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# CALCULATION OF THE STATISTICAL PROPERTIES OF STRANGE ATTRACTORS

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Calculation of the Statistical

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A path integral method is developed for the calculation of the statistical properties of a class of discrete, dissipative mappings which exhibit strange attractors. Exact analytic results are derived for the low order statistical moments. These non-trivial results are verified numerically.

Discrete maps

$$\vec{x}_i = f\,\vec{x}_{i-1} \tag{1}$$

are used to model the dynamics of a wide variety of physical, chemical, and biological systems.[1] These maps exhibit surprisingly complicated dynamical behavior. Under certain conditions these deterministic systems exhibit a transition to chaos.[1-3] In the turbulent or stochastic regime the complexity of the motion suggests that a statistical description is appropriate. The transition from regular to stochastic motion has been studied extensively[4,5]; however, less analytic progress has been made on the statistical description of the evolution of the dynamical system after the transition to stochasticity.[2]

For dissipative systems in two or more dimensions, numerical studies indicate that the long-time stochastic motion lies on a complicated manifold in phase space called a strange attractor.[2,6] Analytic work has been limited to the calculation of the non-integral Hausdorff dimension of the strange attractor.[6,7] However, the dimension of the attractor only indicates the amount of information required to construct, numerically, a coarse-grained probability distribution.[7]

The purpose of this paper is to present a new method for analytically calculating the statistical properties of the dynamics on the strange attractor. This procedure is based on an extension to discrete dynamical systems of earlier work on the functional integral approach to classical statistical dynamics.[8]

To illustrate our path integral method we consider a class of two-dimensional, dissipative maps which exhibit strange attractors[6]

$$x_i = 2x_{i-1} \quad \bmod 1$$

$$y_i = \lambda y_{i-1} + p(x_{i-1}) \qquad (2)$$

where $\lambda < 1$ and $p(x)$ is a periodic function with period 1. This mapping is characteristic of a class of discrete nonlinear systems in which one degree of freedom, $x_i$, evolves independently. A different mapping of this type is studied in the Appendix. We consider more complicated dynamical systems in subsequent publications.

Using a path integral representation of the conditional probability distribution we calculate exact expressions for the statistical moments of $y_T$ averaged over $x_0$. After many iterations of the mapping, $T \to \infty$, the asymptotic moments converge to the time averages. Under the ergodic hypothesis these are identical to the statistical moments of y with respect to the invariant probability measure on the strange attractor. These non-trivial analytic results agree with those derived by numerically advancing the mapping.

We construct a path integral representation of the conditional probability distribution for discrete dynamical systems of the type defined by Eq. (2). The conditional probability for the transition from a point $x_{i-1}$, $y_{i-1}$ in phase space to $x_i$, $y_i$ is

$$P(x_i, y_i | x_{i-1}, y_{i-1}) = \sum_{n=-\infty}^{\infty} \delta(x_i - 2x_{i-1} + n)\, \delta(y_i - \lambda y_{i-1} - p(x_{i-1})) \qquad (3)$$

where the $\sum_{n=-\infty}^{\infty}$ ensures the periodicity of $x_i$. Equation (3) can be used to write any conditional probability

$$P(x_T y_T | x_0 y_0) = \prod_{i=1}^{T-1} \int_0^1 dx_i \int_{-\infty}^{\infty} dy_i \prod_{j=1}^{T} P(x_j y_j | x_{j-1} y_{j-1}) \quad . \qquad (4)$$

Replacing the $\delta$ functions by their Fourier transforms

$$\delta(y_i) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\hat{y}_i \, e^{i\hat{y}_i y_i}$$

$$\sum_{n=-\infty}^{\infty} \delta(x_i + n) = \sum_{\hat{x}_i=-\infty}^{\infty} e^{i2\pi \hat{x}_i x_i} \qquad (5)$$

gives

$$P(x_T y_T | x_0 y_0) = \prod_{k=1}^{T-1} \int_0^1 dx_k \int_{-\infty}^{\infty} dy_k \prod_{j=1}^{T} \sum_{\hat{x}_j=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\hat{y}_j}{2\pi}$$

$$\times \exp i2\pi \sum_{i=1}^{T} \hat{x}_i \, [x_i - 2x_{i-1}]$$

$$\times \exp i \sum_{i=1}^{T} \hat{y}_i [y_i - \lambda y_{i-1} - p(x_{i-1})] \quad . \qquad (6)$$

This path integral representation of the conditional probability distribution is a straightforward application of the functional integral formalism developed for classical dynamical systems.[8]

Reordering the indices we perform the $y_i$ and $x_i$ integrations for $i = 1, \ldots, T-1$. The $y_i$ integrations are trivial giving rise to factors of $\delta(\hat{y}_i - \lambda\hat{y}_{i+1})$ which eliminate the $\hat{y}_i$ integrals for $i = 1, \ldots, T-1$. The $x_i$ integrations give factors of

$$I_i(\hat{x}_i, \hat{x}_{i+1}, \hat{y}_T, \lambda) = \int_0^1 dx_i \; e^{i2\pi x_i[\hat{x}_i - 2\hat{x}_{i+1}] - i\lambda^{T-i-1}\hat{y}_T p(x_i)} \quad . \qquad (7)$$

After these manipulations the conditional probability function can be written

$$P(x_T y_T | x_o y_o) = \int_{-\infty}^{\infty} \frac{d\hat{v}_T}{2\pi} \sum_{\hat{x}_T=-\infty}^{\infty} C(\hat{x}_T \hat{y}_T | x_o y_o) \; e^{i\hat{x}_T x_T} e^{i\hat{v}_T v_T} \qquad (8)$$

where the characteristic function is

$$C(\hat{x}_T \hat{y}_T | x_o y_o) = \prod_{i=1}^{T-1} \sum_{\hat{x}_i=-\infty}^{\infty} I_i(\hat{x}_i, \hat{x}_{i+1}, \hat{v}_T, \lambda) \; e^{-i[4\pi \hat{x}_1 x_o + \lambda^{T-1}\hat{v}_T p(x_o)]} \cdot e^{-i\lambda^T \hat{v}_T y_o} \quad . \qquad (9)$$

The conditional probability distribution $P(x_T y_T | x_o y_o)$ and its Fourier transform $C(\hat{x}_T \hat{y}_T | x_o y_o)$ contain a complete dynamical description of the maps defined by Eq. (2). Since the asymptotic motion lies on a strange attractor Eqs. (8) and (9) can, therefore, be used to derive the statistical properties of the attractor.

If we average over a uniform distribution in $x_o$ Eq. (9) reduces to

$$C(\hat{x}_T \hat{v}_T | v_o) = \prod_{i=1}^{T-1} \sum_{\hat{x}_i=-\infty}^{\infty} I_i(\hat{x}_i, \hat{x}_{i+1}, \hat{v}_T, \lambda) \; I_o(0, \hat{x}_1, \hat{v}_T, \lambda) \; e^{-i\lambda^T \hat{v}_T v_o} \quad . \qquad (10)$$

The statistical moments of $y_T$ averaged over initial $x_o$ are derived from the characteristic function by differentiating with respect to $\hat{v}_T$. For example,

$$\langle y_T y_o \rangle = i y_o \frac{\partial}{\partial \hat{y}_T} C(\hat{x}_T \hat{y}_T | y_o) \Big|_{\hat{x}_T, \hat{y}_T = 0} \qquad (11)$$

$$\langle y_T^{\,2}\rangle = -\frac{\partial^2}{\partial \hat{y}_T^{\,2}} C(\hat{x}_T\hat{y}_T|y_0)\Bigg|_{\hat{x}_T,\ \hat{y}_T = 0} \tag{12}$$

$$\langle y_T^{\,4}\rangle = \frac{\partial^4}{\partial \hat{y}_T^{\,4}} C(\hat{x}_T\hat{y}_T|y_0)\Bigg|_{\hat{x}_T,\ \hat{y}_T = 0} \quad . \tag{13}$$

Since the derivatives of $C(\hat{x}_T\hat{y}_T|y_0)$ are evaluated at $\hat{x}_T$, $\hat{y}_T = 0$, only the first few terms of the Taylor series expansions of the $I_i$ are required to evaluate the low order moments of $y_T$

$$I_i(\hat{x}_i, \hat{x}_{i+1}, \hat{y}_T, \lambda) = \delta_{\hat{x}_i - 2\hat{x}_{i+1},\, 0} + (-i)\lambda^{T-i-1}\widehat{p}(\hat{x}_i - 2\hat{x}_{i+1})\hat{y}_T$$

$$+ (-i)^2 \lambda^{2(T-i-1)}\widehat{p^2}(\hat{x}_i - 2\hat{x}_{i+1})\frac{\hat{y}_T^{\,2}}{2!} + \ldots \tag{14}$$

where $\widehat{f}(k) = \int_0^1 dx\, e^{i2\pi kx} f(x)$.

To illustrate the utility of our path integral respresentation of the probability distribution we consider a specific map

$$x_i = 2x_{i-1} \qquad \text{mod } 1 \tag{15}$$

$$y_i = \lambda y_{i-1} + \sin 2\pi x_{i-1} \quad . \tag{16}$$

This map is similar to the one studied by Kaplan and Yorke.[6] For $\lambda < 1$ it exhibits a strange attractor. In Fig. 1 we have advanced the mapping for $\lambda = .2$ and plotted the phase points for $10^5$ time-steps. The complex structure of the attractor is illustrated in a magnified view, Fig. 2.

An examination of the mapping provides a qualitative understanding of the detailed structure of the strange attractor. Figures 1 and 2 show that the attractor is a multivalued function of x. Consider any $x_T \in (0,1)$. We can calculate the various $y_T$ corresponding to $x_T$ by tracing the mapping backwards using $y_T = \lambda y_{T-1} + \sin 2\pi x_{T-1}$. Since $x_T = 2x_{T-1} \bmod 1$ there are two possibilities for $x_{T-1}$

$$x_{T-1} = \left\{ \begin{array}{l} .5 x_T \\ .5(x_T + 1) \end{array} \right\} .$$

Therefore, for small $\lambda \to 0$, the attractor will appears to consist of two strands corresponding to

$$y_T \approx \left\{ \begin{array}{l} \sin \pi x_T \\ \sin \pi(x_T + 1) \end{array} \right\} .$$

At finer resolution each of these strands appears to consist of a pair of strands corresponding to the two possibilities for $y_{T-1}$. If we continue this analysis we find that each strand of the attractor corresponds to a different branch of a tree. The forks in the branches result from the choices

$$x_{i-1} = \left\{ \begin{array}{l} .5\ x_i \\ .5\ (x_i + 1) \end{array} \right\} .$$

For small $\lambda \to 0$ the values of $y_T$ for different branches are very close together and the attractor appears to be composed of tightly woven one-dimensional strands (see Figs. 1 and 2). As $\lambda \to 1$ the tree opens up, the branches overlap, and the strange attractor appears to fill a two-dimensional volume (see Fig. 3).

We will calculate the statistical averages $\langle y_T y_o \rangle$, $\langle y_T^2 \rangle$, and $\langle y_T^4 \rangle$ for the dynamics on this strange attractor. Higher moments are straightforward. In this case, the integrals $I_i$ have a particularly simple form. They are ordinary Bessel functions[9]

$$I(x_i, x_{i+1}, v_T, \ldots) = I_{x_i - 2x_{i+1}}(\ldots^{T-i-1} v_T) \qquad (17)$$

which have well known power series expansions[9]

$$\ldots \quad \frac{(-1)^r}{r!\,(m+r+1)} \left(\frac{x}{2}\right)^{m+2r} \qquad (18)$$

The characteristic function averaged over $x_0$ for this map is

$$\ldots = \prod_{i=1}^{T-1} \sum_{x_i=-\infty}^{\infty} I_{x_i - 2x_{i+1}}(\ldots^{T-i-1} v_T)\, I_{-2x_1}(\ldots^{T-1} v_T)\, e^{-i\ldots^{T} v_T v_0} \qquad (19)$$

If we differentiate $C(x_T v_T | v_0)$ once with respect to $v_T$ and set $x_T, v_T = 0$, then the only non-zero contribution comes from the derivative acting on the factor $e^{-i\ldots^{T} v_T v_0}$. The two-time correlation function is given by Eq. (11)

$$\langle v_T v_0 \rangle = \ldots^{T} v_0^2 = v_0^2\, e^{-\frac{T}{\tau_c}} \qquad (20)$$

where the decay time is

$$\tau_c = \frac{1}{\ln \frac{1}{\ldots}} \qquad (21)$$

Because of the simplicity of the mapping these results are easily verified by explicitly averaging the mapping.

Since the arguments of the Bessel functions vanish as $\hat{y}_T \to 0$, only Bessel functions of order 0, ± 1, and ± 2 and of order 0, ± 1, ± 2, ± 3, ± 4 can possibly contribute to $\langle y_T^2 \rangle$ and $\langle y_T^4 \rangle$, respectively. In fact a careful study of the various combinations of $\hat{x}_i$ = 0, ± 1, and ± 2 shows that the only nonvanishing contribution to $\langle y_T^2 \rangle$ comes from the order 0 Bessel functions corresponding to $\hat{x}_i$ = 0 for all i = 1,..., T. Using the first few terms of the Taylor expansion of $J_0$

$$[illegible]^{T-i-1} v_{[illegible]} = 1 - (\frac{[illegible]^{T-i-1}}{2} v_T)^2 + \ldots \quad , \qquad (22)$$

we differentiate Eq. (18) twice to get

$$[illegible] \sum_{i=[illegible]}^{[illegible]} ( [illegible]^{T-i-1} [illegible] )^2 + ([illegible]^T v_0)^2$$

$$[illegible] \frac{[illegible] - [illegible]^{2T}}{[illegible] - [illegible]^2} + ([illegible]^T v_0)^2 \quad . \qquad (23)$$

[illegible] exact result for all T for the expectation of [illegible] averaged over initial $x$ [illegible].

A careful examination of the terms in $C(\hat{x}_T \hat{v}_T | v_0)$ corresponding to different combinations of $\hat{x}_i$ shows that nonvanishing contributions to $\langle y_T^4 \rangle$ arise only in two cases. In addition to the contribution from taking all $\hat{x}_i$ = 0, a second contribution comes from the combinations $\hat{x}_j = \pm 1$, $\hat{x}_{j+1} = \pm 1$, and $\hat{x}_{i \neq j,\ j+1} = 0$. Using the Taylor expansions of $J_0$, $J_{\pm 1}$ and $J_{\pm 2}$ we differentiate $C(\hat{x}_T \hat{v}_T | v_0)$ four times to get

$$\langle y_T^4 \rangle = \frac{3}{4}\left(\frac{1-\lambda^{2T}}{1-\lambda^2}\right)^2 - \frac{3}{8}\left(\frac{1-\lambda^{4T}}{1-\lambda^4}\right) - \frac{3}{2}\lambda^5\left(\frac{1-\lambda^{4(T-2)}}{1-\lambda^4}\right) \times \begin{cases}1, & T \geq 3\\ 0, & T < 3\end{cases}$$

$$+ 3\left(\frac{1-\lambda^{2T}}{1-\lambda^2}\right)(\lambda^T y_0)^2 + (\lambda^T y_0)^4 \quad . \qquad (24)$$

The first two terms result from taking all $\hat{x}_i = 0$. The third term comes from the combinations of nonzero $\hat{x}_i$; and the last two terms, which exhibit the decay exponent of the initial conditions, come from the derivatives acting on $e^{-i\lambda^T y_T y_0}$. This result for the average of $y_T^4$ over $x_0$ is also exact for all T.

*Equations (23) and (24) can be verified for small T = 1, 2, 3 ...* by iterating the mapping a few time-steps and explicitly averaging $y_T^2$ and $y_T^4$ over $x_i$. However, even for T = 3 this is a laborious task for $\langle y_T^4 \rangle$. [illegible] the averages of $y_T^2$ and $y_T^4$ approach

$$[illegible] \qquad (25)$$

$$[illegible] \, )^2 - \frac{3}{8}\left(\frac{1+4\lambda^5}{1-\lambda^4}\right) \quad . \qquad (26)$$

These exact asymptotic results are independent of time and initial conditions. Consequently, if the asymptotic dynamics on the strange attractor are described by an invariant measure and the motion is ergodic,[1] then both the time averages of $y_T^2$ and $y_T^4$ and the averages with respect to the invariant measure are given by Eqs. (25) and (26).

$$\lim_{N\to\infty} \frac{1}{N} \sum_{T=0}^{N} y_T^2 = \frac{1}{2}\left(\frac{1}{1-\lambda^2}\right) \qquad (27)$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{T=0}^{N} y_T^4 = \frac{3}{4}\left(\frac{1}{1-\lambda^2}\right)^2 - \frac{3}{8}\left(\frac{1+4\lambda^5}{1-\lambda^4}\right) \quad . \qquad (28)$$

Equations (26) and (27) have been verified numerically by averaging the mapping over $10^5$ time-steps. The analytic and numerical results are compared in Fig. 4.

It is interesting to compare these exact asymptotic results with those derived using a random phase approximation which is frequently used in calculations of the statistical properties of turbulent systems. If we ignore Eq. (15) and assume that the $x_i$ are independent and uniformly distributed, then averaging Eq. (16) we get

$$\langle y^2 \rangle_{R.P.A.} = \frac{1}{2} \left(\frac{1}{1-\lambda^2}\right) \tag{29}$$

$$\langle y^4 \rangle_{R.P.A.} = \frac{3}{4} \left(\frac{1}{1-\lambda^2}\right)^2 - \frac{3}{8} \left(\frac{1}{1-\lambda^4}\right) \quad . \tag{30}$$

The random phase approximation reproduces all the terms in the expressions for the statistical moments which result from taking all $\tilde{x}_i = 0$ in Eq. (19). For $\langle y^2 \rangle$ this approximation recovers the exact asymptotic result. However, for $\langle y^4 \rangle$ and higher moments the random phase approximation fails.

*In the limit of weak damping,* $\lambda \to 1$, *the attractor expands in* $y$ and the asymptotic distribution function approaches a Gaussian. For example, if we replace $\lambda$ in Eq. (25) by $1-\varepsilon$, $\varepsilon \ll 1$,

$$\langle y_\infty^4 \rangle \approx 3 \langle y^2_\infty \rangle^2 \times [1 + O(\varepsilon)] \quad . \tag{31}$$

The low order moments in the random phase approximation are also Gaussian in this limit. Of course, for nonzero $\varepsilon$ the exact high order moments must deviate from Gaussian since the strange attractor remains bounded in y.

Our path integral approach is easily extended to more complicated mappings in higher dimensions; however, in general the analysis is not as easy. This procedure can also be used to calculate the statistical properties of conservative systems. Rechester and White[11] have recently applied a similar method to the Chirikov-Taylor mapping to calculate the diffusion coefficient in the stochastic regime. In a subsequent paper[12] we will consider a dissipative mapping introduced by Zaslavskii[13] which reduces to the Chirikov-Taylor mapping in the absence of damping. Our results provide a statistical description of the strange attractor which occurs in the damped case; and we recover the diffusion coefficient of Rechester and White in the dissipationless limit.

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## Appendix

The periodicity requirements in the previous systems can be relaxed if $x_i$ is bounded on a finite interval. For example, we have applied our method to dynamical systems in which the evolution of $x_i$ is determined by a map on the unit interval which was studied by Kac.[14]

$$x_i = \left\{ \begin{array}{ll} 2x_{i-1} \quad , & x_{i-1} \leq .5 \\ 2 - 2x_{i-1} \quad , & x_{i-1} > .5 \end{array} \right\} \tag{A1}$$

$$y_i = \lambda y_{i-1} + f(x_{i-1}) \tag{A2}$$

where λ < 1.

For $f(x_{i-1}) = \sin 2\pi x_{i-1}$ the $I_j(\hat{x}_j, \hat{x}_{j+1}, \hat{v}_T, \lambda)$ can again be expressed in terms of Bessel functions

$$I_i(\hat{x}_i, \hat{x}_{i+1}, \hat{v}_T, \lambda) = \frac{1}{2} [J_{\hat{x}_i - 2\hat{x}_{i+1}}(\lambda^{T-i-1}\hat{v}_T) + J_{\hat{x}_i + 2\hat{x}_{i+1}}(\lambda^{T-i-1}\hat{v}_T)] \quad . \tag{A3}$$

ing the first few terms in the Taylor expansions of $I_i$, we have derived exact analytic expressions for the statistical moments of $y_T$.

The results for $\langle y_T y_o \rangle$ and $\langle y_T^2 \rangle$ are the same as before, Eqs. (20) and (23). However, for $\langle y_T^4 \rangle$ the contributions from combinations of nonzero $\hat{x}_i$ exactly and the third term in Eq. (24) is eliminated. Consequently, $\langle y_\infty^4 \rangle$ is correctly determined by the random phase approximation in the asymptotic long-time limit. These results for the mapping defined by Eqs. (A1) and (A2) have also been verified numerically.

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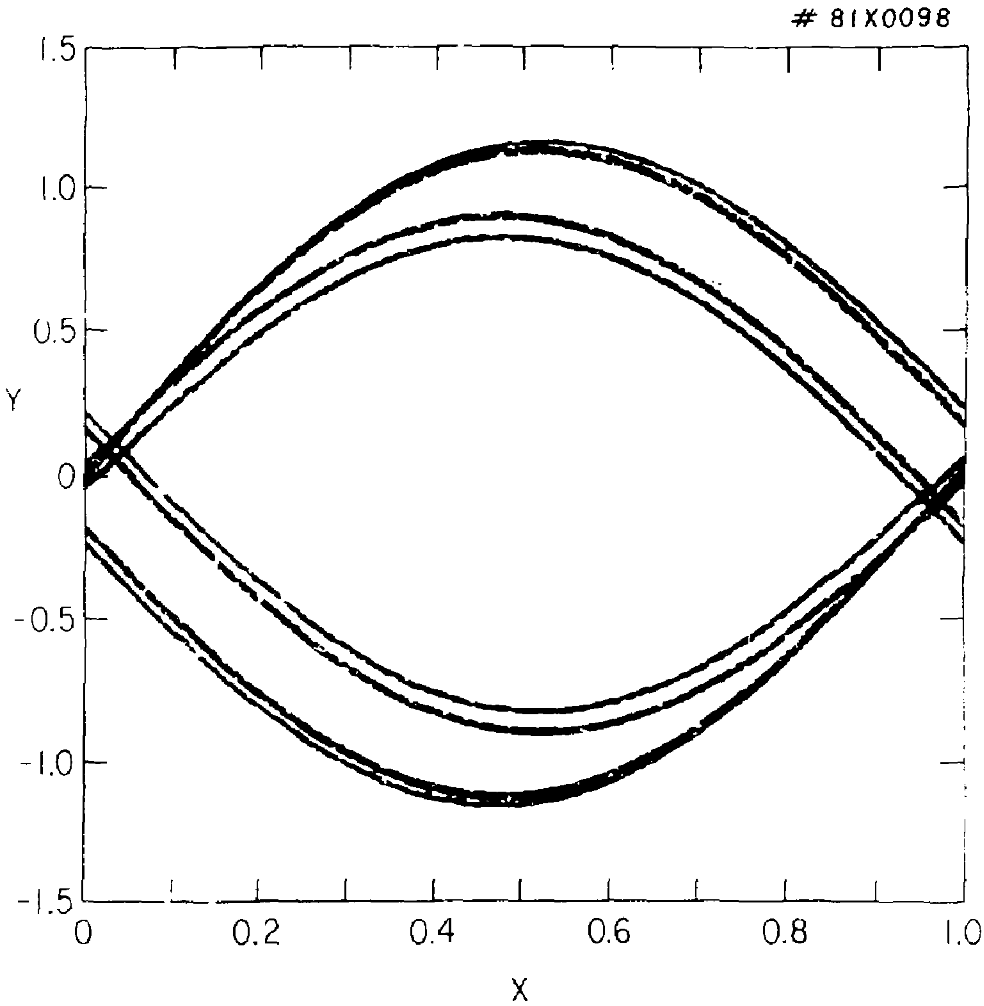


Fig. 1. The strange attraction for the dynamical system defined by Eqs. (15) and (16) with [illegible] = .2. $10^5$ time-steps are plotted.

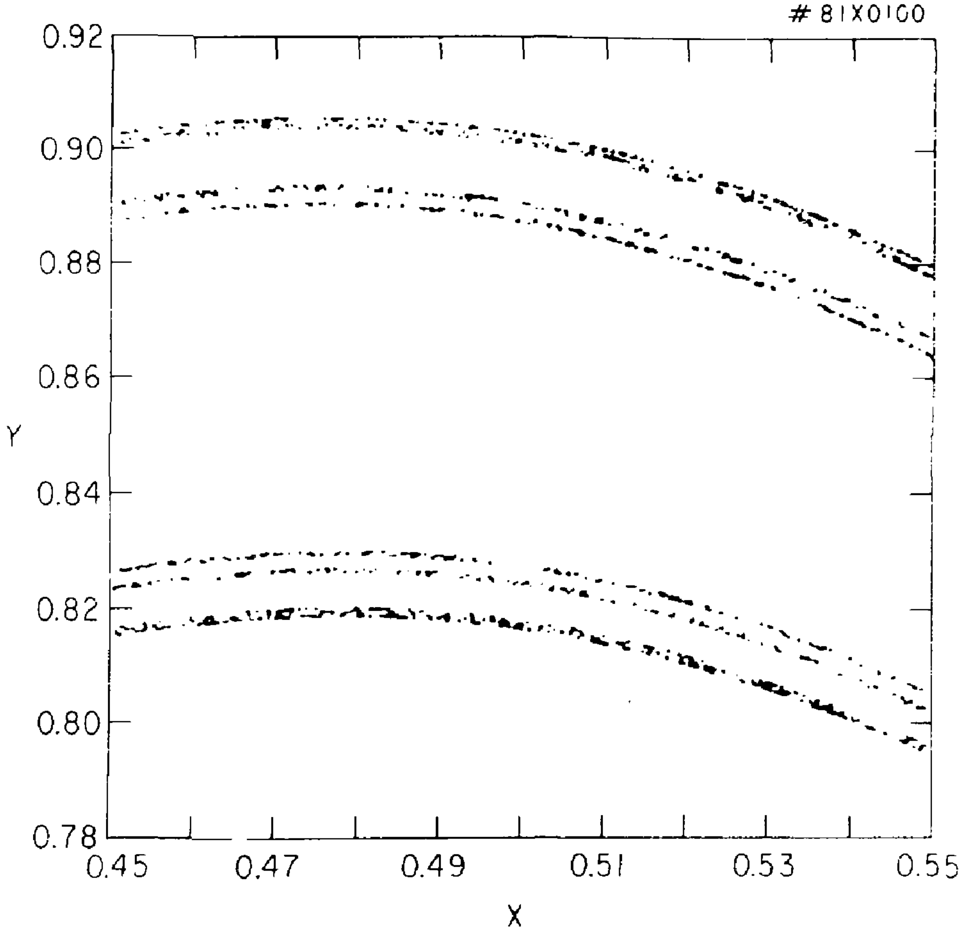


Fig. 2. A magnified view of a section of the strange attractor shown in Fig. 1. illustrating the complex structure of the attractor.

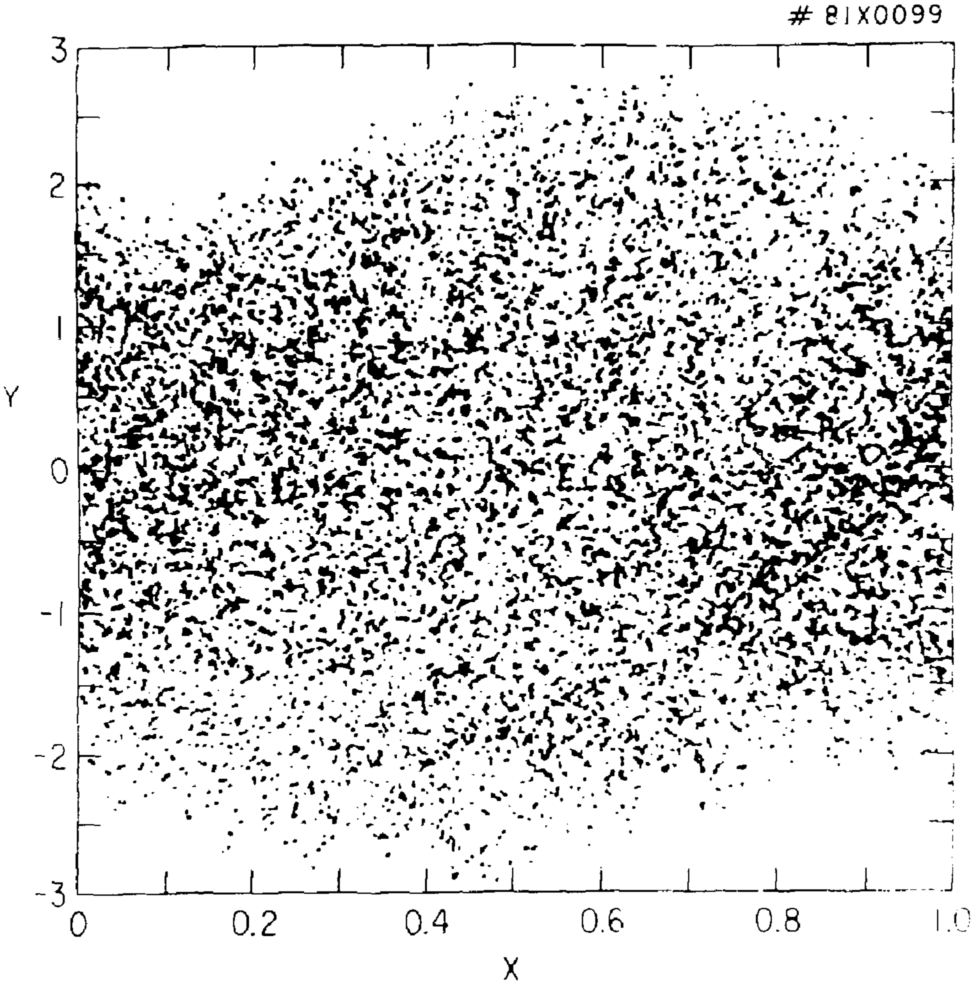


Fig. [illegible]. The strange attractor for [illegible] = .8. The complicated structure appears to fill a two-dimensional area.

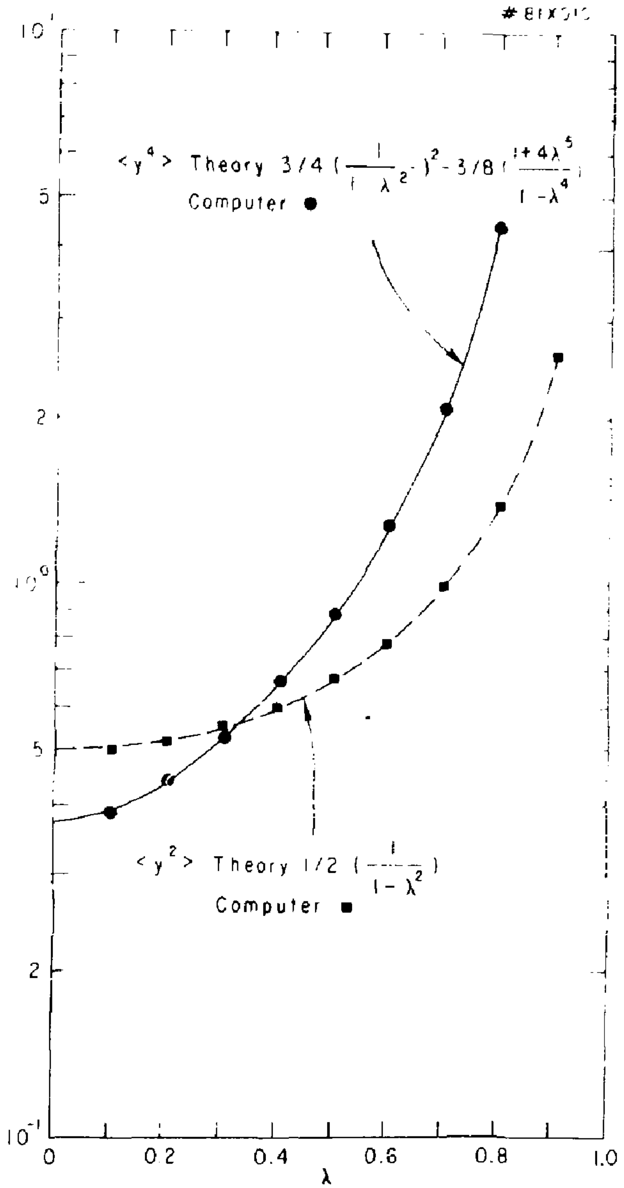


Fig. 4. A comparison of the theoretical values for $\langle y_T^2\rangle$ and $\langle y_T^4\rangle$ for $T = 10^5$ with those calculated by advancing the mapping numerically.