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CALCULATION OF THE MAGNETIC FIELD DUE TO A BIOELECTRIC CURRENT DIPOLE IN AN ELLIPSOID*

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Abstract. The bioelectric current dipole model is important both theoretically and computationally in the study of electrical activity in the brain and stomach due to the resemblance of the shape of these two organs to an ellipsoid. To calculate the magnetic field **B** due to a dipole in an ellipsoid, one must evaluate truncated series expansions involving ellipsoidal harmonics $\mathbb{E}_n^m$, which are products of Lamé functions. In this article, we extend a strictly analytic model (G. Dassios and F. Kariotou, J. Math. Phys. *44* (2003), 220–241), where **B** was computed from an ellipsoidal harmonic expansion of order 2. The present derivations show how the field can be evaluated to arbitrary order using numerical procedures for evaluating the roots of Lamé polynomials of degree 5 or higher. This can be accomplished using an optimization technique for solving nonlinear systems of equations, which allows one to acquire an understanding of the truncation error associated with the harmonic series expansion used for the calculation.

Keywords: magnetic field, dipole, ellipsoid

MSC 2000: 78A25

1. Introduction

The ellipsoidal model is important in magneto- and electrogastrography (MGG, EGG) and magnetoencephalography (MEG) due to the closer resemblance of this geometric shape to the anatomy of both the brain and stomach compared to the spherical and conoidal models. In the case of the brain, electrical activity can be modeled using dipoles whose locations within the ellipsoid are almost unconstrained. In gastric modelling, on the other hand, although the stomach interior does not possess the same anatomical and physiological characteristics as the brain, one can

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nevertheless use dipoles to model gastric electrical activity (GEA) provided that the dipoles within the ellipsoid are sufficiently close to the inner surface of this idealized body [16]. In this manner, one can use dipoles to simulate electrical activity in the smooth muscle syncytium of the stomach and at the same time account for the thickness of the gastric wall in a reasonable fashion that does not compromise the anatomical constraints of the modeling problem.

In human brain studies [12], bioelectric activity has been quantified directly via EEG and MEG [1], [24]. In MGG, the phenomenon studied is GEA, which is generated by the periodic depolarization and repolarization of cells in the stomach [3]. GEA has its origin in the gastric corpus and can be recorded as a slow wave that propagates aborally towards the pylorus through the electric syncytium of the stomach [20]. In the quasistatic approximation, one or several current dipoles can be used to model this phenomenon. A current dipole $\mathbf{Q}$ is an approximation for a current source recorded at a distant field point and represents a concentration of impressed current density $\mathbf{J}_i$ to a point $\mathbf{r}_0$. Anomalies in the characteristics of dipole propagation have been studied [20] and their relevance to the field of medical diagnosis has been the focus of active research [9]. In particular, the use of superconducting quantum interference device (SQUID) magnetometers, pioneered by Cohen et al. in the 1970s [4], [5], [6], has proven to be very suitable for detecting and studying the GEA both in healthy and diseased subjects [16]. One practical advantage of using SQUIDs for recording biological data is the possibility of studying gastrointestinal electromagnetic phenomena noninvasively. Studies of such nature are encouraging because they offer the ability to characterize abnormal current propagation, which is associated with pathological conditions such as gastroparesis and ischemia [15].

The current dipole approximation has been widely employed in biomagnetism to model biologic electrical activity [23]; for this purpose, the stomach has been simulated using cylinders, cones, conoids, ellipsoids and realistic models [7]. Among others, Mirizzi et al. proposed a mathematical model to simulate the extracellular electrical control activity where an annular band polarized by electric current dipoles moves distally from the mid-corpus to the terminal antrum [21], [22]. Mintchev and Bowes later constructed a conoidal dipole model of the electrical field produced by the human stomach, where spontaneous depolarization and repolarization due to ionic exchange were both simulated [19]. In 2003, Irimia and Bradshaw constructed a model of the stomach in which an annular band of dipoles advances along a truncated ellipsoid [14], thus simulating the electric potential and electric field recorded by a nasogastric probe.

To compute the electric potential Φ and magnetic field $\mathbf{B}$ due to an electric current dipole located in an ellipsoid, one must evaluate a truncated expansion of normal ellipsoidal harmonic terms $\mathbb{E}_n^m$. This was first proposed by Kariotou [17] and Dassios [8], who derived formulas for Φ up to order 2 in $\mathbb{E}_n^m$. In our most recent study [15], the electric field due to a dipole in an ellipsoid was computed in a generalized approach where the corresponding expansion can be carried out to arbitrary degree. In this article, we present a similar mathematical derivation for evaluating the magnetic field $\mathbf{B}$. In the next section, a minimal mathematical description of our formalism is provided; we thereafter derive formulas for the magnetic field to arbitrary order in the ellipsoidal harmonics. The reader is referred to [15] for a similar derivation of the electric field and potential.

2. Mathematical formalism

The standard equation of the ellipsoid in the Cartesian coordinate system $(x_1, x_2, x_3) = (x, y, z)$ is given by

$$\frac{x_1^2}{\alpha_1^2} + \frac{x_2^2}{\alpha_2^2} + \frac{x_3^2}{\alpha_3^2} = 1, \tag{1}$$

where $0 < \alpha_3 < \alpha_2 < \alpha_1 < +\infty$ are the ellipsoidal semiaxes. The ellipsoidal system [17], [13], on the other hand, has coordinates (ϱ, μ, ν) with semifocal distances h_1, h_2 and h_3, defined as

$$h_1^2 = \alpha_2^2 - \alpha_3^2, \tag{2}$$

$$h_2^2 = \alpha_1^2 - \alpha_3^2, \tag{3}$$

$$h_3^2 = \alpha_1^2 - \alpha_2^2. \tag{4}$$

Conversion between the two systems is made using

$$x_1 = \frac{\varrho\mu\nu}{h_2 h_3}, \tag{5}$$

$$x_2 = \frac{\sqrt{\varrho^2 - h_3^2}\sqrt{\mu^2 - h_3^2}\sqrt{h_3^2 - \nu^2}}{h_1 h_3}, \tag{6}$$

$$x_3 = \frac{\sqrt{\varrho^2 - h_2^2}\sqrt{h_2^2 - \mu^2}\sqrt{h_2^2 - \nu^2}}{h_1 h_2}, \tag{7}$$

where $\varrho \in [h_2, +\infty)$, $\mu \in [h_3, h_2]$ and $\nu \in [-h_3, h_3]$. In the ellipsoidal system, the Laplace equation assumes the form

$$(\mu^2 - \nu^2)\frac{\partial^2 \Phi}{\partial \beta^2} + (\varrho^2 - \nu^2)\frac{\partial^2 \Phi}{\partial \varphi^2} + (\varrho^2 - \mu^2)\frac{\partial^2 \Phi}{\partial \chi^2} = 0, \tag{8}$$

where Φ is the electric potential and β, γ and χ can be written as definite elliptic integrals of ϱ, μ and ν, respectively.

To calculate the electric potential Φ for an ellipsoid, separation of variables for the Laplace equation in ellipsoidal coordinates leads to the Lamé equation, which involves Lamé polynomials that form the normal interior harmonic function

$$\mathbb{E}_n^m(\varrho,\mu,\nu) = E_n^m(\varrho)E_n^m(\mu)E_n^m(\nu) \tag{9}$$

where $n = 0, 1, 2, \ldots$ and $m = 1, 2, \ldots, 2n+1$. There are four classes or species of Lamé functions, denoted by $K(\eta_i)$, $L(\eta_i)$, $M(\eta_i)$ and $N(\eta_i)$, respectively, where η_i is any of the coordinates ϱ, μ or ν. These are referred to as Lamé functions of the first (as opposed to the second) kind; they can be written as

$$K(\eta_i) = \sum_{k=0}^{r+1} a_k \eta_i^{n-2k}, \tag{10}$$

$$L(\eta_i) = \sqrt{\eta_i^2 - h_3^2}\sum_{k=0}^{n-r} a_k \eta_i^{n-(k+1)}, \tag{11}$$

$$M(\eta_i) = \sqrt{h_2^2 - \eta_i^2}\sum_{k=0}^{n-r} a_k \eta_i^{n-(k+1)}, \tag{12}$$

$$N(\eta_i) = \sqrt{(\eta_i^2 - h_3^2)(\eta_i^2 - h_2^2)}\sum_{k=0}^{r} a_k \eta_i^{n-2(k+1)}. \tag{13}$$

The index r in the above summations is given by

$$r = \begin{cases} \frac{1}{2}n & \text{for } n \text{ even}, \\ \frac{1}{2}(n-1) & \text{for } n \text{ odd}, \end{cases} \tag{14}$$

where n is the degree of the ellipsoidal harmonic $\mathbb{E}_n^m$. For a harmonic of degree n, $2m+1$ associated Lamé functions exist; thus there are $r+1$ functions of type K, $n-r$ functions of type L, $n-r$ functions of type M and r functions of type N for a total of $2m+1$ Lamé functions of degree n.

Surface ellipsoidal harmonics are products of the form $E_n^m(\mu)E_n^m(\nu)$ and they refer to the ellipsoidal surface $\varrho = \varrho_0$. The normalization functions γ_n^m associated with these harmonics are written as

$$\gamma_n^m = \oint_{\varrho=\varrho_0} \frac{[E_n^m(\mu)E_n^m(\nu)]^2}{\sqrt{(\varrho_0^2-\mu^2)(\varrho_0^2-\nu^2)}}\,\mathrm{d}S, \tag{15}$$

where

$$\mathrm{d}S = \mathrm{d}\mu\,\mathrm{d}\nu(\mu^2-\nu^2)\sqrt{\frac{(\varrho^2-\mu^2)(\varrho^2-\nu^2)}{(\mu^2-h_3^2)(h_2^2-\mu^2)(h_3^2-\nu^2)(h_2^2-\nu^2)}} \tag{16}$$

is the ellipsoidal surface element [2]. Lamé showed that the roots of the functions bearing his name must all be real, distinct and located in the interval $(-\alpha_1^2, \alpha_3^2)$. In ellipsoidal coordinates, the harmonics can be written as

$$\mathbb{E}_n^m(\mathbf{r}) = L_{ij} \prod_{k=1}^{m} \Psi_k, \tag{17}$$

where L_{ij} denotes the appropriate entry in the table L given in ellipsoidal coordinates, where

$$L = \left\{ \begin{array}{cccc} & \varrho & \sqrt{\varrho^2 - h_3^2} & \sqrt{\varrho^2 - h_2^2} & \sqrt{(\varrho^2 - h_3^2)(\varrho^2 - h_2^2)} \\ 1 & \mu & \sqrt{\mu^2 - h_3^2} & \sqrt{h_2^2 - \mu^2} & \sqrt{(\mu^2 - h_3^2)(h_2^2 - \mu^2)} \\ & \nu & \sqrt{h_3^2 - \nu^2} & \sqrt{h_2^2 - \nu^2} & \sqrt{(h_3^2 - \nu^2)(h_2^2 - \nu^2)} \end{array} \right\} \tag{18}$$

and

$$\Psi_k = (\varrho^2 - \psi_k^2)(\mu^2 - \psi_k^2)(\nu^2 - \psi_k^2). \tag{19}$$

The columns in L correspond to each of the function types K, L, M and N, while the rows refer to the coordinates in the chosen system, i.e. ϱ, μ or ν. To evaluate $\mathbb{E}_n^m$, one must select appropriate entries in L for each coordinate and multiply the resulting quantity by the product $\prod_k \Psi_k$. To determine the values of i and j based on the value of n, one can make use of the fact that the first $r+1$ functions are of type K, the following $n-r$ are of type L, etc., as explained before.

In the above equations, ψ_k are the roots of the corresponding function $\Psi_k(\varrho, \mu, \nu)$ expressed in ellipsoidal coordinates. The normal unit vector with respect to the ellipsoidal surface is defined as

$$\hat{\mathbf{n}} = D_n \hat{\boldsymbol{\varrho}} \tag{20}$$

with

$$D_n = \frac{\alpha_2 \alpha_3}{\sqrt{(\alpha_1^2 - \mu^2)(\alpha_1^2 - \nu^2)}}. \tag{21}$$

3. Derivation of the magnetic field B

The mathematical theory of ellipsoidal harmonics is of great interest in a variety of scientific areas, including gravitational astrophysics [26], physical geodesy [10] and numerical analysis, e.g. for obtaining solutions to the ellipsoidal Stokes problem [25]. In biophysics, it is useful for computing the electric potential, electric field and magnetic field due to one or several quasistatic current dipoles located in an organ whose shape is approximately ellipsoidal, such as the human brain or stomach.

Consider a point $\mathbf{r}'$ located inside a body of volume V, where a primary current dipole source with moment $\mathbf{Q}$ is also located. The physics of this problem [28], [11] allows one to model the phenomenon at hand as a concentration of the impressed current $\mathbf{J}_i$ to a point $\mathbf{r}_0$ using the Dirac delta functional $\delta(\mathbf{r}-\mathbf{r}_0)$ via the algebraic expression

$$\mathbf{J}_i(\mathbf{r}) = \mathbf{Q}\delta(\mathbf{r}-\mathbf{r}_0). \tag{22}$$

The electric field $\mathbf{E}$ induced by the impressed current creates an induction current

$$\mathbf{J}_d(\mathbf{r}) = \sigma\mathbf{E}(\mathbf{r}), \tag{23}$$

where σ is the tissue conductivity. Since anatomical and physiological characteristics of the human body allow for such currents to be considered quasistatic [12], [18], [28], [29], the electric field is irrotational and Poisson's equation can be used to find the electric potential Φ.

Deriving a generalized expression for the magnetic field $\mathbf{B}$ is somewhat more tedious. Because much of the underlying theory required for this task has already been derived by Sarvas [28], we only summarize it here. The magnetic field due to some current density $\mathbf{J}$ is given by the law of Biot and Savart:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\oint_\Omega \mathbf{J}(\mathbf{r}') \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\,\mathrm{d}^3\mathbf{r}', \tag{24}$$

where Ω is the support of $\mathbf{J}$ and μ_0 is the permeability of free space. An important detail concerning the above formula and the remainder of this section pertains to the difference between $\mathbf{r}'$ and $\mathbf{r}_0$. The variable $\mathbf{r}'$ refers to the integration space Ω, whereas $\mathbf{r}_0$ is related to the position of the dipole. Thus, $\mathbf{r}'$ is used in association with the current density $\mathbf{J}$ over the entire volume of the ellipsoid, whereas $\mathbf{r}_0$ is used for the current dipole that approximates this density. The conceptual differences associated with this aspect of the theory will be explained below in more detail. In the quasistatic approximation—which is justified here [28], [17], [8]—one of Maxwell's equation reads

$$\mathbf{J} = \mathbf{J}_i + \mathbf{J}_d = \mathbf{J}_i + \sigma\mathbf{E}, \tag{25}$$

where $\sigma\mathbf{E}$ is the Ohmic induction current previously described. Replacing $\mathbf{J}$ in Eq. 24 by the quantities on the right-hand side above and using $\mathbf{E}^- = -\boldsymbol{\nabla}\Phi^-$ yields the magnetic field as an integral over the volume V of the ellipsoid:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\oint_V [\mathbf{J}_i(\mathbf{r}') - \sigma\boldsymbol{\nabla}\Phi^-(\mathbf{r}')] \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\,\mathrm{d}^3\mathbf{r}'. \tag{26}$$

Using the definition of a current dipole $\mathbf{Q}$ located at $\mathbf{r}_0$ given before, i.e.

$$\mathbf{J}_i(\mathbf{r}) = \mathbf{Q}\delta(\mathbf{r}-\mathbf{r}_0), \tag{27}$$

it can be shown (see [28] for details) that the magnetic field is given by

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\mathbf{Q} \times \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} - \sigma\frac{\mu_0}{4\pi}\oint_{V^-} \boldsymbol{\nabla}\Phi^-(\mathbf{r}') \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\,\mathrm{d}^3\mathbf{r}'. \tag{28}$$

From the above formula, one can see that the variable $\mathbf{r}_0$ is associated with the dipole location, whereas $\mathbf{r}'$ is used in the integral over the entire ellipsoidal volume, hence the difference between the two. One can apply Stokes' theorem to convert this integral into a surface integral, with the result

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\mathbf{Q} \times \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} - \sigma\frac{\mu_0}{4\pi}\oint_{S} \Phi^-(\mathbf{r}')\hat{\mathbf{n}} \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\,\mathrm{d}S'. \tag{29}$$

Our task is now to express the above equation for $\mathbf{B}$ in terms of normal ellipsoidal harmonic functions $\mathbb{E}_n^m$. Let I_n^m be elliptic integrals of the form

$$I_n^m(\varrho) = \int_\varrho^\infty \frac{\mathrm{d}t}{[E_n^m(t)]^2\sqrt{t^2-h_2^2}\sqrt{t^2-h_3^2}} \tag{30}$$

with $n = 0, 1, 2, \ldots$, and $m = 1, 2, \ldots, 2n+1$. One can use the identity

$$\boldsymbol{\nabla}'\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \tag{31}$$

together with the result

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1}\frac{4\pi}{\gamma_n^m} I_n^m(\varrho)\mathbb{E}_n^m(\mathbf{r}')\mathbb{E}_n^m(\mathbf{r}) \tag{32}$$

derived in [8] to conclude that

$$\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1}\frac{4\pi}{\gamma_n^m} I_n^m(\varrho)\mathbb{E}_n^m(\mathbf{r})\boldsymbol{\nabla}'\mathbb{E}_n^m(\mathbf{r}'). \tag{33}$$

We can also compute the useful quantity

$$\mathbf{p} \equiv \hat{\mathbf{n}} \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \tag{34}$$

$$= D_n(\mathbf{r}') \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} \frac{4\pi}{\gamma_n^m} I_n^m(\varrho) \mathbb{E}_n^m(\mathbf{r}) [\hat{\boldsymbol{\varrho}}' \times \boldsymbol{\nabla}' \mathbb{E}_n^m(\mathbf{r}')]. \tag{35}$$

The unit vector $\hat{\boldsymbol{\varrho}}'$ comes from the definition of $\hat{\mathbf{n}}$ in ellipsoidal coordinates given in Eq. 20. This unit vector refers to the *surface* of the ellipsoid being integrated over in Eq. 29; because the integration variable there is $\mathrm{d}S'$ as a function of (ϱ', μ', ν'), the unit normal $\hat{\mathbf{n}}$ is also a function of the primed variables. This is the motivation for writing $\hat{\boldsymbol{\varrho}}'$ rather than $\hat{\boldsymbol{\varrho}}$ in the above expression.

Before evaluating the integral over the closed surface in Eq. 29, we note that its variable of integration is $\mathbf{r}'$. It was shown by Kariotou [17] that the exterior electric potential due to a dipole in the ellipsoid can be written as

$$\begin{aligned} \Phi^-(\mathbf{r}) = g_0^1 + \sum_{n=1}^{\infty} \sum_{m=1}^{2n+1} \frac{1}{\sigma \gamma_n^m} [\mathbf{Q} \cdot \boldsymbol{\nabla}_{\mathbf{r}_0} \mathbb{E}_n^m(\mathbf{r}_0)] \mathbb{E}_n^m(\mathbf{r}) \\ \times \left[I_n^m(\varrho) - I_n^m(\alpha_1) + \frac{1}{\alpha_2 \alpha_3 E_n^m(\alpha_1)} \left(\frac{\mathrm{d}E_n^m}{\mathrm{d}\alpha_1} \right)^{-1} \right], \end{aligned} \tag{36}$$

where g_0^1 is a constant. The only quantities in $\Phi^-(\mathbf{r}')$ and $\mathbf{p}$ that are dependent on this variable are, respectively, $\mathbb{E}_n^m(\mathbf{r}')$ (from making the substitution $\mathbf{r} \to \mathbf{r}'$ in Eq. 36) and $D_n(\mathbf{r}')[\boldsymbol{\varrho}' \times \boldsymbol{\nabla}' \mathbb{E}_n^m(\mathbf{r}')]$. Thus the constants and functions involving only $\mathbf{r}$ can be factored out from the integral. To ease our calculation, let us define the following functions:

$$a_n^m(\mathbf{r}) = \frac{4\pi}{\gamma_n^m} I_n^m(\varrho) \mathbb{E}_n^m(\mathbf{r}), \tag{37}$$

$$b_n^m(\mathbf{r}') = D_n(\mathbf{r}') [\hat{\boldsymbol{\varrho}}' \times \boldsymbol{\nabla}' \mathbb{E}_n^m(\mathbf{r}')], \tag{38}$$

$$c_n^m(\mathbf{r}_0) = \frac{1}{\sigma \gamma_n^m} \mathbf{Q} \cdot \boldsymbol{\nabla}_{\mathbf{r}_0} \mathbb{E}_n^m(\mathbf{r}_0), \tag{39}$$

$$d_s^t(\varrho; \alpha_i) = I_s^t(\varrho) - I_s^t(\alpha_1) + \frac{1}{\alpha_2 \alpha_3 E_s^t(\alpha_1)} \left(\frac{\mathrm{d}E_s^t}{\mathrm{d}\alpha_1} \right)^{-1}. \tag{40}$$

This allows us to write

$$\Phi^-(\mathbf{r}, \mathbf{r}', \mathbf{r}_0) = b_0^1 + \sum_{n=0}^{\infty} \sum_{m=0}^{2n+1} \mathbb{E}_n^m(\mathbf{r}') c_n^m(\mathbf{r}_0) d_n^m(\varrho; \alpha_i), \tag{41}$$

$$\mathbf{p}(\mathbf{r}, \mathbf{r}') = \sum_{n=0}^{\infty} \sum_{m=0}^{2n+1} a_n^m(\mathbf{r}) b_n^m(\mathbf{r}'). \tag{42}$$

The integrand over the surface of the ellipsoid in Eq. 29 can now be written as

$$\Phi^{-}\mathbf{p} = b_0^1 \sum_{n=0}^{\infty}\sum_{m=0}^{2n+1} a_n^m(\mathbf{r}) b_n^m(\mathbf{r}') + \sum_{n=0}^{\infty}\sum_{m=0}^{2n+1} a_n^m(\mathbf{r}) b_n^m(\mathbf{r}') \sum_{s=0}^{\infty}\sum_{t=0}^{2s+1} \mathbb{E}_s^t(\mathbf{r}') c_s^t(\mathbf{r}_0) d_s^t(\varrho;\alpha_i). \tag{43}$$

From a computational perspective, it is useful to assign the value 0 to the constant b_0^1 because this allows one to drop one infinite summation from the above equation. Carrying out the integration over the ellipsoidal surface for which $\varrho = \varrho_0$, we obtain

$$\oint_{\varrho_0} \Phi^{-}\mathbf{p}\,\mathrm{d}S' = \sum_{n=0}^{\infty}\sum_{m=0}^{2n+1}\sum_{s=0}^{\infty}\sum_{t=0}^{2s+1} a_n^m(\mathbf{r}) c_s^t(\mathbf{r}_0) d_s^t(\varrho_0;\alpha_i) \oint_{\varrho_0} b_n^m(\mathbf{r}') \mathbb{E}_s^t(\mathbf{r}')\,\mathrm{d}S'. \tag{44}$$

Note the dependence of d_s^t on ϱ_0 as a constant rather than on ϱ as a variable since the value of this coordinate is constant on the surface. A simplification to the equation of the magnetic field given above comes from contracting the two infinite summations over n and s in the same equation using the convolution method for sequences that one can derive from Cauchy's product formalism [27]. This method, although typically used in the context of complex analysis for power series, is nevertheless perfectly applicable to real sequences. Therefore, recalling the Cauchy product formula

$$\left(\sum_{n=1}^{\infty} a_n\right)\cdot\left(\sum_{s=1}^{\infty} b_s\right) = \sum_{n=1}^{\infty}\sum_{p=1}^{n} a_{n-p} b_p, \tag{45}$$

we can relabel the subscripts in Eq. 48 appropriately, and simplify our notation by defining the function

$$\zeta_{ns}^{mt} \equiv a_{n-s}^m(\mathbf{r}) c_s^t(\mathbf{r}_0) d_s^t(\varrho_0;\alpha_i) \oint_{\varrho_0} b_{n-s}^m(\mathbf{r}') \mathbb{E}_s^t(\mathbf{r}')\,\mathrm{d}S'. \tag{46}$$

This leads us to the following expression for our integral:

$$\oint_{\varrho_0} \Phi^{-}\mathbf{p}\,\mathrm{d}S' = \sum_{n=0}^{\infty}\sum_{s=0}^{n}\sum_{m=0}^{2(n-s)+1}\sum_{t=0}^{2s+1} \zeta_{ns}^{mt}. \tag{47}$$

In addition to making the notation more compact and easier to follow, this formulation removes the additional and unnecessary degree of freedom s from the double infinite summation for $\mathbf{B}$. The expression involving the magnetic field is given by

$$\frac{1}{\mu_0}\mathbf{B}(\mathbf{r}) = \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1} \frac{1}{\gamma_n^m} I_n^m(\varrho) \mathbb{E}_n^m(\mathbf{r}) [\mathbf{Q} \times \boldsymbol{\nabla}_{\mathbf{r}_0} \mathbb{E}_n^m(\mathbf{r}_0)] - \sum_{n=0}^{\infty}\sum_{s=0}^{n}\sum_{m=0}^{2(n-s)+1}\sum_{t=0}^{2s+1} \zeta_{ns}^{mt}. \tag{48}$$

This can also be written in the elegant factorized form

$$(49)\qquad \frac{1}{\mu_0}\mathbf{B}(\mathbf{r}) = \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1}\left[\frac{1}{\gamma_n^m} I_n^m(\varrho)\mathbb{E}_n^m(\mathbf{r})\mathbf{Q}\times\boldsymbol{\nabla}_{\mathbf{r}_0}\mathbb{E}_n^m(\mathbf{r}_0) - \sum_{s=0}^{n}\sum_{t=1}^{2(n-s)+1}\zeta_{ns}^{mt}\right],$$

which completes our derivation of the magnetic field.

A final remark is in place concerning the computational complexity of evaluating $\mathbf{B}$. Upon examining the formulas for Φ and $\mathbf{B}$, it is not difficult to realize that calculating the magnetic field is the most computationally-intensive task (since it effectively involves two infinite summation even though the Cauchy summation formula can be used to remove one of them). Let u represent the degree of the highest term used in each expansion, i.e. the highest selected value of n, and let τ represent the time needed to compute either one of Φ or $\mathbf{B}$ for some particular set of values for n and m. Using Big-Oh notation, computing Φ and $\mathbf{E}$ requires $\sum_{n=1}^{u}(2n+1)\tau = (u^2+2u)\tau$, i.e. the algorithm is $\mathcal{O}(u^2)$. In the case of computing the magnetic field, there are two degrees of freedom because n and s can be chosen arbitrarily. If the assumption is made that the two are equal, we can let $\tau' \equiv (u^2+2u)\tau$ in the above equation and the algorithm is then found to be $\mathcal{O}(u^4)$. Thus the calculation of the magnetic field is significantly more intensive.

Fig. 1 depicts the results of a simple numerical study regarding the accuracy of the expansion in Eq. 49. Physical parameter values (as specified in the caption) were selected so that the situation described resembles closely the experimental protocol of MGG or MEG (the case discussed is identical to that in [16], where images are also provided). What the figure demonstrates is that, for points located farther and farther away from the ellipsoid, the contribution of higher and higher order terms decreases. Close to the surface of the ellipsoid, however, these terms can have a significant contribution. In conclusion, for applications where reasonable numerical precision is needed for the calculation of $\mathbf{B}$, a large expansion ($n > 10$) may be required. What this implies is that, if the ellipsoidal approximation to the brain or stomach is to be used effectively in similar simulation studies, low-order expansions (e.g. $n < 3$) may be insufficient.

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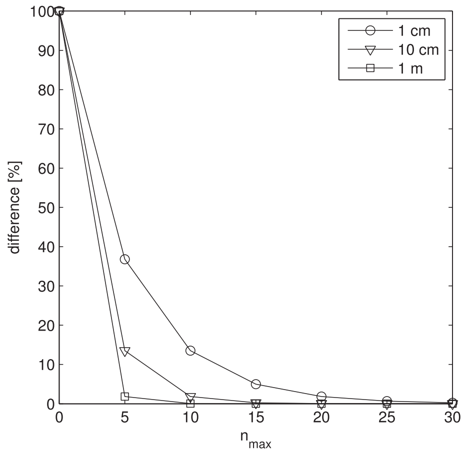

Figure 1. Numerical accuracy results for the expansion in Eq. 49. The independent variable $n_{\max}$ refers to the size of the expansion over n in Eq. 49, i.e. $\mathbf{B}/\mu_0 = \sum_{n=0}^{n_{\max}}(\ldots)$. Let $\mathbf{B}(i)$ refer to the value of the magnetic field computed for an expansion over $n = 0, \ldots, i$. The quantity on the vertical axis was computed according to the formula $[\mathbf{B}(n_{\max}) - \mathbf{B}(i)]/\mathbf{B}(n_{\max})$. In other words, it represents the percentage difference between $\mathbf{B}(i)$ and the most accurate value of $\mathbf{B}$ that was computed for this numerical example, namely $\mathbf{B}(n_{\max})$. The results presented are for an ellipsoid of dimensions $(\alpha_1, \alpha_2, \alpha_3) = (7.5, 5.0, 4.0)$ cm located at the origin (this is identical to the case discussed in [16], where images are also provided). The three curves represent values computed for a point A located on the Cartesian z axis at a distance of 1 cm, 10 cm and 1 m, respectively, from the upper extremity of the ellipsoid (i.e. the point on the surface of the ellipsoid closest to A).

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