CALCULATION OF MEAN AND FLUCTUATING PROPERTIES OF THE INCOMPRESSIBLE TURBULENT BOUNDARY LAYER

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Presented at the 1968 Conference on "Computation Methods in Turbulent Boundary Layers" (Stanford University)

Stanford, California
August 18-24, 1968

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SUMMARY

The conservation equations for mass, mean momentum, and turbulent kinetic energy for the incompressible turbulent boundary layer have been solved by a finite difference procedure. Mathematical models developed by Glushko (1965) for the production, dissipation, and diffusion of the turbulent kinetic energy in the flat-plate boundary layer have been modified and used to calculate a non-equilibrium boundary layer subjected initially to a large adverse pressure gradient which is followed by a run of constant pressure.

Comparisons of both mean and fluctuating flow properties have indicated generally good agreement between the calculated results and experimental measurements of Goldberg's (1966). The best overall agreement with data was obtained by reducing the scale of turbulence in the outer part of the boundary layer to about 70 percent of the flat-plate values as used by Glushko. The calculations have indicated that further simple modifications to the turbulence scale function and to some of the mathematical models for the turbulence terms should improve the accuracy of predictions for the Goldberg data. It is expected that predictions of equal accuracy should be possible for other arbitrary pressure distributions and wall boundary conditions.

INTRODUCTION

The basic difficulty in the calculation or analysis of all turbulent flows is the problem of how to relate the random fluctuating characteristics of the flow to the mean flow properties. In the Reynolds equations of mean motion, for example, this problem has generally been met by introducing more or less arbitrary assumptions for the Reynolds stress terms (eqs. (14)-(17), Reynolds (1968)). In 1945, Prandtl, Nevzgljadov, and Chou reported on independent investigations intended to develop a more rigorous approach to this problem. These investigations were based on the idea of using independent differential equations to describe the dynamics of the correlations for the turbulent velocity fluctuations. These equations are derived (Hinze (1959) pp. 250-260, for example) from the Navier-Stokes equations of motion and contain terms for double velocity correlations (Reynolds stress and turbulence kinetic energy terms), triple velocity correlations, and correlations of velocity and pressure fluctuations.

Rotta (1951) extended the work of Prandtl (1945), and, in particular, Chou (1945), in considerable detail based on advances in theory and new data not available during the earlier investigations. These methods were further developed and applied to various types of simple flows by Emmons (1954), Townsend (1961), Levin (1964), and Spalding (1967). An integral form of the turbulence energy equation was utilized by McDonald (1966) to compute the mean profiles by an integral method.

Kovaszany (1967) and Nee (1967) assumed that the effective total viscosity obeys a "rate equation" expressing the "conservation" of the total viscosity. Harlow (1967) gave a more formal derivation (based on the equation for the turbulence kinetic energy) of a rate equation for the eddy viscosity and also constructed a "transport" equation for the scale of turbulence by analogy with Brownian motion. The same authors (Harlow (1968)) have since derived a new transport equation for the dissipation function. Since this function depends on the scale of turbulence, the new equation together with their previous rate equation for the eddy viscosity, were proposed as the basic equations for a general method of computing turbulent shear flows.

In all of the investigations mentioned above, except the integral method of McDonald (1966), it was necessary to make assumptions regarding the relative magnitudes of the various fluctuating quantities in order to simplify and obtain solutions to the nonlinear equations involved. Consequently, these methods were applied mostly to simple flows or portions of simple flows where some terms in the correlation equations could be neglected. Hence, the mathematical formulations of remaining terms representing the fluctuating quantities could not be tested for their degree of generality. In the integral method of McDonald (1966), it is still not possible to test the detailed spatial variations of these formulations and the turbulent diffusion terms drop out completely upon integration across a shear layer.

The answer to these difficulties is, of course, to obtain numerical solutions of the complete equations with automatic computing machines. Such solutions have been given recently by Glushko (1965), Bradshaw (1967), and Nash (1968). In the methods of Bradshaw (1967) and Nash (1968) the molecular shear is neglected and it was therefore necessary to provide the correct wall boundary condition by incorporating the "law of the wall" relation between velocity and wall shear. Glushko, on the other hand, kept all the viscous terms and used formulations with the correct limiting form at the wall. It might be expected that this latter approach would therefore be somewhat more general than Bradshaw's in that sudden changes in wall-boundary conditions could be negotiated and extension to compressible flows where the "law of the wall" relation may not be generally applicable should give better results. The only computation presented by Glushko, however, was for the flat plate, where again the degree of generality of his assumptions for the fluctuating flow parameters could not be determined.

It is the primary purpose of the present paper, then, to test the method of Glushko in a nonequilibrium, adverse pressure gradient flow and to determine whether his formulations of the turbulence quantities based primarily on flat-plate data result in satisfactory predictions for the mean properties of this more complex flow. Since the ultimate success of these methods depends on the assumptions relating the fluctuating properties, comparisons of computed values of turbulent kinetic energy with experimental data will also be made.

SYMBOLS

C	constant in dissipation function, equation (5)
E	turbulent kinetic energy profile, $\bar{e}/U_\infty^2$
e	instantaneous value of turbulent kinetic energy, $(1/2)\left(u'^2 + v'^2 + w'^2\right)$
$\bar{e}$	mean value of turbulent kinetic energy, $(1/2)\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)$
F	velocity profile, $\bar{U}/U_\infty$
H	function of r or κr in transport functions
l	mean scale of turbulence
n	exponent in definition of η, equation (7b)
Re_x	length Reynolds number, $U_\infty x/\nu$

r	turbulent Reynolds number, $\frac{\sqrt{\bar{e}}\, l}{\nu}$
r_o	constant in transport functions
$\hat{V}$	transformed normal velocity, equation (8)
α	constant in transport functions
η, ξ	transformed variables, equation (7)
κ	constant in transport functions
φ	turbulence scale function, equation (1) and tabulations, p. 6

Subscripts

ave	mean or average value
d	dissipation
δ	nominal edge of boundary layer
e	edge of boundary layer
o	initial condition
T	turbulent
t	transition

THEORY

Assumptions for Fluctuation Terms

The basic physics of the present method (essentially the same as that of Glushko (1965)) have been well stated by Reynolds (1968). Briefly, the equations of continuity, mean momentum, and turbulence energy are solved simultaneously by a finite difference procedure. The expressions for the fluctuating quantities in the turbulence energy equation (production, diffusion, and dissipation of turbulence kinetic energy) as developed by Glushko (1965) were based on the general approach of Rotta (1951) wherein the dissipation and diffusion terms are assumed to be functions of $\bar{e}$, l, and $r = \frac{\sqrt{\bar{e}}\, l}{\nu}$. The form of these functions depends primarily on physical and dimensional reasoning. The mean scale of turbulence l was evaluated from flat-plate data for two point correlation coefficients of the longitudinal velocity fluctuations, and was taken as a "universal" function of the form

$$\frac{l}{\delta} = \varphi\left(\frac{y}{\delta}\right) = \varphi_{.33} \tag{1}$$

Glushko's expression for the Reynolds stress utilizes a modified form of Prandtl's (1945) eddy viscosity relation and was assumed as,

$$\tau_T = -\rho\, \overline{u'v'} = \mu\, \bar{H}(r)\, \alpha r \frac{\partial \bar{U}}{\partial y} \tag{2}$$

where

$$\bar{H}(r) = \begin{cases} \dfrac{r}{r_o} & 0 < \dfrac{r}{r_o} < 0.75 \\ \dfrac{r}{r_o} - \left(\dfrac{r}{r_o} - 0.75\right)^2 & 0.75 < \dfrac{r}{r_o} < 1.25 \\ 1 & 1.25 < \dfrac{r}{r_o} < \infty \end{cases} \tag{3}$$

The turbulence production term for the problems considered herein is then

$$\text{Production} = \tau_T \frac{\partial \bar{U}}{\partial y} = \alpha\, \rho\, \bar{H}(r) \sqrt{\bar{e}}\; l \left(\frac{\partial \bar{U}}{\partial y}\right)^2 \tag{4}$$

The dissipation term was written as,

$$\epsilon \equiv \nu\, \overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial u_i'}{\partial x_j}} = \nu C \left[1 + \bar{H}(\kappa r)\; \alpha\; \kappa r\right] \frac{\bar{e}}{l_d^{\,2}} \tag{5}$$

where $l_d = l$, taken as the same function as in equation (1), and $\bar{H}(\kappa r)$ is the same function as given by equation (3) except that r is replaced by κr.

Glushko assumed that the total diffusion of turbulence energy was due to the gradient of $\bar{e}$ and specified the corresponding diffusion coefficient to be the same quantity given in the square brackets of equation (5). Hence, he obtained for the diffusion terms

$$\frac{\partial}{\partial y}\left[\nu \frac{\partial \bar{e}}{\partial y} - \overline{v'\left(\frac{p'}{\rho} + e\right)}\right] = \frac{\partial}{\partial y}\left\{\nu \left[1 + \bar{H}(\kappa r)\; \alpha\; \kappa r\right] \frac{\partial \bar{e}}{\partial y}\right\} \tag{6}$$

It is obvious that the generality of these assumed expressions for the production, dissipation, and diffusion of turbulent kinetic energy can only be determined by comparison of final results with data.

Computing Equations and Procedure

The computing procedure is essentially the same as that of Glushko (1965) except that the equations are transformed to similarity type coordinates. The main reason for transforming to similarity type coordinates is to provide scale factors that, in terms of the transformed variables ξ and η, reduce or remove the rate of increase in boundary-layer thickness with distance ξ along the surface. The number of computing steps, $\Delta\eta$, required across the boundary layer to obtain desired accuracy in the finite difference procedures can thereby be reduced and kept more nearly constant. Also, in a region of approximate local similarity, the streamwise step size $\Delta\xi$ can be increased since, for this situation, the rate of change of the dependent variables with ξ is much reduced.

The transformed variables are defined as

$$\xi(x) = \int_0^x \frac{U_\infty}{\nu}\, dx \tag{7a}$$

$$\eta(x,y) = \frac{U_\infty}{\nu (2\xi)^{\bar{n}}}\, y \tag{7b}$$

where $\bar{n}$ is, in general, a variable function of ξ and is determined from the requirement that η_δ (the boundary-layer thickness in the transformed coordinates) is constant.

The transformed normal velocity is defined as

$$\hat{V} = \frac{(2\xi)^{2\bar{n}}\, \nu}{U_\infty}\, F \frac{\partial y}{\partial x} + (2\xi)^{\bar{n}} \frac{\bar{V}}{U_\infty} \tag{8}$$

and the final computing equations are then written as

Momentum.-

$$(2\xi)^{2\bar{n}} F \frac{\partial F}{\partial \xi} + \hat{V} \frac{\partial F}{\partial \eta} = \frac{(2\xi)^{2\bar{n}}}{U_\infty} \frac{dU_\infty}{d\xi} (1 - F^2) + \frac{\partial}{\partial \eta}\left(M \frac{\partial F}{\partial \eta}\right) \tag{9}$$

[Tur]bulent kinetic energy.-

$$(2\xi)^{2\bar{n}} F \frac{\partial E}{\partial \xi} + \hat{V} \frac{\partial E}{\partial \eta} = - \frac{2(2\xi)^{2\bar{n}}}{U_\infty} \frac{dU_\infty}{d\xi} FE + (M - 1)\left(\frac{\partial F}{\partial \eta}\right)^2 + \frac{\partial}{\partial \eta}\left(D \frac{\partial E}{\partial \eta}\right) - C \frac{(2\xi)^{2\bar{n}}}{R_\delta^{\,2}} \frac{DE}{\varphi^2} \tag{10}$$

Continuity.-

$$(2\xi)^{2\bar{n}} \frac{\partial F}{\partial \xi} + \frac{\partial \hat{V}}{\partial \eta} + (2\xi)^{2\bar{n}} F\left(\frac{\bar{n}}{\xi} + \frac{d\bar{n}}{d\xi} \ln 2\xi\right) = 0 \tag{11}$$

where

$$F = \frac{\bar{U}}{U_\infty}, \quad E = \frac{\bar{e}}{U_\infty^{\,2}} \tag{12}$$

$$\begin{aligned} M &= 1 + \alpha r\, \bar{H}(r) = 1 + \alpha\, \bar{H}(r)\, \varphi \sqrt{E}\, R_\delta \\ D &= 1 + \alpha\, \kappa r\, \bar{H}(\kappa r) = 1 + \alpha\kappa\, \bar{H}(\kappa r)\, \varphi \sqrt{E}\, R_\delta \\ R_\delta &= \frac{U_\infty \delta}{\nu} \end{aligned} \tag{13}$$

The boundary conditions to be applied to the above system of equation are:

$$\begin{aligned} &\eta = 0: \quad F = E = 0,\ \hat{V} = 0, \text{ or } \hat{V} = (2\xi)^{\bar{n}} \frac{V}{U_\infty} \\ &\eta \to \infty: \quad F \to 1.0,\ E \to 0, \text{ or } E \to E_e \end{aligned} \tag{14}$$

where E_e is one-half the square of the free-stream turbulence intensity.

The system of equations (9) - (11) along with auxiliary functions for M, D, and $\bar{n}(x)$, is solved by a linear implicit finite difference procedure. This procedure combines certain aspects of the methods given in Glushko (1965) and Blottner (1964). The external velocity U_∞ and its derivative dU_∞/dx must be specified functions of x and profiles of $\hat{V}$, F, and E must be specified at the initial station ξ_0. For details of the procedure and results, the reader is referred to a forthcoming NASA publication by the present authors.

RESULT AND DISCUSSION

The effect of some modifications to the method of Glushko (1965) on both mean and fluctuating flow properties will be presented for flat-plate flow and for one of the experimental flows of Goldberg (1966) with a large adverse pressure gradient. The principal modifications considered are to the l/δ function (eq. (1)) and to the dissipation and diffusion terms (eqs. (5) and (6)). Also, as many of the Stanford cases as possible will be run with the "standard" inputs of velocity distribution and initial inputs of C_f, H, θ, and velocity profiles. These results will be presented on the standard output plots of H, C_f, and R_θ.

It should be noted that in all calculations by the present method the skin friction has been computed from the correct limiting form evaluated at the wall as,

$$\tau_w = \mu \left(\frac{\partial \bar{U}}{\partial y}\right)_w \tag{15}$$

The φ functions used in the present calculations were obtained from the following tabulated values, where the $\varphi_{.33}$ function is based on the Glushko (1965) result.

y_2/δ	$\varphi_{.33}$	l/δ $\varphi_{.25}$	$\varphi_{.20}$
0	0	0	0
.2	.20	.20	.20
.4	.30	.25	.20
.5	.33	.25	.20
.6	.32	.25	.20
.7	.30	.25	.20
.8	.26	.20	.20
$\geq$ 1.4	.01	.01	.01

Linear interpolation between the tabulated values was used in the solutions. Prandtl's mixing length relation is thereby recovered in the "law of the wall" region (say $x_2/\delta < 0.2$) where production and turbulent dissipation are approximately equal. That is, by equating production and dissipation there is obtained (with $r \gg 1$)

$$\frac{\tau_T}{\rho} \cong \frac{\alpha}{\sqrt{\kappa C}} \, l^2 \left(\frac{\partial \bar{U}}{\partial y}\right)^2 \tag{16}$$

Then with $l = y$, $\alpha = 0.2$, $\kappa = 0.4$, $C = 3.93$ (Glushko (1965))

$$\frac{\tau_T}{\rho} \cong (0.4y)^2 \left(\frac{\partial \bar{U}}{\partial y}\right)^2$$

which corresponds to the Prandtl mixing length relation for turbulent boundary layers.

Flat-Plate Flow

The calculation was started at $Re_x = 10^4$ with input values of F and $\hat{V}$ from exact numerical solutions to the laminar Blasius flow. The input profile for the turbulent kinetic energy was taken as (see Glushko (1965))

$$E(\xi_o, \eta) = E_o{}^* \left(\frac{\eta}{\eta^*}\right)^2 \exp^2 \left\{\frac{1}{2}\left[1 - \left(\frac{\eta}{\eta^*}\right)^2\right]\right\} \tag{17}$$

where $E_o{}^*$ and η^*/η_e are specified constants. Unless otherwise noted, the results shown herein were computed with $E_o{}^* = 2.5 \times 10^{-4}$ and $\frac{\eta^*}{\eta_e} = 0.4$. Additional required inputs were;

$U_e = 100$ ft/sec , $\nu = 1.58 \times 10^{-4}$ ft^2/sec,

$\eta_{e,o} = 4.95$, $1 - F_e = 1 \times 10^{-4}$, $\left(\frac{\partial F}{\partial \eta}\right)_e = 3.4 \times 10^{-4}$

$\Delta\eta_o = 0.05,$ $\qquad 1 - F_\delta = 0.01$ and $E_{e,o} = 0.82 \times 10^{-6}.$

The $\varphi_{.33}$ function was used in the flat-plate solution. The values of the constants in the transport functions M and D as used by Glushko were $\alpha = 0.2$, $\kappa = 0.4$, $C = 3.93$, $r_o = 110$. These values and the $\varphi_{.33}$ function were adjusted by Glushko to give agreement with flat-plate flow. In the present report, this $\varphi_{.33}$ function as well as the $\varphi_{.25}$ and $\varphi_{.20}$ functions as tabulated on p. 6, will also be applied to adverse pressure gradient flows.

Mean flow quantities.- The variation of the form factor H with Re_x is shown in figure 1. The value of H is at first approximately constant at the initial value of 2.592, corresponding to the Blasius solution for laminar flow, and then H abruptly decreases at some value of Re_x. Increasing the diffusion term by a factor of 3 over the original Glushko form (that is, multiplying equation (6) by 3) increased the value of x (or Re_x) where the mean profiles first began to change from the laminar input shape, by a factor of about 1-1/3. (This value of the Reynolds number will be designated $Re_{x,t}$, and due to the behavior of the mean flow properties can be considered analogous to a transition Reynolds number.) However, the H curve appeared to approach the same asymptotic value of approximately 1.4 which is in agreement with the data of Wieghardt (1951) considered typical of flat-plate flows.

The effects of this modification (to the diffusion term) on C_f as shown in figure 2 are of the same nature as the effects on H. That is, the "transition" Reynolds number is increased by the same factor when the larger diffusion term is used but the final asymptotic variation of C_f is in agreement with that for fully turbulent flow as obtained from Schlicting (1960), p. 540, and the data of Wieghardt (1951). It is also of interest to note that when the input disturbance level E_o^* was reduced to 1×10^{-8} from 2.5×10^{-4}, the "transition" Reynolds number was further increased to about 8×10^4.

The computed values of the mean velocity are plotted in conventional profile form in figure 3 and it is seen that the profiles develop from the laminar input profile at $Re_x = 1 \times 10^4$ to turbulent type profiles for $Re_x \geq 1 \times 10^6$. At this Reynolds number the profile shapes have apparently "settled out" to the shape characteristic of turbulent boundary layers as indicated by the data of Wieghardt (1951). When the turbulent diffusion term is increased by a factor of 3, the agreement with data is improved, particularly in the outer part of the boundary layer.

The "standard" Wieghardt flat-plate flow (IDENT 1400) was computed with the Glushko diffusion (eq. (6)) increased by three, two values of E_o^*, and two l/δ functions. The $\varphi_{.25}$ function appears to give somewhat better agreement with data than the $\varphi_{.33}$ function, except for the values of C_f at $x > 3.0$ meters.

For all the calculations presented herein, the value of $\bar{n}$ was retained as a constant. It was found that appreciable savings in computer time could be obtained by the use of a value of $\bar{n}$ appropriate to the type of flow being computed. For example, for the flat-plate problem discussed in this section, if the value of $\bar{n}$ at $\xi = 5 \times 10^4$ was changed from 0.5 to 0.8 the number of $\Delta\eta$ steps required at $\xi = 10^6$ was reduced from 290 to 125. Actually the problem had to be reinitialized at $\xi = 5 \times 10^4$ to accommodate the new value of $\bar{n}$, but the computer time from that point on was reduced by approximately one-half.

Fluctuating flow quantities.- In figure 4 the computed values of $\sqrt{E}$ (total turbulent intensity divided by $\sqrt{2}$) are plotted against $y/\delta_F = 0.995$ for several values of Re_x and with the same modification to the diffusion term as noted previously. The changes in $\sqrt{E}$ profiles with Reynolds number duplicate those of Glushko with some dependence on the peak turbulent intensity

$E_o{}^*$ at the initial station $(\xi_o = 4 \times 10^4)$. That is, when $E_o{}^*$ is decreased, the peak values of $\sqrt{E}$ at subsequent stations in the transition region are decreased, figure 4(b), but the profiles for $Re_x > 10^6$ are of the same shape and magnitude regardless of the value of $E_o{}^*$. This computed behavior of the $\sqrt{E}$ profiles is qualitatively in agreement with experimental observations in the transition region. The factor of three increase of the turbulent diffusion term improved the agreement with the Klebanoff (1955) data and decreased $\sqrt{E}$ in most of the boundary layer except near the outer edge where $\sqrt{E}$ was increased by factors of two or more. These latter effects would be expected due to the gradient type model (eq. (6)) used to formulate the diffusion term. The simple expedient of increasing this term by a factor of three produced better agreement with data for both the $\sqrt{E}$ and profiles of $\tau/\rho\bar{e}$ (not shown herein). Consequently, the remaining discussion will be limited to those results with the turbulent diffusion term multiplied by three. It is emphasized that these simple modifications appeared to improve the agreement between the calculations and experimental data of the fluctuating and mean characteristics of the "fully" turbulent boundary layer and also improved the agreement with data of the "transition" Reynolds number $Re_{x,t}$. That is, even the minimum transition Reynolds number observed in experiments is still somewhat larger than the maximum value obtained here of about 8×10^4. It therefore appears possible that criteria for selecting the "best" models for the turbulent terms may depend not only on the results for developed turbulent flow but also on the location and behavior of the transitional type flow.

Nonequilibrium Boundary Layers

The data from the investigation of Goldberg (1966) chosen as the test case for detailed discussion (his pressure distribution number 3) should be a particularly severe test of the method because the boundary layer was first driven nearly to separation by a large adverse pressure gradient and then allowed to relax toward a flat-plate flow by imposing a constant pressure. Also, hot-wire measurements of turbulent shear and longitudinal turbulence intensity were available.

The distribution of external velocity and its derivative with respect to x as used to obtain the present results are shown in figure 5. Since there was some uncertainty in reading the small graphs published by Goldberg (1966), two alternate velocity distributions are shown in figure 5(a) and the differences between them are within the reading accuracy of Goldberg's original figures and probably within the experimental errors of the original data. The derivatives of the two velocity curves, however, are considerably different as shown in figure 5(b) where dU_e/dx as used in the calculations, is plotted against x. These differences can become important, as will be shown, when the boundary layer approaches separation. Table I lists the values of $\bar{U}/U_e$ and E used at the input station corresponding to $x = 4$ inches (presumably the distance from the nose of the 10-inch-diameter test cylinder). The initial ξ_o for this station was computed by assuming flat-plate flow at $U_e = 85$ ft/sec with sufficient Reynolds number to give the observed skin friction of 0.00350. The initial velocity profile was taken directly from the data plot of Goldberg (1966) at $x = 4$ inches and the initial E profile was taken from the measured longitudinal intensity at the same station as

$$E = \frac{\overline{u'^2} + \overline{v'^2} + \overline{w'^2}}{2U_\infty^2} \cong \left(\frac{\overline{u'^2}}{U_\infty^2}\right)_{\text{measured}} \qquad (18)$$

since an examination of data where all three components had been measured (Klebanoff (1955), for example) showed that this relation was approximately correct for $0.1 < y/\delta < 0.8$.

Relation between skin friction and $\sqrt{E}$ profiles.- The calculated variation in skin friction is compared with the experimental data in figure 6. The vertical arrows represent the spread in the experimental skin friction data obtained by four methods as discussed by Goldberg (1966). Theoretical results are presented for both velocity distributions of figure 5, three values of C (dissipation constant), and the three l/δ functions. For velocity distribution number 1, C = 3.93, and $\varphi_{.33}$, the agreement with data is good except in the region of the minimum C_f where the present method overpredicts C_f by as much as 100 percent. The agreement is better when the alternate velocity distribution number 2 was used indicating the sensitivity of the results in this region to the imposed velocity distribution.

In order to determine the relative importance of the dissipation term (eq. (5)) for this particular type of flow, additional solutions with C = 5 and 6 for pressure distribution number 2 were obtained. An increase in C increases the dissipation and reduces the skin friction by an almost constant amount over the entire test region and gives improved agreement with data near the minimum C_f region at the expense of poorer agreement elsewhere. Since the turbulent kinetic energy equation can be paraphrased as

$$\frac{D\overline{e}}{Dt} = \text{Production} + \text{Diffusion} - \text{Dissipation} \tag{19}$$

it is evident that an increase in dissipation should decrease $\overline{e}$, and this, in turn, should decrease the turbulent shear which from equation (2) is approximately

$$\tau_T \cong \alpha \rho \sqrt{\overline{e}} \quad l \frac{\partial \overline{U}}{\partial y} \tag{20}$$

for $r > r_o$. Since the skin friction depends directly on the magnitude of τ_T in the wall region, the noted decrease in C_f appears reasonable, although it is somewhat surprising to find the almost linear (inverse) relation between the magnitudes of the dissipation and C_f.

A physical "explanation" for the improved agreement between the computed and experimental skin friction in the minimum C_f region, as caused by larger values of C, is to be found in the behavior of the $\overline{e}$ profiles which are shown in figure 7. The theoretical values in this figure were computed with the diffusion term taken as three times Glushko diffusion (3 × eq. (6)), velocity distributions 1 and 2, different values of C, and different l/δ functions. The first thing to note is that all these various modifications had only minor effects on the magnitude and distribution of $\sqrt{E}$. An increase in C does reduce $\sqrt{E}$, as it should according to equations (19) and (20), but the best agreement with the data is generally obtained with C = 3.93. The input profile of $\sqrt{E}$ (fig. 7(a)) is similar to the flat-plate profiles of figure 5. As the minimum C_f region is approached (fig. 7(b)) the peak in both the computed and measured $\sqrt{E}$ profiles moves away from the wall and increases in magnitude, and there is a corresponding large increase in the average turbulent kinetic energy $(\overline{e})_{ave}$ across the entire boundary layer. That this increase in $(\overline{e})_{ave}$ can be associated with a decrease in a dissipation length scale l_d can be seen by noting that for large r, the dissipation (eq. (5)) is approximately

$$\epsilon = \kappa\alpha \, (\overline{e})^{3/2} \frac{Cl}{l_d^{\,2}} \tag{21}$$

where l_d is defined here as a microscale of the turbulence. Then if C is regarded as a "universal" constant, the larger values of C as used for the solutions of figure 6 should be considered as equivalent to corresponding decreases in the square of the dissipation scale, l_d^2. If l is considered an integral scale, the relation between l and l_d for isotropic turbulence (Hinze (1959), p. 185) may be writt[illegible] as

$$\left(\frac{l_d}{l}\right) \sim \frac{1}{\left(\frac{\sqrt{\bar{e}}\, l}{\nu}\right)_{ave}} \tag{22}$$

and this relation is seen to be in agreement with the results of figure 6 as related to the change with x in $\sqrt{E}$ profiles of figure 7, since as $(\bar{e})_{ave}$ increases, l_d should decrease which, according to the above reasoning, accounts for an increase in dissipation, a reduction in $\sqrt{\bar{e}}$, and finally, the reduction in C_f, as computed. It can then be expected that a functional relation for l_d of the type given by equation (22) would result in good agreement over the entire test length for this case. The above explanation depends on the assumption that the integral scale l is not affected by $(\bar{e})_{ave}$.

Boundary-layer thickness parameters and mean velocity profiles.- The computed values of θ are compared in figure 8 with the experimental data of Goldberg (1966). The differences in the values of θ between the theoretical calculations and the experimental data are small, but, as is well known, these small differences can lead to large effects on C_f. In general, the velocity distribution number 1 gives better agreement with the data but it is obvious that a comparison of θ alone is not sufficient to judge the accuracy of a method when $C_f \to 0$.

The shape parameter H is a more sensitive indicator of the accuracy of a method as shown by figure 9 where the effects of the two velocity distributions, the values of C, and the l/δ functions are shown. The velocity distribution number 2 and the largest value of C give the best agreement with the experimental data for $l/\delta = \varphi_{.33}$. However, the use of the other two l/δ functions gives the best overall agreement with the data, and bracket the data in the vicinity of the peak H. The reason for the better agreement of H with data for $l/\delta = \varphi_{.20}$ and $\varphi_{.25}$ is apparent from comparisons of computed velocity profiles (not shown herein) with the data. In the vicinity of the minimum C_f the agreement between the computed and experimental velocity profiles was poor for $l/\delta = \varphi_{.33}$ regardless of the velocity distribution or the value of C. However, when $\varphi_{.20}$ and $\varphi_{.25}$ were used, the theoretical results bracketed the data.

Discussion of turbulence scale functions.- It has already been noted from figure 7 that $\sqrt{E}$ is relatively insensitive to changes in free-stream velocity distribution, C, and the l/δ function. The computed mean velocity profiles were insensitive to velocity distribution and C, but could be affected considerably by a change in l/δ. Since the turbulent shear, production, and dissipation are the dominant terms in the equations, it is apparent from the above discussion and from the form of these terms (see eqs. (2) - (5)) that the only way to effect the mean velocity profiles, and hence the H values to any appreciable extent is to modify the turbulence scale function l/δ. In the previous section, a change in C was related to a change in a dissipation microscale l_d, but from equation (16), a change in C corresponds also to a change in Prandtl's mixing length constant. Since Prandtl's mixing length relation is known to apply even in an [illegible]erse pressure gradient in the law of the wall region, the l/δ function was not changed for $y/\delta < 0.2$. The Goldberg (1965) experimental values of mixing length [illegible] that l/δ should be decreased in the region of $y/\delta > 0.$[illegible]. [illegible] decrease would reduce the turbulent shear and thereby result i[illegible]ment with the experimental velocity profiles, the l/δ function [illegible]ged in the manner shown by the tabulations on p. 6. The eddy viscosity function ϵ/μ does not depend on free-stream velocity distribution or C (since $\sqrt{E}$ was independent of these parameters) but is directly dependent on l/δ and agrees better with the data for $l/\delta = \varphi_{.20}$. This direct dependence of ϵ/μ on l/δ is, of course, the reason for the marked effect of the l/δ function on both C_f (fig. 6) and $\bar{U}/U_\infty$. The success of these modifications to l/δ indicates that further minor adjustments to these functions should give any desired degree of agreement with experimental data. In particular, the trends in $\sqrt{E}$ (fig. 7) for this problem indicate the magnitude of the change in l/δ should probably depend on the level of $(\bar{e})_{ave}$. Spalding (1967a) has applied to free shear flows a differential equation for the mean scale of turbulence. This equation was based on Rotta's hypothesis (1951, Part II) and relates the scale of turbulence to $(\bar{e})_{ave}$ and τ_T. It is possible that the use of a similar relation may improve the predictions of this present method.

Other adverse pressure gradient cases.- Additional adverse pressure gradient cases have been computed with the "standard" velocity distributions as supplied for the Stanford cases and for various values of C. The l/δ function used for these additional cases was generally the $\varphi_{.25}$ function as tabulated on p. 6. Exceptions are noted on the figures. Some difficulty was experienced in obtaining these solutions since input values of E were not available, except for Bradshaw's (IDENT 2400). When input E values were not available, a trial and error reinitializing procedure to obtain input E was used. This procedure was satisfactory in some cases such as Ludwig Tillman (IDENT 1200) but in other cases such as Moses (IDENT 3800), the final results appeared to be quite sensitive to input E profiles as well as input profiles of $\bar{V}$.

CONCLUDING REMARKS

The equations for the incompressible, turbulent boundary layer with constant fluid properties have been solved by a numerical procedure in similarity type coordinates. Comparisons of calculated values for both mean and fluctuating flow properties with experimental measurements in nonequilibrium boundary layers as well as the flat-plate boundary layer have indicated generally good agreement.

For the flat-plate calculation, the laminar Blasius velocity profile and arbitrary small "disturbance type" profiles for the turbulent kinetic energy were used as initial conditions at a Reynolds number of 10^4. As the calculation proceeded, little change in the mean profiles of velocity and turbulent kinetic energy were noted until at some downstream station, depending on the level of the input disturbance and modifications to the turbulence terms, rather abrupt changes began and the subsequent mean velocity profiles were qualitatively similar to those observed experimentally in the transition region between laminar and fully turbulent flow. The Reynolds number at which these changes in the mean profiles were first obtained in the calculation can therefore be termed a "transition" Reynolds number and its dependence on the level of the input turbulent kinetic energy was shown by Glushko; it is shown herein that modifications to the models of the turbulent terms also affected this "transition" Reynolds number.

It was found that when the turbulent diffusion term of Glushko's was increased by a factor of 3, the agreement with experimental values of mean velocity and the ratio of turbulent shear to turbulent energy was improved in the outer portion of the fully turbulent boundary layer for both the flat-plate and nonequilibrium flows. It was also found that when the dissipation term was changed for the nonequilibrium flow, the skin friction was reduced by almost constant increments that depended directly on reductions in the square of the dissipation scale. Analysis of these results indicated that the micro-(dissipation) scale may be related to the integral scale of turbulence in about the same way as for isotropic turbulence. The best overall agreement with measured values of skin friction, form factor, mean velocity profiles, and fluctuating properties was obtained by reducing the value of the turbulence scale from a peak of 0.33 to 0.20 or 0.25 of the boundary-layer thickness in the outer part of the boundary layer. The linear relation of turbulence scale with distance from the wall, in accordance with Prandtl's mixing length theory and as used by Glushko, was retained in the "law of the wall region." It is concluded that simple modifications to the turbulence scale function and to the turbulent fluctuation terms as modeled by Glushko result in accurate predictions of mean and fluctuating characteristics of turbulent and transitional boundary layers with arbitrary boundary conditions.

TABLE I.- INITIAL VELOCITY AND ENERGY PROFILES AND OTHER

INPUT DATA FOR ADVERSE PRESSURE GRADIENT CASE

Values based on data of Goldberg (1966)

$\xi_o = 8 \times 10^5$, $\nu = 1.8 \times 10^{-4}$ ft^2/sec , $\rho = 0.076$[illegible] lb/ft^3 ,

$U_{e,o} = 77.8$ ft/sec , $\Delta\xi = 625$, $\Delta\eta = 0.025$, $\bar{n} = 0.5$,

$1 - F_e = 8 \times 10^{-5}$, $\left(\frac{\partial F}{\partial \eta}\right)_e = 1 \times 10^{-5}$, $E_e = 1 \times 10^{-4}$

η	F_{ξ_o}	E_{ξ_o}
0	0	0
.1	.215	.22 $\times 10^{-2}$
.2	.415	.54
.3	.540	1.05
.4	.580	1.04
.5	.608	.92
.6	.626	.86
.8	.644	.78
1.0	.662	.72
1.2	.681	.675
1.4	.700	.64
1.8	.731	.615
2.2	.760	.579
2.6	.790	.530
3.0	.818	.474
4.0	.880	.321
5.0	.924	.203
6.0	.956	.108
6.5	.968	.069
7.0	.975	.039
7.5	.982	.010
8.0	.991	.01
8.5	.996	.01
9.0	.999	.01

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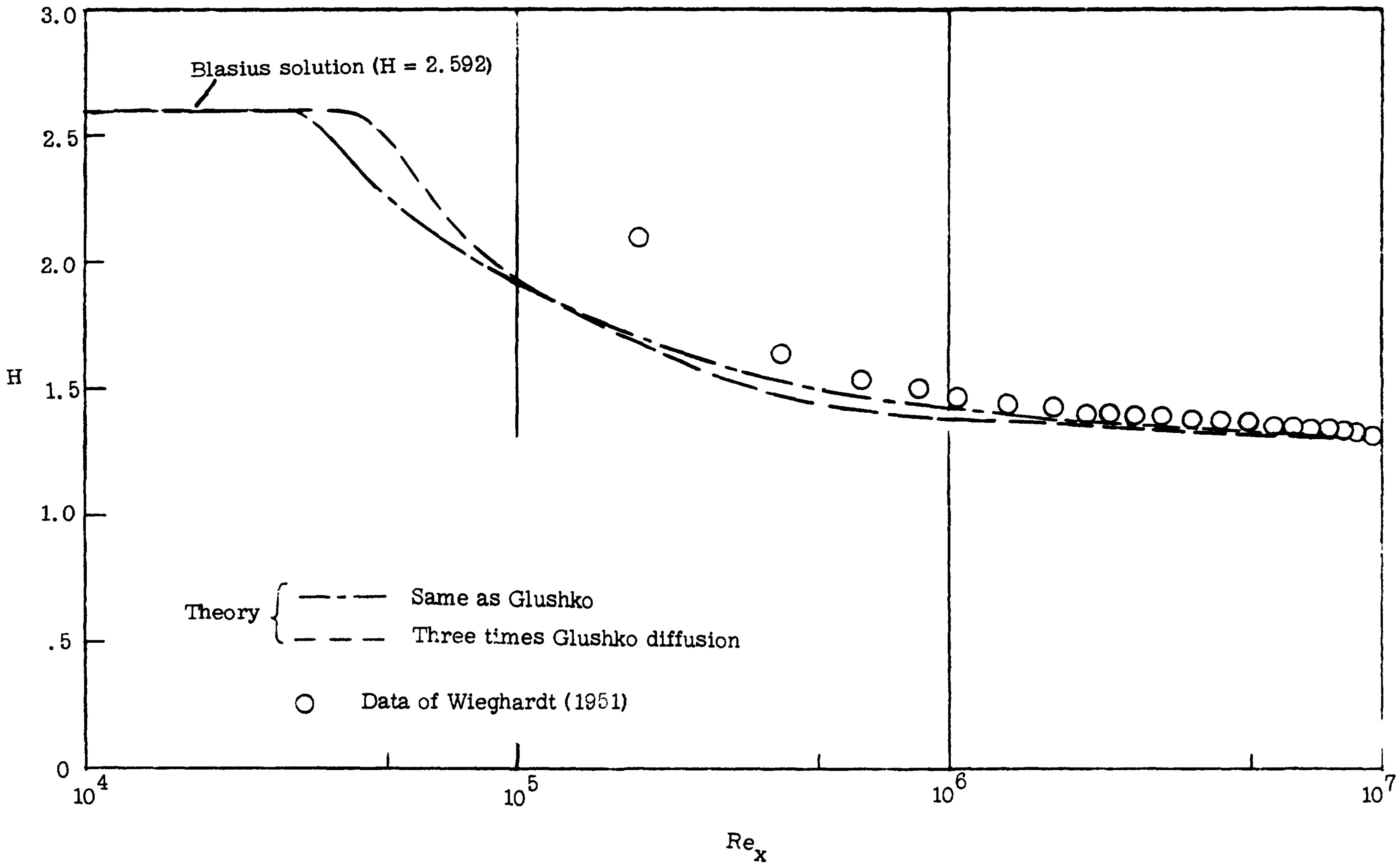


Figure 1.- Effect of modification to diffusion term on form factor H for flat-plate flow.

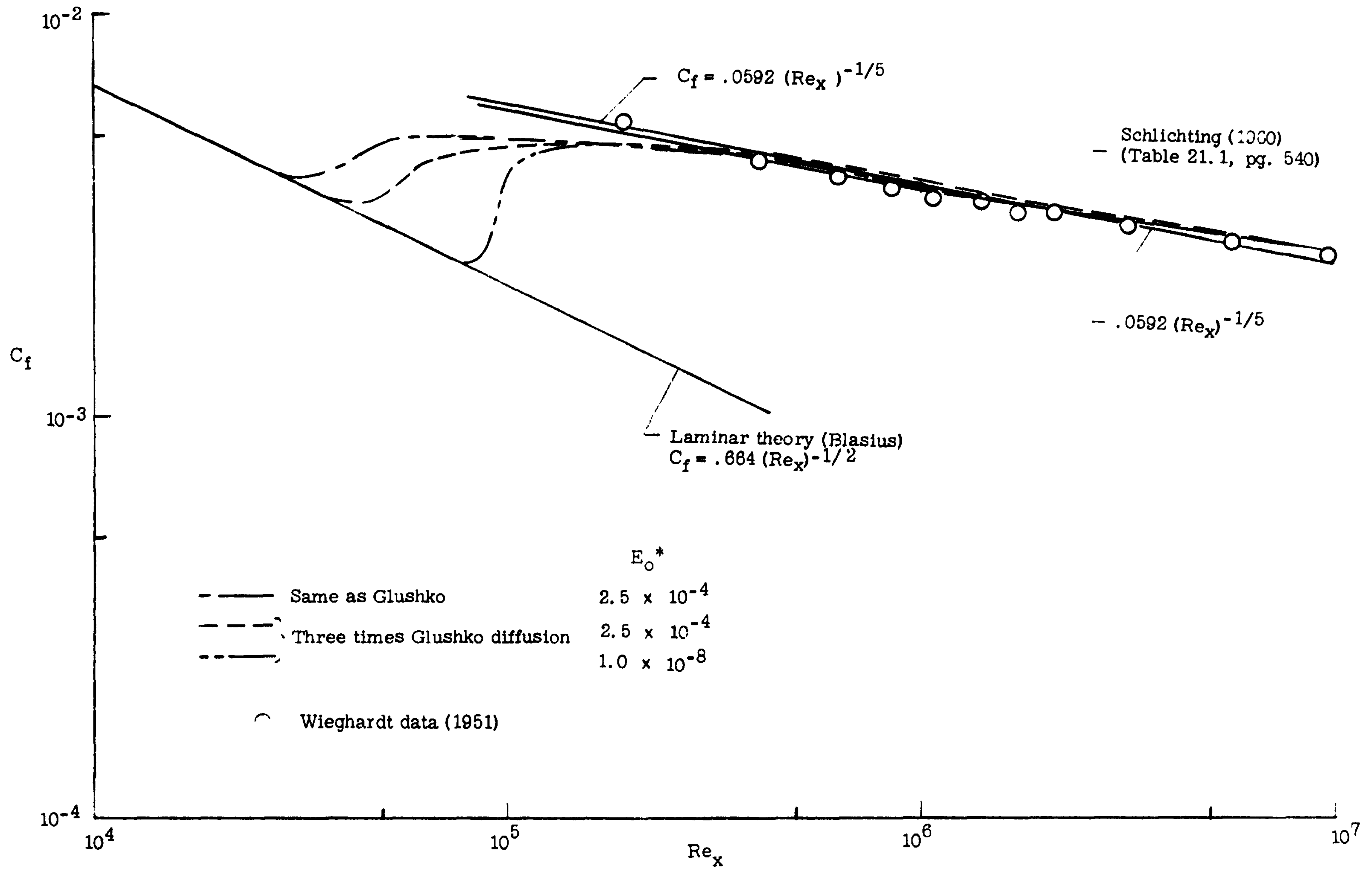


Figure 2.- Effects of modification to diffusion term and of different values of E_o^* (maximum input E at ξ_o) on C_f distribution for flat-plate flow.

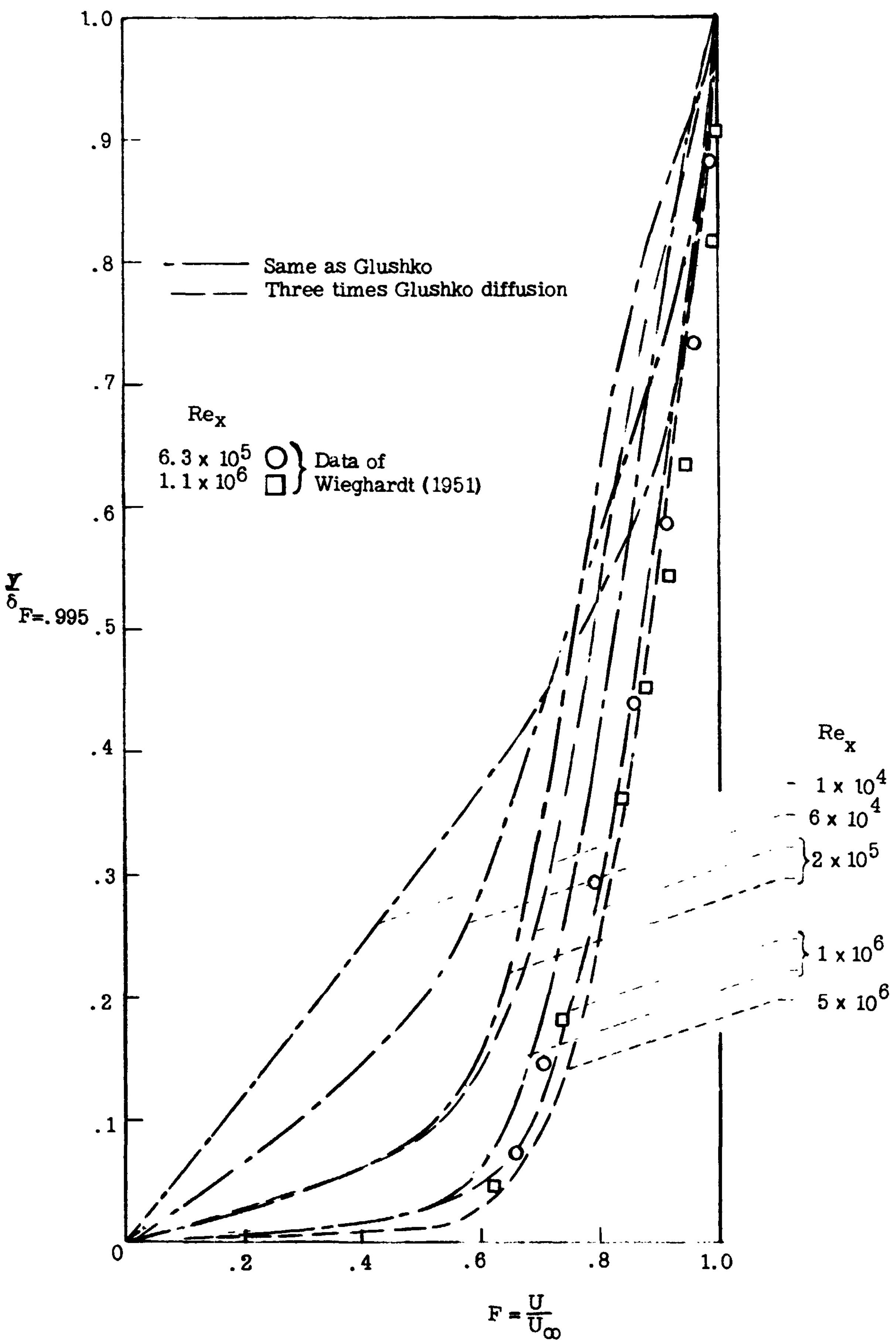
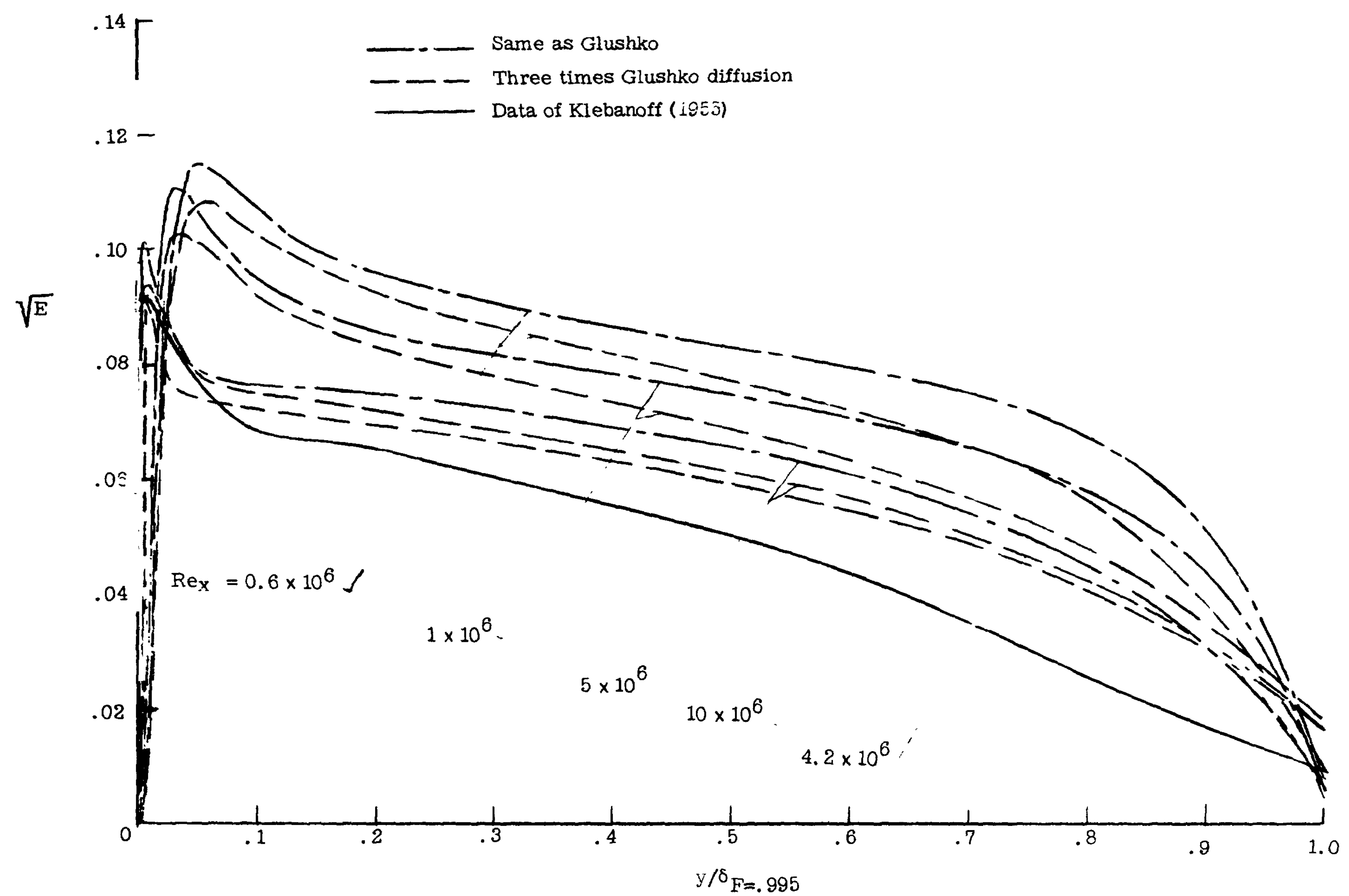
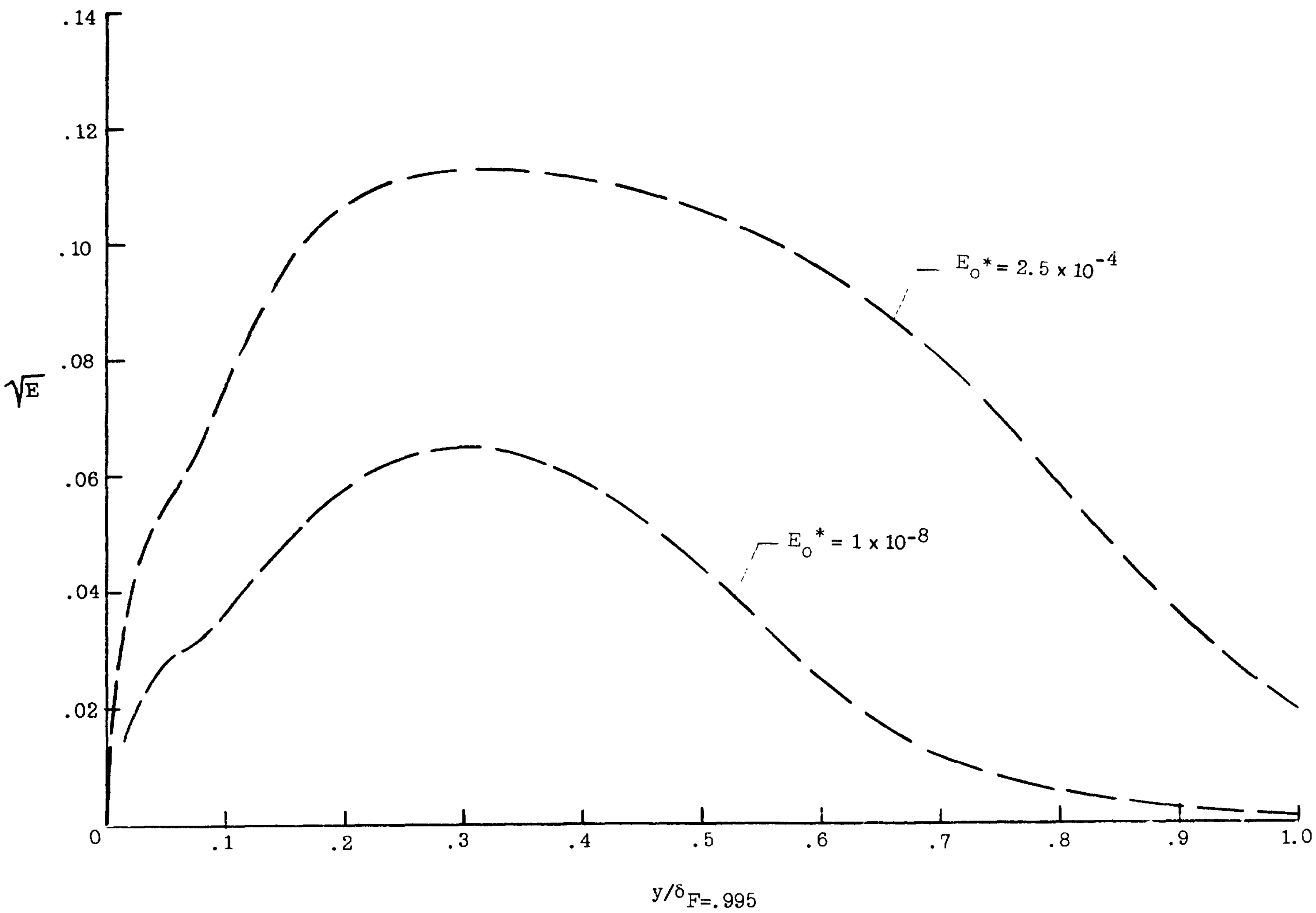


Figure 3.- Development of velocity profiles from laminar input at $Re_x = 1 \times 10^4$ to turbulent at $Re_x \geq 1 \times 10^6$ for $dp/dx = 0$.



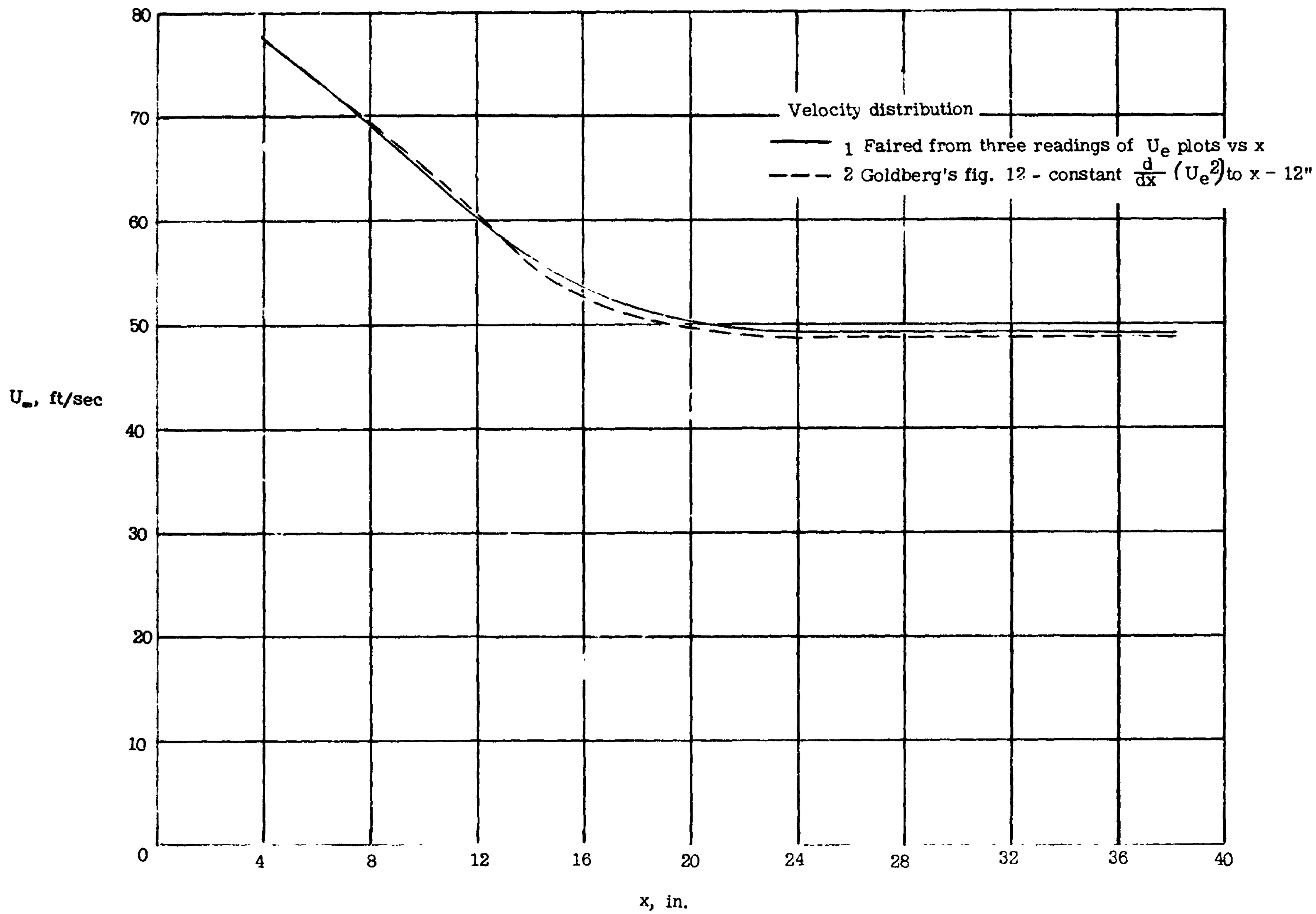
(a) Curves shown are for $E_o* = 2.5 \times 10^{-4}$, but for $Re_x \geq 5 \times 10^6$ the use of $E_o* = 1 \times 10^{-8}$ gives same curves.

Figure 4.- The variation in $\sqrt{E} = \sqrt{\overline{u'}^2 + \overline{v'}^2 + \overline{w'}^2}/2\,U_\infty$ across the flat-plate boundary layer; as computed for various values of Re_x with different assumptions for diffusion, and as measured by Klebanoff (1955).



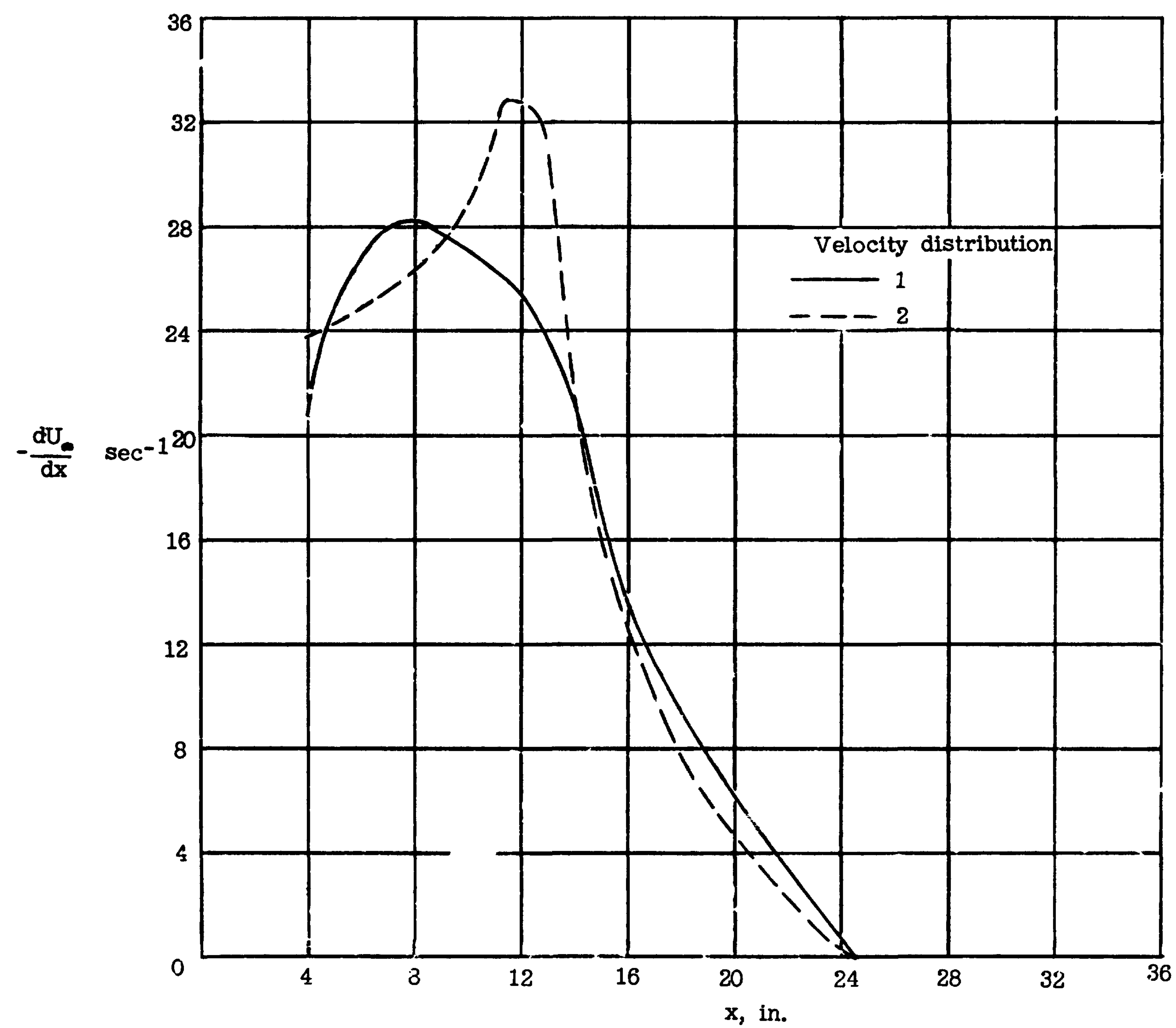
(b) Effect of E_o* in transition region $\left(Re_x = 9 \times 10^4\right)$. Diffusion = three times Glushko diffusion.

Figure 4.- Concluded.



(a) Velocity variation with x.

Figure 5.- Free-stream flow distribution from the Goldberg (1966) pressure distribution number 3 as used in the present calculations.



(b) Variation of velocity derivative.

Figure 5.- Concluded.

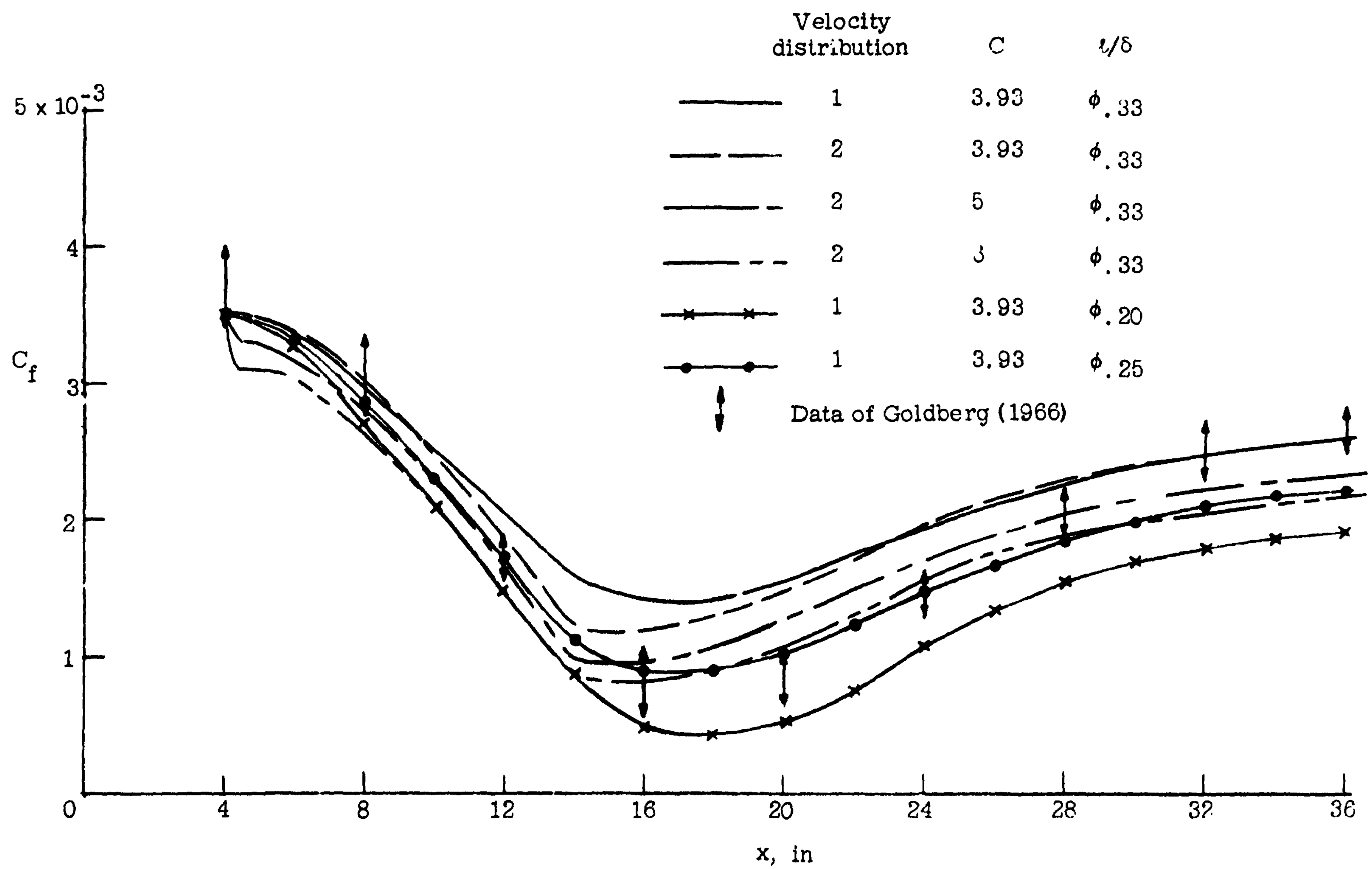
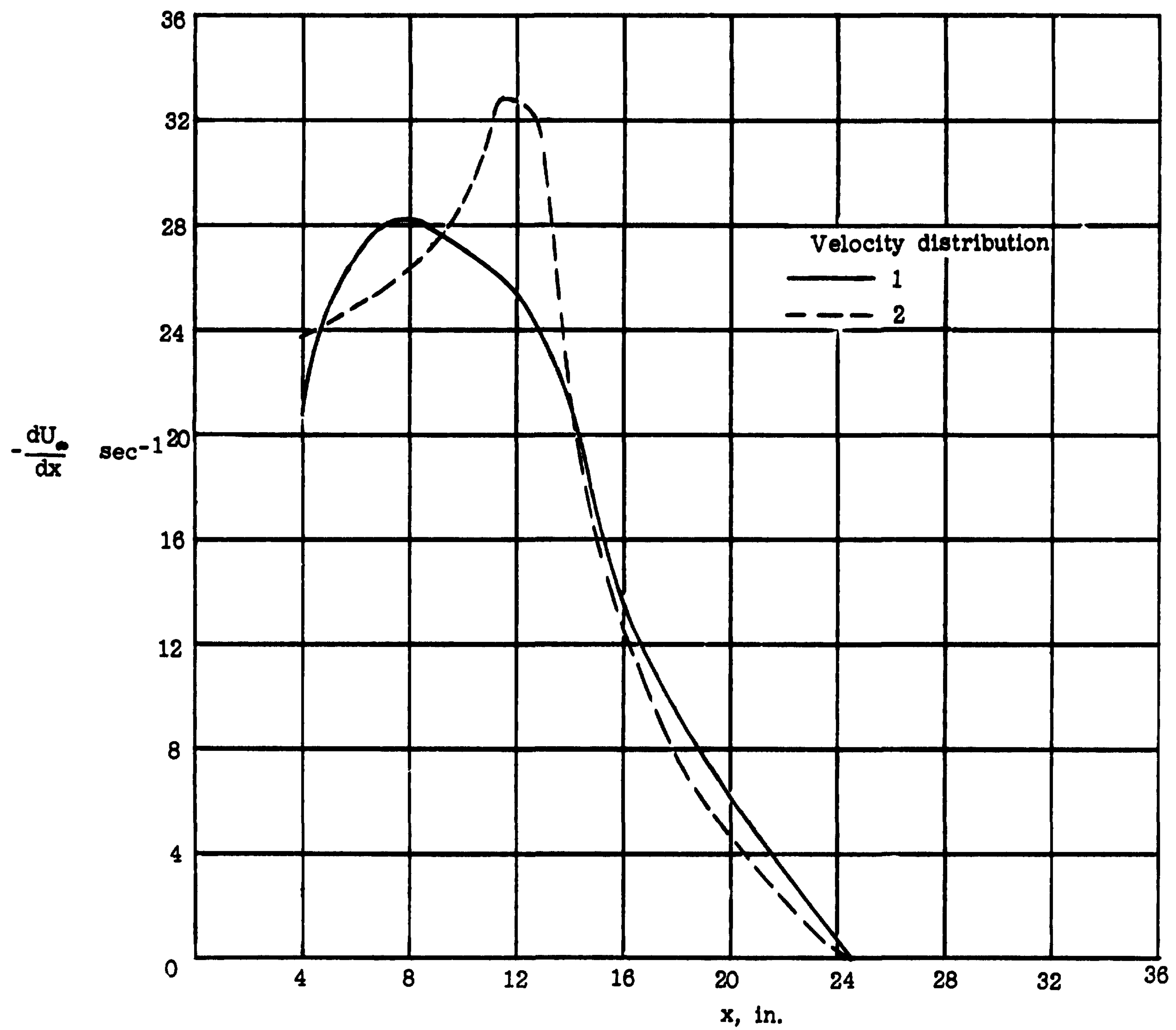
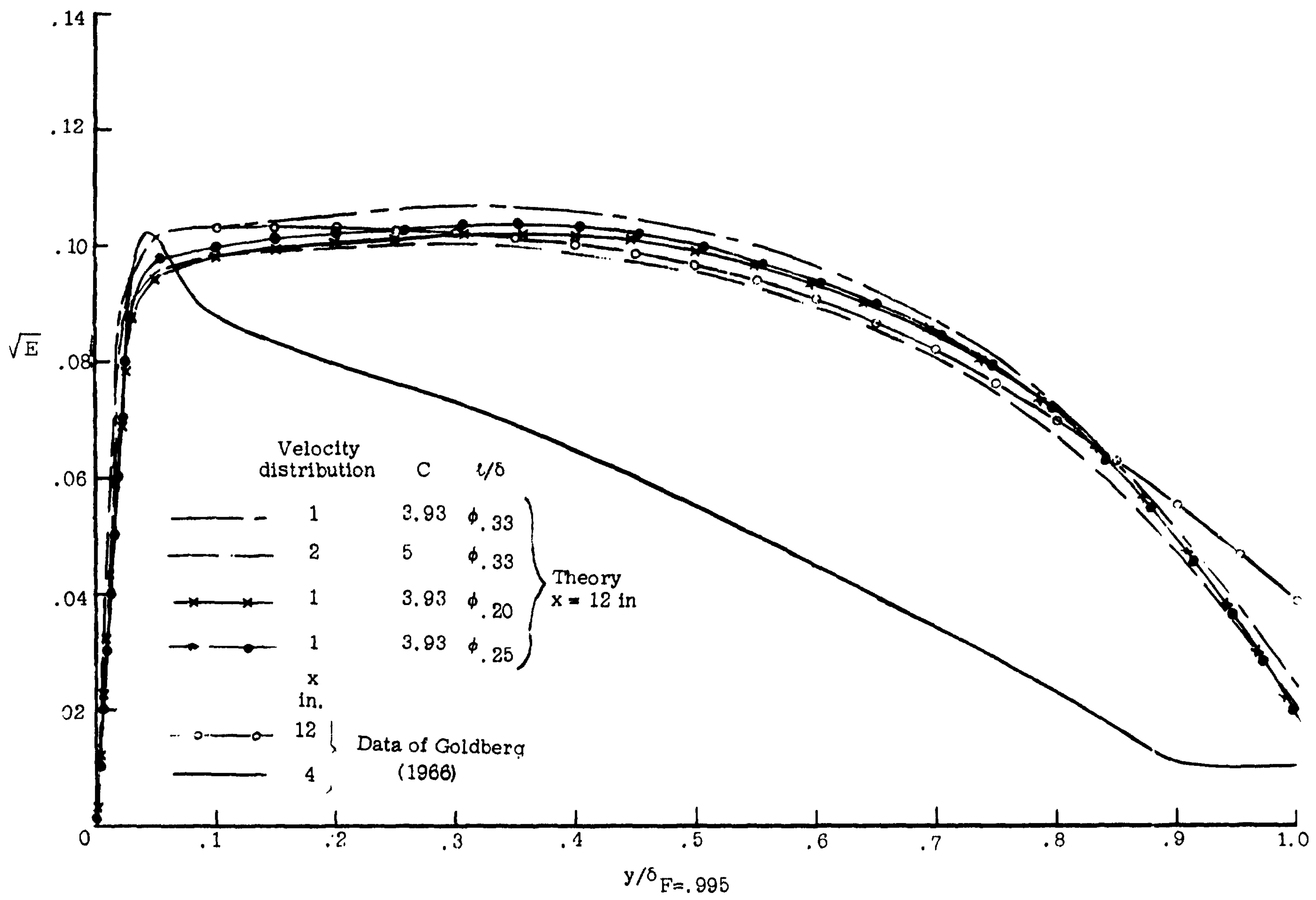


Figure 6.- Comparison of calculated skin friction with measured values of Goldberg (1966) for different velocity distributions, values of C, and l/δ functions. Diffusion = 3 times Glushko diffusion and $\bar{H}(r) = 1$ in outer part of boundary layer.



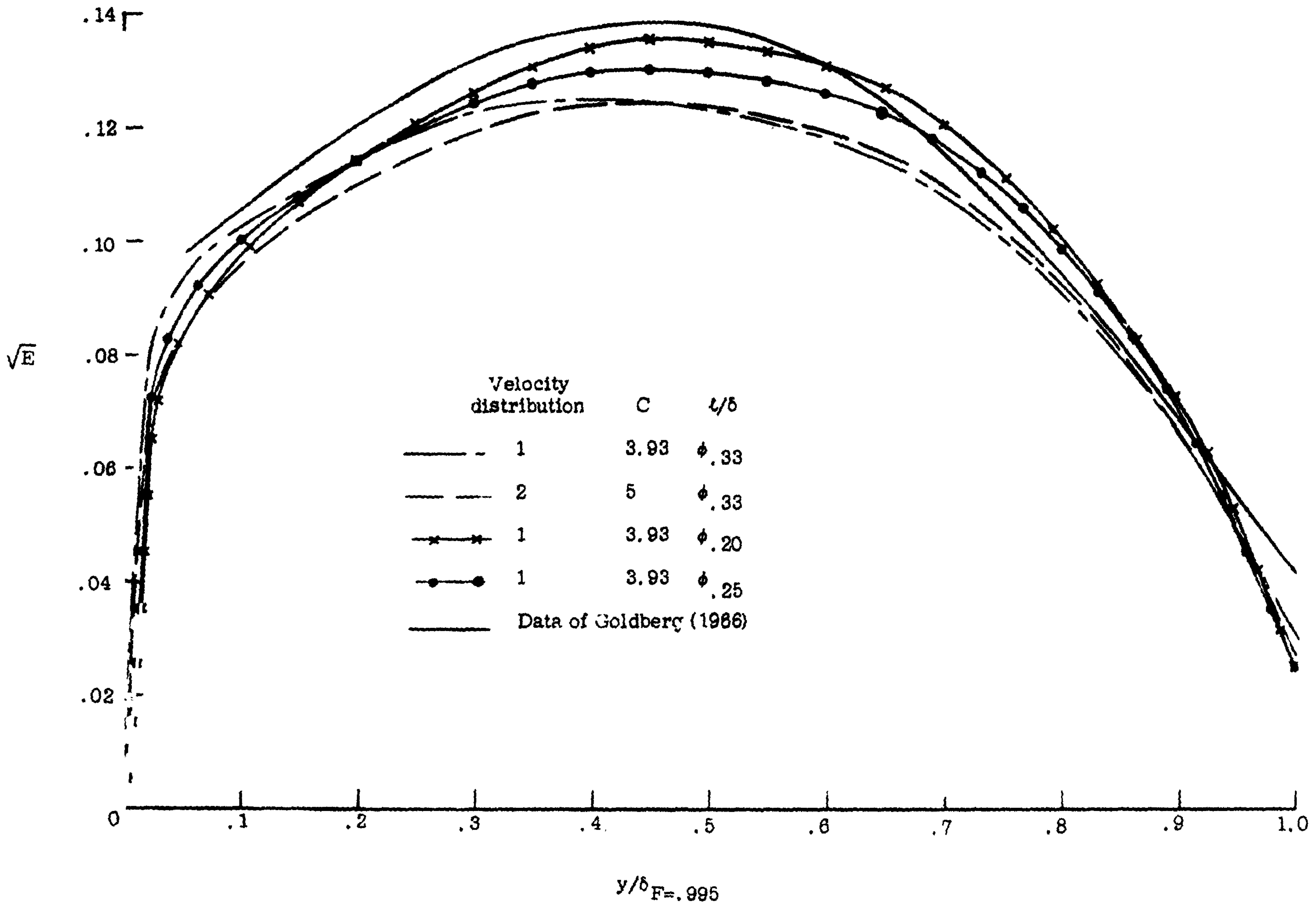
(b) Variation of velocity derivative.

Figure 5.- Concluded.



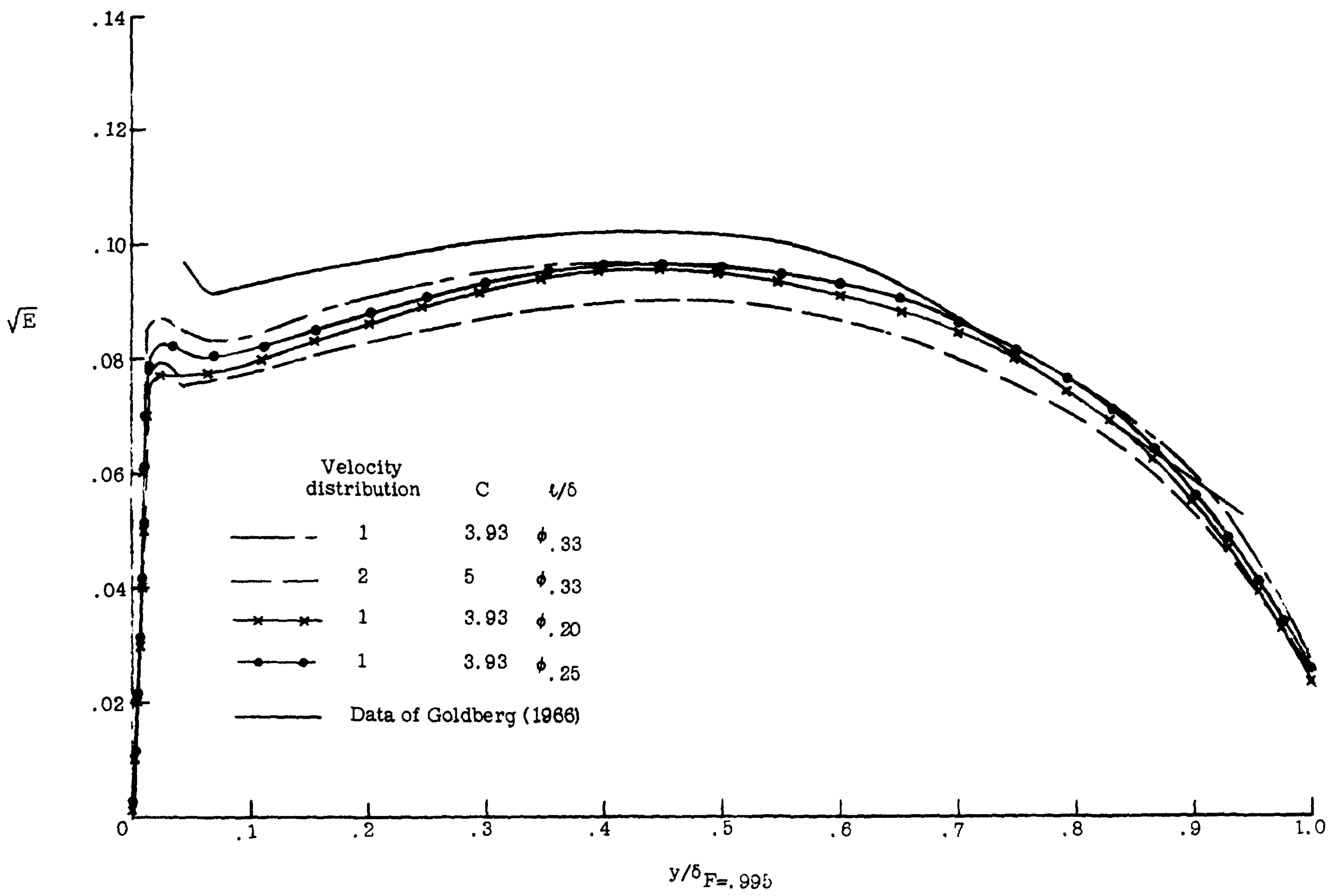
(a) x = 12 inches and input profile used to start solution at x = 4 inches.

Figure 7.- Calculated distribution of $\sqrt{E}$ compared with distribution obtained from measured longitudinal turbulence intensity of Goldberg (1966). Diffusion = 3 times Glushko diffusion, and $\bar{H}(r) = 1$ in outer part of boundary layer.



(b) x = 20 inches.

Figure 7.- Continued.



(c) x = 36 inches.

Figure 7.- Concluded.

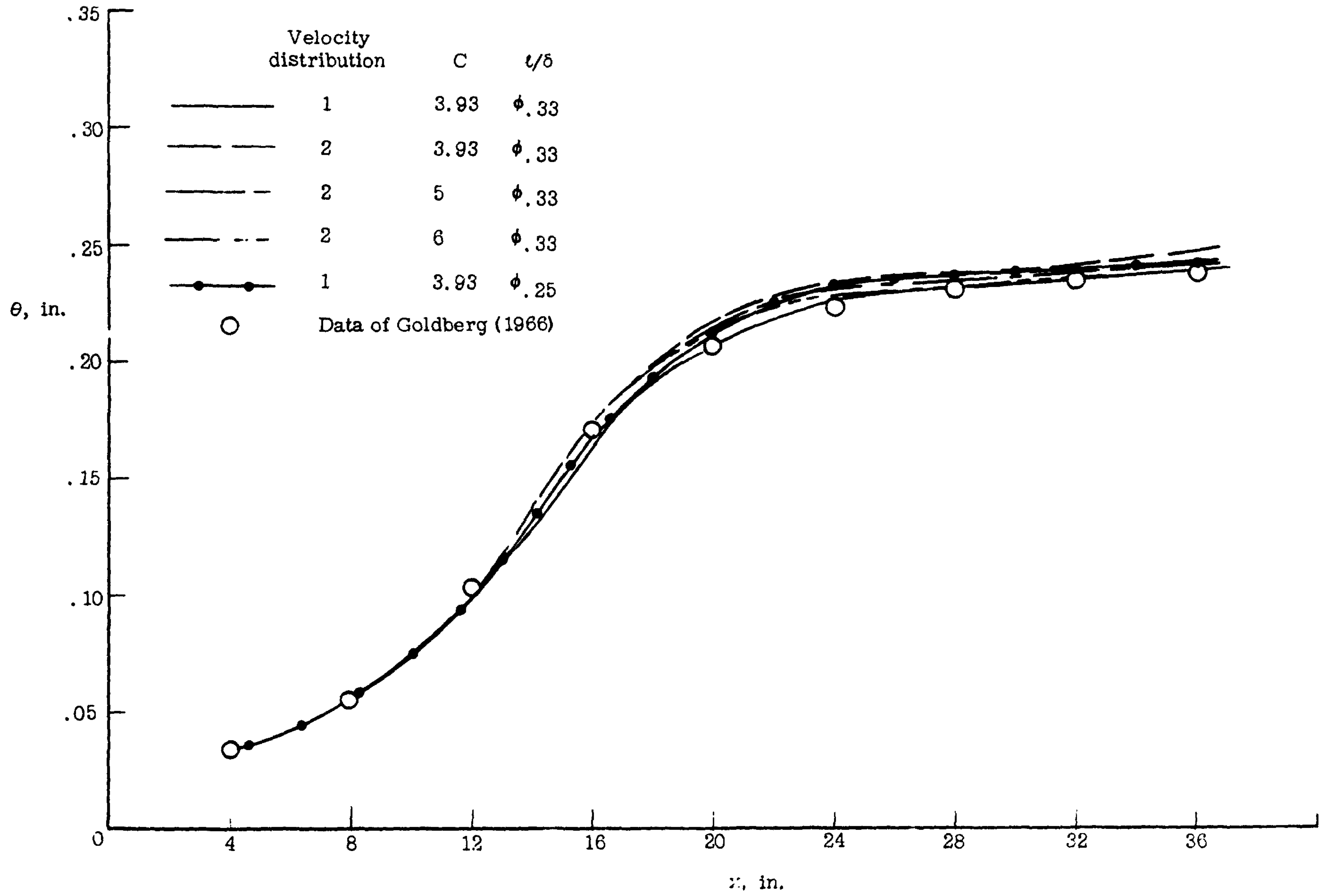


Figure 8.- Effect of velocity distribution, value of dissipation factor, C, and l/δ function on momentum thickness.

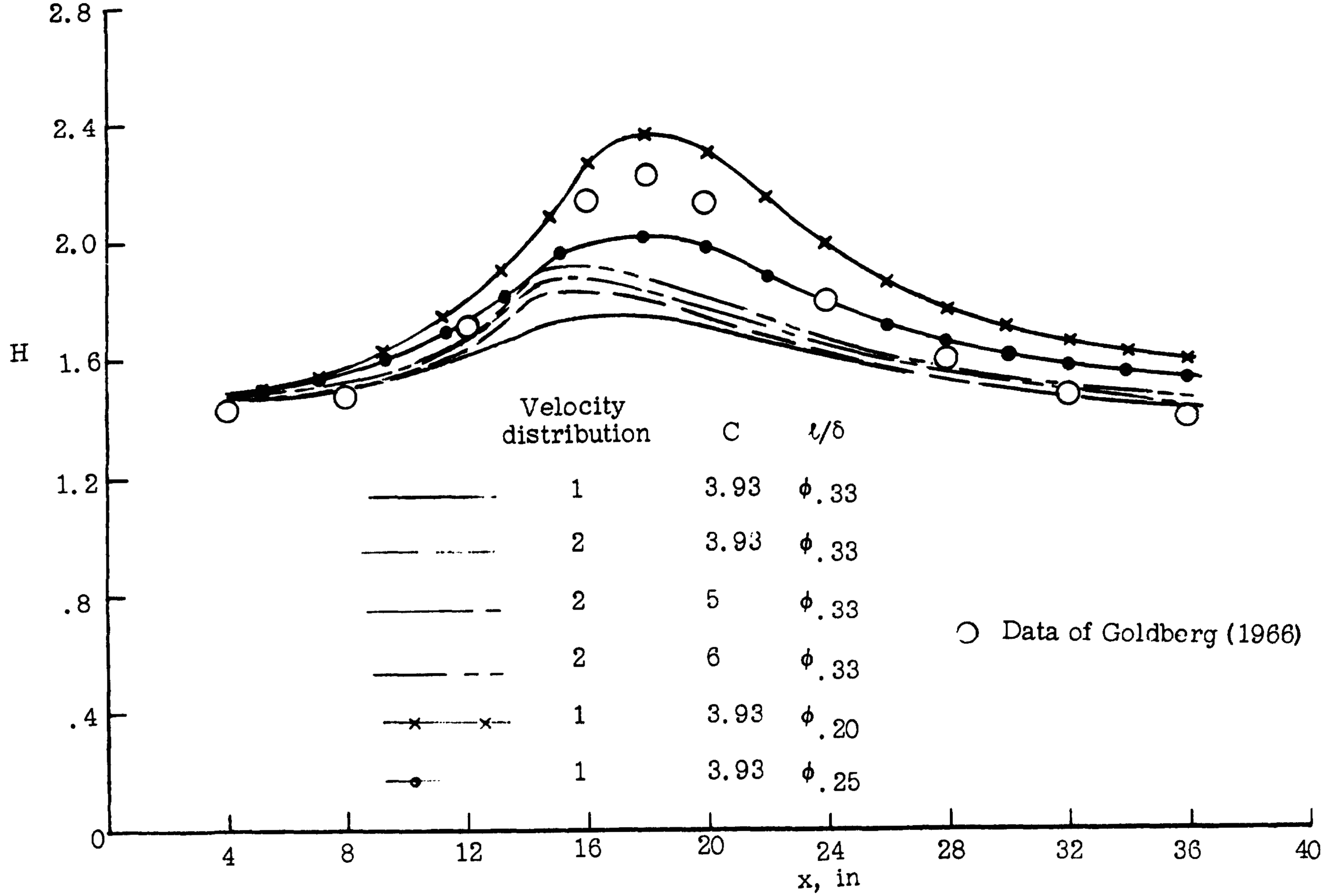


Figure 9.- Comparison of computed values of the shape parameter H, for velocity distributions 1 and 2, various values of C, and various l/δ functions with data of Goldberg (1966).

NASA-Langley, 1968