been given it, there is still a great deal about the solar wind interaction with the magnetosphere that is either unknown or contested. One of the few things that is generally agreed upon is that the surface bounding the magnetosphere is relatively thin.

Ferraro (8) was the first to quantitatively calculate the thickness of this surface by considering an idealized, one-dimensional problem. Dungey (21) streamlined his calculation and eliminated some ambiguities which it contained. The same results can be obtained by a different method used by Davis, Lüst and Schlüter (22) in calculating the structure of hydromagnetic shock waves. This latter method, which stresses more the individual particle approach and enables one to obtain the trajectories of the particles as a function of time, is given in Appendix I. There it is shown that the trajectories of the particles of a cold plasma, whose pair (ion + electron) mass is M_t and pair density is N_o, projected normally with velocity U_o into a region of constant field $B_o=(16\pi M_t N_o U_o^2)^{\frac{1}{2}}$ are as shown in Figure 2. In addition it is shown that the magnetic field falls off in the plasma in direct proportion to the displacement of the particle trajectory from its asymptote.

Thus it is clear from Figure 2 that for a density of 2.5 protons/cc the field has fallen to 5% of its initial

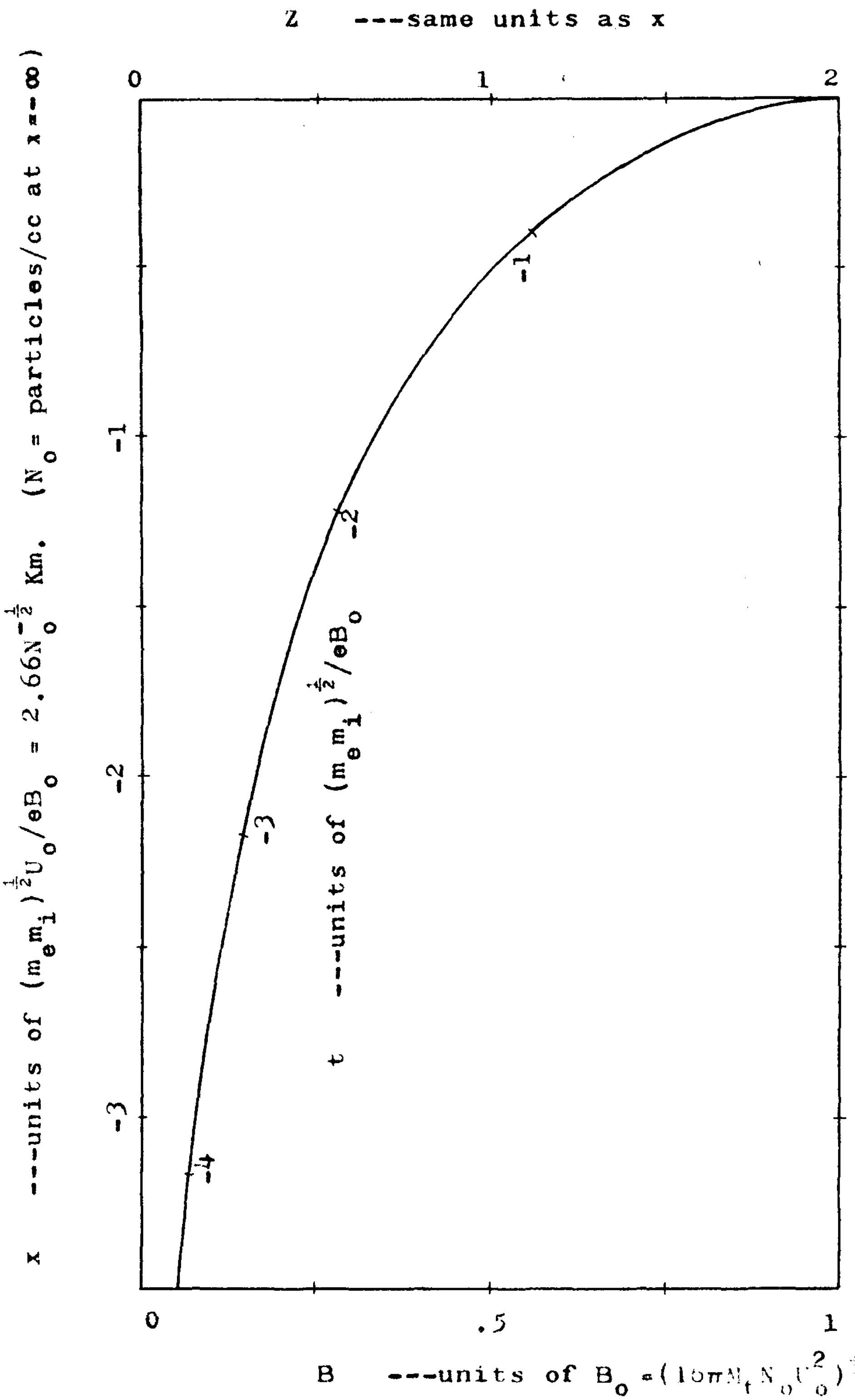


Figure 2. Plot of B(x) and the typical trajectory. Ion and electron trajectories are obtained from this by the relations: $z_i(x)=(m_e/m_i)^{\frac{1}{2}}Z(x)$ & $z_e(x)=-(m_i/m_e)^{\frac{1}{2}}Z(x)$.

value in a distance of only about five kilometers, and equation I-22 shows that thereafter it decreases by a factor of two every 1.65 Km. These distances are of course negligible compared to the scale of the surface.

Knowing that the surface is negligibly thin, it is next necessary to decide what pressure is exerted on the surface by the streaming plasma outside.

For the model assumed in Appendix I (specular reflection of normally directed particles) the pressure is easily inferred by a momentum balance.

$$P = 2(M_i+M_e)N_0U_0^2 \tag{2.1}$$

In general the particles are incident upon the surface obliquely rather than normally but this does not change significantly the results arrived at in Appendix V. A Lorentz transformation based on a relative velocity parallel to the interface will reduce the problem to one of normal incidence. Thus any constant velocity which is parallel to the surface and small compared to the velocity of light may be superimposed on the given solution without altering the scale and structure normal to the surface. The only modification necessary in equation 2.1 is to replace the total velocity U_0 by its normal component $U_0 \cos\psi$, where $\pi-\psi$ is the angle which the wind makes with the normal to the surface. Using the abbreviation $M_t=(M_i+M_e)$ for the total pair mass, the pressure law for arbitrary angle of incidence then becomes:

$$P = 2M_t N_0 U_0^2 \cos^2\psi \tag{2.2}$$

If the surface is actually curved rather than flat, then the tension in the magnetic field lines lying in the surface will help to balance the pressure of the field just inside the surface and equation 2.2 is not precisely correct. However, this correction is clearly very small because the normal force exerted on the surface by the field lines in the surface is proportional to the ratio of the effective surface thickness to its radius of curvature. For the magnetopause this ratio is about 10^{-4}.

A more serious objection to equation 2.2 arises from the assumption made throughout the calculations that the outgoing stream passes unimpeded through the ingoing stream. From an individual particle viewpoint this assumption would certainly be quite valid if there were no magnetic fields in the plasma, for the distance which a single reflected proton would travel back through the stream before it's cumulative deflection approached 90° is of the order of 10^6 A.U.(virtually infinite) for a solar wind of 500 Km/sec and 2.5 protons/cc. Also, using the formula given by Spitzer (23, p.78) for the relaxation time in a plasma (defined as the average time for a typical particle to be deflected 90°), one finds that if the wind has a temperature of 10^5 °K, its own internal relaxation time is of the order of 10^5 seconds.

Since the length of the magnetosphere cavity is of the order of $4x10^5$ Km, the wind passes it in about 10^3 seconds or only one hundredth of its own internal relaxation time.

However, objections do arise from the randomizing effect of any magnetic fields contained in the wind and from the possibility of a collective interaction such as a two stream instability. Parker (24) worked out the problem of two interpenetrating cold plasma streams and came to the conclusion that the solar wind flowing through a stationary interplanetary gas would be unstable and would lead to a shock front only about 100 meters thick between the two. Presumably, then, the counterflowing stream of reflected particles might similarly react with the incoming stream thus providing the dissipative mechanism needed to have a thin standoff shock. Noerdlinger (25) also treated this problem in a very general manner. On the other hand a detailed treatment by Kellogg and Liemohn (26) has shown that two contra streaming plasmas are not necessarily unstable if their internal temperatures are high enough compared to their relative kinetic energy. For instance they show that two equal density plasmas each with internal temperature T and streaming through each other with relative velocity U_0 are stable if

$$kT > 0.2M_e U_0^2 \qquad (2.3)$$

For a 500 Km/sec wind this indicates that there is no interaction with the reflected plasma as long as its temperature is greater than about $3{,}000^{\circ}K$, which is more than an order of magnitude below present estimates of its temperature.

The deflection and randomization of particles by fields contained in the wind is the most serious objection to the hypothesis of interpenetration. It has been shown by spacecraft data (27) that there are fields within the solar wind. However, it is not within the scope of this paper to try to decide if there is or is not a steady state shock enveloping the magnetosphere. We will use the assumption that the particles are specularly reflected (i.e. do not interact with the incoming stream) because the pressure law it gives is as good as any other and it has the further advantage of simplifying the calculations.

As a final defense of the pressure law derived from the assumption of specular reflection it is worthwhile to note that ordinary hypersonic flow past a blunt body results in just such a pressure distribution (28). The only change necessary is the substitution of the pressure at the stagnation point for the factor $(2M_t N_0 U_0^2)$. This change alters only the scale of the solution and not its shape.

Another objection to this simplified model arises from the fact that for a cold plasma the surface must extend to infinity on the night side, whereas the real wind has a temperature of the order of $10^5\ ^{\circ}K$ and therefore would close off the cavity at a finite distance due to its thermal pressure. The maximum radius of the cavity is determined almost entirely by the momentum flux of the wind, but for a given momentum flux the location of this maximum radius is determined by the thermal pressure which must there balance the pressure of the field just inside. Past that point the field inside falls off so rapidly that the shape is determined primarily by the rate at which the gas can expand into vacuum. According to Lees (12) the resulting cavity is about 60 earth radii in length. This pressure resulting from the plasma temperature will be ignored, however, simply because its inclusion would seriously complicate the problem. Therefore the computed surface will have little relation to the actual magnetosphere on the far night side of the earth, but it should still give a good approximation to it on the daylight side.

The question of instabilities in the surface is an important one, but one about which there is no general agreement. Parker (29) considered the two dimensional problem of a tenuous ionized gas incident upon the surface of an incompressible conducting fluid in which is embedded a uniform magnetic field. He found it to be unstable and deduced therefrom that the surface of the magnetosphere is unstable. Dessler (30) concluded from magnetic data at the surface of the earth that the surface must be stable, but Coleman and Sonett (31) took exception with the basis of his argument. Later Dessler (32) advanced an independent and very convincing argument for the stability of the surface.

The present author feels that the instability of Parker's model proves nothing concerning the real surface, first because the outer fringes of the magnetosphere are not loaded with matter like the field in his problem and second because his problem ignores the stabilizing curvature of the field lines. Having this demonstrated the moot nature of the stability problem, we will now ignore it and assume the surface is stable in order to calculate its steady state shape. If later investigations should demonstrate that it is indeed unstable, the "steady state" solution will at least provide a valuable zero order approximation to it.

In the numerical calculations of this paper, the ring current described by Sonett, et al (33) will be ignored. It could be easily included, but it was not felt that it was advisable at this time to expend the computer time which would be required to solve the problem for various ring current strengths and diameters.

In summary, then, it will be assumed that the solar wind problem has a steady state solution in which an infinitely thin current sheath terminates the earth's magnetic field, assumed to be a simple dipole; and that the pressure exerted on this surface by the wind is given by equation 2.2.

3. The Moment Technique.

The moment technique is a general method which can, in principle, be used to determine the shape of the surface of separation in any problem involving an infinitely conducting plasma separated from a magnetic field by an infinitesimally thin current sheath. Of course any such problem involves two "sub-problems." First, one must be able to compute the pressure P exerted by the plasma on the surface for any assumed surface shape. This is a problem in kinetic theory and in the discussion which follows its solution will be taken as given. Second, one must be able to solve for the magnetic field inside any assumed surface shape and ascertain whether its pressure balances the plasma pressure. The boundary conditions on the magnetic field just inside the surface are as follows:

$$B_n = 0 \tag{3.1}$$

which amounts to saying that the field is excluded from the plasma, and

$$\frac{B_t^2}{8\pi} = P \tag{3.2}$$

which is necessary for dynamic equilibrium.

The basic idea of the moment technique is to replace equations 3.1 and 3.2 by two different but equivalent conditions. First, if the field is everywhere zero in the plasma region as equation 3.1 implies, then the surface current at each point of the surface must be $B_t/4\pi$, where B_t is the magnetic field just inside that point. Using this fact, equation 3.2 may be written in terms of the surface current.

$$J^2 = P/2\pi \tag{3.3}$$

This fixes the magnitude of $\underset{\sim}{J}$ at every point on the surface, and then in principle the direction of $\underset{\sim}{J}$ is determined (if we know its direction on one line of the surface) by the requirement that $\underset{\sim}{J}$ be divergence free, However, the details of the process for determining the direction of $\underset{\sim}{J}$ will depend entirely upon the particular problem; for instance, see section 4 for the uniform pressure problem and section 5 for the plasma wind problem.

Finally equation 3.1 is replaced by the condition that the magnetic field vanish everywhere in the plasma region. This will be true if each of the magnetic multipole moments of the sources in the field region is cancelled by the corresponding moment of the surface current. Actually the field will vanish to a very high order of accuracy if only the lower moments cancel, and it is this fact that makes the moment technique useful.

The ~~plasma region~~ magnetosphere is assumed current free so that $\nabla \times \underset{\sim}{B} = 0$, and the magnetic field may be decomposed into multipole moments either in terms of its scalar or vector potential.

The scalar potential φ, defined to be the function whose gradient is $\underset{\sim}{B}$, is certainly a solution of Laplace's equation, since $\nabla\cdot\underset{\sim}{B}=\nabla^2\varphi=0$. Likewise, if we define the vector potential $\underset{\sim}{A}$ to be the function whose curl is $\underset{\sim}{B}$ and choose a gauge in which $\nabla\cdot\underset{\sim}{A}=0$, each of its components will satisfy Laplace's equation since $\nabla\times B=\nabla\times\nabla\times A=$ $=\nabla(\nabla\cdot A)-\nabla^2 A=0$. The following functions form a set of solutions of Laplace's equation in terms of which any solution which vanishes at infinity, such as φ or A_x, may be expanded.

$$D^p_{nm} = \frac{P^m_n(\cos\theta)}{r^{n+1}}\cos(m\phi-p\tfrac{\pi}{2}) \qquad \begin{array}{l} n=0,1,\ldots\infty \\ m=0,1,\ldots n \\ p=0,1 \end{array} \tag{3.4}$$

If the field region surrounds the plasma region, then solutions vanishing at the origin are needed instead, but this case will not be considered further. Thus we may write the following expressions for the potentials.

$$\underset{\sim}{A} = R_n J_o \sum_{n=o}^{\infty}\sum_{m=o}^{n}\sum_{p=o}^{1}(X^p_{nm}\underset{\sim}{e}_x+Y^p_{nm}\underset{\sim}{e}_y+Z^p_{nm}\underset{\sim}{e}_z)D^p_{nm} \tag{3.5}$$

$$\varphi = R_n J_o \sum_{n=o}^{\infty}\sum_{m=o}^{n}\sum_{p=o}^{1} S^p_{nm}D^p_{nm} \tag{3.6}$$

Here R_n has the units of a length and J_o the units of current-per-unit-width. These factors have been written explicitly so that the remainder of the right hand side might be dimensionless. In general, lower case letters will denote dimensionless variables and capital letters will denote dimensioned variables (except for the moments and the functions such as D^p_{nm} and P^m_n which are obviously dimensionless). Thus

$$R = R_n r \qquad\qquad J = J_o j \tag{3.7}$$

Since there are three times as many vector moments as scalar moments and yet either set of moments is adequate to describe the field, it follows that the vector moments cannot all be independent quantities. In Appendix III, (2n+3) relationships are derived which must hold between the vector moments for each value of n, and it is pointed out that there are (2n-1) more relationships which will depend on the gauge of $\underset{\sim}{A}$ (since specifying the curl and divergence of $\underset{\sim}{A}$ still leaves one free to add to $\underset{\sim}{A}$ the gradient of any scalar function which satisfies Laplace's equation). Thus there are really only (2n+1) independent vector moments for each value of n, just as there are (2n+1) scalar moments. The equations relating the scalar moments to the vector moments are also derived in Appendix III. The (4n+4) relationships given by equation III-23 can be summarized as follows:

$$S^p_{nm} = (2p-1)Z^{1-p}_{nm} + \tfrac{1}{2}(2-\delta_{m0})(n-m)\left[(1-2p)X^{1-p}_{nm+1} - Y^p_{nm+1}\right] \qquad 0 \le m \le n \quad p=0,1$$

$$S^p_{nm} = (1-2p)Z^{1-p}_{nm} + \frac{(1+\delta_{m1})}{(n+1-m)}\left[(1-2p)X^{1-p}_{nm-1} + Y^p_{nm-1}\right] \qquad 1 \le m \le n \qquad p=0,1$$

$$Y^p_{nn} = (2p-1)X^{1-p}_{nn} \qquad\qquad p=0,1 \tag{3.8}$$

It is clear from these equations why the scalar moments must be considered. In order that the magnetic field vanish outside the surface it is only necessary that its scalar moments vanish, and this clearly does not imply that its vector moments vanish. The only reason that the vector moments are considered at all is that they are considerably easier to calculate directly than the scalar moments are, and by equation 3.8 the scalar moments can be easily obtained from them.

In summary, then, the basic outline of the moment technique is as follows: First calculate the moments of the scalar potential of the fixed sources within the surface. Then assume a trial shape for the surface and determine the resulting fluid forces (it is assumed that this is possible). Next calculate the surface current which would satisfy equation 3.3 on that surface, and finally calculate the scalar moments of this surface current. If these just cancel the moments of the fixed sources, the problem is solved; if not, vary the surface appropriately and repeat the process until an adequately accurate solution is obtained.

In Section 4 this method will be applied to the test case of a dipole in a uniform pressure plasma, and in succeeding sections it will be applied to the more important case of a dipole in a plasma wind.

4. Solution for the Uniform Pressure Case.

Consider a magnetic dipole of moment $M\underset{\sim}{e}_z$ emu surrounded by a stationary plasma of uniform pressure P dynes/cm^2. The solution of this problem is discussed in a paper written by Leverett Davis, Jr. and the author (20) but it will be repeated here in terms of the more general notation of Section 3.

The unit of length R_n will be chosen to be the radius in the equatorial plane to the point where the magnetic pressure of the undisturbed dipole field equals the gas pressure

$$R_n^3 = M(8\pi P)^{-\frac{1}{2}} \tag{4.1}$$

From equation 3.3 it is clear that $\underset{\sim}{J}$ has the constant magnitude $(P/2\pi)^{\frac{1}{2}}$ so this will be chosen as the unit current J_o. Obviously the bounding surface and $\underset{\sim}{J}$ must have axial symmetry so $\underset{\sim}{J}$ must be in the ϕ direction.

The scalar potential of the dipole at the field point $\underset{\sim}{R}_2 = R_n \underset{\sim}{r}_2$ is:

$$\varphi = 4\pi R_n J_o \cos\theta_2 r_2^{-2} = 4\pi R_n J_o D_{10}^0 \tag{4.2}$$

Obviously, then all scalar moments of the dipole surface must vanish except $S_{10}^0 = -4\pi$ (see equations 3.4 and 3.6).

If the coordinates of surface points are specified by $\underset{\sim}{R} = R_n \underset{\sim}{r}$, the vector potential of the field due to the surface currents is:

$$\underset{\sim}{A}(\underset{\sim}{R}_2) = R_n J_o \int \frac{\underset{\sim}{e}_\phi dS}{|\underset{\sim}{r}_2 - \underset{\sim}{r}|} \tag{4.3}$$

Use equation 6.2 to express $|\underset{\sim}{r}_2-\underset{\sim}{r}|^{-1}$ as an infinite series, each term of which is separable into its $\underset{\sim}{r}$ and $\underset{\sim}{r}_2$ dependence. The symmetry about the polar axis enables the ϕ integration to be done easily with the result:

$$\underset{\sim}{A}(\underset{\sim}{R}_2) = R_n J_o \underset{\sim}{e}_\phi \sum_{n=1}^{\infty} \frac{P_n^1(\cos\theta_2)}{r_2^{n+1}} I_n \tag{4.4}$$

where

$$I_n = \frac{2\pi}{n(n+1)} \int_o^{\pi} r^{n+1}\left[r^2+(\frac{dr}{d\theta})^2\right]^{\frac{1}{2}} P_n^1(\cos\theta)\sin\theta d\theta \tag{4.5}$$

Set $\underset{\sim}{e}_\phi=(\cos\phi\underset{\sim}{e}_y-\sin\phi\underset{\sim}{e}_x)$ in equation 4.4 and it becomes clear by comparison with equation 3.5 that $Y_{n1}^o=-X_{n1}^1=I_n$ and all other vector moments are zero. This means (refer to equation 3.8) that $S_{n0}^0=-nI_n$ and all other scalar moments are identically zero. Actually even $S_{n0}^0=0$ for n even, because $P_n^1(\cos\theta)$ is an odd function for n even. Thus the problem reduces to choosing a function $r(\theta)$ such that:

$$I_1 = 4\pi \qquad I_n=0 \qquad n=3,5,7,9\ \ldots \tag{4.6}$$

Since the surface has cusps at the poles and is symmetric about the equatorial plane it is better to express r as a function of the magnetic latitude α rather than the polar angle θ.

$$r = C\left[1-\sum_{s=1}^{N} c_s\,\alpha^{2s}\right] \tag{4.7}$$

To solve for the parameters, set C=1 at first and ignore I_1. Consider the next N non-trivial I_n (i.e. those for n=3,5,7, ...2N+1). It is easy to differentiate the I_n under the integral sign and obtain analytic expressions for the rates of change of the I_n with respect to the various c_s. Hence the Generalized Newton's Method was used to determine the c_s which reduced the I_n to zero. Finally I_1 is made equal to 4π by adjusting C, which is seen to be the equatorial radius. The computation was carried out on a Burroughs 220 computer for various values of N up to seven. For the case $N=7$ the numerical results are given in Table 1 and the resulting cross section is plotted in Figure 3.

Table 1. Coefficients in the Equation for the Surface.

$C = 1.41395$	$c_3 = 0.001085$	$c_6 = -0.000326$
$c_1 = 0.120039$	$c_4 = -0.000200$	$c_7 = 0.000094$
$c_2 = 0.004180$	$c_5 = 0.000597$	

It is true that at the pole the last few terms of equation 4.7 are of the order of 7% of the first term, but this does not indicate an error of that order there. The coefficients in Table 1 are not the first seven terms in the power series expansion of the true surface. They are the coefficients of the polynomial of degree fourteen which most closely approximates the true surface. There are two reasons for believing that the solution is very accurate even near the pole. First, when the computation was carried out with only four parameters, the radius of the computed surface near $\alpha=\frac{\pi}{2}$, where agreement was worst, was only about one percent greater than the corresponding radius of the

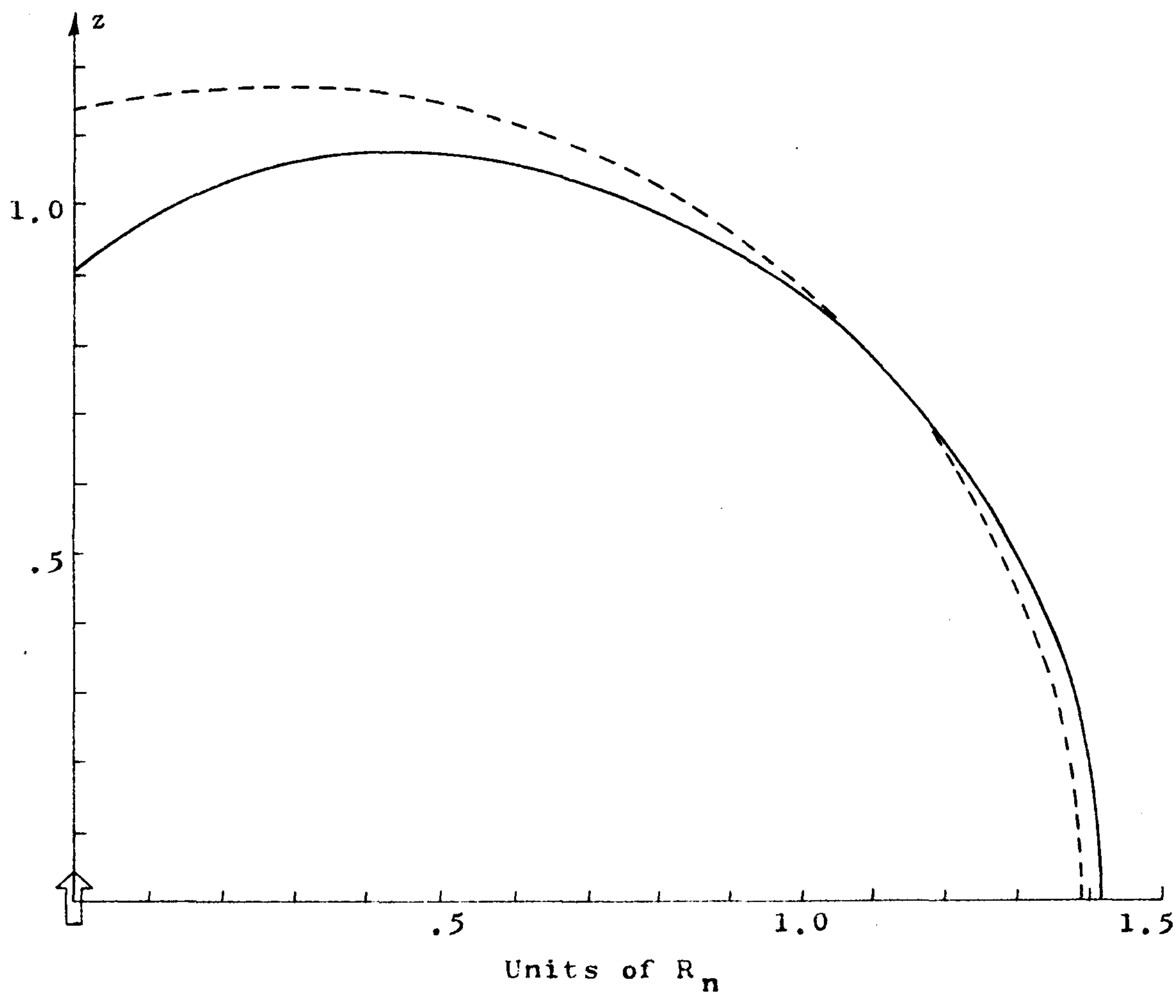


Figure 3 Cross section (one quadrant) of the surface bounding a dipole field in a uniform pressure plasma. The dashed line was calculated by using Beard's condition; the solid line, by using the moment technique.

seven parameter surface. Second, when c_1 was changed so as to decrease the radius to the surface by only 0.1% at the pole, the residual fields at distances greater than $0.3R_n$ outside the surface (calculated as described in the test of the next section) were increased by a factor of ten or more. A major feature of interest in this computation, in addition to providing a test of the moment technique, is that it indicates that the surface very definitely has cusps at the poles and that these cusps do not go clear to the origin as has been suggested, but rather intersect the axis at a finite distance. The cusps undoubtedly intersect the axis tangentially in reality, but such a surface could not be represented by a polynomial with a finite number of terms such as was used. However, the greater the number of parameters that were used the steeper the angle of intersection was. It is easy to see that these are the results that should be expected. Consider a cavity in a medium of zero permeability. If there were a finite angle between the surface and the axis, the field there would be zero, and if the cusp were at the dipole the field would be infinite; in either case the field would not be in equilibrium with the plasma pressure.

If we define the field just inside the surface to be $B_s=(8\pi P)^{\frac{1}{2}}$, then it is a simple matter to see that the change in the field, ΔB_0, at the origin due to the surface currents is

$$\Delta B_o = B_s \int_0^{\frac{\pi}{2}} \cos^2\alpha \quad (1+(\frac{dr}{rd\alpha})^2)^{\frac{1}{2}} \quad d\alpha \qquad (4.8)$$

For a sphere the integral is just $\pi/4$, and for any other surface it would be slightly greater. For the computed surface it is 0.76933. Thus a 10 γ disturbance in the geomagnetic field at the earth could arise from a sudden change of pressure of 2.52×10^{-10} dynes/cm^2 on the surface (i.e. a particle density times temperature of 1.83×10^6 K^o/cm^3 or a kinetic energy density of 1.58×10^2 ev/cm^3).

For comparison purposes the uniform pressure problem was also solved by Beard's differential equation technique. To get the equation for $R(\alpha) = r(\alpha)R_n$, set the magnetic pressure of the tangential component of a field $1/f$ times as strong as the earth's field equal to the plasma pressure

$$(1/8\pi)(f^{-1}\underset{\sim}{e}_t \cdot \underset{\sim}{B}(r,\alpha))^2 = P \qquad (4.9)$$

or in full:

$$\left[(\underset{\sim}{e}_\alpha + \frac{dr}{rd\alpha}\underset{\sim}{e}_r) \cdot (\cos\alpha\underset{\sim}{e}_\alpha - 2\ \sin\alpha\underset{\sim}{e}_r)\frac{M}{r^3R_n^3}\right]^2 = 8\pi f^2 P\left[1+(\frac{dr}{rd\alpha})^2\right] \qquad (4.10)$$

Call $r(0)=r_e$ and note that $dr/d\alpha = 0$ at $\alpha=0$ by symmetry. Inserting these values, and the value of R_n from equation 4.1, into equation 4.10 one obtains the relation

$$f\ r_e^3 = 1 \qquad (4.11)$$

and the differential equation

$$\left[\cos\alpha - 2\sin\alpha\frac{dr}{rd\alpha}\right]^2 = \left(\frac{r}{r_e}\right)^6\left[1+\left(\frac{dr}{rd\alpha}\right)^2\right] \qquad (4.12)$$

When equation 4.12 is solved it gives $r(\alpha)/r_e$. Then r_e is determined by the condition that $I_1=4$. Since equation 4.12 is of second degree there are two such solutions. The appropriate solution is plotted in Figure 3 and it is seen that it differs significantly from the moment technique result near the pole.

There is also an interesting sidelight that can be gleaned from these calculations. There has been some discussion recently as to whether the factor f which Beard assumes to be $\frac{1}{2}$ should not be closer to 1/3. From equation 4.11 we see that in this three dimensional case

$$f = r_e^{-3} = (1.39577)^{-3} = 0.36775 \qquad (4.13)$$

To determine the relative accuracy of the methods, the field due to the surface was calculated (at various radii along the polar axis and in the equatorial plane), subtracted from the field of the dipole located at the origin, and then divided by the dipole field. This gives a number which would be zero everywhere outside the surface for the true surface and would be one everywhere for the dipole field alone. The computations for this test were carried out on a Burroughs 220 computer, replacing the surface by ninety-eight current loops. The results of this test for the two surfaces are given in Table 2. The values on the polar axis may be incorrect by as much as 5% due to truncation error. The truncation error was removed from the equatorial values by subtracting the solution for a sphere with a $\cos\alpha$ current variation, which should theoretically be zero everywhere and which therefore equals the truncation error in practice. Since the surface approximates a sphere near the equator and the $\cos\alpha$ current approximates a uniform current near the equator, the truncation error must be very nearly the same for both cases near the surface at the equator. The inherent roundoff error in the calculation was about $.2 \times 10^{-5}$.

Table 2. Ratio of Net Field to Dipole Field $\times 10^5$

Distance from the surface-- Fraction of Equatorial Radius	Moment Surface		Beard Surface	
	On the Polar Axis	In the Equatorial Plane	On the Polar Axis	In the Equatorial Plane
0.04	-905	-0.4	-61078	7126
0.08	-222	+0.2	-42966	6721
0.16	- 23	0.6	-27676	5997
0.32	- 2.7	0.5	-15913	4844
0.64	- 0.9	0.5	- 7947	3324
1.28	- 0.2	0.2	- 3378	1817
2.56	- 0.1	0.3	0 1222	773
5.12	- 0.0	0.5	- 386	267
10.24	0.2	0.0	- 110	81

Clearly, the moment technique gives a net field outside which is about 0.001 of that given by the surface derived using Beard's boundary condition.

Slutz (19) has also solved this identical problem by an iterative procedure which begins with a trial surface. However, his procedure involved solving for the scalar potential of the field inside the surface, treating it as a cavity in a diamagnetic medium, and then comparing the resultant fields just inside the surface with the fields given by the pressure law to indicate how to change the surface for the next iteration. The result he obtained is very close to that given by the moment technique except near the equator where his cross section is nearly flat and lies about 3% inside the moment result. When the fields for Slutz's surface were calculated they were much larger than those for the moment surface, especially in the equatorial plane.

5. Relationship of the Current and Surface for the Wind Case

It is clear from the proceeding that before the moment technique can be applied to test and improve a surface, the currents flowing on that surface must be known. Consider an axially symmetric source of magnetic field located at the origin and oriented along the y direction and a plasma moving in the -z direction. A surface z(x,y) such as the one shown in Figure 1 will be formed. Choosing x and y as the independent variables enables the surface to be described by a single valued function, restricts the independent variables to a finite range and simplifies certain formulas in the derived later. Adopting the notation $z_x = \frac{\partial z}{\partial x}$, $z_y = \frac{\partial z}{\partial y}$, the outward normal to the surface has the following form.

$$\underset{\sim}{n} = \frac{\text{grad}\ (z-z(x,y))}{|\text{grad}\ (z-z(x,y))|} = \frac{\underset{\sim}{e}_z - z_x \underset{\sim}{e}_x - z_y \underset{\sim}{e}_y}{(1+z_x^2+z_y^2)^{\frac{1}{2}}} \tag{5.1}$$

Therefore, since ψ is defined as the angle between the normal and the earth-sun direction, it follows that

$$\cos\psi = \underset{\sim}{n}\cdot\underset{\sim}{e}_z = (1+z_x^2+z_y^2)^{-\frac{1}{2}} \tag{5.2}$$

Define the unit surface current (see equation 3.7) as $J_o = (M_t N_o U_o^2/\pi)^{\frac{1}{2}}$. Then the magnitude of the dimensionless current is easily obtained from equations 3.3 and 2.2.

$$j = \cos\psi \tag{5.3}$$

The problem now is to determine the direction of $\underset{\sim}{j}$.

Of course since $\underset{\sim}{j}$ must lie in the surface, it must be perpendicular to $\underset{\sim}{n}$, the normal to the surface.

$$\underset{\sim}{n} \cdot \underset{\sim}{j} = 0 \tag{5.4}$$

The last condition necessary to determine j completely is that it must be divergence free or in other words the flux of $\underset{\sim}{j}$ across any closed curve on the surface must be zero. If this is true, then there must exist a flux function, defined by the line integral

$$f(x,y) = \int_{(0,0)}^{(x,y)} + \underset{\sim}{n} \cdot \underset{\sim}{j} \times \underset{\sim}{ds} \tag{5.5}$$

that depends only on x and y and not on the path of integration chosen on the surface.

The usefulness of this flux function arises from the fact that if $f(x,y)$ is specified, then the corresponding $\underset{\sim}{j}$ (which is therefore guaranteed to be divergenceless) can be easily derived from it, using equations 5.5, 5.1, 5.2 and 5.4 in that order.

$$\begin{aligned} f_x = \frac{\partial f}{\partial x} &= \underset{\sim}{n} \cdot \underset{\sim}{j} \times \left[\frac{d\underset{\sim}{s}}{dx}\right]_y = -\underset{\sim}{j} \cdot \underset{\sim}{n} \times (\underset{\sim}{e}_x + z_x \underset{\sim}{e}_z) \\ &= -\cos\psi \; \underset{\sim}{j} \cdot \left[\underset{\sim}{e}_y + z_y \underset{\sim}{e}_z + z_x (z_x \underset{\sim}{e}_y - z_y \underset{\sim}{e}_x)\right] \\ &= -\cos\psi \; \underset{\sim}{j} \cdot \left[\underset{\sim}{e}_y (1 + z_x^2 + z_y^2) + z_y \frac{\underset{\sim}{n}}{\cos\psi}\right] = \frac{-j_y}{\cos\psi} \end{aligned} \tag{5.6}$$

A similar calculation shows that the same result, except for the minus sign, holds with x and y interchanged.

With j_x and j_y known, equation 5.4 can now be used to obtain j_z in terms of derivatives of f and z.

$$\left[\frac{\underset{\sim}{n}}{\cos\psi}\right] \cdot \left[\frac{\underset{\sim}{j}}{\cos\psi}\right] = \frac{j_z}{\cos\psi} - z_x f_y + z_y f_x = 0 \tag{5.7}$$

Hence the surface current is

$$\underset{\sim}{j} = \cos\psi \left[f_y \underset{\sim}{e}_x - f_x \underset{\sim}{e}_y + (z_x f_y - z_y f_x)\, \underset{\sim}{e}_z \right] \tag{5.8}$$

Substitution of this value of $\underset{\sim}{j}$ into equation 5.3 transforms it into a partial differential equation relating the functions $z(x,y)$ and $f(x,y)$.

$$\frac{j^2}{\cos^2\psi} = f_x^2 + f_y^2 + (z_x f_y - z_y f_x)^2 = 1 \tag{5.9}$$

It seems most natural, in using the moment technique to solve any problem, to guess a surface and then compute the currents that should flow on that surface. In other words, assume $z(x,y)$ is known and use equation 5.9 to solve for $f(x,y)$.

Unfortunately, this straightforward way is not tractable. Equation 5.9 as an equation for determining $f(x,y)$ from $z(x,y)$ is non-linear and it appears (from many trials) to be impossible to devise a stable numerical method of solving it. Of course analytical methods can be ruled out from the beginning because of the necessarily complicated functions that must be assumed for $z(x,y)$.

However, there is nothing inherent in the overall method which requires one to begin the process by assuming a surface. If instead a flux function with an appropriate number of parameters is assumed, then equation 5.9 might be used to obtain the surface which satisfies equation 5.3. In fact if $z(x,y)$ is considered to be the unknown function in equation 5.9 it then becomes a linear equation.

$$f_x z_y - f_y z_x = (1-f_x^2 - f_y^2)^{\frac{1}{2}} \tag{5.10}$$

The sign chosen for the square root is the one which is appropriate in the first quadrant.

It turns out that even this linear first order equation seems to be numerically unstable for any straightforward method of solution involving a regularly spaced grid. However, the particular form of the coefficients in this equation make it possible to reduce it to the problem of solving an ordinary differential equation along certain curves. To see why this is so, rewrite equation 5.10 as follows.

$$P(x,y)z_x + Q(x,y)z_y - R(x,y) = 0 \tag{5.11}$$

Referring to equation 5.1 for $\underset{\sim}{n}$, this is clearly equivalent to the equation

$$\underset{\sim}{n}\cdot(P\underset{\sim}{e}_x + Q\underset{\sim}{e}_y + R\underset{\sim}{e}_z) = 0 \tag{5.12}$$

which says that a line with direction numbers (P,Q,R) is perpendicular to the normal to the surface and is therefore tangent to the surface. Thus an infinitesimal line element with direction numbers proprotional to these will lie in the solution surface. Clearly then the differential equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \tag{5.13}$$

determine a line, called an integral curve, which lies entirely in the solution surface if any one point of it lies in the solution surface. Thus we could construct the surface, if we knew the value of $z(x,y)$ along one line which is not an integral curve, by following the integral curves which intersect that line.

The thing which makes this approach feasible in this case is that the integral curves are fairly easy to obtain. Rewriting equation 5.13 explicity,

$$\frac{dx}{-f_y} = \frac{dy}{f_x} = \frac{dz}{(1-f_x^2-f_y^2)^{\frac{1}{2}}} \tag{5.14}$$

it is clear that the first equation takes an especially simple form.

$$f_x dx + f_y dy = df = 0 \tag{5.15}$$

This simply says that along any integral curve of the surface $f(x,y)=f_o$, a constant. Thus in principle for any curve one could write $x=x(y,f_o)$ where f_o is now just a constant parameter. Substitution of this into the second of equations 5.14 gives a simple ordinary differential equation for $z(y,x(y,f_0))$.

In practice it is much better to use the distance s (in the xy plane) along the curve, rather than either x or y, as the independent variable in the solution of equation 5.14. In terms of s, then, equation 5.14 becomes

$$\frac{dz}{ds} = - \left[\frac{1}{f_x^2 + f_y^2} - 1 \right]^{\frac{1}{2}} \tag{5.16}$$

In obtaining this, s has been chosen to increase in the counter-clockwise direction around the upper neutral point.

As pointed out above, determination of the surface uniquely requires specification not only of the flux function $f(x,y)$ but also of one line in the surface. Clearly the best line to use is that part of the intersection of the surface with the $x=0$ plane which lies between the sub-solar point and the upper neutral point. A few of the variable parameters will then be used in specifying the current function.

In passing it may be noted that when only the parameters specifying this line are changed (the flux function remaining unchanged) it is unnecessary to reintegrate equation 5.16 before calculating the new moments. This fact can shorten the computer time required for the problem.

Thus we have a direct method of obtaining a surface and surface current which are consistent with equation 5.3.

6. Calculation of the Moments for the Wind Case.

Consider a source of magnetic field located at the origin and a zero temperature plasma wind moving in the -z direction. Assume that a surface $z(x,y)$ and the flux function $f(x,y)$ for its surface currents are given. In this section the formulae will be derived for the moments of those currents.

The proper unit current density $J_o=(M_t N_o U_o^2/\pi)^{\frac{1}{2}}$ was defined in Section 5. At this time the unit length R_n will be defined to be the distance from either neutral point to the z axis. With the convention that $f=0$ at the subsolar point, that has the double advantage of making both $y=1$ and $f=1$ at the upper neutral point, because $\frac{\partial f}{\partial y}=1$ on the line joining the subsolar point and the upper neutral point (see the first paragraph of Appendix IV).

Let $\underset{\sim}{R}_2=R_n \underset{\sim}{r}_2$ be the coordinates of a field point and $\underset{\sim}{R}=R_n \underset{\sim}{r}$ be the coordinates of a point on the surface. The integral form for the vector potential is:

$$\underset{\sim}{A}(\underset{\sim}{R}_2) = R_n J_o \int \frac{\underset{\sim}{j}(\underset{\sim}{r})dS}{|\underset{\sim}{r}_2-\underset{\sim}{r}|} \tag{6.1}$$

To separate this integral into its moments, make use of the expansion of $1/|\underset{\sim}{r}_2-\underset{\sim}{r}|$, in associated Legendre functions.

$$\frac{1}{|\underset{\sim}{r}_2-\underset{\sim}{r}|} = \sum_{n=o}^{\infty} \sum_{m=o}^{n} (2-\delta_{m0}) \frac{(n-m)!}{(n+m)!} \frac{r^n}{r_2^{n+1}} P_n^m(\cos\theta) P_n^m(\cos\theta_2) \cos m(\phi-\phi_2) \tag{6.2}$$

Upon making this substitution and transferring everything possible through the integral sign, A becomes:

$$\underset{\sim}{A}(\underset{\sim}{R}_2) = R_n J_o \sum_{n=o}^{\infty} \sum_{m=o}^{n} \sum_{p=o}^{1} \underset{\sim}{I}_{nm}^{p} \frac{P_n^m(\cos\theta_2)\cos(m\phi_2 - p\frac{\pi}{2})}{r_2^{n+1}} \tag{6.3}$$

where

$$\underset{\sim}{I}_{nm}^{p} = (2-\delta_{m0})\frac{(n-m)!}{(n+m)!}\int_S \underset{\sim}{j}(\underset{\sim}{r})P_n^m(\cos\theta)\cos(m\phi - p\frac{\pi}{2})r^n dS \tag{6.4}$$

It is clear by comparison of equations 3.5 and 6.3 that the components of the $\underset{\sim}{I}_{nm}^{p}$ are just the vector potential moments defined before.

$$X_{nm}^{p} = \left[I_{nm}^{p}\right]_x \qquad Y_{nm}^{p} = \left[I_{nm}^{p}\right]_y \qquad Z_{nm}^{p} = \left[I_{nm}^{p}\right]_z \tag{6.5}$$

Before proceeding further the source field will be specialized to one which mirrors in the yz plane and mirrors with a change of sign in the xz plane. This is necessary in order for the surface to be symmetric about these two planes and topologically similar to Figure 1. Since the surface current must be perpendicular to the field just inside, it is clearly flowing in the x (or -x) direction as it crosses either of the planes of symmetry, and j_x is an even function about either plane. Visualization of the current flow pattern (with the help of Figure 4) shows that j_y is odd about both planes of symmetry and j_z is even about the xz plane and odd about the yz plane. These symmetries of the surface and current cause three-fourths of the vector moments to vanish identically.

For instance, consider the symmetries about the xz ($\phi=0$) plane. Cos $m\phi$ is an even function of ϕ, while j_y is odd, which causes Y^o_{nm} to vanish, since the rest of the integrand is even. Sin $m\phi$ is an odd function of ϕ, while j_x and j_z are even, which causes X^1_{nm} and Z^1_{nm} to vanish.

Likewise consider the symmetries about the yz($\phi=\frac{\pi}{2}$) plane. About this plane j_z and j_y are odd, so Z^o_{nm} vanishes when m is even (since cos $m\phi$ is then even) and Y^1_{nm} vanishes when m is odd (since then sin $m\phi$ is even). Similarly j_x is even about this plane, which causes X^o_{nm} to vanish when m is odd.

Thus, using the symbol without the superscript to indicate the non-zero moment for that n and m, the only non-zero integrals of equation 6.4 are as follows:

$$\begin{aligned} X_{nm} &= X^o_{nm} && m = 0,2,4 \ldots n \\ Y_{nm} &= Y^1_{nm} && m = 2,4 \ldots\ldots n \qquad n=1,2,3 \ldots \\ Z_{nm} &= Z^o_{nm} && m = 1,3,5 \ldots n \end{aligned} \qquad (6.6)$$

Accordingly, when p=0 or m is even in equations 3.8 <u>all</u> the vector moments vanish, and so the associated scalar moments must vanish. Thus the only non-zero scalar moments are S^1_{nm} (m odd) and these will henceforth be denoted by the symbol S_{nm}. For a properly symmetric source, then, equations 3.8 reduce to:

$$S_{nm} = Z_{nm} - (n-m)(X_{nm+1} + Y_{nm+1}) \qquad Y_{nn} = X_{nn} \qquad (6.7)$$

$$S_{nm} = -Z_{nm} + \frac{(1+\delta_{m1})}{(n+1-m)}(Y_{nm-1} - X_{nm-1}) \quad n=1,2,3\ldots \quad m=1,3,5\ldots n$$

Any values of the vector moments, then, which will zero these scalar moments will necessarily zero the field outside.

Since (as shown in Section 4) z must be calculated by following a flux line on the surface, the integrals of equation 6.4 are most economically calculated by using f as one of the integration coordinates. If the integration were done (for instance) over x and y then every one of the grid points would lie on a different flux line and either each of these lines would have to be followed (consuming a great amount of time) or the points would have to be interpolated from neighboring lines (a difficult procedure introducing its own inaccuracies).

Use as coordinates the flux function f(x,y) and s, the distance (in the xy plane) along the flux line, measured from the line joining the neutral point and the subsolar point. First transform equation 6.4 from an integral over the surface to an integral over the projection of the surface into the xy plane, (i.e. set $dS=dxdy/\cos\psi$).

$$\underset{\sim}{I}^{p}_{nm} = \frac{4(2-\delta_{m0})(n-m)!}{(n+m)!}\int_0^2 dy\int_0^{x_{max}(y)} dx \left[\frac{\underset{\sim}{j}}{\cos\psi}\right] r^n P_n^m\left(\frac{z}{r}\right)\cos\left(m\phi-p\frac{\pi}{2}\right) \qquad (6.8)$$

The fact that the maximum value of y is 2 follows from the considerations in the first paragraph of Appendix IV.

In transforming to the new coordinates (f,s) the elemantal area changes from dxdy to $dfds/|\text{grad } f|$. Referring to equations 5.8, 5.10 and 5.16, one obtains the following identities.

$$\frac{j_x}{\cos\psi|\text{grad } f|} = \frac{f_y}{(f_x^2 + f_y^2)^{\frac{1}{2}}} = \frac{dx}{ds}$$

$$\frac{j_y}{\cos\psi|\text{grad } f|} = \frac{-f_x}{(f_x^2 + f_y^2)^{\frac{1}{2}}} = \frac{dy}{ds} \tag{6.9}$$

$$\frac{j_z}{\cos\psi|\text{grad } f|} = -\left[\frac{1-f_x^2-f_y^2}{f_x^2 + f_y^2}\right]^{\frac{1}{2}} = \frac{dz}{ds}$$

Substituting these identities into equation 6.8, the non-zero moments, equation 6.6, become

$$X_{nm} = 4(2-\delta_{m0})\frac{(n-m)!}{(n+m)!}\int_0^1 df\int_0^{S(f)} ds\left(\frac{dx}{ds}\right)U_{nm}C_m$$

$$Y_{nm} = 8\,\frac{(n-m)!}{(n+m)!}\int_0^1 df\int_0^{S(f)} ds\left(\frac{dy}{ds}\right)U_{nm}S_m \tag{6.10}$$

$$Z_{nm} = 8\,\frac{(n-m)!}{(n+m)!}\int_0^1 df\int_0^{S(f)} ds\left(\frac{dz}{ds}\right)U_{nm}C_m$$

where S(f) is the total length in the first quadrant of the flux curve f=constant, and the U, S and C are defined as follows:

$$U_{nm} = r^n P_n^m\left(\frac{z}{r}\right)(r^2-z^2)^{-\frac{1}{2}m}$$

$$C_m = \cos m\phi\ (r^2-z^2)^{\frac{1}{2}m} \tag{6.11}$$

$$S_m = \sin m\phi\ (r^2-z^2)^{\frac{1}{2}m}$$

In the computer program these functions are easily generated by the following recursion relations:

$$U_{nn}=(2n-1)!! \qquad U_{nn-1}=(2n-1)!!\ z$$

$$U_{nm}=\frac{2(m+1)zU_{nm+1}-(x^2+y^2)U_{nm+2}}{(n-m)(n+m+1)} \tag{6.12}$$

$$C_1=x \qquad S_1=y \qquad C_m=C_{m-1}C_1-S_{m-1}S_1 \qquad S_m=S_{m-1}C_1+C_{m-1}S_1$$

From these relations, it is clear that the $U_{nm}C_m$ and $U_{nm}S_m$ factors of the integrands of equation 6.10 are simply polynomials in x, y and z each term of which is of degree n, The highest degree of z in any of these terms is n-m. Since x and y are bounded while $z\to-\infty$, it is clear that the larger the value of m (for a given n), the more accurately the integral may be evaluated. This leads to the conclusion that the first of equations 6.7 is the better one to use in calculating the scalar moments. Substitution of equations 6.10 into this equation gives the explicit relation.

$$S_{nm}=8\frac{(n-m)!}{(n+m)!}\int_0^1 df\int_0^{S(f)} ds\left[\frac{dz}{ds}U_{nm}C_m-\frac{(1-\delta_{nm})U_{nm+1}}{(n+m+1)}\left(\frac{dx}{ds}C_{m+1}+\frac{dy}{ds}S_{m+1}\right)\right] \tag{6.13}$$

Thus to this point the machinery has been set up for obtaining a surface z(x,y) and calculating all its multipole moments. Before proceeding further it is necessary to specialize to a particular source field.

7. Specific Solution for a Dipole Source.

This section will begin with a summary of all those formulae derived in previous sections which are necessary for programming a computer to obtain a numerical solution. For the case when the source field mirrors in the yz plane and mirrors with a change of sign in the xz plane, the scalar potential of the surface currents is

$$= R_n J_o \sum_{n=1}^{\infty} \sum_{m=1}^{n} S_{nm} D^1_{nm} \qquad \text{(m odd only)} \qquad (7.1)$$

where $J_o = (M_t N_o U_o^2/\pi)^{\frac{1}{2}}$, R_n is the y coordinate (in centimeters) of the neutral point and the S_{nm} are obtained from equation 6.13.

$$S_{nm} = 8\left\{\frac{(n-m)!}{(n+m)!}\right\}\int_0^1 df \int_0^{S(f)} ds \left[U_{nm}\frac{dz}{ds}C_m - U_{nm+1}\frac{(1-\delta_{nm})}{(n+m+1)}\left(\frac{dx}{ds}C_{m+1} + \frac{dy}{ds}S_{m+1}\right)\right] \qquad (7.2)$$

The coordinates (f,s) are the value of the current function and the distance in the xy plane along the lines f= constant, measured from x=0. $S(f_o)$ is the length in the first quadrant of the line $f=f_o$. Equations 6.9 give the derivatives of the coordinates with respect to s.

$$\frac{dx}{ds} = \frac{f_y}{|\text{grad } f|}$$

$$\frac{dy}{ds} = \frac{-f_x}{|\text{grad } f|} \qquad (7.3)$$

$$\frac{dz}{ds} = -\left[\frac{1}{|\text{grad } f|^2} - 1\right]^{\frac{1}{2}}$$

And z at any point on a curve f=constant is found by integrating dz/ds along that curve. Finally the functions U, C and S are given by equation 6.12.

$$U_{nn}=(2n-1)!! \qquad U_{nn-1}=(2n-1)!!\,z$$

$$U_{nm}=\frac{2(m+1)zU_{nm+1}-(x^2+y^2)U_{nm+2}}{(n-m)(n+m+1)} \tag{7.4}$$

$$C_1=x \qquad S_1=y \qquad C_m=C_{m-1}C_1-S_{n-1}S_. \qquad S_m=S_{m-1}C_1+C_{m-1}S_1$$

Consider now a dipole source. The scalar potential of a dipole of moment $M\underset{\sim}{e}_y$ is

$$=\frac{M\sin\theta\sin\phi}{R_n^2r^2}=\frac{M}{R_n^2}D_{11}^1 \tag{7.5}$$

Since the potential of equation 7.1 must be equal and opposite to this, it is clear that for the true surface

$$S_{nm}=0 \qquad n=2,3,4\ldots \qquad m=1,3,5\ldots n \tag{7.6}$$

and equating coefficients of the D_{11}^1 terms gives the scaling relation

$$R_n^3=\frac{M}{S_{11}}\left[\frac{\pi}{M_tN_oU_o^2}\right]^{\frac{1}{2}} \tag{7.7}$$

which will be used to determine R_n after the surface has been made to satisfy equation 7.6 approximately.

The first step in the solution of the problem is the choice of a function of x, y and some parameters A_i, which is sufficiently restricted in functional form that for any reasonable values of the A_i the resulting function $f(x,y)$ has all the qualitative features that the current function must have (as seen from Figure 4). However, at the same time the parameters must permit enough variability in f to bring it sufficiently close to the true function for some set of values of the A_i. Actually the choice of a parametrized form for this function was one of the most time consuming aspects of the entire problem, and it is not here pretended that the best possible function has been developed, only that a satisfactory one has. If any investigator should desire in the future to improve on the results presented in this paper, he could surely do so by working out a different analytic form for f which has the ability to come closer to the true f, whatever that is.

Without further apology then, the current function used in this work will be of the following form:

$$f(\rho,\phi)=a(v)\sin\phi e^{g(u,v)}\left[1-\sqrt{(1-u)^2+h(u,v)}\right] \qquad (7.8)$$

where (ρ,ϕ) are the usual polar coordinates in the xy plane; $v=\cos^2\phi$; $a(v)$ is half the radius of the surface at $z=-\infty$; $u=\rho/a(v)$; and g and h are double power series in u and v, given by equations IV-39 and IV-54 to IV-56. The motivations leading to this form for f, as well as the conditions on g and h and their derivations, are discussed in Appendix IV and will not be considered here, except to note that permitting $h(u,v)$ to contain terms up to u^8 and y^5 allows 22 free parameters among the coefficients after all the conditions are applied. These parameters are denoted by A_i, $1 \leq i \leq 22$. Likewise it is shown in Appendix IV that if $g(u,v)$ contains only terms up to u^4 and v^4 but is otherwise as unrestricted as possible (consistent with the conditions on f) it contains 15 free parameters among its coefficients. These are denoted by A_i, $31 \leq i \leq 45$. The remaining arbitrary function in equation 7.8 will be parametrized as follows:

$$a(v) = 1+A_{61}v+v(1-v)\left[A_{62}+(2v-1)A_{63}+(2v-1)^2A_{64}\right] \qquad (7.9)$$

As pointed out on page 35 this flux function does not uniquely specify an associated surface, but the profile of the surface must also be specified. The profile will be defined to be that part of the cross section of the surface in the meridian plane which lies between the subsolar point and the neutral point. This profile will be parametrized as follows:

$$\begin{aligned} z(y) = A_{71}-A_{72}\Big[&y^2+y^2(y^2-1)(A_{73}+A_{74}y^2) \\ &+A_{72}A_{75}\left[\sqrt{1-y^2}-1+\tfrac{1}{2}y^2\left[1+\tfrac{1}{4}y^2\left(1+\tfrac{1}{2}y^2(1+5y^2)\right)\right]\right]\Big] \end{aligned} \qquad (7.10)$$

The distance from the dipole to the subsolar point of the surface is given by A_{71}. The z distance from the subsolar point to the neutral point is given by A_{72}. A_{75} governs the plateau in the immediate neighborhood of the neutral point. The remaining terms aid in adjusting the overall shape properly.

As already mentioned most of the qualitative restrictions which can be placed on f as a consequence of the physics of the problem have been incorporated automatically by the restrictions placed on h and g in Appendix IV. However, there is one very important restriction which can not be so easily fulfilled. This is the condition, obvious from equation 5.9, that

$$|\nabla f| \leq 1 \tag{7.11}$$

Clearly every parameter will affect the gradient of f in a way which will depend non-linearly on every other parameter. Thus it would be impossible to derive a set of reasonable restrictions which would guarantee that equation 7.11 is satisfied. The best that can be done is to test each trial set os parameters against equation 7.11 and reject those sets which violate it significantly. In practice it was found that it was difficult to find a set of parameters which didn't violate this condition at some point in the xy plane, even when the shape and moments of the resulting surface were ignored. Thus it was decided to tolerate gradients greater than one as long as they occurred over only a small percentage of the surface; and in these cases equation 5.16 was kept from becoming imaginary by the simple expedient of setting the gradient equal to one. This complicated the convergence process in that constant manual adjustments were needed in the parameters to minimize these unphysical gradients, but it couldn't be avoided.

The obvious way to go about reducing the moments is by the generalized Newton method (used in the uniform pressure case). As everyone knows who has used the method extensively, however, it is very prone to wandering when the number of variables exceeds 5 or 6, unless the problem is a well conditioned one. From what has been said already, though, it should be obvious that this problem is not a well conditioned one and, indeed, it was found that Newton's method was virtually useless for as few as five parameters and moments. One thing which contributes heavily to this difficulty is that there is no natural ordering of the parameters as to importance. That is to say: with 46 parameters occurring in four different power series (two of which are double series); which five parameters should be chosen to reduce the first five moments? In all likelihood some 15 or so of these parameters should really be varied in order to reduce the first five moments smoothly to zero.

Therefore since it was unrealistic to work with less than about 15 parameters at a time, but even more unrealistic to try to reduce 15 moments at a time by Newton's method, it was necessary to work out a new method by which N parameters (A_k) could be used to reduce M quantities(V_i), where $M<N$. That is, the following equations for the changes a_i in the parameters A_i must be solved:

$$\sum_{k=1}^{N} H_{ik}a_k=-V_i \qquad i=1,2...M \tag{7.12}$$

where $H_{ki} = \partial V_k/\partial A_i$ is assumed to be a constant. Since $N>M$ this system of equations does not have a unique solution unless an additional condition is imposed. The natural condition is to require that

$$E = \sum_{k=1}^{N} (a_k/w_k)^2 \tag{7.13}$$

be a minimum, where w_k are approximately chosen weighting factors for the parameters. There are two advantages to thus minimizing the length of the a_i vector: 1) the assumption of the constancy of the H_{ki} is more valid, and 2) the conditions such as equation 7.11 which have been manually optomized will be interfered with as little as possible.

To solve equations 7.12 and 7.13 together, first solve equation 7.12 for the first M of the a_i in terms of the remaining a_i.

$$a_i = -\sum_{j=1}^{M} H^{-1}_{ij}\left[V_j + \sum_{k=M+1}^{N} H_{jk}a_k\right] \qquad i=1,2\ldots M \tag{7.14}$$

where H^{-1} is the inverse of the square matrix formed from only the first M columns of H. These expressions can now be inserted into equation 7.13 to give E in terms of only the last N-M of the a_k. It is then a straightforward matter to differentiate the resulting E with respect to each of the a_k. Setting these derivatives equal to zero (the condition for a minimum) gives (N-M) linear equations for the (N-M) desired a_k.

$$\frac{a_k}{w_k^2} + \sum_{n=M+1}^{N} \left[\sum_{i=1}^{M} P_{ki} H_{in} \right] a_n = - \sum_{i=1}^{M} P_{ki} V_i \qquad k=M+1, \cdots N \quad (7.15)$$

where
$$P_{ki} = \sum_{j=1}^{M} \sum_{m=1}^{M} H_{jm}^{-1} H_{mk} H_{ji}^{-1}/w_j^2 \quad (7.16)$$

After these equations are solved for the last N-M of the a_k, these values may be substituted into equation 7.14 to obtain the first M of the a_i. While it may appear that M of the a_i are treated essentially differently than the remaining, it is clear that the result does not depend on how the a_i are apportioned into the two groups, because the basic equations 7.12 and 7.13 completely determine the nature of the solution and they are completely symmetric in the a_i.

As expected it was found in practice that this method was very much more stable than Newton's method, which simply amounts to a special case of equation 7.14 with M=N (which eliminates the second term).

Even with this improved method of convergence, however, it was found that it was unadvisable to try to "zero" more than the first 5 to 8 moments (n=4 or 5) by this method. Experience with the uniform pressure problem on the other hand indicated that it would be necessary to at least reduce considerably the moments up to about n=7 in order to achieve much accuracy in the surface. Thus as it finally worked out the convergence process itself became semi-manual. That is between each cycle, in which the moments up to n=4 or 5 were "zeroed" by the above technique, it was necessary to study the H_{ij} for the n=6 and n=7 moments, as well as the gradients of the current function, in an attempt to vary the parameters in such a way as to reduce these moments and the excessive gradients. The process is a laborious and difficult one, but with some skill is a convergent one.

All of the numerical calculations were carried out on an IBM 7090 and the final version of the program for these calculations is given in Appendix V together with an explanation of the program and flow diagrams of the major subroutines. Therefore, it is unnecessary to go into that in any detail here except to mention one fact which is significant in the interpretation of the results. Since the purpose is to zero all the moments (except the dipole moment) it would not in principle affect things if all the moments were multiplied by arbitrary finite factors. However, having accepted our inability to actually zero all the moments, and desiring rather to reduce them all to some common low level, it becomes significant what factors the moments are multiplied by as this will affect their relative reduction. The thing which finally governed the choice of the proper factor was the accuracy with which the various moments could be calculated. It was found that if the factor $(2n-1)!!$ is dropped from the definition of U_{nm}, and the factor $(n-m)!/(n+m)!$ in equation 7.2 is replaced by $1/n!!$, then all the calculated moments will have about the same number of decimal places of accuracy before truncation error sets in. Also this change of factor clearly deemphasizes the higher moments as rightfully they should be.

The final solution (that is, the solution beyond which further improvement was judged too difficult to be worth while) is illustrated pictorially in Figure 1 and topographically in Figure 7, and projections of its current lines are given in Figures 4, 5 and 6, which likewise give silhouettes of the surface. Table 3 gives the values of the various parameters for this surface, and Table 4 gives the calculated values of the moments up to m=7. The integrations were done using 30 curves and a basic interval size of 0.07 (see Appendix V).

It should be noted that the surface plotted in these figures is not exactly the one calculated, though it differs from it only slightly. First of all, over about 3.6% of the projected area of the surface in the xy plane (mostly near the subsolar point) the gradient of f exceeded one. These regions then were considered by the computer to be perpendicular to the wind, but in plotting them I smoothed them out to conform to the slope of neighboring regions. The second change consisted of smoothing out the surface in the region near the dipole-sun meridian plane above the neutral point. There were local oscillations of the surface there resulting probably from a defective current function. The extent of these corrections on the cross sections in the two planes of symmetry is shown in Figure 8, and an indication of their effect on the surface as a whole is given in Figure 7.

TABLE 3. Parameters for the solution surface.

Parameters defining the asymptotic cross section.

A_{61}=-0.0154 A_{62}= 0.0700 A_{63}= 0.0200 A_{64}= 0.0600

Parameters defining the moridian plane profile.

A_{71}= 1.0166 A_{72}= 0.7480 A_{73}= 0.3370 A_{74}= 0.1970
A_{75}= 0.0300

Non-zero parameters in g(u,v).

A_{34}= 0.3000 A_{36}= 0.7500 A_{40}=-0.1400 A_{45}= 0.7200
A_{35}= 0.1000 A_{38}= 0.2000 A_{43}= 0.0900

Non-zero parameters in h(u,v).

A_1 = 1.5388	A_6 =-0.0737	A_{11}= 1.0400	A_{16}=-0.0136
A_2 = 0.0277	A_7 =-0.7844	A_{12}=0.0720	A_{17}= 0.0230
A_3 =-1.6113	A_8 = 0.3817	A_{13}= 0.5470	A_{18}= 1.7700
A_4 =-0.2435	A_9 =-0.0076	A_{14}= 1.1320	A_{19}= 1.3230
A_5 =-0.0184	A_{10}= 0.1060	A_{15}=-0.0243	A_{20}= 0.0540
			A_{21}= 0.0300

TABLE 4. Residual moments for the solution surface.

n	m	Moment	n	m	Moment	n	m	Moment
2	1	-0.00003	5	1	0.00174	6	5	0.00389
3	1	-0.00004	5	3	-0.00198	7	1	0.00026
3	3	0.00002	5	5	-0.00097	7	3	0.00009
4	1	-0.00008	6	1	-0.00064	7	5	0.00163
4	3	0.00003	6	3	-0.00026	7	7	-0.00077

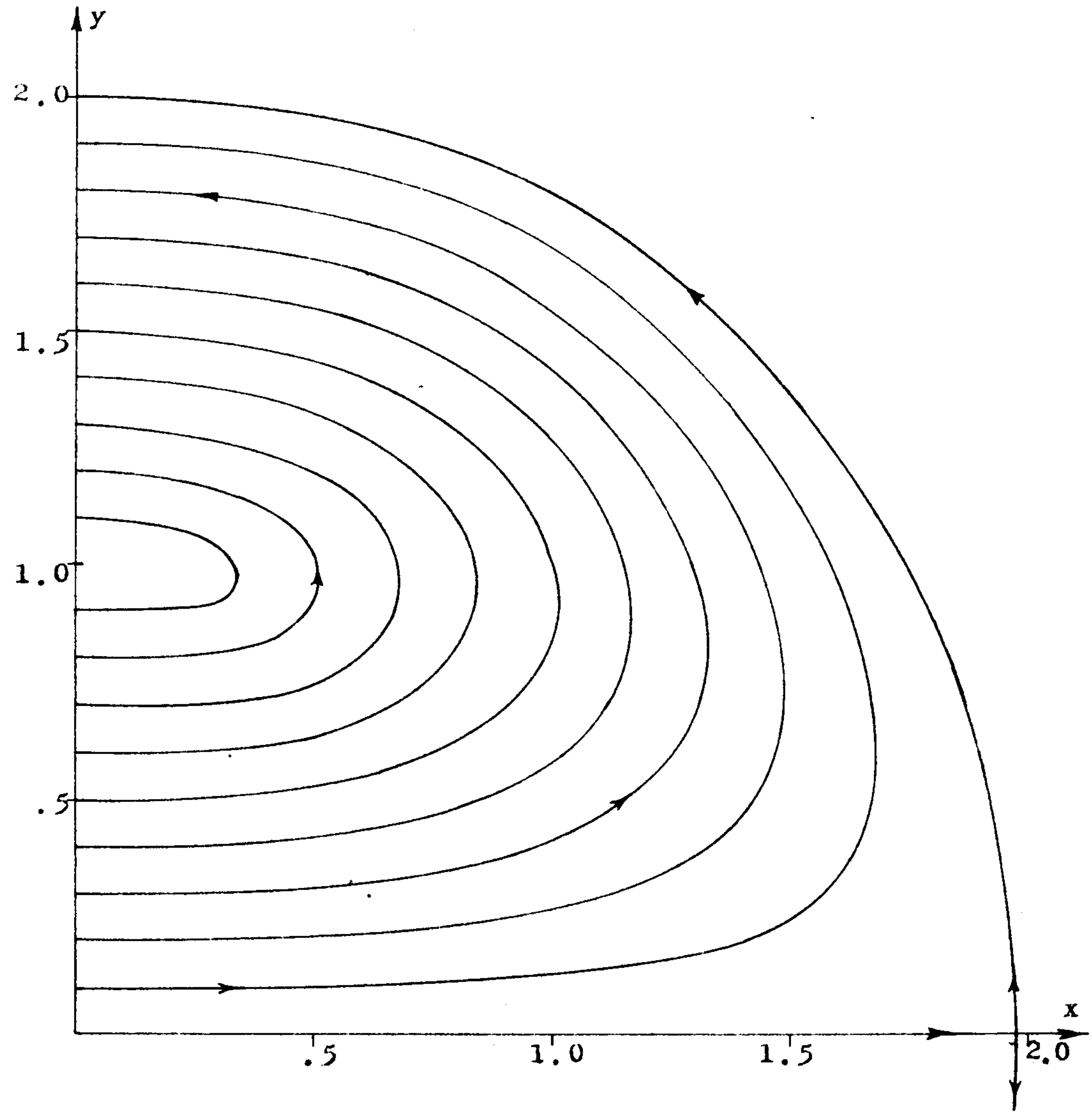


Figure 4 Front view of surface showing current lines. Units of $R_n = 0.680(M_t N_o U_o^2)^{-1/6} R_e$ (earth radii). For a plasma of 2.5 protons/cc & velocity 500 Km/sec $R_n = 9.16 R_e$.

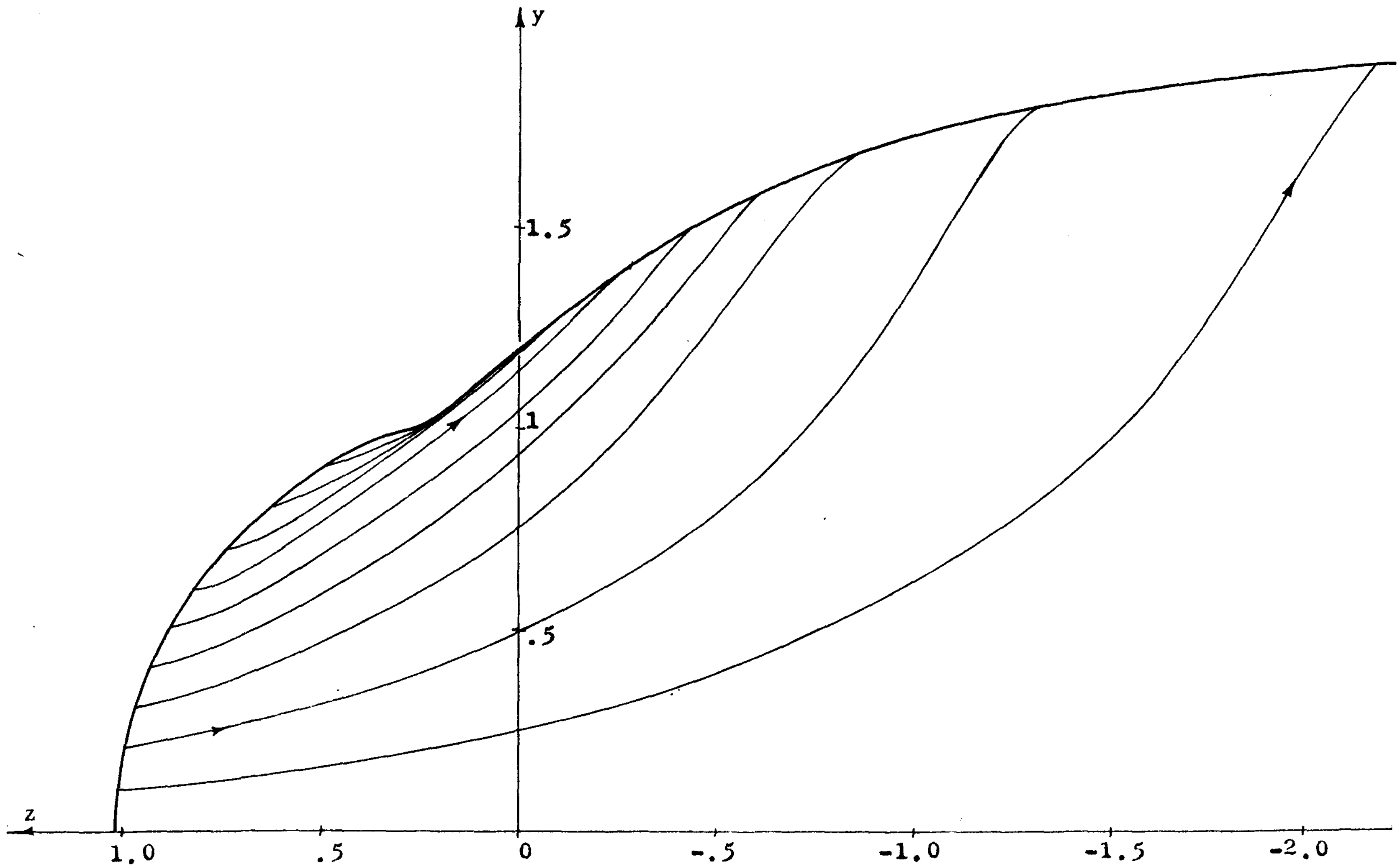


Figure 5 Side view of surface showing current lines. Units of R_n (see Figure 4).

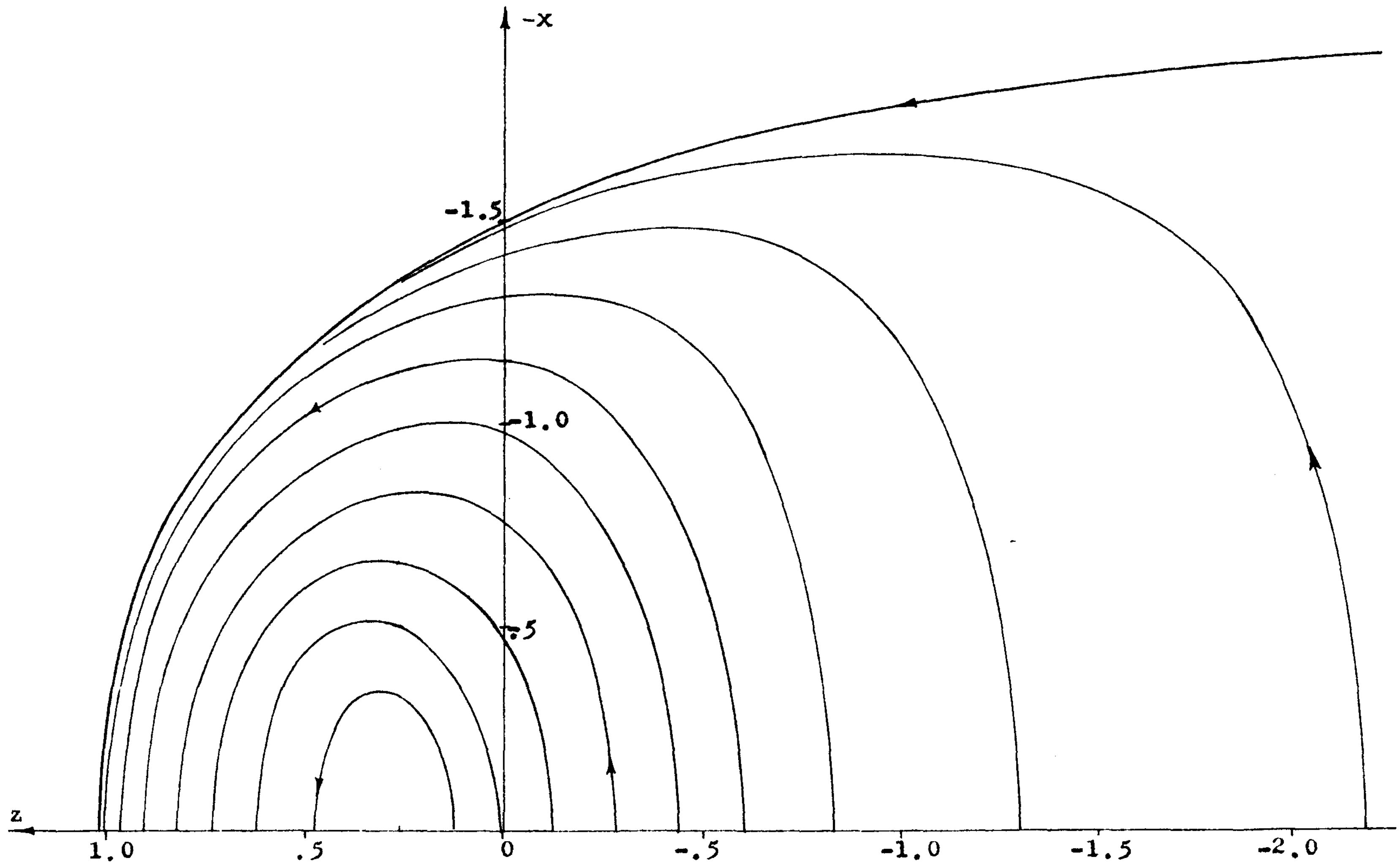


Figure 6 Top view of surface showing current lines. Units of R_n (see Figure 4).

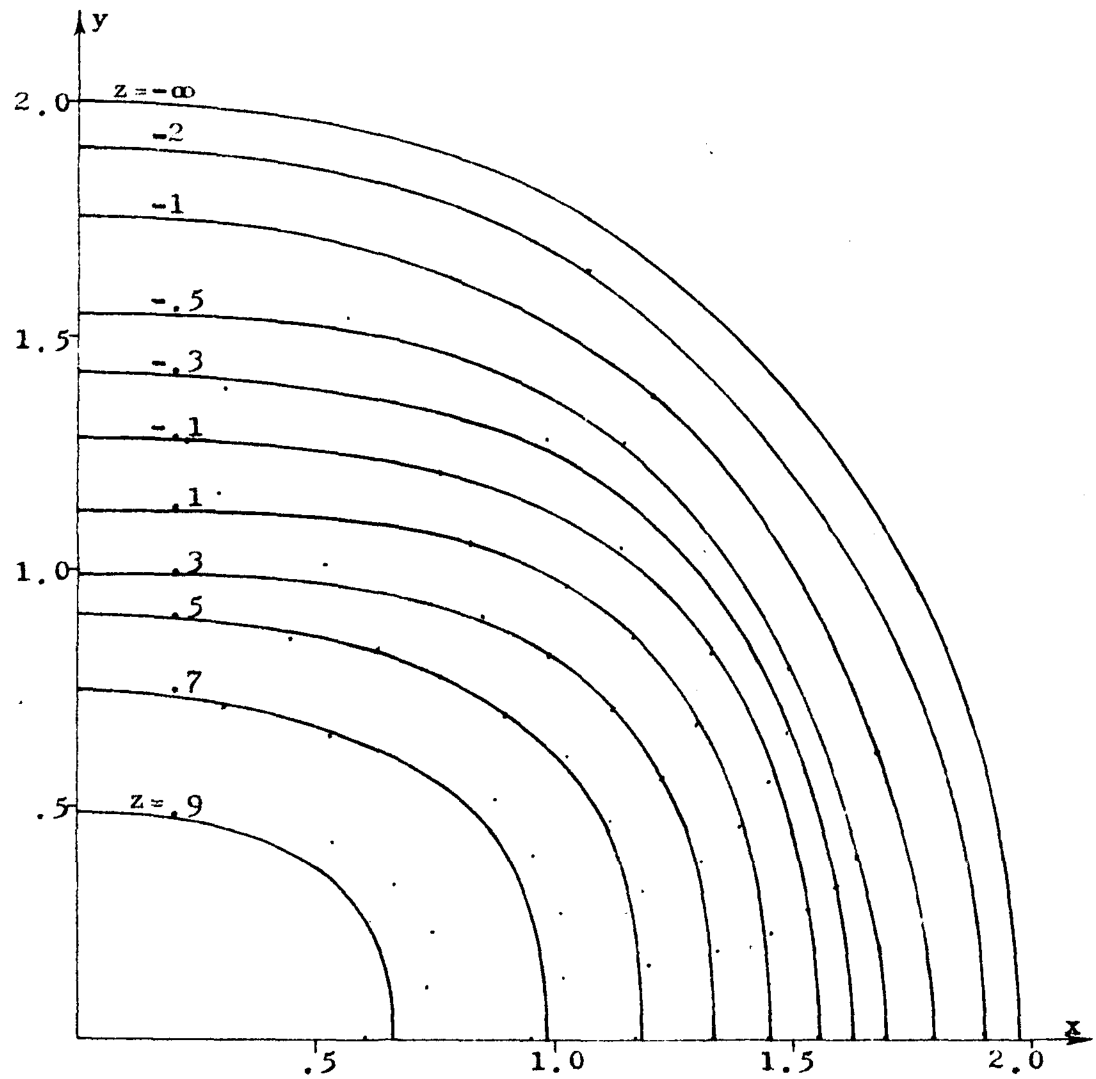


Figure 7 Contours of constant z.
Units of R_n (see Figure 4).

Dots show calculated points, indicating extent to which surface was modified by smoothing.

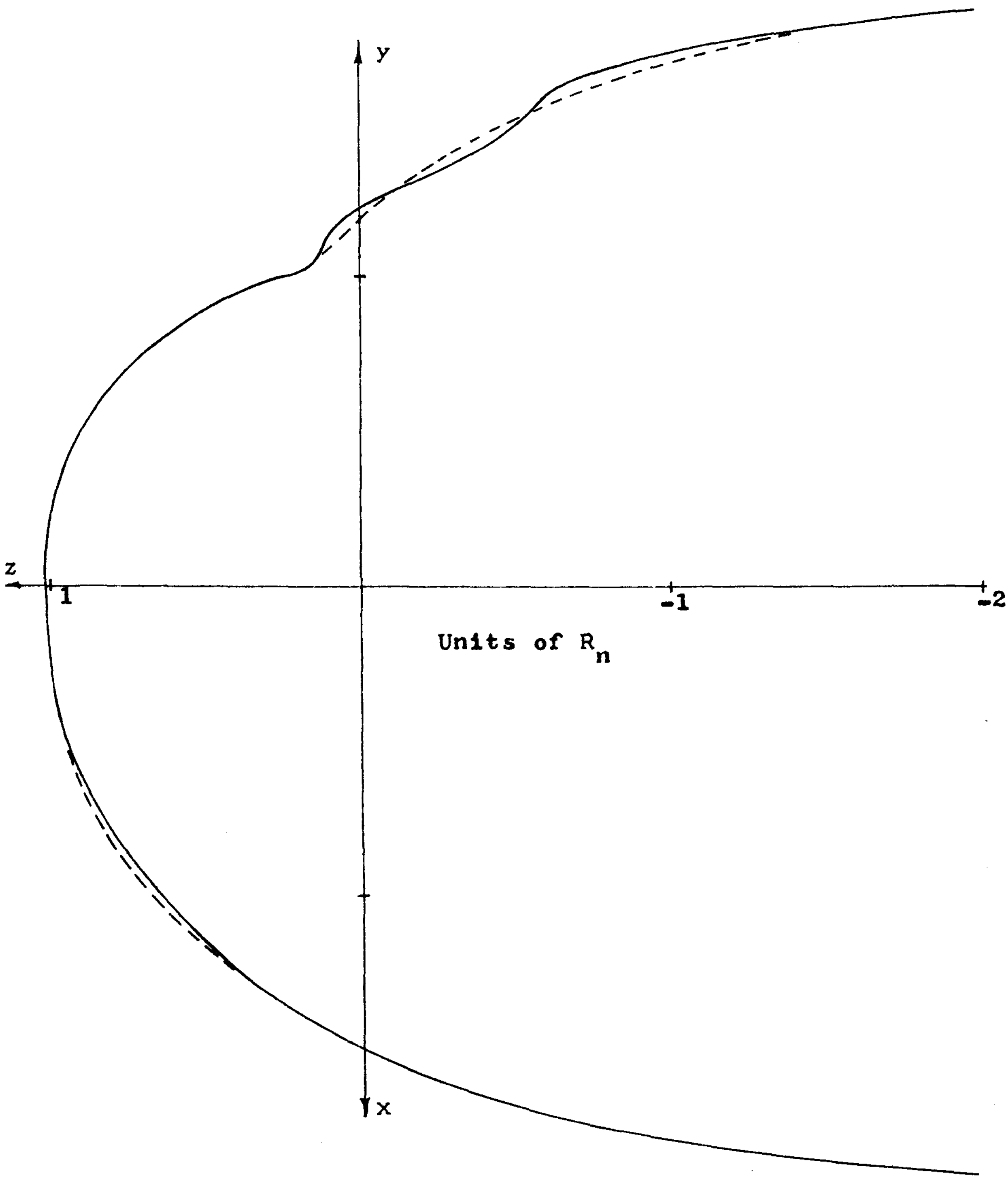


Figure 8 Cross sections (in yz and xz planes) of actual surface generated by parameters of Table 3 (solid lines) showing the alterations (dotted lines) made in smoothing.

8. <u>Results & Conclusions</u>.

The solution of the problem of a magnetic dipole in a cold field-free plasma wind, as obtained in Section 7, is illustrated in Figures 1, 4, 5, 6 and 7.

To relate this to the geomagnetic case, take the dipole moment M to be .311 gauss-(earth radii)^3 and the plasma to be ionized hydrogen. Then, using the computed moment $S_{11}=-7.0030$, equation 7.7 becomes

$$R_n = 84.8\ (N_o U_o^2)^{-1/6} \quad \text{earth radii} \qquad (8.1)$$

where N_o is in proton/cc and U_o is in Km/sec. Mariner II data (6) suggests $N_o=2.5$ and $U_o=500$, which gives $R_n=9.16$, or in other words 9.3 earth radii out to the subsolar point. This is entirely consistent with the experimental values (34).

Since the moment technique is the first approximate method of solution for this problem which also specifies the surface currents, it is the first which can be used to calculate the magnetic field everywhere. Appendix II develops the integrals necessary to calculate the field $\underset{\sim}{B}$ in the two planes of symmetry where B_x vanishes.

These integrals have been evaluated numerically at various points, for the surface calculated in Section 7, and plots have been made of the resulting magnetic fields. The heavy lines in Figure 9 show some representative magnetic field lines in the meridian plane and the lighter lines in that figure and in Figure 10, which shows the equatorial plane, show contours of constant field strength. The dotted lines in each figure give the contours of constant field strength for the unperturbed dipole field. Of course for the exact solution the field strength outside the surface should be zero everywhere, so the field strengths which were calculated outside the surface in these two figures give some idea of the accuracy of the indicated surface. To translate the relative field strengths multiply them by the factor $J_o = (M_t N_o U_o^2/\pi)^{\frac{1}{2}}$, which equals 5.77 γ ($1\gamma = 10^{-5}$ gauss) for a 500 Km/sec wind with 2.5 protons/cc.

For field strengths greater than about 64 the contours do not depart from the original dipole contours sufficiently to show the difference. The field near the origin $\underset{\sim}{B} = J_o B(x,y,z)\underset{\sim}{e}_y$ is approximately:

$$
\begin{aligned}
B(x,0,0) &= 4.30-0.80x^2 \\
B(0,y,0) &= 4.30+2.17y^2 \qquad (8.2) \\
B(0,0,z) &= 4.30+3.32z
\end{aligned}
$$

Thus the compression of the magnetosphere (again using Mariner II data) increases the earth's field at the equator by 26.9 γ at noon and 22.8 γ at midnight and decreases it at the pole by 25.0 γ.

Before concluding this discussion of the field a few remarks concerning the topology of the field are in order.

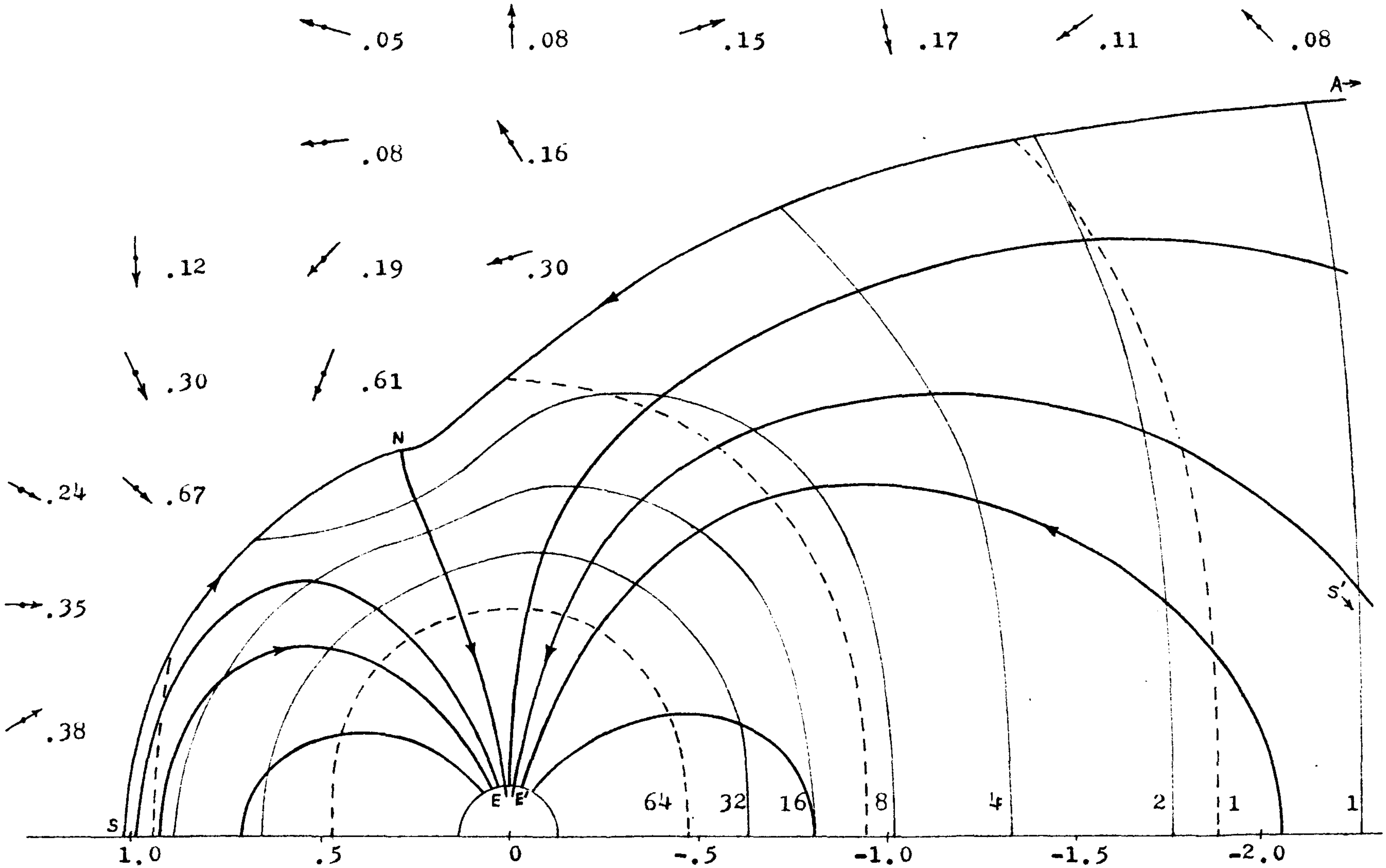


Figure 9 Field lines & magnitude contours in the "meridian" plane, units of $(M_t N_o U_o^2/\pi)^{\frac{1}{2}}$. Dashed contours are for the dipole alone. The computed field magnitudes and directions at several points outside the surface are included to indicate the accuracy of the solution.

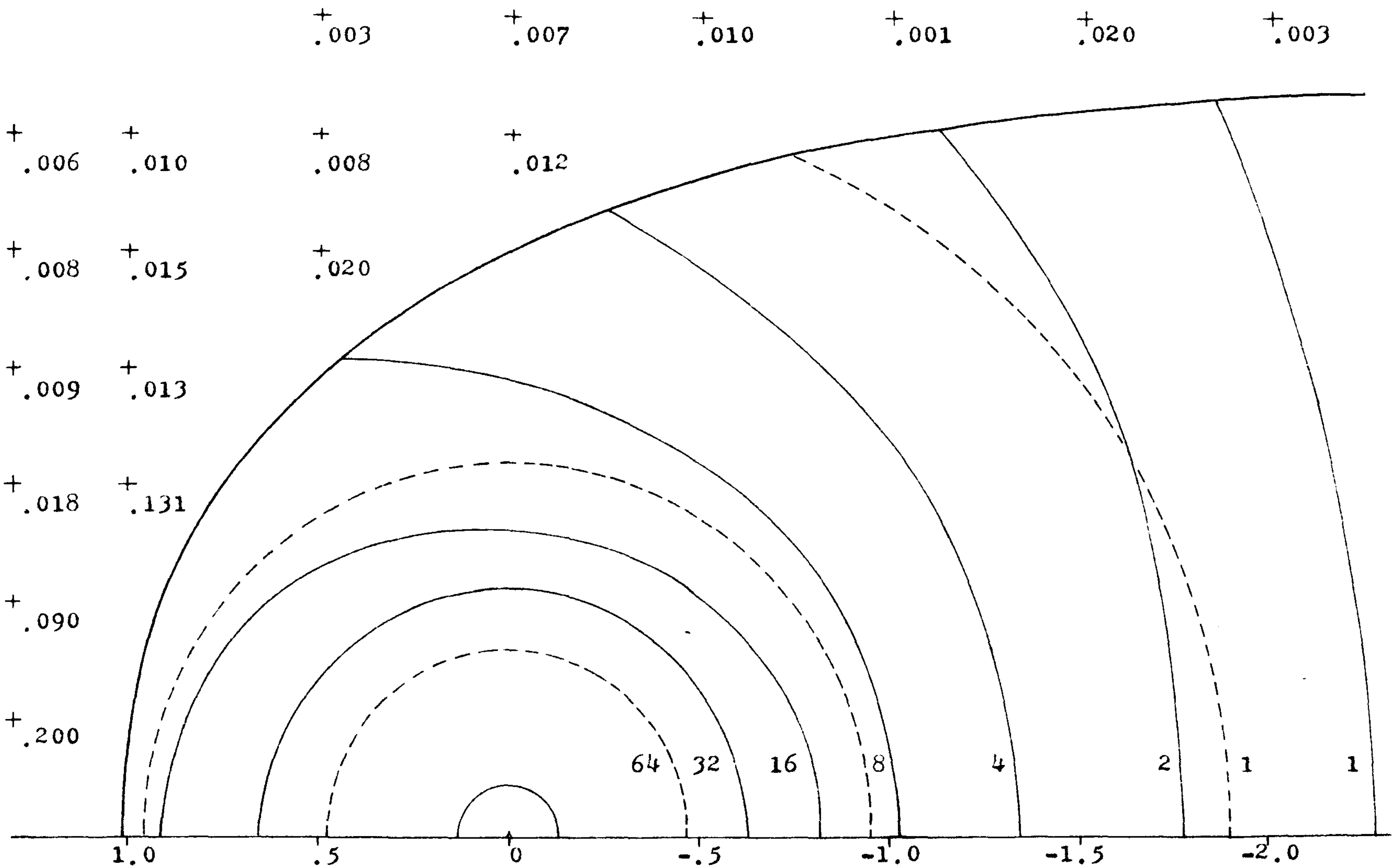


Figure 10 Field strength contours in the equatorial plane, units of $(M_t N_o U_o^2/\pi)^{\frac{1}{2}}$. Dashed contours are for the dipole alone. The computed field strengths at several points outside the surface are included to indicate the accuracy of the solution.

As pointed out by Johnson (10) the field lines divide into two essentially different groups: those that co-rotate with the earth and those that always extend into the tail of the cavity. To see why this must be so consider the line which passes through the neutral point N (see Figure 9); it fans out at that point over the entire surface and in particular passes through the subsolar point S and the antisolar point A at $z=-\infty$. This line intersects the earth at some point E on the noon meridian. Since the earth is rotating, however, the line which intersects at the particular point E can be the neutral line for only an instant, and twelve hours later must intersect the earth at the point labeled E' and make a simple loop in the tail of the cavity, intersecting the equatorial plane at S'. The family of all such lines which pass through N at some instant each 24 hours form an envelope which divides the lines into two groups: 1) those which intersect the earth at a latitude lower than E and therefore pass through the region outlined by SNE once each 24 hours, and 2) those which intersect the earth nearer the pole than E and therefore can pass through the meridian plane only in the region outlined by S'E'ENA, Topologically the two regions occupied by these two groups of lines form interlocking tori (donuts). The field lines of the first group rotate rigidly with the earth, but the second group is confined to the tail of the cavity and therefore rotates instead about its own centerline. This type of motion is referred to by Dungey (21) as twiddling. In reality, of course, the earth's axis of rotation does not coincide with the dipole axis, and neither are perpendicular to the wind direction, as in the present idealized case, but this does not qualitatively change the picture.

Since Beard's approximate boundary condition is the only other method described in the literature for obtaining a solution to this problem, it is naturally of interest to compare the two solutions.

Figure 11 gives half the equatorial cross section (below the z axis) as given in the original article by Beard (15), and half the meridian cross section (above the z axis) as given in a later treatment by Spreiter and Briggs (16). The Spreiter and Briggs section was used because Beard gives only a hand drawn guess of the night side shape in the meridian plane in his original article. The dashed lines in Figure 11 represent the corresponding cross sections of the surface calculated by the moment technique.

Figure 11 is plotted in units of R_n so the height of the neutral point coincides (by definition) for the two cases, but in order for Beard's solution to correspond to the same plasma momentum flux density it is necessary to choose an approximate value for f (defined to be the fraction of the field just inside the surface which is

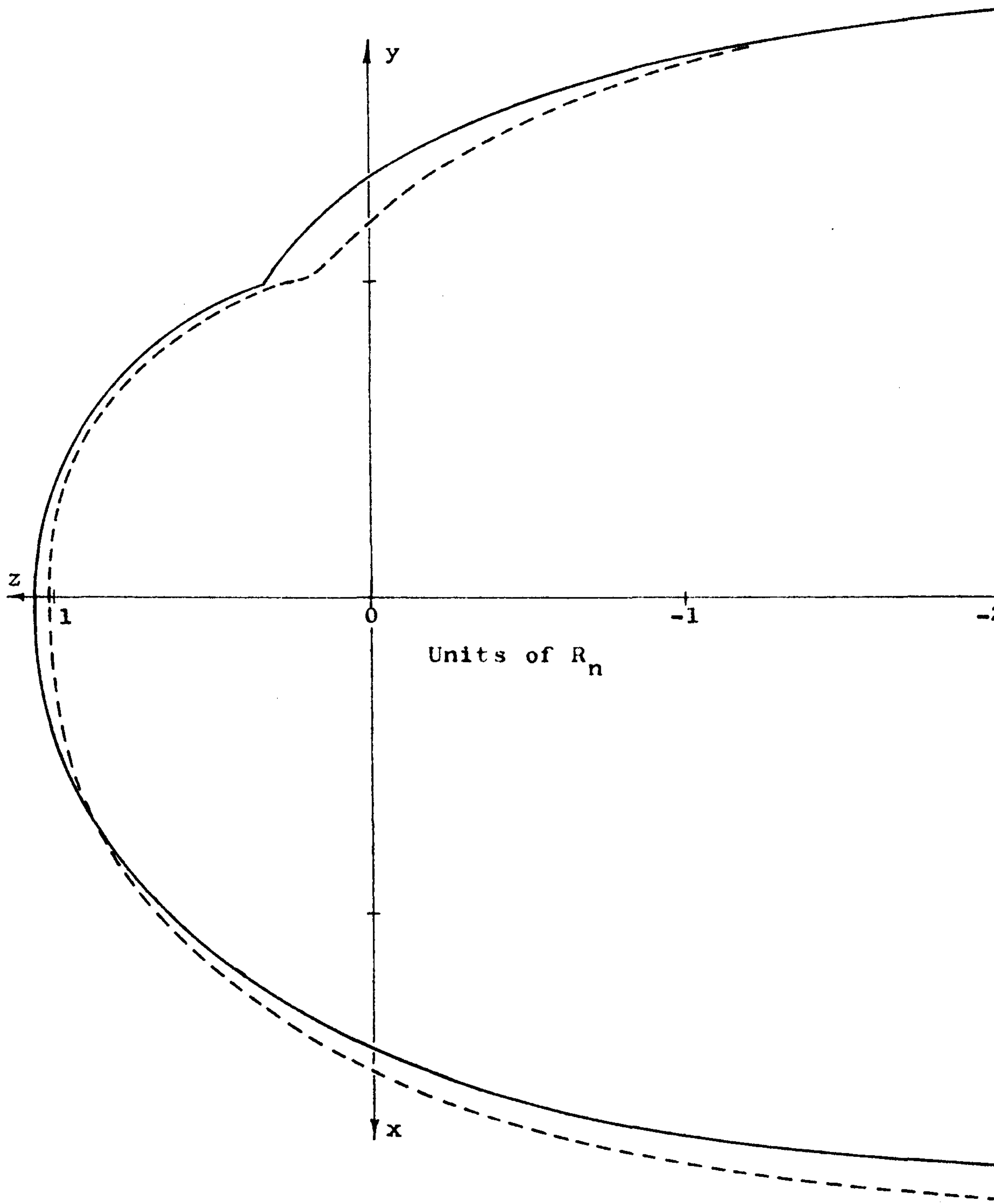


Figure 11 Cross sections of Beard's surface and the moment surface (dashed) in the equatorial plane (below line) and the meridian plane (above line).

contributed by the dipole). The subsolar point for Beard's surface is at 1.058 R_n, and the earth's field at that point is

$$\frac{M}{(1.058)^3 R_n^3} = \frac{7.003}{(1.058)^3}\left[\frac{M_t N_o U_o^2}{\pi}\right]^{\frac{1}{2}} = f(16\pi M_t N_o U_o^2)^{\frac{1}{2}} \qquad (8.3)$$

where equation 7.7 with $S_{11}=-7.0030$ is used to obtain the center expression. Solving this for f gives $f=0.4714$. For comparison, the corresponding f for the moment solution (obtained by using 1.0166 rather than 1.058) is $f=0.5303$.

In a later article (36) Beard refined his calculations by taking into account some of the surface current. He indicates that this makes the cross section between the subsolar point and the neutral point slightly elliptical, decreasing the radius to the subsolar point by .8% and increasing the height of the neutral point about 3%. This makes the shape (with proper choice of f) more nearly the same as the dashed curve in this region, but Beard has not yet extended his second approximation to any other parts of the surface.

It was not possible to compare fields outside as a test of the relative accuracy (as was done for the uniform pressure case) because the full three dimensional solution by Beard's technique has not been published. Neither does Beard's method yield the surface currents, and these are necessary to calculate the fields.

It is unfortunate that the complications encountered in this problem made it impossible to achieve the sort of accuracy obtained in the uniform pressure problem, but more accuracy in the calculations is probably not justifiable anyway considering the inaccuracy in the model. In addition to all the possible objections mentioned in Section 2, there is one effect which makes the pressure law of equation 3.3 inaccurate even if the plasma were truly collisionless, field free, stable and therefore free of any shock transitions. This is the fact that a particle which glances off the surface just below the neutral point will be traveling at such an angle that it may glance off the surface again just above the neutral point. Thus the pressure in this region above the neutral point would exceed that given by equation 2.2.

In conclusion, the moment technique is in principle a completely general approach for determination of the surface of separation between a perfectly conducting plasma and a magnetic field. However, in practice it can entail almost prohibitive difficulty except in cases of considerable symmetry, such as the dipole in a uniform pressure plasma. An example of another problem of like symmetry for which the moment technique should be useful is that of a gravitating plasma cloud surrounded by a magnetic field which is uniform at infinity.

APPENDIX I Determination of the Surface Thickness

Consider a cold plasma flowing in the X direction (with velocity U_o at $X=-\infty$) from a field free region into a region of magnetic field $\underset{\sim}{B}=B(X)\underset{\sim}{e}_y$. Since a steady state solution is desired, the electric field must be able to be expressed as the gradient of a scalar $-\Phi$. Further, since nothing varies in the Y or Z directions, all quantities are functions only of X. Clearly the trajectories described by the particles will be symmetrical with respect to their ingoing and outgoing sections, so we need consider explicitly only the ingoing particles. Let the velocity of these particles be

$$\underset{\sim}{V}_p = U_p(X)\underset{\sim}{e}_x + W_p(X)\underset{\sim}{e}_z \qquad \text{(I-1)}$$

where $p=e$ for the electrons, or $p=i$ for the ions. The Y component of velocity does not enter the problem and so may be assumed zero without loss of generality. Further we will assume normal incidence, i.e. $W_p(-\infty)=0$.

Due to the absence of thermal motions, all particles of the same sign must penetrate to the same value of X, and so the flux of particles must be independent of X for all X less than this maximum X. Our last assumption concerning the boundary conditions on the problem will be that the velocity of the protons and electrons are equal at $-\infty$, as are the densities. Therefore we may write

$$N_i(X)U_i(X) = N_o(X)U_o(X) = N_oU_o \qquad \text{(I-2)}$$

where $N_o = N_i(-\infty) = N_e(-\infty)$

The equations of motion for the particles are

$$M_p d\underset{\sim}{V}_p/dT = q_p(-\nabla\Phi + \underset{\sim}{V}_p \times \underset{\sim}{B}) \qquad \text{(I-3)}$$

and the Maxwell's equation relating the field and current becomes:

$$\frac{dB}{dX} = 8\pi \sum_p q_p N_p W_p \qquad \text{(I-4)}$$

The extra factor of two has been inserted here because both the ingoing and outgoing particles contributed equally to the current in the z direction, but N_p will be used to refer to the particle density of the ingoing stream only. If we were now to impose the remaining Maxwell's equation, $d^2\Phi/dX^2 = 4\pi e c^2(N_i - N_e)$, we would have an exact set of equations for the system. However this system of equations would be too difficult to solve. The system of equations that results if this condition is replaced by the approximate relation

$$N_e(x) = N_i(x) = N(x) \qquad \text{(I-5)}$$

is very much simpler to solve. This approximation is certainly a good one, for the ratio of the Debye shielding length to the gyroradius for electrons, $0.12\ B\ N_e^{-\frac{1}{2}}$ (emu), is small in the cases of interest here. In fact, the solution of this approximate set of equations is probably more meaningful physically than the solution of the exact but idealized (no thermal motions) set of equations.

In order to put the remaining equations in dimensionless form, define a set of units in terms of N_o, U_o and the ion and electron masses. Let the unit magnetic field be the field necessary to balance the pressure of the plasma flux. The natural unit velocity is U_o, and the unit length will be chosen as the geometrical mean of the Larmor radii of an ion and electron each traveling with unit velocity in unit magnetic field. The dimensionless variables (lower case letters) are thus defined as follows:

$$X = x(M_i M_e/(16\pi M_t N_o))^{\frac{1}{2}}/e \qquad M = m(M_i M_e)^{\frac{1}{2}}$$

$$T = t(M_i M_e/(16\pi M_t N_o U_o^2))^{\frac{1}{2}}/e \qquad V = vU_o \tag{I-6}$$

$$B = b(16\pi M_t N_o U_o^2)^{\frac{1}{2}} \qquad \Phi = \varphi U_o^2 (M_i M_e)^{\frac{1}{2}}/e$$

In terms of these new variables equation I-3 becomes:

$$m_p d\underset{\sim}{v}_p/dt = s_p(\nabla\varphi + \underset{\sim}{v}_p \times \underset{\sim}{b}) \tag{I-7}$$

where $m_i = (M_i/M_e)^{\frac{1}{2}}$, $m_e = 1/m_i$, $s_i = 1$ and $s_e = -1$. Likewise using equation I-5, equation I-4 becomes:

$$2(m_i + m_e)\frac{db}{dx} = \frac{n}{n_o}(w_i - w_e) \tag{I-8}$$

Combining equations I-2 and I-5:

$$u_i(x) = u_o(x) = u(x) = n_o/n(x) \tag{I-9}$$

Sum the z component of equation I-7 over p and integrate over t (using the boundary condition $w(-\ )=0$) to obtain the equation expressing conservation of z momentum:

$$m_e w_e + m_i w_i = 0 \tag{I-10}$$

Multiply the z component of equation I-7 by w_i, the x component by u, add and integrate over t to obtain the energy equations.

$$\tfrac{1}{2}m_p(u^2 + w_p^2) + s_p\varphi = \tfrac{1}{2}m_p \tag{I-11}$$

The constant of integration was fixed by the condition that $w_p=0$ and $\varphi=0$ when $u=1$. To express φ as a function of u multiply the x component of equation I-7 by $s_p m_p$, sum over p, use equation I-10 to eliminate the terms in b, write d/dt as ud/dx and integrate with respect to x.

$$\varphi = \tfrac{1}{2}(m_i - m_e)(1-u^2) \tag{I-12}$$

Eliminating φ between these two equations, one may write w_p as a function of u

$$m_p w_p = s_p(1-u^2)^{\frac{1}{2}} \tag{I-13}$$

The factor s_p gives the proper sign to the square root.

To obtain the conservation law for the x momentum flux, sum the x component of equation I-7 over p, multiply the right hand side by nu/n_o ($=1$ by equation I-9), eliminate $(w_i - w_e)$ by equation I-8 and integrate with respect to t.

$$1 - u = b^2 \tag{I-14}$$

The constant of integration is determined by the condition that $b=0$ when $u=1$.

Multiply equation I-8 by u, set $nu/n_o=1$, differentiate with respect to t and use equation I-7 to eliminate dw_p/dt.

$$2\frac{d^2b}{dt^2} = ub \tag{I-15}$$

Using equation I-14 to express u as a function of b, rewrite this equation as:

$$2\frac{db}{dt}\frac{d}{db}\left(\frac{db}{dt}\right) = \frac{d}{db}\left(-\tfrac{1}{4}u^2\right) \tag{I-16}$$

Integrate this with respect to b and take the square root,

$$\frac{db}{dt} = \tfrac{1}{2}(1-u^2)^{\frac{1}{2}} \tag{I-17}$$

The constant of integration was determined by the fact that the derivative of B must vanish at $x=-\infty$ where $u=1$, and the sign of the square root was determined by the fact that b must increase with time on an ingoing orbit. If equation I-17 is multiplied by $2b$, equation I-14 may be used to eliminate b.

$$\frac{du}{dt} = -(1-u)(1+u)^{\frac{1}{2}} \tag{I-18}$$

Adopting the convention that $u=0$ at $t=0$, this can be integrated exactly.

$$\sqrt{2}t = \ln\left[c\,\frac{\sqrt{2}-\sqrt{1+u}}{\sqrt{2}+\sqrt{1+u}}\right] \qquad c=\frac{\sqrt{2}+1}{\sqrt{2}-1} \tag{I-19}$$

The explicit expression for u is easily derived.

$$u=2\left[\frac{c-e^{\sqrt{2}t}}{c+e^{\sqrt{2}t}}\right]^2-1 \qquad \text{(I-20)}$$

and integrated exactly to give

$$x=t-2+2\sqrt{2}\left[\frac{c-e^{\sqrt{2}t}}{c+e^{\sqrt{2}t}}\right] \qquad \text{(I-21)}$$

where the convention is adopted that $x=0$ at $t=0$. Of course these formulae apply only for negative t, since they were derived for ingoing particles.

Use equation I-20 in equation I-14 to obtain b.

$$b=\frac{2\sqrt{2c}\; e^{\frac{t}{\sqrt{2}}}}{(c+e^{\sqrt{2}t})} \qquad \text{(I-22)}$$

The simplest way to obtain the trajectories is to note from equations I-13 and I-17 that db/dt and $w_p=dz_p/dt$ are proportional. Thus, choosing the convention that $z_p=0$ at $t=-\infty$ where $b=0$, one can write:

$$z_p = \frac{2s_p}{m_p}\, b \qquad \text{(I-23)}$$

In plotting these results graphically in Figure 2, an artificial displacement z, which is the geometric mean of z_i and $-z_e$, is used because it is identical to $2b$. A further advantage is that the total velocity on this artificial trajectory is just unity (see equation I-13) so that the time is equal to the arc length.

APPENDIX II Field Inside the Cavity.

Once the proper surface and its current function have been determined, it is then a straightforward matter to calculate the magnetic field at any point in space. Taking the curl of equation 6.1 and adding to it the gradient of equation 7.5 one obtains the following expression for the field anywhere:

$$\underset{\sim}{B}(\underset{\sim}{R}_2) = J_o \int_S \frac{\underset{\sim}{j}(\underset{\sim}{r}) \times (\underset{\sim}{r}_2 - \underset{\sim}{r})}{|\underset{\sim}{r}_2 - \underset{\sim}{r}|^3} \, dS + J_o S_{11} \frac{3y_2 \underset{\sim}{r}_2 - r_2^2 \underset{\sim}{e}_y}{r_2^3} \qquad \text{(II-1)}$$

Equation 7.7 has been used to eliminate the M/R_n^3 in the second term.

For simplicity the field will be calculated only in the equatorial and meridian planes where B_x vanishes and it is necessary to integrate over only half the surface because of symmetry. The y and z components of equation II-1 are explicitly:

$$\begin{aligned} B_y &= J_o \int_S \frac{j_z(x_2-x) - j_x(z_2-z)}{|\underset{\sim}{r}_2 - \underset{\sim}{r}|^3} \, dS + \frac{J_o S_{11}(3y_2^2 - r_2^2)}{r_2^5} \\ B_z &= J_o \int_S \frac{j_x(y_2-y) - j_y(x_2-x)}{|\underset{\sim}{r}_2 - \underset{\sim}{r}|^3} \, dS + \frac{3J_o S_{11} y_2 z_2}{r_2^5} \end{aligned} \qquad \text{(II-2)}$$

By the same sort of coordinate transformations and substitutions which led from equation 6.4 to equation 6.10 these equations become:

$$B_y = J_o S_{11} \frac{2y_2^2 - x_2^2 - z_2^2}{\left[x_2^2 + y_2^2 + z_2^2\right]^{\frac{5}{2}}} + \tag{II-3}$$

$$+2J_o \int_0^1 df \int_0^S ds \left[\frac{\frac{dz}{ds}(x-x_2) + \frac{dx}{ds}(z_2-z)}{\left[(x-x_2)^2 + (y-y_2)^2 + (z-z_2)^2\right]^{\frac{3}{2}}} + \frac{\frac{dz}{ds}(x+x_2) + \frac{dx}{ds}(z_2-z)}{\left[(x+x_2)^2 + (y+y_2)^2 + (z-z_2)^2\right]^{\frac{3}{2}}} \right]$$

$$B_z = 3J_o S_{11} \frac{y_2 z_2}{y_2^2 + z_2^2} + \tag{II-4}$$

$$+2J_o \int_0^1 df \int_0^S ds \left[\frac{\frac{dx}{ds}(y-y_2) - x\frac{dz}{ds}}{\left[x^2 + (y-y_2)^2 + (z-z_2)^2\right]^{\frac{3}{2}}} + \frac{-\frac{dx}{ds}(y+y_2) + x\frac{dy}{ds}}{\left[x^2 + (y+y_2)^2 + (z-z_2)^2\right]^{\frac{3}{2}}} \right]$$

The factor of two has been introduced because the integrals cover only half the surface. The first term in the integrand of equation II-3 covers the first quadrant and the second term covers the second or fourth quadrant depending upon whether $y=0$ or $x=0$ (i.e. depending upon whether the field is being calculated in the equatorial or meridian plane). Equation II-4 is to be used only for the meridian plane because B_z vanishes in the equatorial plane.

APPENDIX III Relation of Vector Moments to Scalar Moments

As was pointed out in Section 3, any solution of Laplace's equation which vanishes at infinity may be expanded in terms of the functions

$$D^p_{nm} = \frac{P^m_n(\cos\theta)}{r^{n+1}} \cos(m\phi - p\tfrac{\pi}{2}) \qquad \text{(III-1)}$$

Therefore the vector potential of the localized current system is

$$\underset{\sim}{A}(\underset{\sim}{r}) = R_n J_o \sum_{n=o}^{\infty} \sum_{m=o}^{n} \sum_{p=o}^{1} \left[X^p_{nm}\underset{\sim}{e}_x + Y^p_{nm}\underset{\sim}{e}_y + Z^p_{nm}\underset{\sim}{e}_z\right] D^p_{nm} \qquad \text{(III-2)}$$

And likewise, for some appropriately defined parameters S^p_{nm}, the scalar potential for the same current system is

$$\phi(\underset{\sim}{r}) = R_n J_o \sum_{n=o}^{\infty} \sum_{m=o}^{n} \sum_{p=o}^{1} S^p_{nm} D^p_{nm} \qquad \text{(III-3)}$$

Now, if these two potentials are to give the same field, the following equation must be true.

$$\nabla\phi(\underset{\sim}{r}) - \nabla\times\underset{\sim}{A}(\underset{\sim}{r}) = 0 \qquad \text{(III-4)}$$

We will use this equation to derive certain relationships among the X's, Y's, Z's and S's. The equation

$$\nabla\cdot\underset{\sim}{A}(\underset{\sim}{r}) = 0 \qquad \text{(III-5)}$$

will not introduce any additional relationships among the vector moments, because equation III-2 assumes that $\nabla^2\underset{\sim}{A} = 0$, equation III-4 assures that $\nabla\times\nabla\times\underset{\sim}{A}=0$, $\nabla\cdot\underset{\sim}{A}(\infty)=0$ and,

$$\nabla(\nabla\cdot\underset{\sim}{A}) = \nabla\times\nabla\times\underset{\sim}{A} + \nabla^2\underset{\sim}{A} \qquad \text{(III-6)}$$

However the derivation of the relationships is simpler and more symmetrical if both equations III-4 and III-5 are used together.

Clearly the derivative operations will cause mixing between components of different m but the same n, but will not mix components with different n. Thus equations III-4 and III-5 together are really four scalar equations for <u>each</u> value of n.

Explicitly these four equations are as follows:

$$\begin{aligned}
&\sum_{m=o}^{n}\sum_{p=o}^{1} S_{nm}^{p}\frac{\partial D_{nm}^{p}}{\partial x} - Z_{nm}^{p}\frac{\partial D_{nm}^{p}}{\partial y} + Y_{nm}^{p}\frac{\partial D_{nm}^{p}}{\partial z} = 0\\
&\sum_{m=o}^{n}\sum_{p=o}^{1} S_{nm}^{p}\frac{\partial D_{nm}^{p}}{\partial y} - X_{nm}^{p}\frac{\partial D_{nm}^{p}}{\partial z} + Z_{nm}^{p}\frac{\partial D_{nm}^{p}}{\partial x} = 0\\
&\sum_{m=o}^{n}\sum_{p=o}^{1} S_{nm}^{p}\frac{\partial D_{nm}^{p}}{\partial z} - Y_{nm}^{p}\frac{\partial D_{nm}^{p}}{\partial x} + X_{nm}^{p}\frac{\partial D_{nm}^{p}}{\partial y} = 0\\
&\sum_{m=o}^{n}\sum_{p=o}^{1} X_{nm}^{p}\frac{\partial D_{nm}^{p}}{\partial x} + Y_{nm}^{p}\frac{\partial D_{nm}^{p}}{\partial y} + Z_{nm}^{p}\frac{\partial D_{nm}^{p}}{\partial z} = 0
\end{aligned} \qquad \text{(III-7)}$$

In order to solve these equations and be able to express the S_{nm}^{p} in terms of the X_{nm}^{p}, Y_{nm}^{p} and Z_{nm}^{p}, we must investigate the derivatives of the D_{nm}^{p} and be able to express them in terms of linearly independent functions. To make the algebra simpler, define the following new function,

$$D^{\pm}_{nm} = D^{o}_{nm} \pm iD^{1}_{nm} = \frac{P^{m}_{n}(\cos\theta)}{r^{n+1}} e^{\pm im\phi} \qquad \text{(III-8)}$$

and the operators:

$$\partial_{\pm} = \left[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right] \qquad \partial_{o} = \left[\frac{\partial}{\partial z}\right] \qquad \text{(III-9)}$$

These operators, operating on the coordinates, give the following relations,

$$\partial_{\pm} r = \sin\theta e^{\pm i\phi} \qquad \partial_{\pm}\cos\theta = \frac{-\cos\theta\ \sin\theta}{r} e^{\pm i\phi} \qquad \partial_{\pm}\phi = \frac{\pm ie^{\pm i\phi}}{r\sin\theta}$$

$$\partial_{o} r = \cos\theta \qquad \partial_{o}\cos\theta = \frac{\sin^2\theta}{r} \qquad \partial_{o}\phi = 0 \qquad \text{(III-10)}$$

and the partials of the $D^{\pm}_{nm}$ with respect to the coordinates are as follows:

$$\frac{\partial D^{\pm}_{nm}}{\partial r} = \frac{-(n+1)}{r} D^{\pm}_{nm} \qquad \frac{\partial D^{\pm}_{nm}}{\partial \phi} = \pm\, im\, D^{\pm}_{nm}$$

$$\frac{\partial D^{\pm}_{nm}}{\partial \cos\theta} = \frac{-(n+1-m)P^{m}_{n+1} + (n+1)\cos\theta\ P^{m}_{n}}{\sin^2\theta} \frac{e^{\pm im\phi}}{r^{n+1}} \qquad \text{(III-11)}$$

The derivative with respect to z can now be determined by inspection.

$$\partial_{o} D^{\pm}_{nm} = -\,(n+1-m)\ D^{\pm}_{n+1\ m} \qquad \text{(III-12)}$$

To determine the other derivatives we need two recursion relations:

$$(n+1-m)\cos\theta P_{n+1}^{m} - (n+1+m)P_{n}^{m} = -\sin\theta\ P_{n+1}^{m+1} \qquad \text{(III-13a)}$$

$$\cos\theta\ P_{n+1}^{m} - P_{n}^{m} = (n+2-m)\ \sin\theta\ P_{n+1}^{m-1} \qquad \text{(III-13b)}$$

Combining equations III-11 and III-10 for the first operator, one obtains in full:

$$\partial_{\pm} D_{nm}^{\pm} = \frac{e^{\pm i(m+1)\phi}}{r^{n+2}} \times \qquad \text{(III-14)}$$

$$\times \left[-(n+1)\sin\theta P_{n}^{m} + \frac{\cos\theta}{\sin\theta} \left\{ (n+1-m)P_{n+1}^{m} - (n+1)\cos\theta P_{n}^{m} \right\} - \frac{mP_{n}^{m}}{\sin\theta} \right]$$

Introducing equation III-13a reduces this to:

$$\partial_{\pm} D_{nm}^{\pm} = -\ P_{n+1}^{m+1}\ \frac{e^{\pm i(m+1)\phi}}{r^{n+2}} = -\ D_{n+1\ m+1}^{\pm} \qquad \text{(III-15)}$$

The equation corresponding to equation III-14 but with the $\pm$ reversed in the operator is the same as equation III-14 except that the last term is positive and (m-1) appears in the exponent. Making these changes and introducing III-13b reduces the equation to the following form.

$$\partial_{\mp} D_{nm}^{\pm} = \left[(n+1-m)(n+2-m)P_{n+1}^{m-1} \right] \frac{e^{\pm i(m-1)\phi}}{r^{n+2}}$$

$$= (n+1-m)(n+2-m)\ D_{n+1\ m-1}^{\pm} \qquad \text{(III-16)}$$

Clearly this latter equation is not valid when m=0, but since $D^+_{n0} = D^-_{n0}$, equation III-15 can be used in this case.

Before proceeding further, it is now necessary to reconvert to the ordinary cartesian derivatives. This is easily done; consider for example the x derivative of D^o_{nm}.

$$\frac{\partial}{\partial x}(D^0_{nm}) = \tfrac{1}{4}(\partial_+ + \partial_-)(D^+_{nm}+D^-_{nm})$$
$$= \tfrac{1}{2}\left[(n+1-m)(n+2-m)D^0_{n+1\ m-1}-D^0_{n+1\ m+1}\right] \qquad \text{(III-17)}$$

The results of this conversion in compact notation are as follows:

$$\frac{\partial}{\partial z}D^p_{nm} = -(n+1-m)D^p_{n+1,m} \qquad 0\le m \le n$$

$$\frac{\partial}{\partial x}D^p_{nm} = \tfrac{1}{2}\left[(n+1-m)(n+2-m)D^p_{n+1\ m-1} - D^p_{n+1\ m+1}\right] \qquad 1\le m \le n$$

$$\frac{\partial}{\partial y}D^p_{nm} = (p-\tfrac{1}{2})\left[(n+1-m)(n+2-m)D^{1-p}_{n+1\ m-1} + D^{1-p}_{n+1\ m+1}\right] \qquad 1\le m \le n$$

$$\frac{\partial}{\partial x}D^0_{n0} = - D^0_{n+1,1} \qquad \frac{\partial}{\partial y}D^0_{n0} = - D^1_{n+1,1} \qquad p=0,1$$

(III-18)

These last two equations were obtained using the identity $D^+_{n0} = D^-_{n0}$.

These equations express the (6n+3) derivatives of the D^p_{nm} in terms of (2n+3) linearly independent functions. If these are substituted into equations III-7, then those equations will be satisfied if and only if the coefficient of each of these functions vanishes in each of the equations.

Clearly each of the equations III-7 is of the form

$$\sum_{m=0}^{n} \sum_{p=0}^{1} U^{p}_{nm} \frac{\partial}{\partial x} D^{p}_{nm} + V^{p}_{nm} \frac{\partial}{\partial y} D^{p}_{nm} + W^{p}_{nm} \frac{\partial}{\partial z} D^{p}_{nm} = 0 \qquad \text{(III-19)}$$

which implies the following relationships among its coefficients:

$$2(n+1-m)W^{p}_{nm} = (1+\delta_{m1})\left[-U^{p}_{nm-1}+(1-2p)V^{1-p}_{nm-1}\right]$$

$$+(n+1-m)(n-m)\left[U^{p}_{nm+1}+(1-2p)V^{1-p}_{nm+1}\right] \qquad \text{(III-20)}$$

$$W^{0}_{n0} = \frac{n}{2}(U^{0}_{n1}+V^{1}_{n1}) \qquad 1 \le m \le n+1 \qquad p=0,1$$

The four equations of III-7 can now be characterized by substituting S, X, Y and Z for U, V and W according to the following table.

	U	V	W
1	S	-Z	Y
2	Z	S	-X
3	-Y	X	S
4	X	Y	Z

To eliminate the redundancies which exist among the relationships provided by these four equations, rewrite III-20 by writing 1-p for p and multiplying the equation by (1-2p).

$$2(1-2p)(n+1-m)W^{1-p}_{nm} = (1+\delta_{m1})\left[-(1-2p)U^{1-p}_{nm-1} - V^{p}_{nm-1}\right] + (n+1-m)(n-m)\left[(1-2p)U^{1-p}_{nm+1} - V^{p}_{nm+1}\right] \quad \text{(III-21)}$$

Now if we use equation III-20 for equation 1 and equation III-21 for equation 2, then first adding and then subtracting the two equations gives the following equivalent but simpler equations.

$$(1+\delta_{m1})\left[S^{p}_{nm+1}+(1-2p)Z^{1-p}_{nm-1}\right] = (n+1-m)\left[(1-2p)X^{1-p}_{nm} - Y^{p}_{nm}\right] \qquad 1\leq m\leq n+1 \quad \text{(III-22a)}$$

$$(n-m)\left[S^{p}_{nm+1}-(1-2p)Z^{1-p}_{nm+1}\right] = \left[(1-2p)\ X^{1-p}_{nm} + Y^{p}_{nm}\right] \qquad 1\leq m\leq n \quad \text{(III-22b)}$$

Do the same for equations 3 and 4 to obtain the simpler equations corresponding to that pair.

$$(n-m)\left[(1-2p)X^{1-p}_{nm+1}-Y^{p}_{nm+1}\right] = \left[S^{p}_{nm}+(1-2p)Z^{1-p}_{nm}\right] \qquad 1\leq m\leq n \quad \text{(III-22c)}$$

$$(1+\delta_{m1})\left[(1-2p)X^{1-p}_{nm-1}+Y^{p}_{nm-1}\right] = (n+1-m)\left[S^{p}_{nm}-(1-2p)Z^{1-p}_{nm}\right] \qquad 1\leq m\leq n+1 \quad \text{(III-22d)}$$

Rewrite equations III-22a and III-22d putting $(m+1)$ for m.

$$(1+\delta_{m0})\left[S^{p}_{nm}+(1-2p)Z^{1-p}_{nm}\right] = (n-m)\left[(1-2p)X^{1-p}_{nm+1} - Y^{p}_{nm+1}\right]$$

$$(1+\delta_{m0})\left[Y^{p}_{nm}+(1-2p)X^{1-p}_{nm}\right] = (n-m)\left[S^{p}_{nm+1}-(1-2p)Z^{1-p}_{nm+1}\right] \quad \text{(III-23)}$$

$$o\leq m\leq n \qquad p=0,1$$

It is now obvious that equations III-22b and III-22c are redundant and may be ignored. It is also easy to verify that the last of equations III-20, which we have thus far ignored, is also included in equation III-23. Thus if and only if these (4n+4) relationships are satisfied, equations III-7 are satisfied.

These (4n+4) relationships specify the (2n+1) S's in terms of the vector moments and give (2n+3) identities which must hold between the vector moments if they are to describe a curl-free magnetic field. Since there are (6n+3) vector moments, however, (2n-1) of them remain unspecified. This is just what one would expect. We can add to $\underset{\sim}{A}$ the gradient of any function ψ for which

$$\nabla^2 \psi = 0 \qquad \text{(III-24)}$$

without changing either the divergence or curl of A. Since the D^p_{nm} are solutions of Laplace's equation the following form for ψ will satisfy equation III-24.

$$\psi = \sum_{n=o}^{\infty} \sum_{m=o}^{n} \sum_{p=o}^{1} G^p_{nm} D^p_{nm} \qquad \text{(III-25)}$$

It is clear from equation III-18 that if $\nabla\psi$ is added to $\underset{\sim}{A}$, a linear combination of the $G^p_{n-1,m}$ will be added to each of the X^p_{nm}, Y^p_{nm} and Z^p_{nm}. Since there are just (2n-1) of the $G^p_{n-1,m}$ this accounts for the (2n-1) free parameters for each value of n. When the equations are derived which give in terms of the $G^p_{n-1,m}$ the quantities which can be added to the vector moments, they merely state that anything can be added as long as equations III-23 are not violated.

In practice, of course, every moment is fixed by the particular integral form of equation 6.3, and so there must be an additional (2n-1) equations which finish specifying them completely. For instance, for $n=1$ equations III-23 give the following eight equations.

$$\begin{aligned} S^0_{10} &= X^1_{11} = -Y^0_{11} & S^0_{11} &= Y^0_{10} = -Z^1_{11} \\ S^1_{11} &= Z^0_{11} = -X^0_{10} & X^0_{11} &= Y^1_{11} = Z^0_{10} \end{aligned} \qquad \text{(III-26)}$$

Study of the integral forms reveals that the last three moments are not only equal to each other, they are identically zero. Thus the (2n-1) additional equations for $n=1$ is $Z^0_{10} = 0$; however, there seems to be no simple way to obtain these additional equations in general.

APPENDIX IV Choice of the Trial Flux Function

The symmetry of the wind problem dictates that the current must be flowing in the x direction as it crosses the plane $x=0$. Thus on this plane $\frac{\partial f}{\partial x} = 0$. Likewise the symmetry of the surface about this plane requires that $\frac{\partial z}{\partial x} = 0$ there. Substituting these values into equation 5.9, it is clear that on the $x=0$ plane $\frac{\partial f}{\partial y} = \pm 1$. Although cartesian coordinates have been used in the body of the paper, throughout most of this appendix polar coordinates (ρ, ϕ) will be used, because the current function is more easily expressed in terms of them. The condition just derived then becomes $\frac{\partial f}{\partial \rho}(\rho, \frac{\pi}{2}) = \pm 1$. The front view of the surface in Figure 4 is in reality just a plot of the contour lines of $f(\rho, \phi)$. The current line flowing along the x axis and then dividing to go around the outside edge is the line $f(\rho, \phi)=0$. Then since $\frac{\partial f}{\partial \rho} = \pm 1$ on the y axis, $f(\rho, \frac{\pi}{2})$ must increase linearly from zero at $\rho=0$ to one at $\rho=1$ and then decrease linearly to zero again at $\rho=2$. It is also clear from Figure 4 that $f(\rho, \phi)$ must have the same symmetry as $\sin\phi$. Defining $a(\phi)$ as half the asymptotic value of ρ, the simplest function satisfying all these conditions is the function

$$f = \sin\phi \left[a - \sqrt{(a-\rho)^2} \right] \qquad \text{(IV-1)}$$

Note that $a(\frac{\pi}{2})=1$ and that a is symmetric about both the x and y axes. Thus a must be a power series in $v = \cos^2\phi$

whose leading term is unity. Also define $u = \rho/a$, $0 \leq u \leq 2$.

Obviously the f defined by equation IV-1 is too simple. Among other things the ridge, which must occur at the neutral point, also occurs at $u=1$ for all v. To remove this ridge, except at the neutral point, take

$$f = a \sin\phi \left[1 - \sqrt{(1-u)^2 + h}\right] \qquad \text{(IV-2)}$$

where $h \equiv h(u,v)$ must have the same symmetry as $a(v)$ and hence depends on ϕ only through v. Further, h must vanish at $v=0$ to preserve the linearity of f on the y axis and be greater than $-(1-u)^2$ to keep the square root real.

The condition for the current lines to be parallel and uniformly spaced near the origin is that

$$\frac{\partial f}{\partial \rho}(0,\phi) = \sin\phi \left[1-\tfrac{1}{2}h_u(0,v)\right]\left[1+h(0,v)\right]^{-\frac{1}{2}} = \sin\phi \qquad \text{(IV-3)}$$

This means that h must satisfy the following condition:

$$h(0,v) = -h_u(0,v)\left[1-\tfrac{1}{4}h_u(0,v)\right] \qquad \text{(IV-4)}$$

A further restriction on h is provided by another condition which is satisfied automatically by equation IV-1 but not necessarily by equation IV-2. Since the magnetic field just inside the neutral point goes to zero, the surface there could support no pressure and must be parallel to the wind. Since z_y is very large in that neighborhood, $\cos\psi \cong 0$, but $z_y \cos\psi \cong 1$. Reference to equation 5.8 shows, then, that unless $f_x = 0$ at the neutral point, j_z will be finite there.

Therefore

$$\frac{\partial f}{\partial \phi}(a, \tfrac{\pi}{2}) = 0 \qquad \text{(IV-5)}$$

Note that $a=1$ for $\phi = \frac{\pi}{2}$. In order to see if this condition is satisfied, it is necessary to take the limit of $\frac{\partial f}{\partial \phi}(a,\phi)$ as $\phi \to \frac{\pi}{2}$ because $\frac{\partial f}{\partial \phi}(a, \frac{\pi}{2})$ is indeterminate. Also $\frac{\partial f}{\partial \phi}(\rho, \frac{\pi}{2}) = 0$ at every point except $\rho = a$, so the limit can not be taken in the ρ direction. Thus condition IV-5 becomes

$$\lim_{\phi \to \frac{\pi}{2}} \frac{\partial}{\partial \phi} \left(a - ah(1,v)^{\frac{1}{2}}\right) = 0 \qquad \text{(IV-6)}$$

Assuming that the leading term of $h(1,v)$ is v^n, we know that $n \geq 1$ since $h(u,0)=0$, and it is easy to see that equation IV-6 will be satisfied if and only if $n \geq 2$. Before attempting to specify $h(u,v)$ further, it is necessary to consider the asymptotic properties of the surface.

Since the true surface is well represented asymptotically by a circular cylinder of radius 2, consider such a cylindrical cavity in a diamagnetic medium with a magnetic dipole located at the origin and oriented along the y axis. The scalar potential of the magnetic field inside can be found by straightforward analytic procedures analogous to those used by Smyth (35, p 177). The thing which is of interest is the potential just inside the surface for negative z.

$$\varphi_1 = 4\pi m' x in\phi \left[e^{bz} - c' e^{dz} + \cdots\right] \qquad \text{(IV-7)}$$

where $b=.920597$, $c'=1.228$ and $d=2.666$; b and d are equal to half the values of the first and second zeros respectively of the derivative of the Bessel function of index one. The value of the constant m' will depend on the dipole strength. If we introduce an image dipole at $z=2$, thus effectively putting a diamagnetic plane at $z=1$, this formula becomes:

$$\varphi_2 = 4\pi m \sin\phi \left[e^{bz} - c\, e^{dz} + \dots \right] \qquad \text{(IV-8)}$$

where $c=.868c'=1.066$ and $m=1.158m'$.

Clearly then in order to use this formula for the approximate description of the field just inside the true surface for large negative z it is only necessary to change slightly the values of m and c. We can write immediately the currents which will flow on the circular cylinder at large negative z.

$$j_z = -\frac{B\phi}{4\pi} = \frac{1}{8\pi}\frac{\partial \varphi_2}{\partial \phi} = \tfrac{1}{2}m \cos\phi \left[e^{bz} - c\, e^{dz} + \cdots \right] \qquad \text{(IV-9)}$$

$$j_\phi = \frac{Bz}{4\pi} = -\frac{1}{4\pi}\frac{\partial \varphi_2}{\partial z} = -m \sin\phi \left[be^{bz} - cde^{dz} + \dots \right] \qquad \text{(IV-10)}$$

Consider first the equatorial plane for the true surface. In this plane $\frac{\partial f}{\partial \rho} = \frac{\partial f}{\partial x} = 0$, so equation 4.8 gives

$$j_z = -\cos\psi \frac{\partial z}{\partial x} \frac{\partial f}{\partial y} \qquad \text{(IV-11)}$$

where $\frac{\partial f}{\partial y} = \frac{\partial f}{\rho \partial \phi}$, since we are in the equatorial plane.

Equation 5.10 simplifies to

$$-\frac{\partial f}{\partial y}\,\frac{\partial z}{\partial x} = \left[1 - \left(\frac{\partial f}{\partial y}\right)^2\right]^{\frac{1}{2}} \tag{IV-12}$$

Likewise, since $\frac{\partial z}{\partial y} = 0$, equation 5.4 becomes

$$\cos\psi = \left[1 + \left(\frac{\partial z}{\partial x}\right)^2\right]^{-\frac{1}{2}} \tag{IV-13}$$

If equation IV-12 is solved for $\frac{\partial f}{\partial y}$ one obtains

$$\frac{\partial f}{\partial y} = \left[1 + \left(\frac{\partial z}{\partial x}\right)^2\right]^{-\frac{1}{2}} \tag{IV-14}$$

Substituting equations IV-13 and IV-14 into equation IV-11 gives the following rigerous expression for the surface current in terms of the surface.

$$j_z = \frac{-z_x}{1+z_x^{\,2}} \qquad \text{where} \qquad z_x = \frac{\partial z}{\partial x} \tag{IV-15}$$

Thus, to the extent that the surface current of the real problem can be equated to the current on a circular cylinder (equation IV-9, the surface must satisfy the following equation in the equatorial plane.

$$-\frac{z_x}{1+z_x^{\,2}} \cong \tfrac{1}{2}m\left[e^{bz} - ce^{dz}\right], \quad z(2) = -\infty \tag{IV-16}$$

Of course we only want a solution of this equation for large negative z, where z_x is also large and negative, because it is only in this region that the equation is approximately true. Rewriting this last equation in the form

$$-\frac{1}{z_x}\left[1+\frac{1}{z_x^{\,2}}\right]^{-1} = \tfrac{1}{2}m\, e^{bz}\left[1 - ce^{(d-b)z}\right] \tag{IV-17}$$

and using the fact that both z and z_x are large and negative, it is clear that a first order approximation to the solution of this equation is given by the solution of

$$-\frac{1}{z_x} = \tfrac{1}{2}m\, e^{bz} \tag{IV-18}$$

which is

$$bz = \ln \frac{2b}{m}(2-x) \tag{IV-19}$$

Thus to first order we may write

$$\frac{1}{z_x} \cong -\,b(2-x) \tag{IV-20}$$

and using this to replace $1/z_x^{\,2}$ in equation IV-17 gives the second order equation:

$$\int_2^x \frac{-dx'}{1+b^2(2-x')^2} \cong \tfrac{1}{2}m\int_{-\infty}^{z}\left[e^{bz'} - c e^{dz'}\right] dz' \tag{IV-21}$$

which has the solution

$$b(2-x) - \frac{b^3(2-x)^3}{3} + \cdots \cong \tfrac{1}{2}m\left[e^{bz} - \frac{cb}{d}\, e^{dz}\right] \tag{IV-22}$$

or explicitly for z_x

$$z_x \cong \frac{-1}{b(2-x)} + \frac{2b(2-x)}{3} - \frac{2c}{m}\left(1-\frac{b}{d}\right)\left[\frac{2b(2-x)}{m}\right]^{\frac{d}{b}-2} + \ldots \tag{IV-23}$$

This equation is exact up to terms of order $(2-x)$. Since we do not have any knowledge of the proper values for m or c, the last term is of no direct use. However, since it is only of order $(2-x)^{-.896}$, it does indicate that the term of order $(2-x)^0$ vanishes. From equations IV-14 and

IV-23 we can now obtain the desired condition on f.

$$\frac{\partial f}{\partial y} = (z_x{}^2+1)^{-\frac{1}{2}} \cong b(2-x) + \mathcal{O}\left[(2-x)^3\right] \quad \text{(IV-24)}$$

We will also consider the region for $z = -\infty$ but $0 < \phi$, so that $f_\rho < 0 = f_\phi$. From equations 5.8 and 5.10 again the following relationship holds for the true surface.

$$\frac{J_\phi}{J_z} = \frac{\partial f}{\partial \rho}\left[1-\left(\frac{\partial f}{\partial \rho}\right)^2\right]^{-\frac{1}{2}} \quad \text{(IV-25)}$$

When this is equated to the same ratio of currents for the cylinder at $z = -\infty$ and solved, the following relationship is obtained.

$$\frac{\partial f}{\partial \rho}(z = -\infty) = -\left[1 + \frac{1}{4b^2}\cot^2\phi\right]^{-\frac{1}{2}} \quad \text{(IV-26)}$$

At least for the case when $a = 1$, equation IV-24 may be written

$$\frac{\partial f}{\partial \phi}(\rho, 0) = 2bw + \mathcal{O}(w^3) = \frac{1}{u}\left[1 - \left[(1-u)^2 + h(u,1)\right]^{\frac{1}{2}}\right] \quad \text{(IV-27)}$$

where $w = (1-\frac{1}{2}u)$. Since this must vanish at $u = 2 (w = 0)$, we can say first of all that

$$h(2,1) = 0 \quad \text{(IV-28)}$$

Then expanding the right hand side in terms of w

$$\left[1+\tfrac{1}{2}h_u(2,1)\right] w + \left[\left[1+\tfrac{1}{2}h_u(2,1)\right]\left[2+\tfrac{1}{2}h_u(2,1)\right] - \left[1+\tfrac{1}{2}h_{uu}(2,1)\right]\right] w^2 + \mathcal{O}(w^3)$$

(IV-29)

and equating coefficients gives the following relations

$$h_u(2,1) = 4b-2 \qquad h_{uu}(2,1) = 2(4b^2+2b-1) \qquad \text{(IV-30)}$$

Likewise for $a=1$, equation IV-26 may be written

$$\frac{\partial f}{\partial \rho}(2a,\phi) = -\sin\phi(1-ev)^{-\frac{1}{2}} = -\sin\phi\left[1+\tfrac{1}{2}h_u(2,v)\right]\left[1+h(2,v)\right]^{-\frac{1}{2}} \qquad \text{(IV-31)}$$

where $e=1-1/(4b^2)=.705014$, and this gives rise to the condition

$$(1-ev)h_u(2,v)\left[1+\tfrac{1}{4}h_u(2,v)\right] = h(2,v) + ev \qquad \text{(IV-32)}$$

Next consider what additional restrictions may be imposed on h by the cross section of the surface in the equatorial plane near $\rho=0$. Since we desire that $z(\rho)$ be symmetric in ρ, it is reasonable to ask that $\frac{\partial z}{\partial \rho}$ contain no even powers of degree lower than four. Equation IV-12 shows that for this to be true $\frac{\partial f}{\rho\partial\phi}(\rho,0)$ must have a leading term of one and no odd powers lower than five. However, referring to equation IV-2,

$$\frac{\partial f}{\rho\partial\phi}(\rho,0) = \frac{1}{u}\left[1 - \sqrt{(1-u)^2 + h(u,1)}\right] \qquad \text{(IV-33)}$$

Thus the square root must have the following functional form for small ρ

$$\left[(1-u)^2 + h(u,1)\right]^{\frac{1}{2}} = 1-u+\tfrac{1}{2}A_1u^3+\tfrac{1}{2}A_2u^5+\mathcal{O}(u^6) \qquad \text{(IV-34)}$$

Which on squaring both sides gives the equation

$$h(u,1) = A_1u^3(1-u) + A_2u^5 + \mathcal{O}(u^6) \qquad \text{(IV-35)}$$

In order to see the quantitative relationship between A_1 and A_2 and the surface shape, insert equations IV-34 and IV-33 into equation IV-12 to obtain

$$\frac{\partial z}{\partial \rho} = -A_1^{\frac{1}{2}}u\left(1+\tfrac{1}{2}\left(\frac{A_2}{A_1} + \tfrac{3}{4}A_1\right)u^2\right) + \mathcal{O}(u^4) \qquad \text{(IV-36)}$$

If we define r_c as the radius of curvature of the equatorial cross section at $\rho = 0$ then it follows immediately that $A_1 = (\frac{a}{r_c})^2$ and equation IV-36 may be written as:

$$\frac{\partial z}{\partial \rho} = -\frac{\rho}{r_c}\left(1+\frac{\rho^2}{2r_c^2}\right) - \frac{\rho^3 r_c}{2a^4}\left(A_2 - \frac{a^4}{4r_c^4}\right) \qquad \text{(IV-37)}$$

Part of the ρ^3 term has been combined with the ρ term because the combination represents a truly circular cross section up to terms of order ρ^5. The remaining ρ^3 term then represents the lowest order deviation of the cross section from a circle. If we postulate that this is zero, that fixes the value of A_2 for any given r_c, and condition IV-35 becomes

$$h(u,1) = A_1u^3 - A_1u^4 + \tfrac{1}{4}A_1^2\, u^5 + \mathcal{O}(u^6) \qquad A_1 = (a/r_c)^2 \qquad \text{(IV-38)}$$

We will now assume that $h(u,v)$ can be written as a double power series in u and v. The following unusual form for the series was chosen to simplify the application of the previous conditions, and to make the higher numbered parameters have a lesser effect on the surface.

$$h=H_0+uH_1+u^2H_2+u^2(2-u)H_3+ \qquad \text{(IV-39)}$$
$$+u^2(2-u)^2\left[H_4+(u-1)H_5+(u-1)^2H_6+(u-1)^3H_7+(u-1)^4H_8\right]$$

where $H_i=vh_{i1}+v(1-v)\left[h_{i2}+(2v-1)h_{i3}+(2v-1)^2h_{i4}+(2v-1)^3h_{i5}\right]$

There are no v^o terms in this series because $h(u,0)=0$, and the remaining conditions on this series are obtained from equations IV-4, IV-32, IV-38, IV-6 and IV-30.

$$\begin{aligned}
&1)\ h(o,v)= -h_u(0,v)\left(1-\tfrac{1}{4}h_u(0,v) \right)\\
&2)\ h(2,v)= (1-ev)h_u(2,v)\left(1+\tfrac{1}{4}h_u(2,v) \right) -ev\\
&3)\ h(u,1)= A_1u^3-A_1u^4+\frac{A_1^2}{4}u^5+ \mathcal{O}(u^6)\\
&4)\ h_v(1,0)=0\\
&5)\ h(2,1)=0\\
&6)\ h_u(2,1)=4b-2\\
&7)\ h_{uu}(2,1)=8b^2+4b-2
\end{aligned} \qquad \text{(IV-40)}$$

Equation 2) is partially redundant with 5) and 6) but is completely consistent with them.

Consider first equations 3), 5), 6) and 7). Since $H_i=h_{i1}$ when $v=1$, these equations affect only the h_{i1}. If we wish to have A_1 (which determines the radius of curvature at the origin) as a free parameter, then these equations put nine conditions on the h_{i1}, and i must go up to at least 8 as it does in equation IV-39. It is an easy matter then to show that these nine conditions are satisfied if and only if the h_{ij} are as follows:

$$h_{01}=h_{11}=h_{21}=0 \qquad h_{31}=\tfrac{1}{2}-b$$
$$h_{41}=(-12.5(1-2b)+A_1(A_1+6)+2b^2)/32$$
$$h_{51}=(-(1-2b)+2A_1(A_1+1)+8b^2)/32$$
$$h_{61}=(10(1-2b)-6A_1+12b^2)/32 \qquad \text{(IV-41)}$$
$$h_{71}=(9(1-2b)-2A_1(A_1+1)+8b^2)/32$$
$$h_{81}=(2.5(1-2b)-A_1^2+2b^2)/32$$

Consider now condition 1). Since $h(0,v)$ contains no powers of v higher than v^5, $h_u(0,v)$ can contain no powers higher than v^2. Thus, using $h_{01}=h_{11}=0$, we have that $h_u(0,v)=v(1-v)h_{12}$, and condition 1) becomes explicitly:

$$h_{02}+sh_{03}+s^2h_{04}+s^3h_{05} = -h_{12}(1-\tfrac{1}{4}v(1-v)h_{12}) \qquad \text{(IV-42)}$$

where $s=(2v-1)$ for brevity. At this point we will set $h_{12}=A_2$, since it will be convenient to use it as one of the variable parameters. Equating coefficients of the various powers of v and solving gives:

$$h_{02}=-A_2+A_2^2/16 \qquad h_{04}=-A_2^2/16$$
$$\text{(IV-43)}$$
$$h_{03}=h_{05}=0 \qquad h_{13}=h_{14}=h_{15}=0$$

The last three equations just state explicity the conclusion reached prior to equation IV-42. Define now the quantities $F_j = h_{0j} + 2h_{1j} + 4h_{2j}$ and $D_j = h_{1j} + 4h_{2j} - 4h_{3j}$ in terms of which the quantities appearing in condition 2) may be written explicitly (note that $s = 2v-1$).

$$h(2,v) = vF_1 + v(1-v)(F_2 + sF_3 + s^2F_4 + s^3F_5)$$
$$h_u(2,v) = vD_1 + v(1-v)(D_2 + sD_3 + s^2D_4 + s^3n_5) \qquad \text{(IV-44)}$$

Set $v=1$ and compare with equations 5) and 6) to see that $F_1 = 0$ and $D_1 = 4b-2$. Further by the same sort of argument used in conjunction with condition 1), it is clear that $D_3 = D_4 = D_5 = 0$, and condition 2) becomes explicitly:

$$(1-v)(F_2 + sF_3 + s^2F_4 + s^3F_5) = (1-ev)\left[D_1 + D_2 - vD_2\right]\left[1 + \tfrac{1}{4}v(D_1 + D_2 - vD_2)\right] - e \qquad \text{(IV-45)}$$

Equating coefficients of the powers of v gives the relationships:

$$F_2 = (2-e)(D_2 + 2D_1)(D_2 + 2D_1 + 16)/32 - 2e$$
$$F_3 = D_2(-eD_2 + 8D_1(1-e) - 16e)/32 + eD_1^2/8$$
$$F_4 = D_2(D_2(e-2) - 4eD_1)/32 \qquad \text{(IV-46)}$$
$$F_5 = eD_2^2/32$$
$$D_3 = D_4 = D_5 = 0$$

The last three equations, inferred earlier, are given for completeness. As a result of the redundancy of 2), 5) and

6) only four new relations were needed to guarantee that the coefficients of all five different powers of v in equation IV-45 vanish. If we add to these seven equations the equation $D_2=A_3$ (i.e. make it one of the arbitrary parameters), the resultant eight equations can easily be used to obtain more coefficients in terms of A_2 and A_3.

$$\begin{aligned}
&h_{22}=(2-e)(A_3+8b-4)(A_3+8b+12)/128-A_2(16+A_2)/64-\tfrac{1}{2}e \\
&h_{32}=h_{22}+(A_2-A_3)/4 \\
&h_{33}=h_{23}=A_3(-eA_3+32b(1-e)-16)/128+\tfrac{1}{2}e(b-\tfrac{1}{2})^2 \qquad \text{(IV-47)} \\
&h_{34}=h_{24}=A_3(A_3(e-2)-8e(2b-1))/128+A_2^2/64 \\
&h_{35}=h_{25}=eA_3^2/128
\end{aligned}$$

Finally, consider condition 4). Since $h(1,v)=H_0+H_1+H_2+H_3+H_4$, this condition becomes:

$$-h_v(1,0)=\sum_{i=0}^{4}(h_{i1}+h_{i2}-h_{i3}+h_{i4}-h_{i5})=0 \qquad \text{(IV-48)}$$

The unknowns in this equation are h_{42}, h_{43}, h_{44} and h_{45}, one of which may be taken as determined by the equation in terms of the other three. We will choose to determine h_{42}.

$$h_{42}=-h_{41}+h_{43}-h_{44}+h_{45}+2(-h_{22}+h_{23}-h_{24}+h_{25})+(A_3-A_2)/4+b-\tfrac{1}{2} \qquad \text{(IV-49)}$$

Now all the equations IV-40 are satisfied and 23 of the original 45 parameters h_{ij} are fixed by the equations

IV-41, IV-43, IV-47 and IV-49 and the remaining 22 free parameters are A_1, A_2, A_3 and h_{ij} (i=4,5,6,7,8 and j=2,3,4,5 omitting h_{42}).

Despite the care taken to keep h as general as possible, it appeared when convergence was attempted that there was still not sufficient flexibility in f to approach the true f very closely. In order to provide additional flexibility equation IV-2 was multiplied by e raised to the power of an arbitrary function.

$$f = a \sin\phi \, e^{g(u,v)}\left(1 - \sqrt{(1-u)^2 + h(u,v)}\right) \qquad \text{(IV-50)}$$

This new factor is non-negative, as it must be, but of course some conditions must be placed on $g(u,v)$ so that f will still satisfy the conditions for which we so laboriously adjusted $h(u,v)$. A review of these conditions shows that none will be violated if $g(u,v)$ obeys the following simple restrictions:

$$\begin{array}{lll} g(u,0)=0 & g(0,v)=0 & g(2,v)=0 \\ \multicolumn{3}{l}{g(u,1)=(1-\frac{u}{2})^2 u^2\left[g_0(1+u) + g_1 u^2 + \ldots\right]} \end{array} \qquad \text{(IV-51)}$$

In a sense the condition described in equations IV-37 and IV-38 is still "tampered with" in that if $g_0 \neq 0$ then r_o (the radius of curvature at the subsolar point) is altered. If $r_c(g_0=0)=r_{c0}$, then

$$r_c = r_{c0}\left[1-2g_0(r_{c0}/a)^2\right]^{-\frac{1}{2}} \qquad \text{(IV-52)}$$

Also the postulated condition that the cross section is circular to terms of order u^5 will be violated unless the following relation is satisfied.

$$g_1 = g_0(A_1 + \tfrac{3}{4} - g_0) \qquad \text{(IV-53)}$$

With these considerations in mind we will choose g to be of the form:

$$g(u,v) = vu^2(1-\tfrac{1}{2}u)^2\left[g_0\left[1+u+(A_1+\tfrac{3}{4}-g_0)u^2\right] + g_2u^2+g_3u^3\right] + \\ + v(1-v)u(1-\tfrac{1}{2}u)\sum_{i=0}^{3}\sum_{j=0}^{2} g_{ij}u^iv^j \qquad \text{(IV-54)}$$

The g_{ij} are related to the A_i as follows:

$$g_0 = A_{31} \quad g_2 = A_{32} \quad g_3 = A_{33} \quad g_{ij} = A_{3i+j+34} \qquad \text{(IV-55)}$$

The h_{ij} in equation IV-39 which are not given by equations IV-41, IV-43, IV-47 or IV-49 are related to the A_i according to the following scheme:

j \ i	4	5	6	7	8
2		A_4	A_7	A_{12}	A_{19}
3	A_5	A_6	A_8	A_{13}	A_{20}
4	A_9	A_{10}	A_{11}	A_{14}	A_{21}
5	A_{15}	A_{16}	A_{17}	A_{18}	A_{22}

(IV-56)

Note that the A_1 in equations IV-40 to IV-54 is not equal to the parameter A_1 but rather to 1 over the parameter squared. This was done so that the parameter A_1 might equal the radius of curvature in the xz plane at the subsolar point.

APPENDIX V Program for Numerical Calculations.

This problem is one for which the numerical calculations are quite involved and lengthy. Therefore considerable effort was expended in attempting to optimize the method of calculation. There are two basic time consuming operations: 1) tracing the curves f=constant and storing the coordinates and their derivatives at the points to be used in the integrations, and 2) performing the double integration over these points to obtain the moments. To give some idea of the time required, it was found that when the first 16 moments were calculated simultaneously, each of the two operations took roughly 25 seconds (of 7090 time) when about 1200 points were used in the integrands. Thus it takes about 18 minutes to calculate the H_{ij} (partial derivatives of the moments with respect to the parameters) for 20 different parameters.

Clearly then it is important to determine the most efficient way to obtain a given accuracy in the moments of equation 7.2. First of all, consider the integration over f. This has a well defined range (0 to 1) and the integrand (the inner integral over s) can be just as easily calculated at any value of f as at any other. Therefore some sort of Gaussian quadrature is obviously called for. However, it was found that even Gaussian quadrature, which tends to concentrate points near the ends of the range, did not concentrate enough points near the f=0 end of the range, because this region contains the current lines which go far back into the tail and contribute heavily to many of the moments. To improve the situation the following transformation of variables was made:

$$f = \left(\frac{k+1}{2}\right)^2, \quad df = \left(\frac{k+1}{2}\right) dk \qquad \text{(V-1)}$$

Now the integral (as a function of k) runs from -1 to 1 and also has the factor $\frac{1}{2}(k+1)$ in its integrand. The integrand already vanished at k=1 (f=1) but now it vanishes at k=-1 as well. Thus a further advantage can be gained by integrating over k by means of Radau quadrature (a type of Gaussian quadrature which includes the end points of the range). The formula for Radau quadrature, when $F(\pm 1)=0$, is as follows:

$$\int_{-1}^{1} F(k)dk = \sum_{j=1}^{N} H_j F(k_j) \qquad \text{(V-2)}$$

where
$$H_j = \frac{2}{(N+1)(N+2)(P_{N+1}(k_j))^2}$$

$P_N(k)$ is the N^{th} order Legendre polynomial, and k_j are the roots of $dP_{N+1}(k)/dk$. The formula becomes exact when F(k) is a polynomial of degree less than or equal to (2N+1).

Restate the above formula in terms of f, and absorb the factor $\frac{1}{2}(k+1)$ into the weights to obtain:

$$\int_{0}^{1} F(f)df = \sum_{j=1}^{N} W_j F(f_j) \qquad \text{(V-3)}$$

where
$$W_j = \frac{(k_j+1)}{(N+1)(N+2)(P_{N+1}(k_j))^2}, \quad f_j = \left(\frac{k_j+1}{2}\right)^2$$

and k_j are the roots of $dP_{N+1}(k)/dk$.

At the beginning of the program a subroutine called START computes and stores for later use the f_j and W_j which are appropriate for the specified number of curves (N).

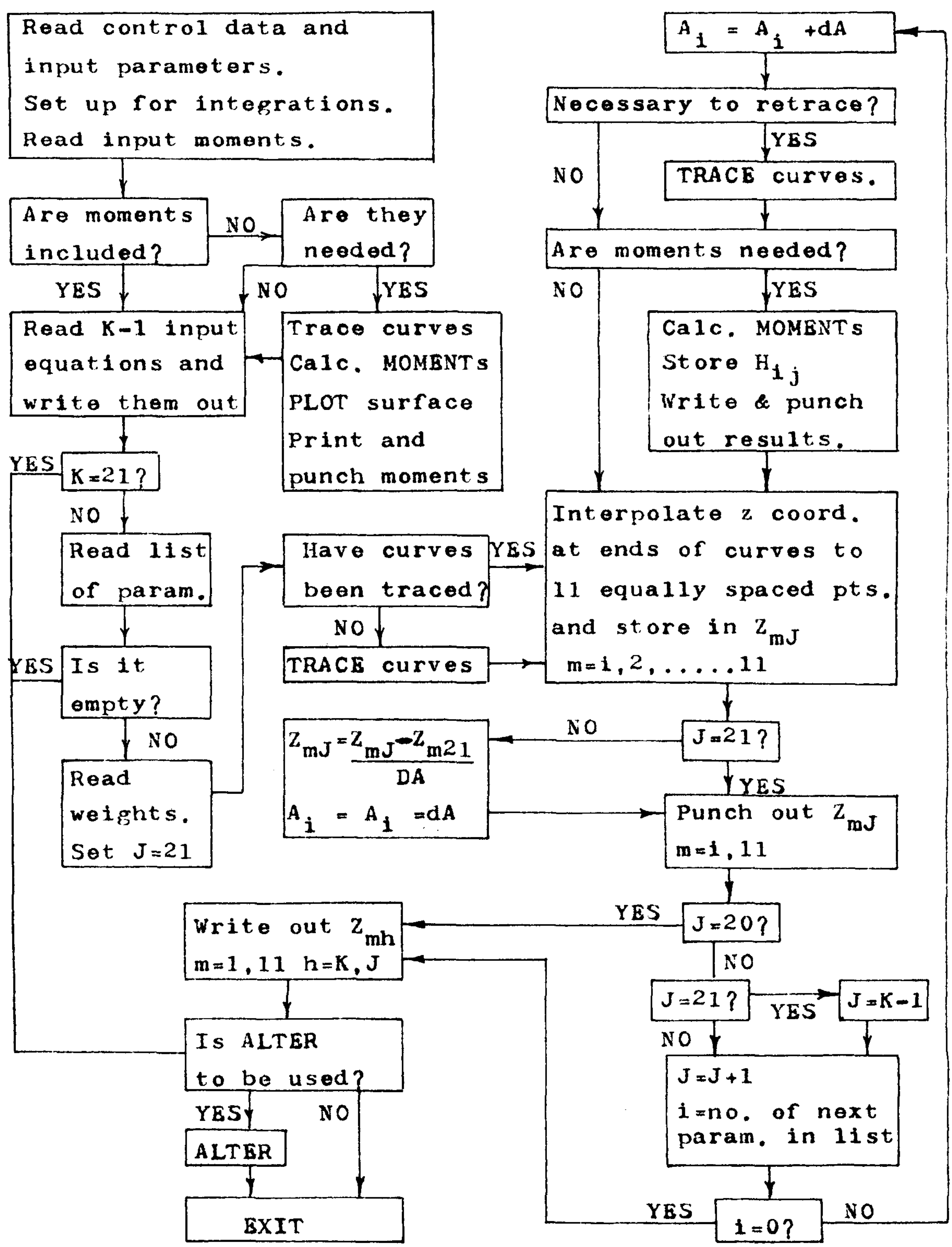
The integrals over S (along the curves), however, can not be done by a Gaussian guadrature, because the points of the integrand must be set up as the curve is traced and the length of the curve is not known before it has been traced. All things considered, it seemed best to do this integration by Simpson's method, varying the interval size over each successive pair of intervals. Starting from some initial interval size, the interval was halved (if it was greater than some minimum) whenever the curve was turning too rapidly to be followed accurately enough or z was changing too rapidly to be calculated accurately enough. Likewise it was doubled (if it was less than the initial interval) whenever the curve was flat enough and z was changing slowly enough. The exact criteria for these changes are best understood by reference to the flow diagram for the TRACE subroutine on page 104. These criteria were determined quantitatively by trial and error. It was further found, by analysis of the integrals along individual curves, that the same initial interval size gave different accuracy for different curves, so a formula was developed (again by trial and error) which gives the optimum initial interval for each curve as a function of a basic interval size (called DSI in the program).

In performing the integrations over the curves, either for the moments or the magnetic fields, the integral over the last partial interval can be done with the same accuracy as the rest of the curve by fitting a parabola to it, since it is known that the rate of change of the integrand with respect to s is zero at the ends of the curves.

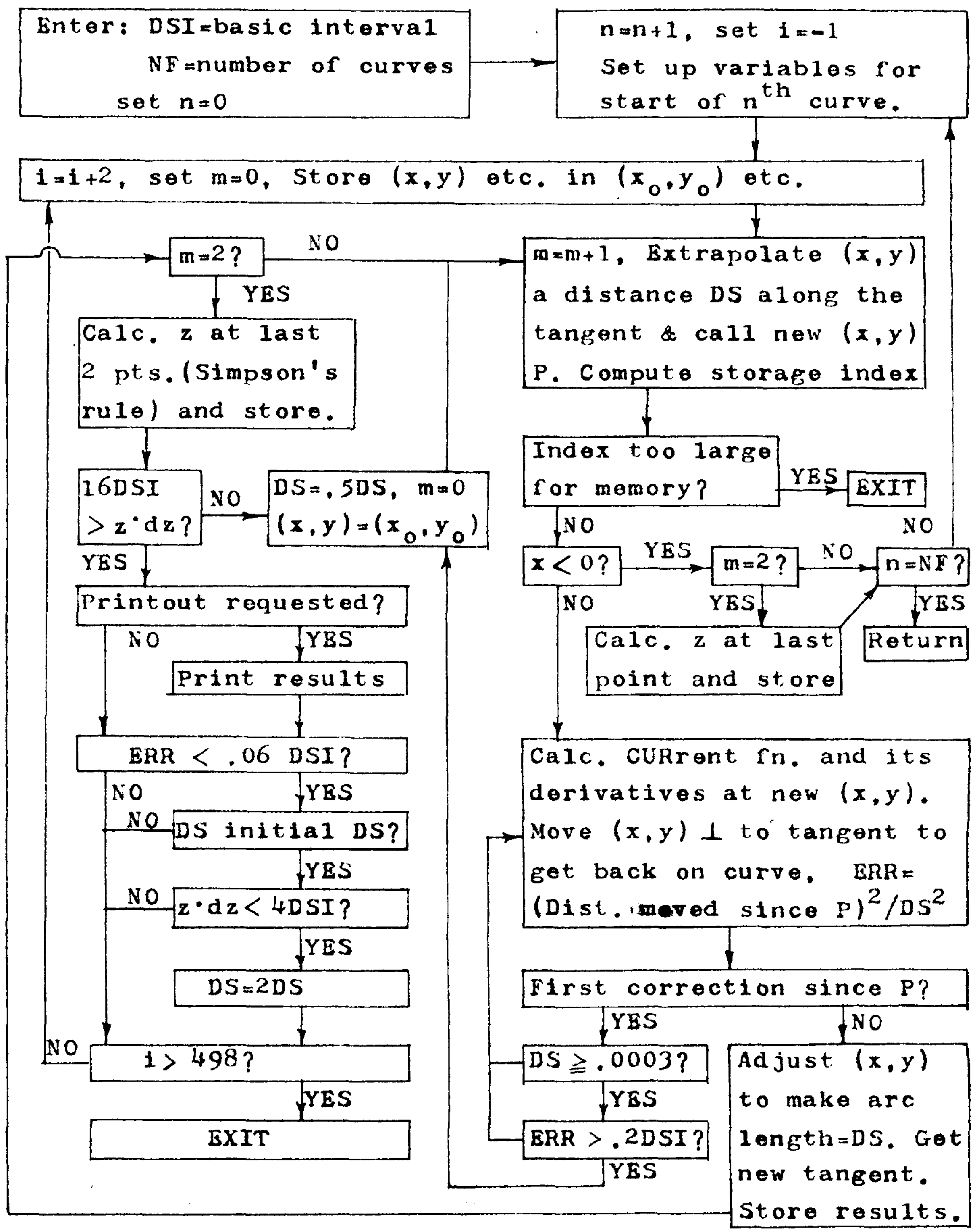
The entire program uses about 12,000 storage locations, so this leaves about 20,000 locations for storage of TRACE results. When 30 curves are traced it is appropriate to use a basic interval of 0.07 and this requires about 2750 points in the integrand or 19,250 locations for storage. Thus the program was written for a maximum of 30 curves. When this maximum is used the moments are probably accurate to four decimal places. During most of the convergence process it is more economical to use 20 curves and DSI=0.11, as this requires only 1200 points in the integrands and takes a proportionately shorter time to run. The resulting accuracy is still slightly more than three decimal places.

The next four pages contain flow diagrams of the three programs used and all the subroutines with any significant logical structure, and the four pages after that contain a reproduction of the final form of the Fortran program used.

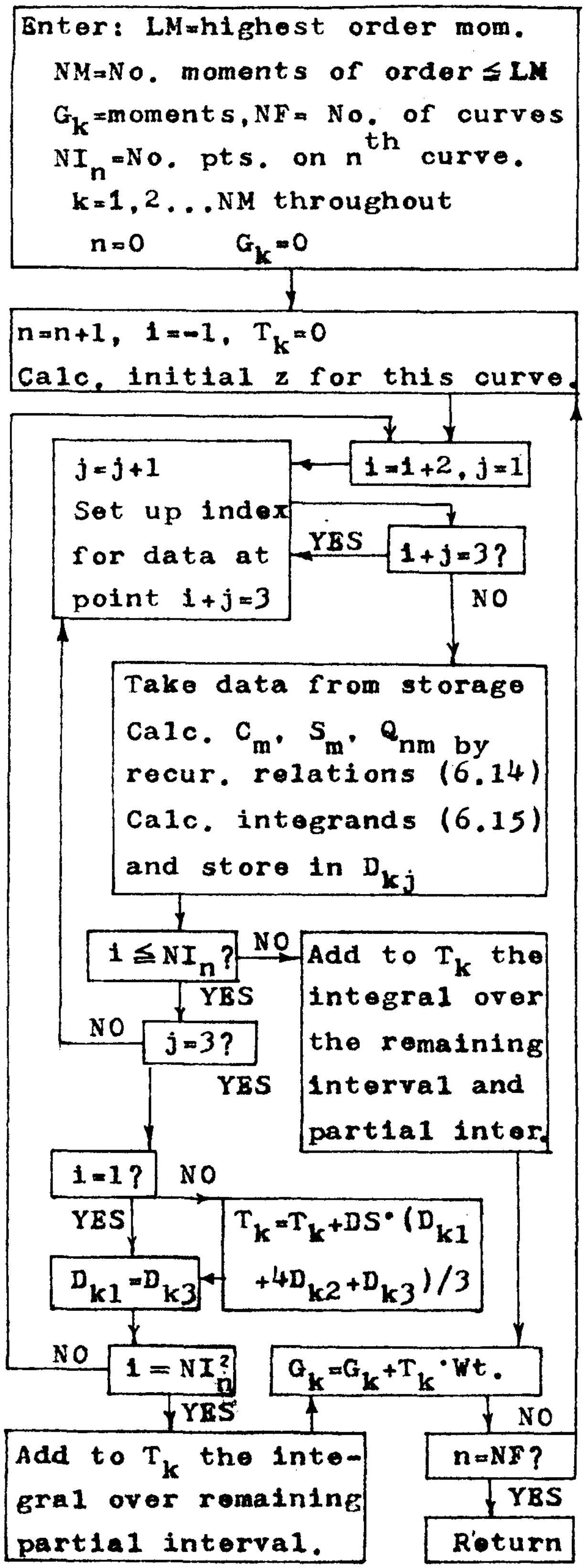
FLOW DIAGRAM OF CONTROL PROGRAM



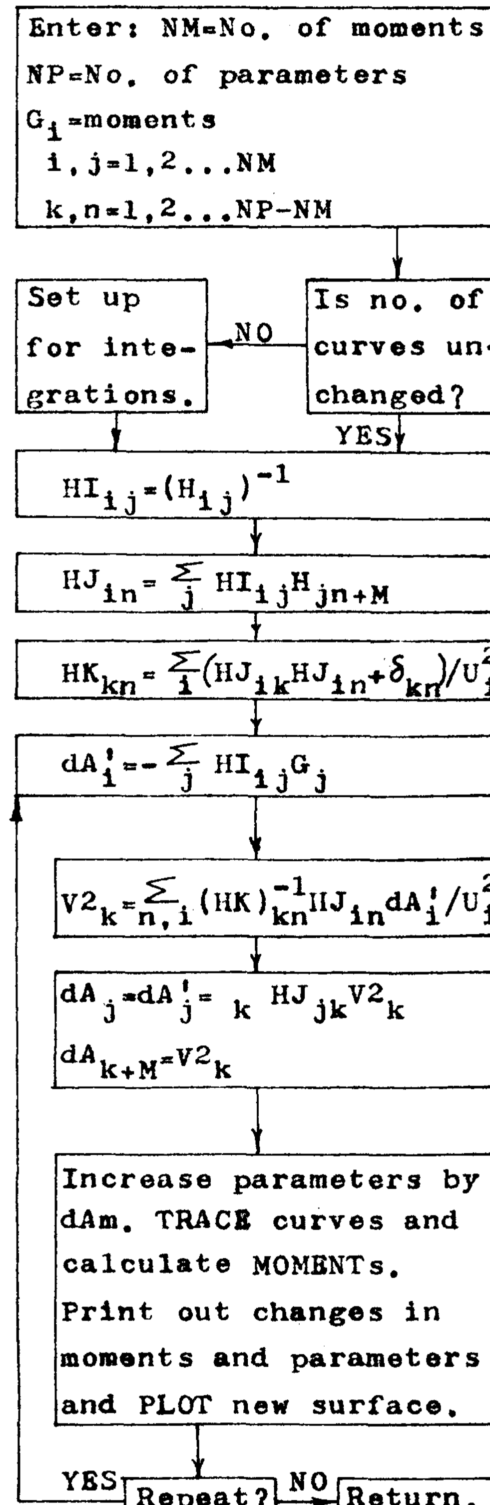
FLOW DIAGRAM FOR TRACE



FLOW DIAGRAM OF MOMENT

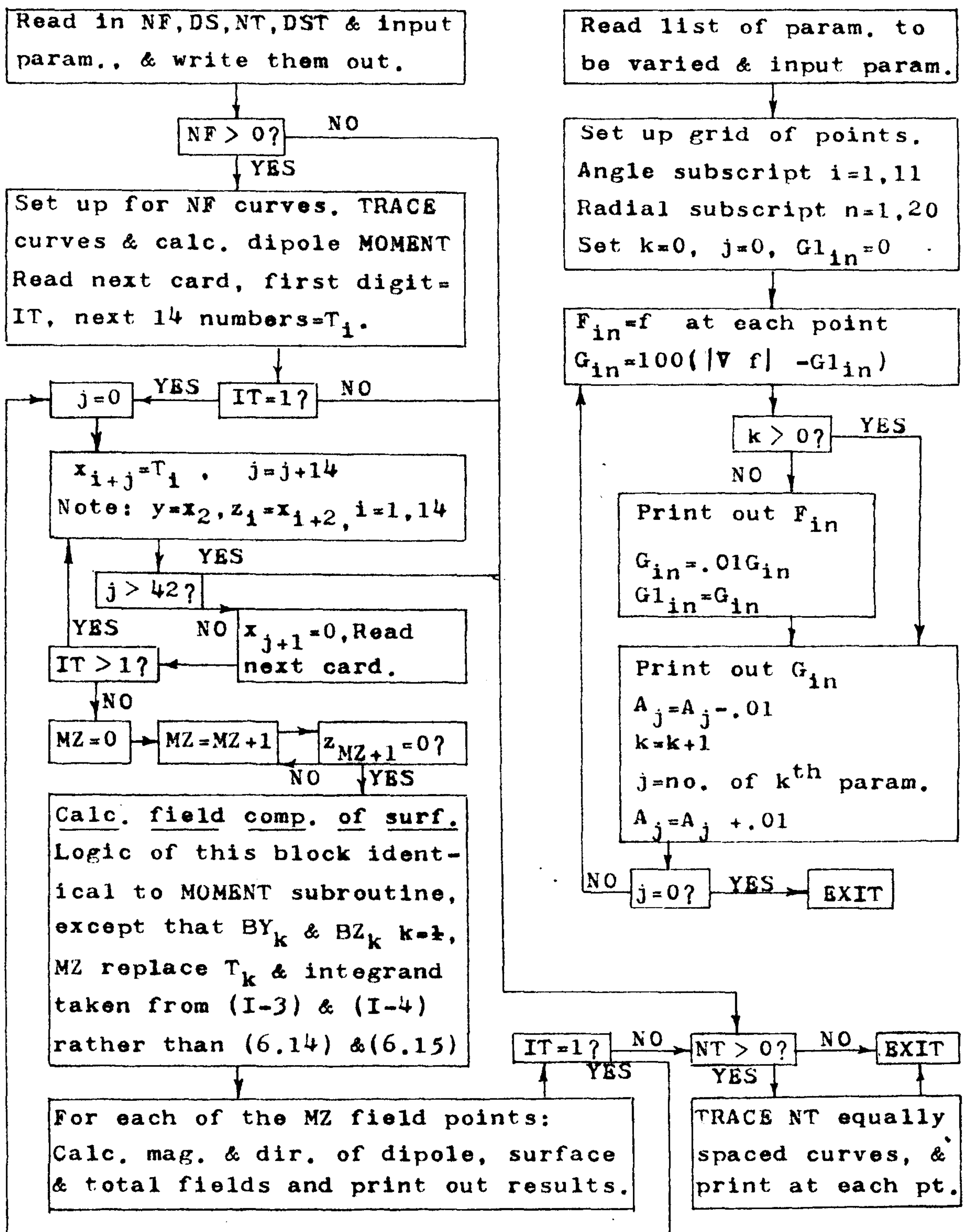


FLOW DIAGRAM OF ALTER



FLOW DIAGRAM OF FIELD

FLOW DIAGRAM OF SAMPLE



CONTROL PROGRAM

```
      COMMON W,A,IA,P,NL,NF,FR,WR,G,H,U,V,NI,IK,Q
      DIMENSION W(100),A(80),IA(20),P(90),FR(30),WR(30),G(30),H(20,20)
     1,U(20),V(15),NI(30),IK(31),Q(10000),Z(30),Y(30),ZE(11,21)
      READ INPUT TAPE 5,1,NN,NP,NM,NS,NF,DS,DA,NF2,DS2,NL
    1 FORMAT(5I3,2F5.3,I3,F5.3,I6)
      DO 3 I=1,80
      READ INPUT TAPE 5,2,IJ,A1
    2 FORMAT(I2,F8.5)
      IF(IJ) 3,4,3
    3 A(IJ)=A1
    4 CALL START
      CALL PSET
      READ INPUT TAPE 5,6,IB,DUM,(V(I),I=1,15)
    6 FORMAT(I2,F5.3,F11.5,6F9.5/8F9.5)
      IF(IB) 14,7,12
    7 CALL TRACE(DS,-1)
      CALL MOMENT(NS)
      DO 8 I=1,15
    8 V(I)=G(I)
      IC=99
      WRITE OUTPUT TAPE 7,6,IC,DUM,(V(I),I=1,15)
      CALL PLOT
      WRITE OUTPUT TAPE 6,9,G(30),(V(I),I=1,15)
    9 FORMAT(15H0DIPOLE MOMENT=F9.6,61H AND MOMENTS 21 31 33 41 43 51 53
     1 55 61 63 65 71 73 75 77 ARE /1H0F12.5,7F9.5,7F8.5)
      IF(NS-8) 12,10,10
   10 NW=15+4*(NS/8)+5*(NS/10+NS/9)
      WRITE OUTPUT TAPE 6,11,(G(I),I=16,NW)
   11 FORMAT(16H0HIGHER MOMENTS= 4F9.5,10F8.5)
   12 WRITE OUTPUT TAPE 6,13,(V(I),I=1,15)
   13 FORMAT(21H1CORRECTION EQUATIONS //1H0F12.5,7F9.5,7F8.5/3H0IA)
   14 DO 15 K=1,20
      READ INPUT TAPE 5,6,IA(K),U(K),(H(I,K),I=1,15)
      U(K)=1.-U(K)
      IF(IA(K)) 17,17,15
   15 WRITE OUTPUT TAPE 6,16,IA(K),(H(I,K),I=1,15)
   16 FORMAT(1H0I2,F10.5,7F9.5,7F8.5)
      GO TO 46
   17 READ INPUT TAPE 5,18,(IA(I),I=K,20)
   18 FORMAT(20I2)
      IF(IA(K)) 46,46,19
   19 KM=18+2*(K/3)/(1+(K-3)/3)
      READ INPUT TAPE 5,20,(U(I),I=K,KM)
   20 FORMAT(15F4.3)
      DO 21 I=K,20
   21 U(I)=1.-U(I)
      J=21
      IF(IB) 22,30,22
   22 CALL TRACE(DS,-1)
      GO TO 30
   23 J=K-1
   24 J=J+1
      JI=IA(J)
      IF(JI) 44,44,25
   25 A(JI)=A(JI)+DA
      IF(JI-70) 26,26,27
   26 CALL PSET
      CALL TRACE(DS,-1)
   27 IF(IB) 30,28,28
   28 CALL MOMENT(7)
      DO 29 I=1,15
   29 H(I,J)=(G(I)-V(I))/DA
      UP=1.-U(J)
      WRITE OUTPUT TAPE 6,16,IA(J),(H(I,J),I=1,15)
      WRITE OUTPUT TAPE 7,6,IA(J),UP,(H(I,J),I=1,15)
   30 DO 31 N=1,NF
      NJ=NF+1-N
      INDEX=IK(N+1)-4
      Z(NJ)=Q(INDEX)+.5*Q(INDEX-2)*Q(INDEX+1)
   31 Y(NJ)=1.-FR(N)
      X=0.
      DO 39 M=1,11
      X=X+.083333333
      DO 32 N=1,NF
      IF(Y(N)-X) 32,36,33
   32 CONTINUE
   33 IF(N-2) 36,36,34
   34 IF(N-NF) 35,37,37
   35 IF(Y(N)-X-X+Y(N-1)) 36,36,37
   36 X3=Y(N+1)
      Z3=Z(N+1)
      GO TO 38
   37 X3=Y(N-2)
      Z3=Z(N-2)
   38 X2=Y(N)
      Z2=Z(N)
      X1=Y(N-1)
   39 ZE(M,J)=Z2-(X2-X)*((X1-X)*(Z2-Z3)/(X2-X3)-(X3-X)*(Z(N-1)-Z2)/
     1 (X1-X2))/(X1-X3)
      IF(J-21) 40,42,42
   40 A(JI)=A(JI)-DA
      DO 41 M=1,11
   41 ZE(M,J)=(ZE(M,J)-ZE(M,21))/DA
   42 WRITE OUTPUT TAPE 7,43,JI,(ZE(M,J),M=1,11)
   43 FORMAT(I2,11F6.3)
      IF(J-20) 24,44,23
   44 WRITE OUTPUT TAPE 6,45,(IA(I),(ZE(M,I),M=1,11),I=K,J)
     1 ,IC,(ZE(M,21),M=1,11)
   45 FORMAT(48H1RATE OF CHANGE OF CROSS SECTION WITH RBSPECT TO  //
     1 (1H0I2,11F10.4))
   46 IF(IB) 49,47,47
   47 IF(NN) 48,49,48
   48 CALL ALTER(NN,NP,NM,NS,NF2,DS2)
   49 CALL EXIT
      END
```

DATA CARDS FOR CONTROL PROGRAM

```
NO. OF
CARDS  COLUMNS
 1       1-3   NUMBER OF  ALTER  CYCLES DESIRED.
         4-6   NUMBER OF PARAMETERS (NP) VARIED BY  ALTER  (≤20)
         7-9   NUMBER OF MOMENTS CONSIDERED BY  ALTER  (≤15)
         10-12 ORDER OF HIGHEST MOMENTS COMPUTED  (≤10)
         13-15 NUMBER OF CURVES TRACED BY  CONTROL (≤30)
         16-20 BASIC INTERVAL ON THESE CURVES  (≈0.110)
         21-25 PARAMETER INCREMENTS USED TO CALCULATE H(I,J)  (≈0.030)
         26-28 NUMBER OF CURVES TRACED BY  ALTER  (≤30)
         29-33 BASIC INTERVAL ON THESE CURVES  (≈0.110)
         34-39 HIGHEST LOCATION OF PROGRAM IN STORAGE  (≈11000)
 P     ONE CARD FOR EACH NON-ZERO PARAMETER.  COLUMNS 1-2 CONTAIN ITS
       NUMBER AND 3-10 CONTAIN ITS VALUE.  (FORMAT 00.00000).
 1     BLANK CARD.
 2     COLUMNS 1-2 OF THE FIRST CARD CONTAIN IB AND A SET OF MOMENTS
       FOLLOW.  IF IB>0 THE PROGRAM WILL USE THESE MOMENTS, IF IB=0
       ITWILL CALCULATE ITS OWN, AND IF IB<0 IT WILL ONLY FIGURE
       CROSS SECTIONS.  NEW CARDS OF THIS AND THE FOLLOWING TYPE ARE
       PUNCHED BY THE PROGRAM WHENEVER CALCULATED.
 2N    ONE PAIR OF CARDS FOR EACH PARAMETER (FOR WHICH THE H(I,J) ARE
       KNOWN) TO BE USED BY  ALTER.  THE FIRST CARD CONTAINS THE
       PARAMETER NUMBER (K) IN COLUMNS 1-2 AND ONE MINUS THE PARAMETER
       WEIGHT IN 3-7 (FORMAT 0.000).  THE H(I,K) FOLLOW.
 2     BLANK CARDS.
 1     LIST OF PARAMETERS (2 COLUMNS EACH) FOR WHICH NEW H(I,J) ARE
       TO BE CALCULATED.  ONLY THE FIRST (20-N) WILL BE DONE, AND
       ONLY THE FIRST (NP-N) WILL BE USED BY  ALTER.
 1     RESPECTIVE WEIGHTS (ACTUALLY ONE MINUS WEIGHTS) TO BE USED
       WITH ABOVE PARAMETERS (FORMAT .000).
```

CUR - COMPUTES CURRENT FUNCTION AND ITS DERIVATIVES AT A POINT

```
      SUBROUTINE CUR(N,J,R,S,C,F,FX,FY,FS)
      COMMON W,A,IA,P
      DIMENSION W(100),A(80),IA(20),P(90),G(11),H(82)
      U=R
      IF(N-1) 1,1,4
    1 V=C*C
      D=V+V-1.
      D2=2.*D
      D3=1.5*D
      V2=2.*V
      T=A(62)+D*(A(63)+D*A(64))
      AB=1.+V*(A(61)+(1.-V)*T)
      GF=(A(61)-D*(+2.*V*(1.-V)*(A(63)+D2*A(64))))/AB
      G1=A(31)*(1./A(1)**2+.75-A(31))+A(32)
      DO 2 K=1,10,3
      G(K)=A(K+33)+V*(A(K+34)+V*A(K+35))
    2 G(K+1)=A(K+34)+V2*A(K+35)
      DO 3 K=1,81,10
      T=P(K+1)+D*(P(K+2)+D*(P(K+3)+D*P(K+4)))
      H(K)=P(K)+(1.-V)*T
    3 H(K+1)=H(K)+V*((1.-V)*2.*(P(K+2)+D2*(P(K+3)+D3*P(K+4)))-T)
    4 IF(J) 6,6,5
    5 U=U/AB
    6 T=U-1.
      B=2.-U
      E=(1.-.5*U)*U
      G2=A(31)+U*(A(31)+U*(G1+U*A(33)))
      G3=G(1)+U*(G(4)+U*(G(7)+U*G(10)))
      G4=E*(E*G2+(1.-V)*G3)
      EG=EXPF(V*G4)
      GV=G4+V*E*((1.-V)*(G(2)+U*(G(5)+U*(G(8)+U*G(11))))-G3)
      GU=V*(E*(E*(A(31)+2.*U*(G1+1.5*U*A(33)))-2.*T*G2)
     1 +(1.-V)*(E*(G(4)+2.*U*(G(7)+1.5*U*G(10)))-T*G3))
      T1=H(41)+T*(H(51)+T*(H(61)+T*(H(71)+T*H(81))))
      T2=H(42)+T*(H(52)+T*(H(62)+T*(H(72)+T*H(82))))
      HH=V*(H(1)+U*(H(11)+U*(H(21)+B*(H(31)+B*T1))))
      HV=   H(2)+U*(H(12)+U*(H(22)+B*(H(32)+B*T2)))
      HU=V*(H(11)+2.*U*(H(21)+(2.-1.5*U)*H(31)+(2.*U*U-6.*U+4.)*T1)
     1 +U*U*B*B*(H(51)+2.*T*(H(61)+1.5*T*(H(71)+H(81)*T/.75))))
      SQ=SQRTF(T*T+HH)
      IF(SQ) 7,10,10
    7 WRITE OUTPUT TAPE 6,8,SQ,U,C,W(98)
    8 FORMAT(4H SQ=F10.7,7H FOR U=F10.7,9H AND COS=F10.7,7H W(98)=F6.0)
    9 CALL EXIT
   10 FU=GU*(1.-SQ)-(T+.5*HU)/SQ
      TT=(1.-SQ)/U
      FR=S*EG*FU
      F=AB*S*EG*(1.-SQ)
      FQ=C*EG*(TT-S*S*(2.*(TT*GV+GF*(TT-FU))-(HV/SQ)/U))
      FX=FR*C-FQ*S
      FY=FQ*C+FR*S
      FS=FR*FR+FQ*FQ
      IF(J) 14,14,11
   11 IF(FS-1.) 13,13,12
   12 FS=1.
      W(98)=W(98)+1.
      IF(W(98)-1000.) 14,14,7
   13 W(99)=W(99)+1.
   14 RETURN
      END
```

TRACE - FOLLOWS A SPECIFIED SET OF CURVES STORING DATA AT EACH PT.

```
      SUPROUTINE TRACE(DSI,J)
      COMMON W,A,IA,P,NL,NF,FR,WR,G,H,U,V,NI,IK,Q
      DIMENSION W(100),A(80),IA(20),P(90),FR(30),WR(30),G(30),H(20,20)
     1,U(20),V(15),NI(30),IK(31),Q(10000)
      W(98)=0.
      W(99)=0.
      IK(1)=1
      DO 32 N=1,NF
      KO=IK(N)
      X=0.
      Y=FR(N)
      PJ=0.
      PI=-9999999999.
      DS=SQRTF(1.-Y)*(1.-1./(2.+250.*Y*Y))*DSI
    1 Q(KO)=DS
      Q(KO+1)=0.
      Q(KO+2)=Y
      Q(KO+3)=0.
      Q(KO+4)=0.
      Q(KO+5)=1.
      Q(KO+6)=0.
      DSA=DS-.00001
      DO 28 I=1,500,2
      XO=X
      YO=Y
      PO=PJ
      PIO=PI
      GO TO 3
    2 DS=.5*DS
      X=XO
      Y=YO
      PJ=PO
      PI=PIO
    3 DO 18 M=1,2
      K=KO+7*(I+M-1)
      IF(K+NL-31680) 4,4,33
    4 XI=X
      YI=Y
      X=X+Q(K-2)*DS
      Y=Y+Q(K-1)*DS
      XJ=X
      YJ=Y
      ASSIGN 6 TO NB
      IF(X) 30,30,5
    5 R=SQRTF(X*X+Y*Y)
      S=Y/R
      C=X/R
      CALL CUR(1,1,R,S,C,F,FX,FY,FS)
      DF=FR(N)-F
      X=X+DF/(FX-PI*FY)
      Y=Y+DF/(FY-PJ*FX)
      ERR=((X-XJ)**2+(Y-YJ)**2)/DS**2
      GO TO NB, (6,8)
    6 ASSIGN 8 TO NB
      IF(DS-.0003) 5,7,7
    7 IF(ERR-.2*DSI) 5,5,2
    8 X=X-ERR*(2.*(X-XI)-DS*Q(K-2))/1.5
      Y=Y-ERR*(2.*(Y-YI)-DS*Q(K-1))/1.5
      PI=-FY/FX
      PJ=-FX/FY
      IF(FX) 10,9,10
    9 PI=9999999999.
   10 IF(FY) 12,11,12
   11 PJ=9999999999.
   12 IF(ABSF(FY)-ABSF(FX)) 13,13,14
   13 Q(K+6)=1./SQRTF(1.+PI*PI)
      Q(K+5)=PI*Q(K+6)
      GO TO 17
   14 Q(K+5)=1./SQRTF(1.+PJ*PJ)
      IF(FY) 15,16,16
   15 Q(K+5)=-Q(K+5)
   16 Q(K+6)=PJ*Q(K+5)
   17 Q(K)=DS
      Q(K+1)=X
      Q(K+2)=Y
      Q(K+4)=-SQRTF(1./FS-1.)
   18 CONTINUE
      Q(K-4)=Q(K-11)+DS*(5.*Q(K-10)+8.*Q(K-3)-Q(K+4))/12.
      Q(K+3)=Q(K-11)+DS*(    Q(K-10)+4.*Q(K-3)+Q(K+4))/3.
      IF(16.*DSI-DS*Q(K+3)*Q(K+4)) 2,29,29
   29 DSA=DSA-DS
      IF(J) 22,19,20
   19 IF(DSA) 20,20,22
   20 WRITE OUTPUT TAPE 6,21,N,I,DS,ERR,F,FS,X,Y,Q(K+3)
   21 FORMAT(3H N=I2,3H I=I2,4H DS=F8.7,5H ERR=F12.9,3H F=F8.7,4H FS=F9.
     17,3H X=F11.8,3H Y=F11.8,3H Z=F11.8)
   22 IF(DSA) 23,23,24
   23 DSA=Q(KO)-.00001
   24 IF(ERR-.06*DSI) 25,28,28
   25 IF(DS+.00001-Q(KO)) 26,28,28
   26 IF(4.*DSI-DS*Q(K+3)*Q(K+4)) 28,28,27
   27 DS=2.*DS
   28 CONTINUE
      GO TO 33
   30 NI(N)=I+M-1
      IK(N+1)=K
      IF(M-2) 32,31,31
   31 D=Q(K-7)+XI
      Q(K-4)=Q(K-11)+(D-XI)*(Q(K-10)*(XI+2.*D)/D+Q(K-3)*(2.*XI+D)/XI)/6.
   32 CONTINUE
      RETURN
   33 NI(N)=I+M-1
      WRITE OUTPUT TAPE 6,34,NL,(NI(K),K=1,N)
   34 FORMAT(7HOIF NL=I6,24H, TRACE EXCEEDED MEMORY. /25HOINTERVALS PER
     1CURVE WERE (25I4))
      CALL EXIT
       END
```

PLOT - COMPUTES AN ARRAY OF GRADIENTS AND PLOTS THE SURFACE

```
      SUBROUTINE PLOT
      COMMON W,A,IA,P,NL,NF,FR,WR,G,H,U,V,NI,IK,Q
      DIMENSION W(100),A(80),IA(20),P(90),FR(30),WR(30),G(30),H(20,20)
     1,U(20),V(15),NI(30),IK(31),Q(10000),Y(80),Z(80),UG(20),G1(11,20)
      DO 1 N=1,20
    1 UG(N)=.1*FLOATF(21-N)
      DO 2 I=1,11
      Q=.1570798*FLOATF(11-I)
      S=SINF(Q)
      C=COSF(Q)
      DO 2 N=1,20
      R=UG(N)
      CALL CUR(N,0,R,S,C,F,FX,FY,FS)
    2 G1(I,N)=SQRTF(FS)
      WRITE OUTPUT TAPE 6,3,(A(I),I=61,64),(A(I),I=1,22),(A(I),I=31,45)
    3 FORMAT(34H1GRADIENTS OF THE CURRENT FUNCTION/11H ASYMPTOTE=4F11.5/
     111H SQ. ROOT =11F11.5/F22.5,10F11.5/11H EXPONENT =11F11.5/F22.5/
     23F11.5/)
      WRITE OUTPUT TAPE 6,4 ,((G1(I,N),I=1,11),N=1,20)
    4 FORMAT(1H011F9.4)
B     SPACE=606060606060
B     DOT=006060606060
B     DASH=406060606060
B     VERT=316060606060
      K=2*NF+1
      DO 5 N=1,NF
      R=FR(N)**2
      NJ=K-N+1
      Z(N)=A(71)-A(72)*(R*(1.+(R-1.)*(A(73)+R*A(74)))-A(75)*(SQRTF(1.-R)
     1 -1.+R*(.5+.125*R*(1.+R*(.5+2.5*R))))))
      INDEX=IK(N+1)-4
      Z(NJ)=Z(N)+Q(INDEX)+.5*Q(INDEX-2)*Q(INDEX+1)
      Y(N)=FR(N)
    5 Y(NJ)=2.-FR(N)
      Y(NF+1)=1.
      Z(NF+1)=A(71)-A(72)
      WRITE OUTPUT TAPE 6,6 ,(A(I),I=61,64),(A(I),I=71,75),(A(I),I=1,22)
     1,(A(I),I=31,45)
    6 FORMAT(40H1GRAPH OF MERIDIAN AND EQUATORIAL PLANES/11H ASYMPTOTE=
     14F11.5,11H PROFILE  =5F11.5/11H SQ. ROOT =11F11.5/F22.5,10F11.5/
     211H EXPONENT =11F11.5/F22.5,3F11.5/)
      X=2.
      DX=-.08333333
      LM=23
    7 DO 19 L=1,LM
      X=X+DX
      DO 8 N=1,K
      IF(Y(N)-X) 8,12,9
    8 CONTINUE
      IF(Y(1)) 18,23,18
    9 IF(N-2) 12,12,10
   10 IF(N-K) 11,13,13
   11 IF(Y(N)-X-X+Y(N-1)) 12,12,13
   12 X3=Y(N+1)
      Z3=Z(N+1)
      GO TO 14
   13 X3=Y(N-2)
      Z3=Z(N-2)
   14 X2=Y(N)
      Z2=Z(N)
      X1=Y(N-1)
      ZL=Z2-(X2-X)*((X1-X)*(Z2-Z3)/(X2-X3)-(X3-X)*(Z(N-1)-Z2)/(X1-X2))/
     1(X1-X3)
      LI=XINTF(20.*(2.-ZL)+.5)
      IF(LI) 19,18,15
   15 IF(132-LI) 18,18,16
   16 WRITE OUTPUT TAPE 6,17,(SPACE,I=1,LI),DOT
   17 FORMAT(132A1)
      GO TO 19
   18 WRITE OUTPUT TAPE 6,17,SPACE
   19 CONTINUE
      IF(Y(1)) 20,23,20
   20 WRITE OUTPUT TAPE 6,17,SPACE,(DASH,I=1,39),VERT,(DASH,I=1,91)
      Z(1)=A(71)
      Y(1)=0.
      KM=NI(1)
      DO 21 K=2,KM
      Z(K)=Q(7*K-3)+A(71)
      Y(K)=Q(7*K-5)
      IF(Y(K-1)-Y(K)) 21,22,22
   21 CONTINUE
   22 X=0.
      DX=.08333333
      LM=28
      GO TO 7
   23 FRAC=W(98)/(W(98)+W(99))
      WRITE OUTPUT TAPE 6,24,FRAC,IK(NF+1)
   24 FORMAT(40H FRACTION OF GRADIENTS GREATER THAN ONE= F7.5,16H DIMENS
     1ION OF Q= I6)
      RETURN
       END
```

SAMPLING PROGRAM FOR CURRENT FUNCTION

```
      COMMON W,A
      DIMENSION W(100),A(80), UG(20),ST(11),CT(11),IA2(42),F1(11,20),
     1 G1(11,20),G2(11,20)
      READ INPUT TAPE 5,1,(IA2(I),I=1,41)
    1 FORMAT(35I2)
      DO 3 I=1,80
      READ INPUT TAPE 5,2,IJ,A1
    2 FORMAT(I2,F8.5)
      IF(IJ) 3,4,3
    3 A(IJ)=A1
    4 DO 5 I=1,11
      Q=.1570798*FLOATF(11-I)
      ST(I)=SINF(Q)
    5 CT(I)=COSF(Q)
      DO 6 N=1,20
    6 UG(N)=.1*FLOATF(21-N)
      K=0
      KI=0
    7 CALL PSET
      DO 8 I=1,11
      S=ST(I)
      C=CT(I)
      DO 8 N=1,20
      R=UG(N)
      CALL CUR(N,O,R,S,C,F,FX,FY,FS)
      F1(I,N)=F
    8 G2(I,N)=100.*(SQRTF(FS)-G1(I,N))
      IF(K) 9,9,12
    9 WRITE OUTPUT TAPE 6,10,(A(I),I=61,64),(A(I),I=1,22),(A(I),I=31,45)
     1 ,((F1(I,N),I=1,11),N=1,20)
   10 FORMAT(21H1CURRENT FUNCTION FOR/11H ASYMPTUTE=4F11.5/11H SQ. ROOT
     1=11F11.5/F22.5,10F11.5/11H EXPONENT =11F11.5/F22.5,3F11.5/
     2(1H011F9.4))
      DO 11 I=1,11
      DO 11 N=1,20
      G2(I,N)=.01*G2(I,N)
   11 G1(I,N)=G2(I,N)
   12 WRITE OUTPUT TAPE 6,13,KI,(A(I),I=61,64),(A(I),I=1,22),(A(I),I=31,
     1 45),((G2(I,N),I=1,11),N=1,20)
   13 FORMAT(50H1GRADIENTS OR VARIATION THEREOF WITH RESPECT TO A(I2,1H)
     1/11H ASYMPTOTE=4F11.5/11H SQ. ROOT =11F11.5/F22.5,10F11.5/11H EXPO
     2NENT =11F11.5/F22.5,3F11.5/(1H011F9.4))
      A(KI)=A(KI)-.01
      K=K+1
      KI=IA2(K)
      A(KI)=A(KI)+.01
      IF(KI) 7,14,7
   14 CALL EXIT
      END
```

DATA CARDS FOR SAMPLE PROGRAM

```
 1   PARAMETERS (TWO COLUMNS EACH) FOR WHICH THE DERIVATIVES OF
     THE GRADIENTS ARE TO BE CALCULATED.
 P   ONE CARD FOR EACH NON-ZERO PARAMETER.  COLUMNS 1-2 CONTAIN ITS
     NUMBER AND 3-10 CONTAIN ITS VALUE.  (FORMAT 00.00000).
 1   BLANK CARD.
```

PSET - COMPUTES MODIFIED PARAMETER SET TO SAVE TIME IN CUR

```
      SUBROUTINE PSET
      COMMON W,A,IA,P
      DIMENSION W(100),A(80),IA(20),P(120)
      A1=1./A(1)**2
      P(4)=-A(2)**2/16.
      P(2)=-A(2)-P(4)
      P(12)=A(2)
      P(22)=.01011706*A(3)*(A(3)+22.729552)-A(2)*(16.+A(2))/64.+.3067029
      P(25)=.005507922*A(3)**2
      P(23)=.062359133 -.0571092*A(3)-P(25)
      P(24)=A(2)**2/64.-.01011708*A(3)*(A(3)+3.6636909)
      P(31)=-.420597
      P(32)=P(22)+.25*(A(2)-A(3))
      P(33)=P(23)
      P(34)=P(24)
      P(35)=P(25)
      P(41)=A1*(A1+6.)/32.+.38156008
      P(51)=A1*(A1+1.)/16.+.23816202
      P(61)=-.1875*A1+.054938937
      P(71)=.21345092 -P(51)
      P(81)=-A1*A1/32.-.012749604
      P(52)=A(4)
      P(53)=A(6)
      P(62)=A(7)
      P(63)=A(8)
      P(54)=A(10)
      P(64)=A(11)
      P(72)=A(12)
      P(73)=A(13)
      P(74)=A(14)
      P(55)=A(16)
      P(65)=A(17)
      P(75)=A(18)
      P(92)=A(19)
      P(83)=A(20)
      P(84)=A(21)
      P(85)=A(22)
      P(43)=A(5)
      P(44)=A(9)
      P(45)=A(15)
      P(42)=-P(41)+P(43)-P(44)+P(45)+2.*(P(23)-P(22)-P(24)+P(25))
     1 +.25*(A(3)-A(2))+.420597
      RETURN
      END
```

MOMENT - COMPUTES ALL MOMENTS UP TO A GIVEN ORDER

```
      SUBROUTINE MOMENT(LM)
      COMMON W,A,IA,P,NL,NF,FR,WR,G,H,U,V,NI,IK,Q
      DIMENSION W(100),A(80),IA(20),P(90),FR(30),WR(30),G(30),H(20,20)
     1,U(20),V(15),NI(30),IK(31),Q(10000),T(30),X(10),Y(10),B(10,10)
     2 ,DT(30,3)
      DO 1 M=1,30
    1 G(M)=0.
      DO 18 N=1,NF
      R=FR(N)**2
      ZI=  A(71)-A(72)*(R*(1.+(R-1.)*(A(73)+R*A(74))))-A(75)*(SQRTF(1.-R)
     1 -1.+R*(.5+.125*R*(1.+R*(.5+2.5*R)))))
      DO 2 M=1,30
    2 T(M)=0.
      IM=NI(N)+1
      DO 13 I=1,IM,2
      DO 9 IS=2,3
      K=IK(N)+7*(I+IS-4)
      IF(I+IS-3) 4,9,4
    4 DS=Q(K)
      X(1)=Q(K+1)
      Y(1)=Q(K+2)
      Z=ZI+Q(K+3)
      R=X(1)**2+Y(1)**2
      DO 5 M=2,LM
      X(M)=X(M-1)*X(1)-Y(M-1)*Y(1)
    5 Y(M)=Y(M-1)*X(1)+X(M-1)*Y(1)
      B(1,1)=1.
      B(2,2)=.5
      B(2,1)=.5*Z
      DO 7 L =3,LM
      FAC=1.
      DO 6 IL=2,L,2
    6 FAC=FAC*FLOATF(L-IL+2)
      B(L,L)=1./FAC
      B(L,L-1)=Z/FAC
      DO 7 MI=3,L
      M=L-MI+1
    7 B(L,M)=(FLOATF(2*M+2)*Z*B(L,M+1)-R*B(L,M+2))/FLOATF(L*L-M*M+L-M)
      IN=0
      DO 8 L=1,LM
      DO 8 M=1,L,2
      IN=IN+1
    8 DT(IN,IS)=Q(K+4)*B(L,M)*X(M)-B(L,M+1)*(Q(K+5)*X(M+1)+Q(K+6)*Y(M+1)
     1)*FLOATF(1-M/L)/FLOATF(L+M+1)
      IF(I-IM) 9,15,15
    9 CONTINUE
      IF(I-1) 12,12,10
   10 DO 11 M=1,IN
   11 T(M)=T(M)+DS*(DT(M,1)+4.*DT(M,2)+DT(M,3))/3.
   12 DO 13 M=1,IN
   13 DT(M,1)=DT(M,3)
      Q1=X(1)**2/(DS*(X(1)+.5*DS))
      DO 14 M=1,IN
   14 T(M)=T(M)+X(1)*(DT(M,1)*(3.+Q1)-Q1*DT(M,2))/3.
      GO TO 17
   15 Q1=X(1)**2/(DS*(X(1)+.5*DS))
      DO 16 M=1,IN
   16 T(M)=T(M)+(X(1)+DS)*(DT(M,2)*(2.+Q1)+DT(M,1)*(1.-Q1))/3.
   17 G(30)=G(30)+WR(N)*T(1)
      DO 18 M=2,IN
   18 G(M-1)=G(M-1)+WR(N)*T(M)
      RETURN
      END
```

START - SETS UP CURVES TO BE TRACED AND WEIGHTS FOR INTEGRATIONS

```
      SUBROUTINE START
      COMMON W,A,IA,P,NL,NF,FR,WR
      DIMENSION W(100),A(80),IA(20),P(90),FR(30),WR(30),T(31)
      M=NF+1
      FAC=1./FLOATF(M*M+M)
      NH=M/2
      DO 10 J=1,NH
      JI=NF+1-J
      IF(J-2) 1,2,3
    1 X=1.-7.4/FLOATF(NF*(NF+6))
      GO TO 6
    2 X=3.3*X-2.3
      GO TO 6
    3 IF(J-4) 4,5,5
    4 X=2.5*X-1.5*W(J-2)
      GO TO 6
    5 X=2.2*X-1.2*W(J-2)
    6 DO 8 K=1,10
      Z=1./(X*X-1.)
      T(1)=X
      T(2)=1.5*X*X-.5
      DO 7 N=3,M
    7 T(N)=(FLOATF(2*N-1)*X*T(N-1)-FLOATF(N-1)*T(N-2))/FLOATF(N)
      Q=T(M)
      U=FLOATF(M)
      D=U*Z*(X*Q-T(M-1))
      DD=U*Z*Z*((U-1.)*(X*X*Q+T(M-2))-Q+(3.-2.*U)*X*T(M-1))
      X=X-D/DD
      IF(ABSF(D/DD) -.000001) 9,9,8
    8 CONTINUE
    9 FR(J)=.25*(1.-X)**2
      W(J)=X
      FR(JI)=.25*(1.+X)**2
      WR(J)=FAC*(1.-X)/Q**2
   10 WR(JI)=FAC*(1.+X)/Q**2
      RETURN
      END
```

FIELD CALCULATIONS AND/OR PLOT TRACE

```
      COMMON W,A,IA,P,NL,NF,FR,WR,G,H,U,V,NI,IK,Q
      DIMENSION W(100),A(80),IA(20),P(90),FR(30),WR(30),G(30),H(20,20)
     1,U(20),V(15),NI(30),IK(31),Q(10000),T(14),X(1),Y(1),Z(54),BY(40)
     2 ,BZ(40),TY(40),TZ(40),DY(40,3),DZ(40,3)
      READ INPUT TAPE 5,1,NF,DS,NT,DST
    1 FORMAT(I3,F5.3,I3,F5.3)
      DO 3 I=1,80
      READ INPUT TAPE 5,2,IJ,A1
    2 FORMAT(I2,F6.5)
      IF(IJ) 3,4,3
    3 A(IJ)=A1
    4 CALL PSET
      WRITE OUTPUT TAPE 6,5,(A(I),I=61,64),(A(I),I=71,75),(A(I),I=1,22)
     1 ,(A(I),I=31,45)
    5 FORMAT(11H1ASYMPTOTE=4F11.5,11H   PROFILE=5F11.5/11H SQ. ROOT =
     111F11.5/F22.5,10F11.5/11H EXPONENT =11F11.5/F22.5,3F11.5)
      IF(NF) 36,36,6
    6 CALL START
      CALL TRACE(DS,-1)
      CALL MOMENT(3)
      READ INPUT TAPE 5,7,IT,(T(I),I=1,14)
    7 FORMAT(I1,14F5.2)
      IF(IT-1) 36,8,36
    8 J=0
    9 DO 10 I=1,14
      J=J+1
   10 X(J)=T(I)
      IF(J-42) 11,11,36
   11 X(J+1)=0.
      READ INPUT TAPE 5,7,IT,(T(I),I=1,14)
      IF(IT-1) 12,12,9
   12 MZ=0
   14 MZ=MZ+1
      IF(Z(MZ+1)) 14,15,14
   15 DO 16 M=1,MZ
      BY(M)=0.
   16 BZ(M)=0.
      DO 30 N=1,NF
      R=FR(N)**2
      ZI=  A(71)-A(72)*(R*(1.+(R-1.)*(A(73)+R*A(74))))-A(75)*(SQRTF(1.-R)
     1 -1.+R*(.5+.125*R*(1.+R*(.5+2.5*R)))))
      DO 17 M=1,MZ
      TY(M)=0.
   17 TZ(M)=0.
      IM=NI(N)+1
      DO 25 I=1,IM,2
      DO 21 IS=2,3
      K=IK(N)+7*(I+IS-4)
      IF(I+IS-3) 19,21,19
   19 DS=Q(K)
      X1=Q(K+1)
      Z1=Q(K+3)+ZI
      T1=Q(K+4)*(X1-X)
      T2=(X1-X)**2+(Q(K+2)-Y)**2
      T3=(X1+X)**2+(Q(K+2)+Y)**2
      T4=2.*X*Q(K+4)
      U2=Q(K+5)*(Q(K+2)-Y)-X1*Q(K+6)
      T5=U2+2.*Y*Q(K+5)
      DO 20 M=1,MZ
      Z2=Z(M)-Z1
      U1=T1+Z2*Q(K+5)
      D1=(SQRTF(T2+Z2*Z2))**3
      D2=(SQRTF(T3+Z2*Z2))**3
      DY(M,IS)=U1/D1+(U1+T4)/D2
   20 DZ(M,IS)=U2/D1-T5/D2
      IF(I-IM) 21,27,27
   21 CONTINUE
      IF(I-1) 24,24,22
   22 DO 23 M=1,MZ
      TY(M)=TY(M)+DS*(DY(M,1)+4.*DY(M,2)+DY(M,3))/3.
   23 TZ(M)=TZ(M)+DS*(DZ(M,1)+4.*DZ(M,2)+DZ(M,3))/3.
   24 DO 25 M=1,MZ
      DY(M,1)=DY(M,3)
   25 DZ(M,1)=DZ(M,3)
      Q1=X1**2/(DS*(X1+.5*DS))
      DO 26 M=1,MZ
      TY(M)=TY(M)+X1*(DY(M,1)*(3.+Q1)-Q1*DY(M,2))/3.
   26 TZ(M)=TZ(M)+X1*(DZ(M,1)*(3.+Q1)-Q1*DZ(M,2))/3.
      GO TO 29
   27 Q1=X1**2/(DS*(X1+.5*DS))
      DO 28 M=1,MZ
      TY(M)=TY(M)+(X1+DS)*(DY(M,2)*(2.+Q1)+DY(M,1)*(1.-Q1))/3.
   28 TZ(M)=TZ(M)+(X1+DS)*(DZ(M,2)*(2.+Q1)+DZ(M,1)*(1.-Q1))/3.
   29 DO 30 M=1,MZ
      BY(M)=BY(M)+WR(N)*TY(M)
   30 BZ(M)=BZ(M)+WR(N)*TZ(M)
      WRITE OUTPUT TAPE 6,31,X,Y
   31 FORMAT(36H0MAGNETIC FIELDS AND TANGENTS FOR X=F5.2,4H  Y=F5.2)
      DO 34 M=1,MZ
      DY=2.*G(30)*(X*X-2.*Y*Y+Z(M)*Z(M))/(SQRTF(X*X+Y*Y+Z(M)*Z(M)))**5
      DZ=-6.*G(30)*Y*Z(M)/(SQRTF(Y*Y+Z(M)*Z(M)))**5
      DB=2.*DY*SQRTF(1.+(DZ/DY)**2)
      DT=-DZ/DY
      IF(X) 32,33,32
   32 BZ(M)=0.
   33 SB=2.*BY(M)*SQRTF(1.+(BZ(M)/BY(M))**2)
      ST=-BZ(M)/BY(M)
      TB=2.*(DY+BY(M))*SQRTF(1.+((DZ+BZ(M))/(DY+BY(M)))**2)
      TT=-(DZ+BZ(M))/(DY+BY(M))
   34 WRITE OUTPUT TAPE 6,35,DB,DT,SB,ST,TB,TT,Z(M)
   35 FORMAT(8H  DIPOLE2F9.4,9H  SURFACE2F9.4,8H   TOTAL2F9.4,3H Z=F5.2)
      IF(IT-1) 36,8,36
   36 IF(NT) 39,39,37
   37 DF=1./FLOATF(NT)
      DO 38 N=1,NT
   38 FR(N)=1.001-DF*FLOATF(N)
      NF=NT
      CALL TRACE(DST,0)
   39 CALL EXIT
      END
```

DATA CARDS FOR FIELD PROGRAM

```
NO. OF
CARDS  COLUMNS
 1       1-3   NO. OF CURVES TRACED FOR FIELD CALCULATIONS (≤30)
         4-8   BASIC INTERVAL ON THESE CURVES  (≈0.110)
         9-11  NUMBER OF PLOT CURVES TO BE TRACED  (≤30)
         12-16 BASIC INTERVAL ON THESE CURVES  (≈0.200)
 P    ONE CARD FOR EACH NON-ZERO PARAMETER.  COLUMNS 1-2 CONTAIN ITS
      NUMBER AND 3-10 CONTAIN ITS VALUE.  (FORMAT 00.00000).
 1    BLANK CARD.
 N    ANY NUMBER OF FIELD POINT CARDS EACH CONTAINING A SINGLE DIGIT
      IN COLUMN 1 AND 14 NUMBERS FOLLOWING IT (FORMAT 00.00).  IF
      THE DIGIT IS 1, THE FIRST TWO NUMBERS ON THAT CARD WILL BE
      TAKEN AS X AND Y COORDINATES AND THE REMAINING NUMBERS AS A
      LIST OF Z COORDINATES.  EITHER X OR Y SHOULD BE ZERO.  IF THE
      DIGIT IS GREATER THAN 1, THE CARD WILL BE TREATED AS A CONTIN-
      UATION CARD, BUT NO MORE THAN TWO SUCH CARDS (40 Z VALUES
      ALTOGETHER) MUST FOLLOW ANY GIVEN 1 CARD.  A ZERO IN A Z LIST
      WILL TERMINATE THE LIST UNLESS IT OCCURS FIRST.
 1    BLANK CARD.
```

ALTER - THEORETICALLY ZEROS MOMENTS BY THE SHORTEST ROUTE

```
      SUBROUTINE ALTER(NN,NP,NM,NS,NF2,DS2)
      COMMON W,A,IA,P,NL,NF,FR,WR,G,H,U,V
      DIMENSION W(100),A(80),IA(20),P(90),FR(30),WR(30),G(30),H(20,20)
     1,U(20),V(15),V2(15),V3(20),DA(20),HI(20,15),HJ(15,15),HK(20,15)
      ND=NP-NM
      NMP=NM+1
      IF(NF2-NF) 1,2,1
    1 NF=NF2
      CALL START
    2 DO 3 I=1,NM
      DO 3 J=1,NM
    3 HI(I,J)=H(I,J)
      CALL MATINV(HI,NM,V3,1,DET)
      DO 4 K=1,NM
      DO 4 J=1,ND
      JI=J+NM
      HJ(K,J)=0.
      DO 4 I=1,NM
    4 HJ(K,J)=HJ(K,J)+HI(K,I)*H(I,JI)
      DO 6 J=1,ND
      DO 5 N=1,ND
      HK(J,N)=0.
      DO 5 K=1,NM
    5 HK(J,N)=HK(J,N)+HJ(K,J)*HJ(K,N)/U(K)**2
      JI=J+NM
    6 HK(J,J)=HK(J,J)+1./U(JI)**2
      DO 20 NAI=1,NN
      DO 7 K=1,NM
      DA(K)=0.
      DO 7 L=1,NM
    7 DA(K)=DA(K)-HI(K,L)*V(L)
      DO 8 J=1,ND
      V3(J)=0.
      DO 8 K=1,NM
    8 V3(J)=V3(J)+HJ(K,J)*DA(K)/U(K)**2
      CALL MATINV(HK,ND,V3,1,DET)
      DO 9 K=1,NM
      DO 9 J=1,ND
    9 DA(K)=DA(K)-HJ(K,J)*V3(J)
      DO 10 J=1,ND
      JI=J+NM
   10 DA(JI)=V3(J)
      DO 11 J=1,NP
      IJ=IA(J)
   11 A(IJ)=A(IJ)+DA(J)
      CALL PSET
      CALL TRACE(DS2,-1)
      CALL MOMENT(NS)
      E1=0.
      E2=0.
      E3=0.
      E4=0.
      DO 12 J=1,NM
      E1=E1+ABSF(V(J))
   12 E2=E2+ABSF(G(J))
      DO 13 J=NMP,15
      E3=E3+ABSF(V(J))
   13 E4=E4+ABSF(G(J))
      WRITE OUTPUT TAPE 6,14,E1,E2,E3,E4,(IA(I),U(I),DA(I),I=1,NP)
   14 FORMAT(32H1SUM OF MOMENTS BEFORE AND AFTER /11H0  PRIMARY=2F10.5/
     111H0SECONDARY=2F10.5/35H0PARAMETERS, WEIGHTS AND INCREMENTS /
     2 (1H0I3,F6.3,F10.5))
      DO 15 I=1,15
   15 V(I)=G(I)
      IC=99-NAI
      WRITE OUTPUT TAPE 7,16,IC,(V(I),I=1,15)
   16 FORMAT(I2,F16.5,6F9.5/8F9.5)
      CALL PLOT
      WRITE OUTPUT TAPE 6,17,(V(I),I=1,15)
   17 FORMAT(13H AND MOMENTS= /1H0F12.5,7F9.5,7F8.5)
      IF(NS-8) 20,19,18
   18 NW=15+4*(NS/8)+5*(NS/10+NS/9)
      WRITE OUTPUT TAPE 6,19,(G(I),I=16,NW)
   19 FORMAT(16H0HIGHER MOMENTS=4F9.5,10F8.5)
   20 CONTINUE
   21 RETURN
      END
```

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