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CALCULATION OF THE WAKEFIELD WITH THE OPTICAL RESONATOR MODEL

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Abstract

The wakefield of a single charge traversing a resonant cavity can be calculated from an infinite sum over the loss factors at all resonances. Since usually only a finite number of modes is found by computation the wakefield may be quite inadequate, in particular for very short distances behind the exciting particle. An estimate of the missing terms is obtained from the "optical resonator model" using the fields in a laser cavity obtained by advanced diffraction theory.

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Introduction

An estimate of bunch lengthening and beam stability in high-energy storage rings can be obtained by "tracking" a large number of (super)particles over many revolutions on a computer. The charged particles influence each other, both directly and via coupling to the surroundings. While the direct effect becomes negligible at ultra-relativistic energies, the second effect may become very strong in the presence of high-Q, high-frequency cavities required for acceleration and/or compensation of energy loss. The interaction with the surroundings can be described by the "wake field" - which should better be called "wake potential" as it is the integrated effect on a particle following a certain distance behind a bunch. If the particle distribution in the bunch is known, the wake field can be obtained directly by numerical methods such as the program BCI. However, the particle distribution itself is usually influenced by the interaction with the surroundings, and one then needs the wake field of each (super)particle ("delta-function wake"), i.e. the Green's function for the particular cavity. The wake field of a single particle cannot be obtained with the program BCI, and can only be approximated insufficiently by taking the wake field of a short bunch - in particular because the computational effort becomes rapidly prohibitive when the bunch length is small compared to the cavity dimensions.

In this report, we discuss the alternative method of obtaining the delta-function wake by computation of the resonances of a cavity. Since this procedure is also limited to some maximum frequency, the missing part of the infinite sum may be important - in particular for large, low-frequency cavities as will be used for LEP. An estimate of this missing part can be obtained with the "optical-resonator model".

Discussion

1. The longitudinal wake field of a single charge traversing a resonant cavity can be obtained from the resonant frequencies ω_n and the loss factors k_n of all (longitudinal) modes by the expression[1)]

$$w(\tau) = 2 \sum_{n=1}^{\infty} k_n \cos\omega_n\tau \qquad (1)$$

where τ is the distance (in units of time) behind the exciting charge. In this report, we shall discuss only longitudinal ($m = 0$) effects caused by particles on the axis of a cavity with revolution symmetry, although similar expressions have been obtained for transverse wake fields in the limit of small displacements from the axis. The factor 2 in equation (1) is due to the fact that we consider only resonances with positive frequencies $\omega_n > 0$.

2. Except for the (unrealistic) case of a closed "pill-box" cavity, the resonant frequencies and loss factors have to be found numerically with computer programs such as SUPERFISH or KN7C. Thus only a finite number of resonances will be known, and it is often impractical to extend the calculations to frequencies where the loss-factors (actually their sum) become negligible. Rather than simply truncating the infinite series, it is preferable to replace the missing terms by an integral

$$\begin{aligned} w(\tau) &= 2 \sum_{n=1}^{\hat{n}} k_n \cos\omega_n\tau + \Delta w(\tau) \\ \Delta w(\tau) &= 2 \int_{\hat{\omega}}^{\infty} k(\omega) \frac{dn}{d\omega} \cos\omega\tau \, d\omega \end{aligned} \qquad (2)$$

in which we have replaced the discrete loss-factors k_n by a continuous function $k(\omega)$ multiplied by the density of modes $dn/d\omega$. This continuous function can be approximated numerically by "binning", i.e. by averaging the values of k_n in frequency-bins of a certain width. Again this technique is limited to the region for which the discrete resonances have been computed. However, the high-frequency behaviour of this function can be estimated by advanced diffraction theory.

3. The energy loss of a charged particle traversing a cavity (per unit frequency interval) has been obtained by Sessler[2)] and reported by Keil[3)], based on results due to Vainshtein[4)] who calculated the fields in a laser resonator - hence the name "optical resonator model". Vainshtein solved the problem of two circular mirrors of radius "a",

separated by a distance "2ℓ" along their common axis, by applying Wiener-Hopf techniques. Sessler argued that this problem is equivalent to two (parallel) infinite planes with circular holes, and hence also to a cylindrical resonator with sideholes of radius a and a gap-length $g = 2\ell$ when the outer radius is assumed to be large. Vainshtein solved an integral equation for the current density in the mirrors and obtained an approximation for the fractional decrease of energy (see Appendix A)

$$\Lambda = 8\nu_{mn}^2 \frac{\sigma(M+\sigma)}{[(M+\sigma)^2 + \sigma^2]^2} \qquad (3)$$

where ν_{mn} is the n^{th} zero of the m^{th} order Bessel-function (it is sufficient to limit the loss calculation to the lowest longitudinal mode with $m = 0$, $n = 1$ and to take $\nu_{01} = 2.405$). $\sigma = -\zeta(^1/_2)/\sqrt{\pi} = 0.8237$ is a constant (ζ is Riemann's zeta-function), and

$$M = a\sqrt{2\omega/c\ell} \qquad (4)$$

is proportional to the square root of the frequency.

Sessler multiplies equation (3) with the incident energy (per unit frequency), for which he takes[3] the Poynting vector at $r = a$ multiplied with the cross-section πa^2

$$\pi a^2 \times \tilde{H}_\theta \tilde{E}_r^* \Big|_{r=a} = \frac{Q^2\omega^2 a^2}{4\pi^2\varepsilon_o\beta^3\gamma^2c^3} K_1^2\left(\frac{\omega a}{\beta\gamma c}\right) \qquad (5)$$

which is obtained[5] from expressions for the Fourier components of the fields of a charge Q moving with velocity βc.

The energy loss (for a unit charge and per unit frequency interval) can then be written

$$\frac{dU}{d\omega} = C_{sv} F(\nu) \cdot G(\overline{\nu}) \qquad (6)$$

where (for $\beta = 1$)

$$C_{sv} = \frac{2\, \nu_{01}^2\, Z_o}{\pi^2 \sigma^2 \beta} \approx 650\ (\Omega) \tag{7}$$

will be called the "Sessler-Vainshtein constant". The first function

$$F(\nu) = \frac{\sqrt{\nu} + 1}{(\nu + 2\sqrt{\nu} + 2)^2}$$

with
$$\nu = \frac{M^2}{\sigma^2} = \frac{4a^2\omega}{\sigma^2 gc} \tag{8}$$

describes the frequency variation of the energy-loss at intermediate to high frequencies. The second function

$$G(\overline{\nu}) = \overline{\nu}^2 K_1^2(\overline{\nu})$$

with
$$\overline{\nu} = \frac{\omega a}{\beta\gamma c} \tag{9}$$

shows the reduction of the energy-loss at extremely high frequencies since $G(\overline{\nu}) \rightarrow \overline{\nu}\cdot\exp(-2\overline{\nu}) \rightarrow 0$ for $\overline{\nu} \gg 1$. However, for LEP with $a \approx 5$ cm, $\gamma \approx 10^5$, $\overline{\nu}$ remains smaller than unity up to 10^{14} Hz, and we may put $G(\overline{\nu}) \approx 1$ for all practical purposes.

4. In reference 3), the gap length $g = 2\ell$ is replaced by the period "d" in the expression for M [equation (4)]. Indeed, in Vainshtein's case of infinitely thin mirrors there is no distinction between these two quantities. We prefer to take the geometric mean $\sqrt{gd}$, and we also find better numerical agreement with computation by putting $\sigma \approx 1$ in the definition of ν [equation (8)], which we rewrite as

$$\nu = \frac{\omega}{\omega_{sv}} \qquad \text{where } \omega_{sv} = \frac{c\sqrt{gd}}{4a^2} \tag{10}$$

is the "Sessler-Vainshtein" frequency.

We now identify the expression for the energy loss [equation (6)] with the function in the integrand of equation (2)

$$k(\omega)\,\frac{dn}{d\omega} = \frac{C_{sv}}{2}\,F\left(\frac{\omega}{\omega_{sv}}\right) \tag{11}$$

The RHS has been divided by two since only positive frequencies are taken under the integral [which therefore has been multiplied in equation (2) by the factor 2 just as for the truncated series, in order to take account of the contribution of negative frequencies].

The function $F(\nu)$ is shown in Fig. 1 on a log-log scale. This shows that it approaches $\nu^{-3/2}$ for very large ν while it is constant ($^1/_4$) for very small ν. In between it can be approximated by various (negative) powers of ν, and in particular in the region of $1 < \nu < 10$ (a few GHz for typical storage-ring parameters) it is approximated quite well by $\nu^{-0.7}$ (see Fig. 2b) close to the value found experimentally in SPEAR[6].

We also evaluate the "average impedance" $\bar{Z}_L$ which has been obtained[7] by "binning" of the discrete resonances calculated for the LEP cavity

$$\bar{Z}_L(\omega) = \frac{\pi}{\Delta f}\sum_{n_1}^{n_2} k_n \tag{12}$$

where $\Delta f = \Delta\omega/2\pi$ is the width of a frequency bin centred on ω. The loss factors are summed over all modes whose resonant frequencies lie within the bin ($\omega_1 > \omega - \Delta\omega/2$, $\omega_2 < \omega + \Delta\omega/2$). We can express this sum also as the product of an average $\langle k \rangle$ and the number of modes in the bin $\Delta n = n_2 - n_1$

$$\sum_{n_1}^{n_2} k_n = \langle k \rangle \Delta n$$

$$\bar{Z}_L(\omega) = 2\pi^2 \langle k \rangle \frac{\Delta n}{\Delta\omega} \tag{13}$$

and thus

We now identify the average $\langle k \rangle$ with the continuous function $k(\omega)$ to obtain with equation (11)

$$\bar{Z}_L(\omega) = \pi^2 C_{sv} F \left(\frac{\omega}{\omega_{sv}}\right) \tag{14}$$

This function is compared with the (discrete) values obtained by binning in Fig. 2. The agreement is excellent above a few GHz.

5. The wakefield of a cavity is obtained by combining equations (2) and (11)

$$\Delta w(\tau) = C_{sv} \int_{\hat{\omega}}^{\infty} F(\nu) \cos\omega\tau \, d\omega \tag{15}$$

with $\nu = \omega/\omega_{sv}$ and $F(\nu)$ given by equation (8).

Unfortunately, this integral cannot be evaluated analytically except for $\tau = 0$ where one finds

$$\Delta w(0) = \omega_{sv} C_{sv} \left[\frac{\pi}{2} - \operatorname{atan}\left(1 + \sqrt{\hat{\nu}}\right) + \frac{\sqrt{\hat{\nu}}}{\hat{\nu} + 2\sqrt{\hat{\nu}} + 2} \right] \tag{16}$$

with $\hat{\nu} = \dfrac{\hat{\omega}}{\omega_{sv}}$

Even numerical evaluation of the integral in equation (15) is difficult because the integrand has an infinite number of oscillations in the integration interval. For large enough values of $\hat{\nu}$, the function $F(\nu)$ can be approximated by $B\cdot\nu^{-3/2}$ (see Fig. 1) where the constant B is determined in such a way that $F(\hat{\nu}) = B\cdot\hat{\nu}^{-3/2}$. Then the integral in equation (15) can be evaluated analytically[1])

$$\Delta w(\tau) = \frac{2B}{\sqrt{\hat{\nu}}}\,\omega_{sv}C_{sv}\left\{\cos\hat{x} - \sqrt{\frac{\pi\hat{x}}{2}}\left[1 - 2S\left(\sqrt{\frac{2\hat{x}}{\pi}}\right)\right]\right\} \tag{17}$$

where $\hat{x} = \hat{\omega}\tau$ and $S(x)$ is the Fresnel integral.

For a small SLAC cavity (S-band) over 400 modes have been computed[1]) up to $\hat{\omega} = 6\cdot10^{11}s^{-1}$. Since $\omega_{sv} < 10^{10}s^{-1}$ (see Table I) $\hat{\nu}$ is over 30 and $F(\nu)$ is quite well approximated by $0.6\nu^{-3/2}$ (see Fig. 1). For the much bigger LEP cavities (353 MHz) 565 modes brought $\hat{\omega}$ only to $7\cdot10^{10}s^{-1}$. Since ω_{sv} is still over $10^{10}s^{-1}$, $\hat{\nu}$ is well below 10 and the approximation $F(\nu) \approx 0.32\nu^{-3/2}$ is not adequate (Fig. 1).

Numerical evaluation of the integral in equation (15) has become possible after an analytic expression has been found for the integral from zero to infinite (see Appendix B).

$$\tilde{F}_o(x) = \int_0^\infty F(\nu)\cos\nu x\cdot d\nu = \frac{\pi}{4}(1 + 4x)e^{2x}\cdot\mathrm{erfc}(\sqrt{2x}) - \sqrt{\frac{\pi x}{2}} \tag{18}$$

where $x = \omega_{sv}\tau$ and erfc(z) is the complementary error-function.

With this integral one can express the required one by

$$\int_{\hat{\omega}}^\infty F\left(\frac{\omega}{\omega_{sv}}\right)\cos\omega\tau\cdot d\omega = \omega_{sv}\cdot\tilde{F}_o(x) - \int_0^{\hat{\omega}} F\left(\frac{\omega}{\omega_{sv}}\right)\cos\omega\tau\cdot d\omega \tag{19}$$

The number of oscillations of the integrand in the integral on the RHS is now only of the order of $\hat{\omega}\tau/\pi$, and numerical evaluation is no problem if the time delay τ is not too large.

Some results of these calculations for a typical LEP cavity are shown in Figs. 3a and b (for time delays up to 80 and 500 ps respectively) based on $\hat{n}$ = 565 ($\hat{\omega}/2\pi$ = 11.75 GHz). For very small time delays ($\omega_{SV}\tau \ll 1$), one finds to a first approximation

$$\Delta w(\tau) = \Delta w(o) - \sqrt{2\pi w_{SV}\tau} \qquad (20)$$

where w(o) is given in equation (16). The wakefield falls initially with infinite slope, and the large initial contribution is reduced by a factor two in less than 1 ps. On a longer time scale (of several 100 ps), the strong fluctuations of the wakefield obtained by summing over 565 modes completely vanish when the optical resonator correction is added. This is all the more gratifying as there is no adjustable parameter in this correction. A comparison of this wakefield with the one obtained by adding the optical resonator correction to the sum over only 20 modes ($\hat{\omega}/2\pi$ = 2.3 GHz) is shown in Fig. 4. For very short times the two curves agree to better than 1%, and stay within 10% for times up to several 100 ps.

Conclusions

The delta-function wake potential (i.e. the Green's function) of a periodic sequence of cavities can be obtained by summing over a finite number of resonances and correcting for the missing modes with an integral. The Sessler-Vainshtein "optical resonator model" yields an excellent fit to the averaged impedance in that part of the frequency spectrum for which resonances have been calculated for the LEP cavity (565 modes up to 12 GHz). However, due to the large size of the cavity, the integrand of the correction term cannot be approximated by a simple -3/2 power law (which was quite acceptable for the much smaller SLAC cavities). Because the integrand has an infinity of oscillations in the integration interval, numerical integration is difficult. However, it has become possible after the main part has been done analytically and yields a correction to the wake field which is initially bigger than the sum over 565 modes. Furthermore it completely removes the wiggles which appear in the truncated sum. Thus we have good reason for our confidence that this wake field is a better approximation to reality and should be used to study bunch lengthening and instabilities in LEP.

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TABLE I

Cavity Parameters

		SLAC[1] (cavity no. 45)	LEP[6]	
Dimensions				
a	(cm)	1.163	5.00	
b	"	4.134	32.50	
g	"	2.915	34.49	
d	"	3.499	42.42	
Fundamental mode				
f_o	(GHz)	2.84	0.353	
ω_o	$(10^{10} s^{-1})$	1.785	0.22	
k_o	(V/pC)	0.683	0.112	(0.158)*
Higher modes				
$\hat{n}$		416	565	(20)*
$\hat{\omega}$	$(10^{10} s^{-1})$	60	7.5	(1.45)*
$2 \sum_{n=0}^{\hat{n}} k_n$	(V/pC)	-	2.25	(0.851)*
Optical resonator model				
ω_{sv}	$(10^{10} s^{-1})$	1.77	1.15	
$\hat{\nu} = \hat{\omega}/\omega_{sv}$		34	6.6	
$F(\hat{\nu})$		0.003	0.019	
$B = \hat{\nu}^{3/2} F(\hat{\nu})$		0.60	0.32	
$\Delta w(o)$	(V/pC)	-	3.44	

* SUPERFISH results

APPENDIX A

The Optical Resonator Model

We give a short description of the results obtained by Vainshtein which form the basis for the analytic extension of the truncated series for the wake function. Vainshtein considers several geometries and introduces his results step by step:

a) Diffraction at the open end of a planar waveguide
(semi-infinite plates $z > 0$ at $y = \pm a$).
The (spatial) Fourier-transform $F(w)$ of the surface current density $f(z)$ in the upper plate is determined by the integral equations

$$\begin{aligned} \int F(w) e^{iwz}\, dz &= 0 \quad ; \quad z < 0 \\ \int L(w) F(w) e^{iwz}\, dz &= 0 \quad ; \quad z > 0 \end{aligned} \tag{A1}$$

Under the assumption $|w| \ll k$ the kernel L can be approximated by

$$L(w) \approx 1 - \exp\left[i\left(2\pi p - a\,\frac{\omega^2}{k}\right)\right] \tag{A2}$$

The frequency is assumed to be high enough for the integer q to be large compared to unity in the expression

$$ka = \pi(q/2 + p) \quad , \tag{A3}$$

where $k = \omega/c$, $|p| < {}^1/_2$.
(i.e. wavelength $\ll$ distance between plates).

The approximate kernel vanishes for $w = \pm w_j$, where

$$w_j = s_j\sqrt{k/2a} \quad \text{with} \quad s_j = \sqrt{4\pi(p + j)} \tag{A4}$$

are the wave numbers of the modes E (or $H)_{o,q+2j}$ ($j > 0$ propagating, $j < 0$ damped, $j = 0$ incident wave).

In general, the Wiener-Hopf technique consists of finding a "factorization" of the kernel

$$L(w) = L_1(w) \cdot L_2(w) \qquad (A5)$$

for which L_1 is "holomorphic" (= analytic) in the upper half-plane Im $w \geq 0$ (and L_2 in the lower one), and tends to unity for $|w| \to \infty$ in this half-plane. Usually, the factors can only be expressed as infinite products. Without derivation, Vainshtein gives an explicit expression

$$L_1(w) = \exp[U(\sqrt{2a/k}\, w,\ p)]$$
$$L_2(w) = L_1(-w) \qquad (A6)$$

where $$U(s,p) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\ln[1 - \exp(2\pi i p^{-t^2/2})]}{t - s\cdot e^{i\pi/4}}\,dt \qquad (A7)$$

With these factors, the solution of the integral equations is

$$F(w) = \frac{A}{2\pi i}\,\frac{L_1(w_o)}{(w+w_o)\,L_2(w)} \qquad (A8)$$

and thus

$$f(z) = A[\bar{e}^{iw_oz} + \sum_{j=0}^{\infty} R_{oj} e^{iw_jz}] \qquad (A9)$$

with $$R_{oj} = -\frac{i\,\exp[U(s_o,p) + U(s_j,p)]}{(s_o + s_j)s_j}$$

A is the amplitude of the incident wave (with mode-number $w_o = s_o\sqrt{k/2a}$, $s_o = \sqrt{4\pi p}$).

R_{oo} is the "reflection coefficient", while R_{oj} ($j \neq 0$) are "transformation coefficients". For small values of $|p|$ the transformation to modes with $j \neq 0$ is negligible and R_{oo} is dominant. It is then claimed to be approximated by

$$R_{oo} \approx -\exp[i\sigma(1+i)s_o] \qquad \text{(A10)}$$

with $$\sigma = -\zeta(^1/_2)/\sqrt{\pi}$$

where $\zeta(z)$ is Riemann's zeta-function. Equation [A10) is exact for $p \to 0$, but remains qualitatively useful up to $|p| = {}^1/_2$.

b) Diffraction at the open end of a semi-infinite circular waveguide (radius a)

The frequency is now assumed to be given by

$$k_a = \nu_{mn}^{(1)} + \pi p \qquad \text{(A11)}$$

where $\nu_{mn}^{(1)}$ is the n^{th} zero of the (derivative of the) m^{th} order Bessel function. It is assumed that $n \gg 1$ and $|p| < {}^1/_2$. For $m \neq 0$ there will be an azimuthal component of the current density in addition to the longitudinal one and the integral equations become rather complicated. However, for $m \ll ka$ and $|w| \ll k$ the current density is determined approximately by the relations derived for the planar waveguide in the previous section.

c) Open cylindrical resonator (radius a, length 2ℓ)

The wave-number $k = \omega/c$, given by equation (A11), is now complex for $q = 1, 2, \ldots$ and p complex and small.

The waves are reflected at the open ends of the resonator $z = \pm\ell$ practically without conversion. This leads to the "characteristic equations"

$$R_{oo} = -(-)^q e^{-2iw_o\ell} \qquad \text{(A12)}$$

Since equation (A4) yields for j = 0

$$2w_o\ell = Ms_o \tag{A13}$$

with $$M = \sqrt{\frac{2k\ell^2}{a}}$$

Equation (A12) with (A10) yields the solutions

$$s_o = \sqrt{4\pi p} = \frac{\pi q}{M + (1+i)\sigma} \tag{A14}$$

from which one can determine $p = p' - ip''$.

The fractional decrease of energy (during the time of traversal 2a/c) is given by

$$\Lambda = 1 - e^{-4\pi p''} \approx 4\pi p''$$

$$= 2\pi^2 q^2 \frac{\sigma(M+\sigma)}{[(M+\sigma)^2 + \sigma^2]^2} \tag{A15}$$

d) <u>Open resonator formed by circular mirrors</u>

$(0 < r < a,\ z = \pm\ell)$

where now $k\ell = \frac{\pi}{2}(q + 2p)$; $q \gg 1,\ |p| < {}^1/_2$ (A16)

The characteristic equation becomes

$$R_{oo} = -e^{-2i\Omega_m(w_o a)} \tag{A17}$$

where $\Omega_m(x)$ is a "phase-function" for which

$$\Omega_m(\nu_{mn}) = n\pi .$$

Expanding this function yields

$$\Omega_m(x) = n\pi + \Omega'(\nu_{mn})(x - \nu_{mn}) \tag{A18}$$

R_{oo} is still given by equation (10), but with s_o replaced by $s_o\Omega'(\nu_{mn})$. Thus the solution of the characteristic equation becomes

$$s_o = \sqrt{4\pi p} = \frac{2\nu_{mn}}{M + (1+i)\sigma} \tag{A19}$$

with $M = \sqrt{\dfrac{2ka^2}{\ell}}$

The fractional decrease of energy becomes [reference 4), equation (80)]

$$\Lambda = 4\pi p'' = \frac{8\nu_{mn}^2\sigma(M+\sigma)}{[(M+\sigma)^2 + \sigma^2]^2} \tag{A20}$$

This expression is asymptotically valid [$q \gg 1$ in equation 16)] for $m > 0$, $n > 1$. The fields obtained with this theory are claimed to agree quite well with those obtained by direct numerical solution of the integral equation[8].

APPENDIX B

The optical resonator wake field

a) We want to evaluate the integral in equation (15) when $F(\nu)$ is given by equation (8). We introduce the dimensionless variables

$$\nu = \frac{\omega}{\omega_{sv}} \quad , \quad x = \omega_{sv}\tau \quad , \quad \hat{\nu} = \frac{\hat{\omega}}{\omega_{sv}} \tag{B1}$$

to get

$$\Delta w(\tau) = C_{sv} \cdot \omega_{sv} \cdot \tilde{F}(x) \tag{B2}$$

with

$$\tilde{F}(x) = \int_{\hat{\nu}}^{\infty} F(\nu) \cos \nu x \cdot d\nu$$

(B3)

where $$F(\nu) = \frac{1 + \sqrt{\nu}}{[(1+\sqrt{\nu})^2 + 1]^2}$$

The substitution $\nu = u^2$ removes the square roots in the integrand and - with $s = \sqrt{\hat{\nu}}$ - one finds

$$\tilde{F}(x) = 2 \int_{s}^{\infty} \frac{u(u+1)\cos xu^2}{[(u+1)^2 + 1]^2} du \tag{B4}$$

b) The integral (B4) can be evaluated analytically only for $x = 0$. In this case we substitute $u+1 = v$ and use partial fraction decomposition to get

$$\tilde{F}(o) = 2 \int_{s+1}^{\infty} \frac{(v-1)vdv}{(v^2+1)^2} = 2 \int_{s+1}^{\infty} \left[\frac{1}{v^2+1} - \frac{v+1}{(v^2+1)^2}\right] dv$$

These terms can be integrated to yield

$$\tilde{F}(o) = \left[\operatorname{atan} v - \frac{v-1}{v^2+1}\right]_{s+1}^{\infty} = \frac{\pi}{2} - \operatorname{atan}(s+1) + \frac{s}{s^2+2s+2} \tag{B5}$$

For $x \neq 0$, however, the integral (B4) could not be done analytically, and even numerical integration is not possible because of the infinite number of oscillations of the integrand in the integration interval. It is then expedient to rewrite equation (B4) as

$$\tilde{F}(x) = \tilde{F}_o(x) - 2 \int_o^s \frac{u(u+1)\cos xu^2}{(u^2 + 2u + 2)^2} du \tag{B6}$$

with

$$\tilde{F}_o(x) = 2 \int_o^{\infty} \frac{u(u+1)\cos xu^2}{(u^2 + 2u + 2)^2} du \tag{B7}$$

The integral in (B6) can be done numerically - at least for not too large time delays - since the integrand has only a finite number of oscillations (of the order of $xs^2/\pi = \hat{\omega}\tau/\pi$). For the definite integral (B7) we attempt to find an analytic expression. First we use again partial fraction decomposition

$$\frac{2u(u+1)}{(u^2+2u+2)^2} = \frac{1}{2j}\left[\frac{u}{(u+1-j)^2} - \frac{u}{(u+1+j)^2}\right] = \operatorname{Im} \frac{u}{(u+1-j)^2} \tag{B8}$$

to rewrite the integral as

$$\tilde{F}_o(x) = \operatorname{Im} \int_o^{\infty} \frac{u \cdot \cos xu^2}{(u+1-j)^2} du \tag{B9}$$

c) Next we use two definite integrals listed in GR[9] 3.853.2

$$I_1(\beta) = \int_0^\infty \frac{\cos(\alpha u)^2}{u^2+\beta^2}\,du = \frac{\pi}{2\beta}\left[c - (c-s)C - (c+s)S\right]$$

and GR 3.722.3 (B10)

$$I_2(\beta) = \int_0^\infty \frac{u\cdot\cos(\alpha u)^2}{u^2+\beta^2}\,du = -\frac{1}{2}\left[c\cdot ci + s\cdot si\right]$$

where $c,s = \cos,\sin(\alpha\beta)^2$;
$C,S = C,S(\alpha\beta)$ are the Fresnel integrals
and $si,ci = si,ci(\alpha\beta)^2$ are the integral sine/cosine.

Both integrals (B10) are valid for α real and Re $\beta < 0$. With these we can calculate

$$I_3(\beta) = \int_0^\infty \frac{\cos(\alpha u)^2}{u-j\beta}\,du = j\beta I_1 + I_2 =$$

$$= j\,\frac{\pi}{2}\left[c - (c-s)C - (c+s)S\right] - \frac{1}{2}\left[c\cdot ci + s\cdot si\right] \qquad \text{(B11)}$$

or, if we put $\beta = j\gamma$ and use $ci(-x) = ci(x) + j\pi$, $si(-x) = -si(x)-\pi$ and $C(jx) = jC(x)$, $S(jx) = -jS(x)$

$$I_4(\gamma) = \int_0^\infty \frac{\cos(\alpha u)^2}{u+\gamma}\,du = \frac{\pi}{2}\left[(c+s)C - (c-s)S - s\right] - \frac{1}{2}\left[c\cdot ci + s\cdot si\right] \qquad \text{(B12)}$$

which is valid for α real, Im$\gamma < 0$.

d) By taking the limit

$$\frac{u}{(u+\gamma)^2} = \lim_{\varepsilon\to 0}\left[\frac{\gamma+\varepsilon}{u+\gamma+\varepsilon} - \frac{\gamma}{u+\gamma}\right] \qquad \text{(B13)}$$

we obtain the integral

$$I_5(\gamma) = \int_0^\infty \frac{u\cos(\alpha u)^2}{(u+\gamma)^2}\,du = I_4(\gamma) + \gamma\,\frac{\partial I_4}{\partial\gamma} \qquad \text{(B14)}$$

Using the derivatives $\frac{\partial}{\partial\gamma}(C,S)_{\alpha\gamma} = \alpha\sqrt{\frac{2}{\pi}}\,(c,s)$

and

$$\frac{\partial}{\partial\gamma}(ci,si)_{(\alpha\gamma)^2} = \frac{2}{\gamma}\,(c,s)$$

one then obtains

$$I_5(\gamma) = \frac{\pi}{2}\overbrace{\left[(c+s)C - (c-s)S - s\right]}^{T_1} + \pi\alpha^2\gamma^2\overbrace{\left[(c-s)C + (c+s)S - c\right]}^{T_2} + \qquad \text{(B15)}$$

$$- \frac{1}{2}\underbrace{\left[c\cdot ci + s\cdot si\right]}_{T_3} + \alpha^2\gamma^2\underbrace{\left[s\cdot ci - c\cdot si\right]}_{T_4} + \alpha\gamma\sqrt{\frac{\pi}{2}} - 1$$

e) In order to evaluate (B9) we need I_5 for $\gamma = 1-j$. For the Fresnel integrals of complex argument we find (GR 8.256)

$$(C + jS)_z = \sqrt{j/2}\,\mathrm{erf}(z/\sqrt{j})$$

$$\text{(B16)}$$

$$(C - jS)_z = \mathrm{erf}\,(z\sqrt{j})/\sqrt{2j}$$

From these one obtains

$$(C,S)_{(1-j)\alpha} = \frac{1\mp j}{4}\left[\operatorname{erf}(\alpha\sqrt{2}) \mp e^{-2\alpha^2} D(\alpha\sqrt{2})\right] \qquad \text{(B17)}$$

where $D(z)$ is $2/\sqrt{\pi}$ times the "Dawson integral" (AS[10] 7.1.17).
For the first two terms in equation (B15) we get, with $b = a^2 = 2\alpha^2$

$$\begin{aligned} 2T_1 &= -D(a)e^{-b} - j(e^b \operatorname{erf} a - 2\sinh b) \\ 2T_2 &= (e^b \operatorname{erf} a - 2\cosh b) + jD(a)e^{-b} \end{aligned} \qquad \text{(B18)}$$

Actually, we need only $\operatorname{Im}(I_5)$ for (B9), i.e.

$$\begin{aligned} \frac{\pi}{2}\operatorname{Im}(T_1) &= -\frac{\pi}{4}\left[e^b \operatorname{erf} a - 2\sinh b\right] \\ \pi \operatorname{Im}(\alpha^2\gamma^2 T_2) &= -\frac{\pi b}{2}\left[e^b \operatorname{erf} a - 2\cosh b\right] \end{aligned} \qquad \text{(B19)}$$

f) Similarly, we use AS 5.22/24 to derive (for $x > 0$)

$$\begin{aligned} \operatorname{si}(-jx) &= -\frac{\pi}{2} - \frac{j}{2}\left[\operatorname{Ei}(x) - \operatorname{Ei}(-x)\right] \\ \operatorname{ci}(-jx) &= \frac{1}{2}\left[\operatorname{Ei}(x) + \operatorname{Ei}(-x)\right] - j\pi \end{aligned} \qquad \text{(B20)}$$

where $\operatorname{Ei}(z)$ is the exponential integral.
With these we get for the next two terms in (B9)

$$\begin{aligned} 2T_3 &= e^{-b} \operatorname{Ei}(b) + e^b \operatorname{Ei}(-b) - j\pi e^{-b} \\ 2T_4 &= \pi\cdot e^{-b} + j\left[e^{-b} \operatorname{Ei}(b) - e^b\cdot \operatorname{Ei}(-b)\right] \end{aligned} \qquad \text{(B21)}$$

For the calculation of $\tilde{F}_o(x)$ we need only

$$\frac{1}{2}\,\mathrm{Im}\,(T_3) = \frac{\pi}{4}\,e^{-b} \tag{B22}$$

and

$$\mathrm{Im}(\alpha^2\gamma^2 T_4) = -\frac{\pi b}{2}\,e^{-b}$$

Combining all these expressions then leads to the final result (with $b = 2\alpha^2 = 2x$)

$$\tilde{F}_o(x) = \frac{\pi}{4}\,(1 + 4x)e^{2x}\,\mathrm{erfc}(\sqrt{2x}) - \sqrt{\frac{\pi x}{2}} \tag{B23}$$

where erfc(z) = 1 - erf(z) is the complementary error-function.

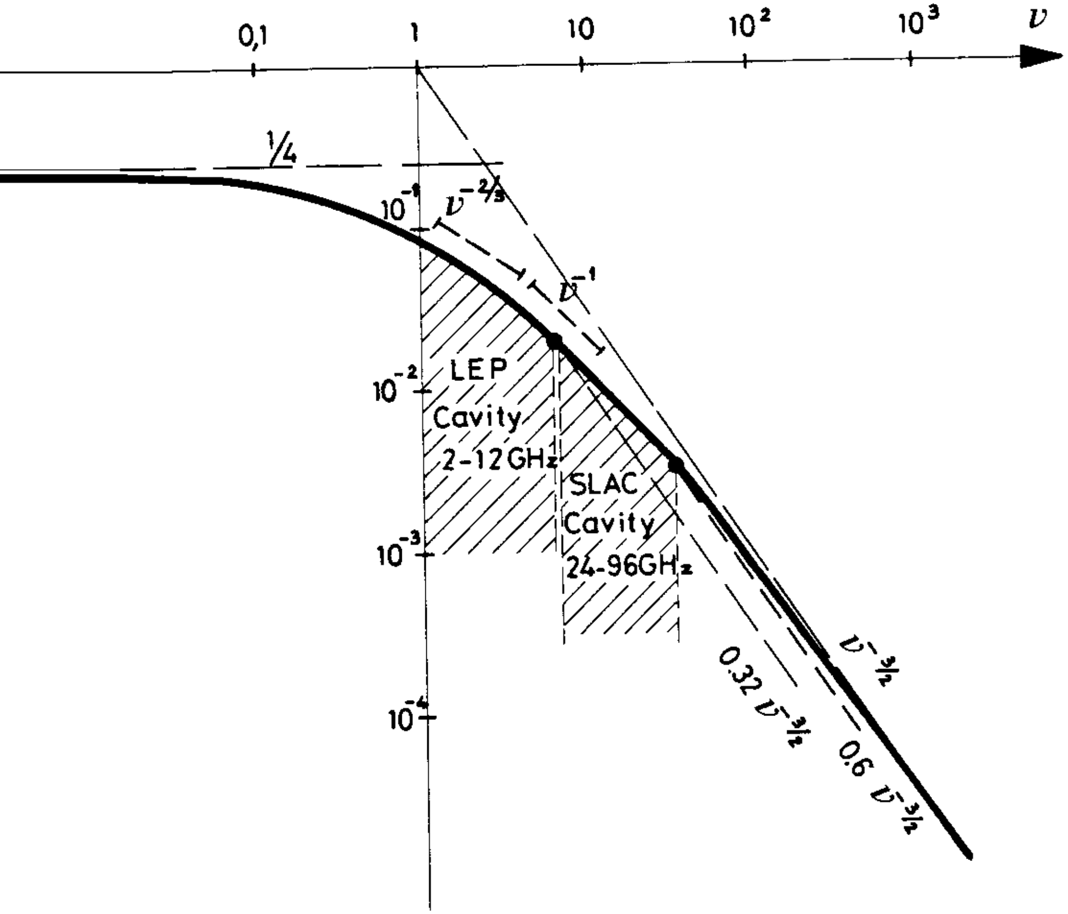

Fig. 1

Optical resonator function $F(\nu)$ on a log-log plot

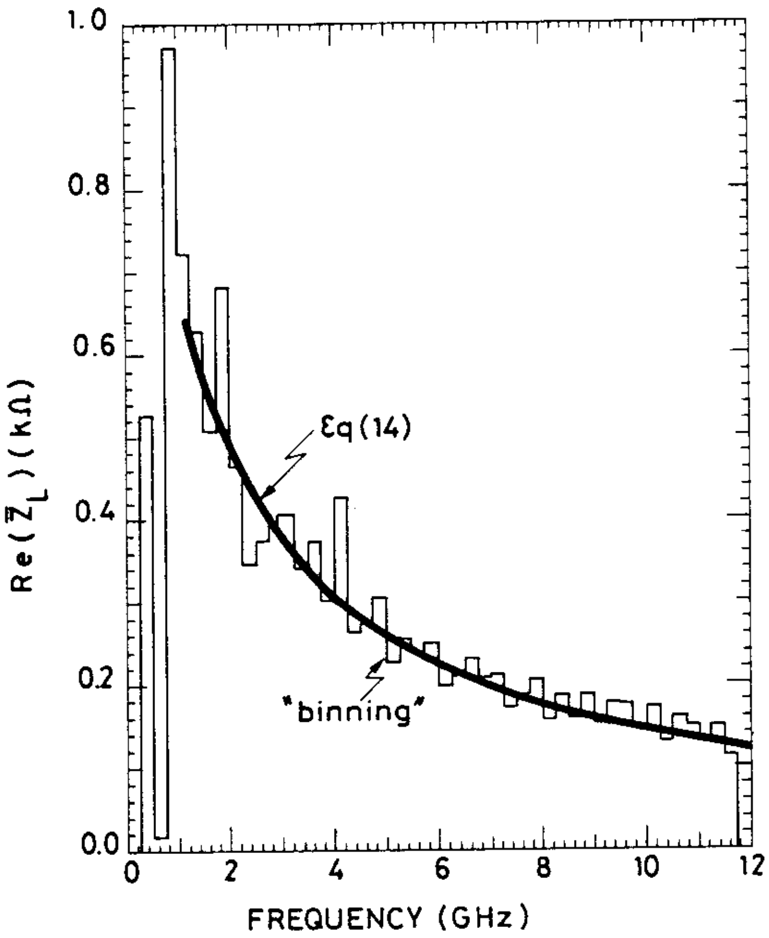

Fig 2a: The average longitudinal impedance of a LEP cavity

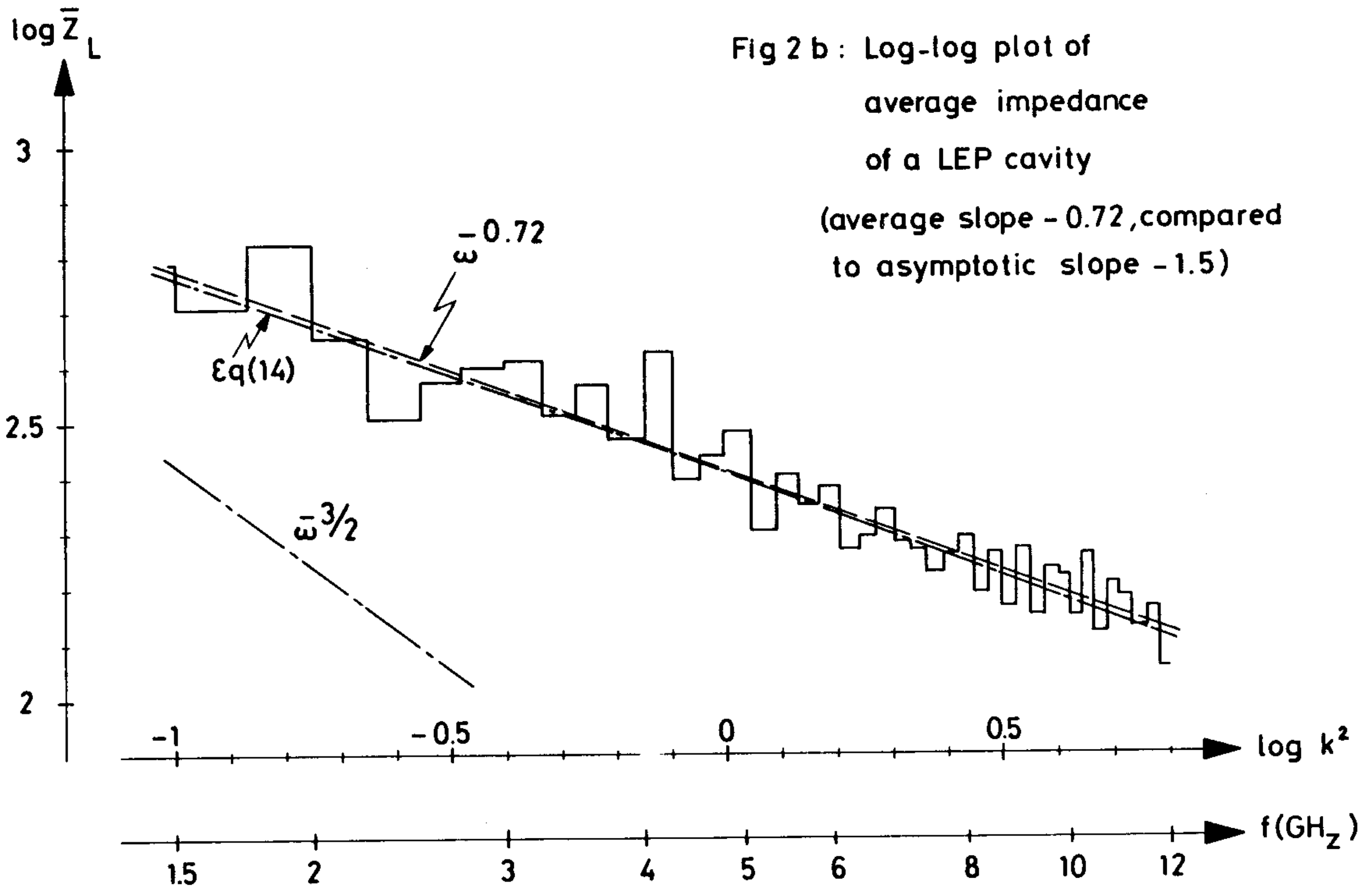

Fig 2 b : Log-log plot of average impedance of a LEP cavity (average slope −0.72, compared to asymptotic slope −1.5)

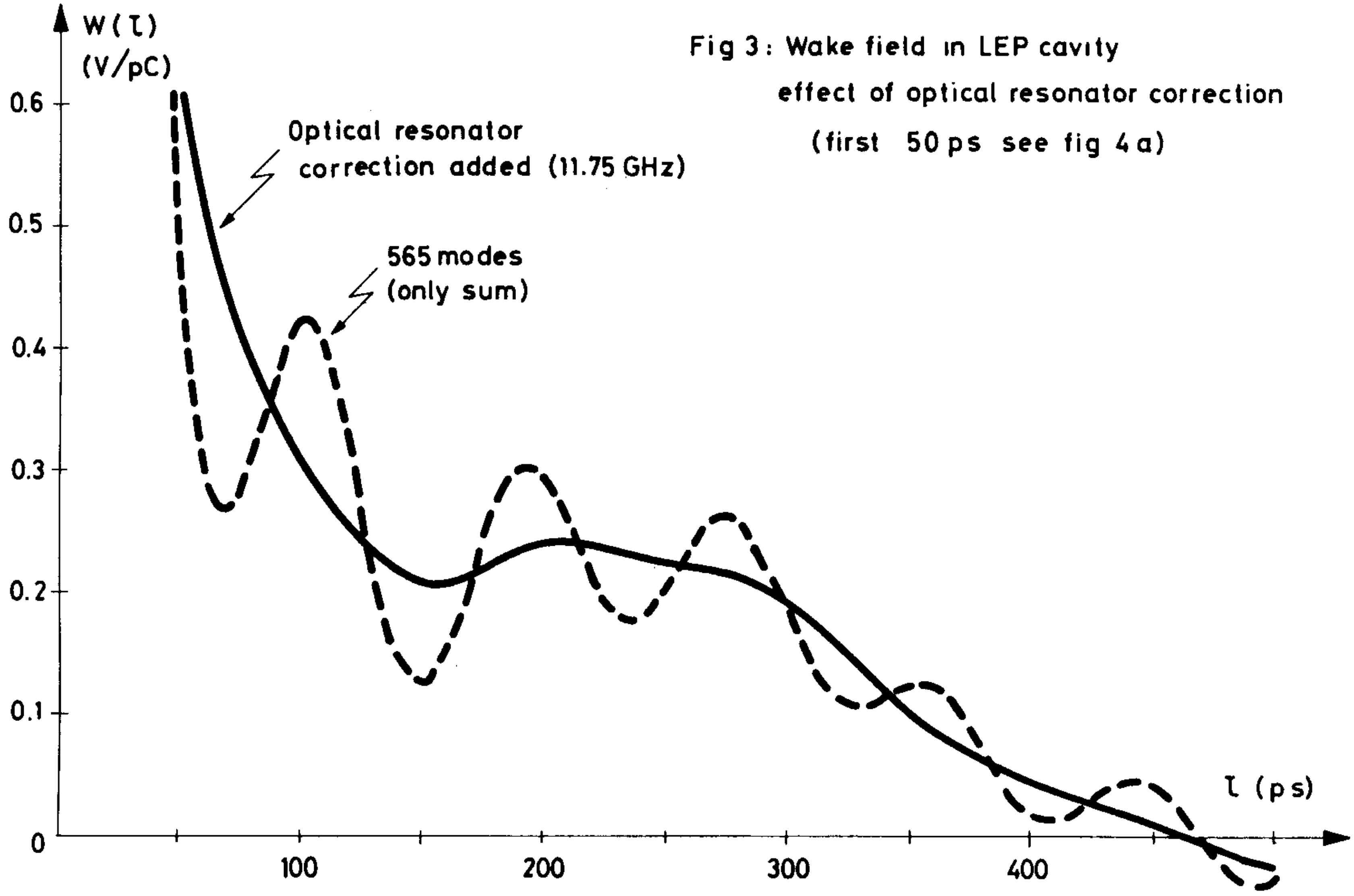

Fig 3: Wake field in LEP cavity
effect of optical resonator correction
(first 50 ps see fig 4 a)

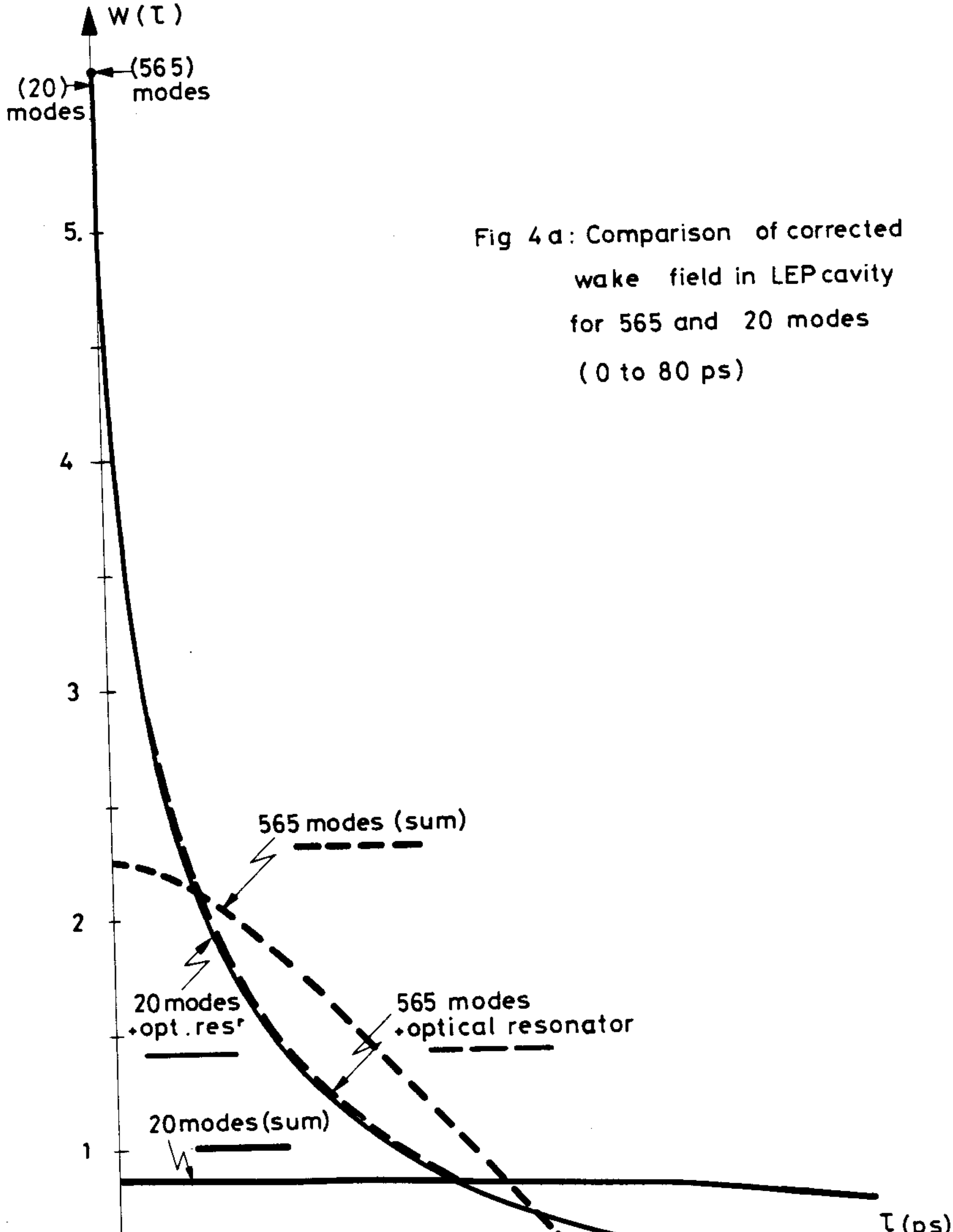

Fig 4a: Comparison of corrected wake field in LEP cavity for 565 and 20 modes (0 to 80 ps)

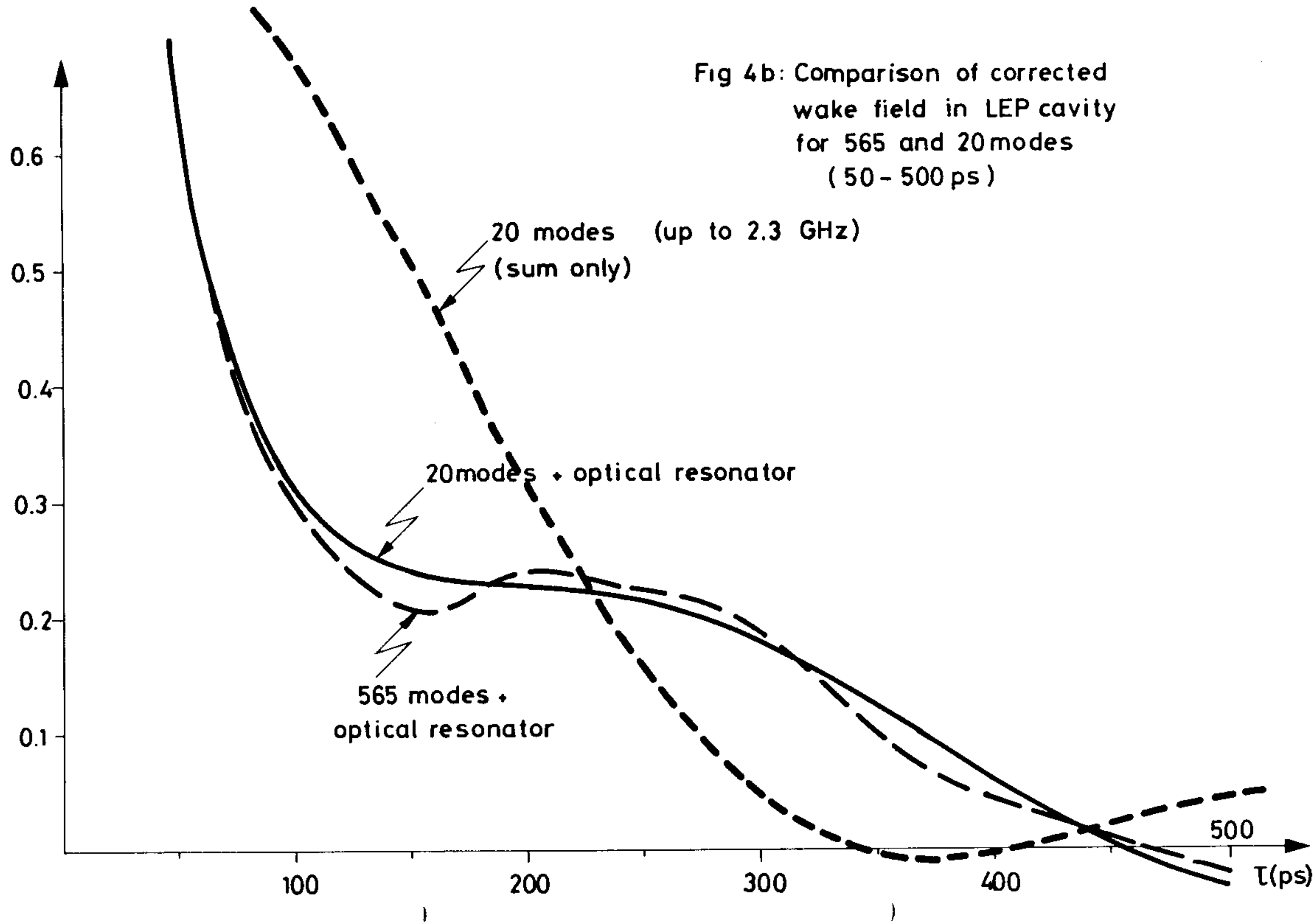

Fig 4b: Comparison of corrected wake field in LEP cavity for 565 and 20 modes (50 - 500 ps)