

CALCULATION OF TRANSMISSION ZEROS
USING QZ TECHNIQUES*

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The application of the numerically stable QZ algorithm is discussed to provide a reliable method for the computation of the transmission zeros of a linear system. Robust numerical software and implementation details are suggested.

Key Word Index-Computational methods; computer software; linear systems; multivariable control systems; numerical analysis; stability of numerical methods; transmission zeros.

Summary-The computation of various "system zeros" is investigated through application of QZ-type algorithms for the nonsymmetric generalized eigenvalue problem. Such algorithms use unitary similarities to efficiently reduce the problem to one where the zeros may be determined in a useful, accurate, and dependable manner. Recent reliable and sophisticated analysis and software (specifically, EISPACK) developed by numerical linear algebra specialists is used. EISPACK, moreover, is widely available and can be applied directly to the transmission zero problem. Examples and timing estimates are given and the associated generalized eigenvector problem is noted with its application to the computation of supremal (A,B)-invariant and controllability subspaces.

1. INTRODUCTION

Consider the following linear time-invariant system:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t)\end{aligned}\tag{1}$$

Here $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the (control) input, and $y(t) \in \mathbb{R}^r$

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is the output. The matrices A , B , C , and D are of appropriate dimensions and no assumptions are made concerning controllability, observability, or rank. We shall investigate in this paper the application of QZ-type algorithms to the computation of various "system zeros", particularly the so-called transmission zeros as defined by Davison [1] (as an amendment to [2]) or the invariant zeros as defined by MacFarlane and Karcanias [3]. Very roughly speaking these zeros are certain complex frequencies at which "transmission" through the system (1) is blocked (cf. [3]).

The role of transmission zeros in the investigation of, for example, regulation, structural stability, decoupling, and servomechanism design is by now well-established. We refer the reader to papers such as [2], [3], [4], [5], and [6] for more details and references.

We shall concentrate instead on numerical aspects. Specifically, we apply the QZ-algorithm [7] to the determination of transmission zeros. This algorithm has the advantages of being efficient, numerically stable, reliable, and, most important, widely available. No assumptions are needed on A , B , C , and D and the application (in a programming sense) to the problem at hand is absolutely trivial. More details are given in Section 3.

This is, to varying degrees, in definite contrast to various other proposed algorithms such as [2], [5] and [8] which rely heavily on certain factorizations or rank conditions or introduce additional sources of potentially destructive numerical errors by some preprocessing of the system data (see e.g. Example 4 of this paper). The numerical determination of rank for example, is a notoriously difficult problem but such a question is never addressed in these papers. Basically, the algorithms are presented devoid of any analysis of a specific numerical nature: perturbation bounds, condition numbers, etc.--analysis which attempts to account for the presence of roundoff errors in the calculations. We feel it is important to at least acknowledge the potential impact of numerical errors. While we do not present detailed analysis here directly,

the QZ-algorithm does rest on very firm theoretical and practical foundations as found, for example, in [7], [9], or [10].

We must point out though, that other algorithms may be computationally efficient and may well be for well-conditioned problems, quite accurate. However, in the presence of roundoff error, from the point of view of general-purpose, reliable mathematical software, we would still argue strongly for the QZ approach, at least on numerical grounds.

That the computation of transmission zeros can be cast as a generalized eigenvalue problem (see Section 4) was recognized by Patel [11] in an article based on a fine paper by Kaufman [12]. The latter paper [12] is consistent with our general philosophy of using "expertly-written" software but is unfortunately "pre-QZ". The former paper [11], though, besides proposing no useful software, suffers from many of the symptoms mentioned in the previous paragraph.

Our other main motivation for using the QZ algorithm is that the associated eigenvectors (see details below) provide a basis for V^* or V^*/R^* (see Wonham [13] for notation).

2. SOME DEFINITIONS

Let $L = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, $M = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}$. We shall define transmission

zeros in terms of the pencil of matrices $P(\lambda) = L - \lambda M$.

Definition 1 [14]: The rank of the pencil $L - \lambda M$ is the order of the largest minor that does not vanish identically.

We shall henceforth assume that $m \geq r$. The case $r > m$ can be handled either in the obvious analogous way to the case $m > r$ or by consideration of the appropriate transpose matrices (see [2]). We shall not need, for the purposes of our algorithm, any a priori assumption on the rank of $P(\lambda)$. However, the set T of all transmission zeros (multiplicities included) of (1) is normally most conveniently defined in the full rank case and we shall make this assumption here. It will be seen later, however,

that the QZ algorithm detects degeneracy (see [2]) and, equally importantly, near - degeneracy in a numerically stable fashion.

We first discuss our definition of transmission zeros in terms of equivalent definitions in the literature.

Definition 2: T is the set of zeros (multiplicities included) of $\phi(\lambda)$ where $\phi(\lambda)$ is the greatest common divisor of all (nonidentically zero) $(n+r) \times (n+r)$ minors of $P(\lambda)$.

This definition is equivalent to Davison's [1]. It is also equivalent, under our nondegeneracy assumption, to MacFarlane's "invariant zeros" [3]. In the degenerate case we will simply adopt the convention, consonant with Davison, that $T = \mathcal{T}$. A fairly extensive comparison of various other definitions of system zeros is given in [4]. We find Definition 2 useful because of the ease and reliability of computation afforded by the QZ algorithm. Moreover, it will be noted how other definitions can be recovered by appropriate additional calculations.

Definition 2, our working definition of transmission zeros, will subsequently be shown to be equivalent to a constructive definition of transmission zeros derived from the QZ algorithm. The relevant aspects of this algorithm are now presented in Section 3.

3. THE QZ ALGORITHM

The QZ algorithm is concerned with solving the so-called generalized eigenvalue problem: find all finite λ for which there exist nontrivial solutions of the equation

$$Lz = \lambda Mz \quad (2)$$

where L and M are general square matrices. The theoretical aspects of this and the analogous "rectangular" problem have been studied extensively

and may be consulted in Gantmacher [14] or Thompson and Weil [15], [16]. Gantmacher gives thorough historical references. Lancaster [17] gives not only comprehensive theoretical treatment of the subject but also physical applications such as to problems of vibrating systems.

The first definitive numerical treatment which squarely addressed the problem of singular M was published by Moler and Stewart [7] in 1973. Their algorithm is based on the following theorem [9] which helps explain its pleasant numerical properties.

THEOREM 1: There exist unitary matrices Q and Z such that QLZ and QMZ are both upper triangular.

The QZ algorithm has been coded in FORTRAN and is now widely available in the EISPACK [18] package. We now briefly outline the four main steps in the algorithm (along with the names of the corresponding EISPACK sub-routines). The reader is referred to [7] for details.

1. **QZHES** : L and M are simultaneously reduced to upper Hessenberg and upper triangular form respectively.
2. **QZIT** : L is reduced to quasi-upper triangular form while the upper triangular form of M is maintained.
3. **QZVAL** : L is effectively reduced to upper triangular form while the upper triangular form of M is maintained; the generalized eigenvalues can then be extracted from the triangular forms.
4. **QZVEC** : the generalized eigenvectors of the reduced problem are found by a "back-substitution" process; the original eigenvectors are found by applying the accumulated Z 's.

Two remarks are crucial for our application so they will be emphasized here.

Remark 1: Since unitary transformations are used the computed generalized eigenvalues are the exact generalized eigenvalues of the "slightly perturbed" problem $(L+G)z = \lambda(M+H)z$ where G and H are perturbation matrices whose norms can usually be bounded by a modest multiple of the machine precision (machine precision is defined to be the smallest number ϵ for which $1+\epsilon > 1$ using floating point addition). Moreover, all well-conditioned generalized eigenvalues are computed accurately independently of the singularity of M (i.e., the "infinite" generalized eigenvalues).

Remark 2: QZVAL does not actually compute the λ_i but rather returns α_i and β_i , the diagonal elements of QLZ and QMZ respectively. All the important information in the problem is contained in the α_i and the β_i and it is our responsibility as users to judiciously compute the λ_i from them. For example, if the elements of L are determined experimentally and are known exactly only to within, say 10^{-3} , then, since we are using unitary transformations, we may wish to call any λ_i corresponding to a $\beta_i < 10^{-3}$ and $\alpha_i \geq 10^{-3}$ an infinite generalized eigenvalue. Details are discussed in the next section.

It is also appropriate at this point to acquaint the reader with certain other developments and extensions related to the QZ algorithm. Ward [19] has developed the combination shift QZ algorithm and, in fact, this extension is implemented in the EISPACK version. L. Kaufman [20] uses stabilized elementary transformations instead of orthogonal (unitary) transformations. The resulting LZ algorithm is better for complex matrices and is about four times as fast as QZ. She also claims to have a better way to compute the eigenvectors. Van Loan [21] has the most

general algorithm yet of those in the unitary similarity family. His VZ algorithm includes the QR, QZ, and singular value decomposition algorithms as special cases.

4. APPLICATION OF THE QZ ALGORITHM TO THE TRANSMISSION ZERO PROBLEM

4.1 Case 1: m = r

In this case we have an alternate definition of the transmission zeros of (1):

Definition 3: T is the set of generalized eigenvalues (multiplicities

included) of the problem (2) where $L = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, $M = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$.

This definition is clearly equivalent to our previous one for by Theorem 1 there exist unitary Q and Z such that

$$Q(L - \lambda M)Z = U_1 - \lambda U_2$$

where U_1, U_2 are upper triangular with diagonal elements $\alpha_1, \dots, \alpha_q$ and β_1, \dots, β_q respectively ($q = n+m$). Obviously $L - \lambda M$ and $U_1 - \lambda U_2$ have the same Smith-McMillan form. Moreover, $U_1 - \lambda U_2$ is upper triangular so $\phi(\lambda)$ is the product of the (nonzero) diagonal terms.

Clearly the application of the QZ algorithm is straightforward. There is also one unexpected bonus for this particular application of QZ and that is that balancing may be applied (e.g. subroutine BALANC in EISPACK) to the first n rows and columns of L because of the special form of M . This can occasionally enhance the accuracy of the computed solution. The problem of balancing for the general generalized eigenvalue problem is still an open area of research.

We would emphasize the fact that no preprocessing (rank tests, matrix multiplication, inversions, etc.) of the system matrices is required.

The data are input directly and one call to one subroutine (EISPAC or RGG) is made. The transmission zeros are then determined once one decides what to numerically use as "zero" for the α_i 's and β_i 's. Typically one would call zero anything less than some ϵ related to the precision to

which the system data are known (or, from purely numerical considerations, one would use an ϵ on the order of the square root of the machine precision). The point is - and this is standard for orthogonal similarity algorithms - that "a quantity may be set to zero if a perturbation of the same size can be tolerated in the original matrix" [7]. There are then three cases to consider:

CASE (a): $\beta_i \geq \epsilon$.

$$\lambda_i = \frac{\alpha_i}{\beta_i} \text{ is a transmission zero.}$$

CASE (b): $\beta_i < \epsilon$, $\alpha_i \geq \epsilon$.

This corresponds to a generalized eigenvalue at infinity. There will be $r+s$ ($s \geq 0$) of these: r of them arise because the rank deficiency of M is r (and they usually appear with "hard zeros" for β_i), while the other s correspond to transmission zeros at infinity.

CASE (c): $\beta_i < \epsilon$, $\alpha_i < \epsilon$.

This is the degenerate case where $L - \lambda M$ is already of less than full rank; $T = \mathbb{C}$. Note that in the near-degenerate case (α_i and β_i simultaneously small, i.e., near ϵ , but, for example, β_i somewhat greater than ϵ) the computed $\lambda_i = \frac{\alpha_i}{\beta_i}$ is "ill-conditioned, however reasonable it may appear" [7]. Of course, a decision has still been made concerning ϵ but, again, the point is that since the α_i 's and β_i 's are derived via orthogonal similarities, an intelligent decision - related to the original data - can be made.

The test for degeneracy in Case (c) is thus also a very reliable and stable way of determining left or right invertibility of a system (see Section 4.2 of this paper and Remark 7 of [2]).

It will be noted that our M has a particularly simple structure which obviates the need for part of the reduction performed in QZHES. Advantage is taken of this feature and is reflected in our timing estimates in Section 7.

4.2 Case 2: $m \neq r$

Without loss of generality we will consider only the case $m > r$. The case $r > m$ can either be recast in this form (by considering the appropriate transpose matrices) or can be handled in the appropriate analogous way to that discussed below.

The procedure is as follows: Augment L and M with $(m-r)$ rows as shown:

$$L_1 = \begin{bmatrix} A & B \\ C & D \\ E_1 & F_1 \end{bmatrix}, \quad M_1 = \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Here E_1 and F_1 are conformably-sized pseudo-random matrices whose elements are, say, uniformly distributed in $[-||L||_1, ||L||_1]$. The QZ algorithm can then be applied to the square matrices L_1, M_1 to give a set of generalized eigenvalues T_1 . Repeat with different random E_2 and F_2 to get a set T_2 . Then almost surely $T = T_1 \cap T_2$.

That this random augmentation method gives T is clear from row rank considerations. The idea is due to Davison and [2] may be consulted for details. It would, of course, be nice to have a more direct way of handling this case while working only with unitary transformations. At this time we know of no appropriate algorithm. There is some measure of consolation in the knowledge that generically this case is rather less interesting than the case $m=r$ (see Section 4.3 below).

4.3 Number of Transmission Zeroes

It is sometimes useful to know how many transmission zeroes a linear system has. This can be particularly important a priori information (if reliably computable) in the fuzzy case of some large transmission zeros (β_i 's near ϵ).

A complete treatment is given by Thompson and Weil in [15] and [16] for the general problem. In our system-theoretic context analytical results are determined by Karcanias [22] and Kouvaritakis and Shaked [23]. However, all these results are quite complicated in general and require the solution of problems which are numerically as difficult as the original problem. Thus it is probably generally not worth the effort.

What are often as useful and very easy to obtain (see [2], [3], [22], [23] for details) are upper bounds or generic numbers. Some upper bounds are reproduced below for reference:

n	if $D \neq 0$
$n - \max(r, m)$	if $D \equiv 0$; B, C full rank
$n - m - d$	if $D \equiv 0$; B, C full rank; $m=r$; CB of rank deficiency d .

Assuming rank B = m and rank C = r the generic results [2] are:

n	if $D \neq 0$; $m = r$
$n - m$	if $D \equiv 0$; $m = r$
0	if $m \neq r$.

4.4 Advantages of the QZ Approach

The major advantage of the QZ approach is reliability. The QZ algorithm computes the transmission zeros of (1) about as accurately as the numerical conditioning of the problem will allow. The most significant benefits derive from the determination of the α_i and β_i (by unitary similarities), the ratios of which determine the finite transmission zeros, if any. There are no controllability or observability assumptions;

there are no initial rank assumptions which need to be checked; degeneracy (or, just as important, near-degeneracy) is detected "automatically" (i.e., without a separate test) in a stable way. In short, no preliminary analysis of the system matrices is needed at all. The algorithm proceeds directly on the raw system data. All the difficult (if done properly) programming and analysis has been done by the specialists.

We might also mention that while a superficial examination of the problem might indicate that the QZ approach would be slightly more CPU-time consuming, in some cases, than other theoretical approaches, in practice it is usually faster because of its reliability and direct applicability.

5. DECOUPLING ZEROS

Definition 4: The set I of input decoupling zeros of (1) is the set of zeros (multiplicities included) of $\psi_{in}(\lambda)$ where $\psi_{in}(\lambda)$ is the greatest common divisor of all $n \times n$ minors of $[(A, B) - (I, 0)]$.

Definition 5: The set O of output decoupling zeros of (1) is the set of zeros (multiplicities included) of $\psi_{out}(\lambda)$ where $\psi_{out}(\lambda)$ is the greatest common divisor of all $n \times n$ minors of $\begin{bmatrix} (A) \\ (C) \end{bmatrix} - \lambda \begin{bmatrix} (I) \\ (0) \end{bmatrix}$.

Making no assumptions on controllability or the rank of B we suggest the following algorithm for determining I . Determine sets I_1, I_2 from the two generalized eigenvalue problems

$$\begin{bmatrix} A & B \\ E_i & F_i \end{bmatrix} z = \lambda \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} z, \quad i = 1, 2$$

with pseudo-random E_i, F_i . Then almost surely $I = I_1 \cap I_2$. The analogous remarks apply for the determination of O .

Once I and O have been determined, various other definitions of transmission zeros (particularly those in terms of the transfer function of (1)) can be recovered by the appropriate additions to or removals from T . Of course one should note that all reasonable definitions of transmission zeros coincide anyway in the common special case of $m = r$, (A, B) controllable, and (C, A) observable. We again refer to [2], [3], [4] for details.

6. EXAMPLES

All our computing was done on the IBM 370/165 system at the University of Toronto with the FORTRAN H Extended, OPTIMIZE (2) compiler. We used the EISPAC control program to call the Generalized Real Non-Symmetric Matrix System Package available on Release 2 of EISPACK. All computations were done in double precision (REAL*8) arithmetic.

Example 1: Our first example concerns a linearized model of the F100-PW-100 jet engine at zero altitude and power lever angle of 83 degrees. This is the Basic Operating Point Number 5 in [24]. As the data are well-known, being the canonical model used for the International Forum on Alternatives for Linear Multivariable Control (Chicago, October 1977) sponsored by the National Engineering Consortium, Inc., we will not reproduce here the 441 parameters of $A \in \mathbb{R}^{16 \times 16}$, $B \in \mathbb{R}^{16 \times 5}$, $C \in \mathbb{R}^{5 \times 16}$, and $D \in \mathbb{R}^{5 \times 5}$. Rounded to 12 significant figures the transmission zeros are:

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+789.898582816
+141.229455020
- 0.665956161639
- 2.00340315558
- 6.71065103680
- 13.7653073045 ± 9.11021474755 j
- 18.9585018941
- 20.5560274938 ± 1.41735335001 j
- 23.1336651689
- 49.6376010324
- 50.4675747639 ± 1.03191416032 j
-829.249095565

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Since $D \neq 0$ and D has rank deficiency 1 we expect to find $16-1 = 15$ transmission zeros. This computation required approximately 1/6 second of CPU time. We also used the method of Section 4.2 with the same data as above but with only the first three inputs. As expected there are no transmission zeros.

Example 2: This low-order example is chosen from Davison and Wang [25]:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \\
 C = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

It can be verified analytically that $\det \begin{pmatrix} A-\lambda I & B \\ C & D \end{pmatrix} = (\lambda - 1)(\lambda^3 + \lambda + 1)$ so this system has the four transmission zeros 1.0, -0.6823278038280192 , $0.3411639019140096 \pm 1.161541399997252 j$. (The analytic solutions are

$$1, p+q, \frac{-(p+q)}{2} \pm \frac{(p-q)}{2} \sqrt{-3} \text{ where } p = \sqrt[3]{-\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{27}}}, q = \sqrt[3]{-\frac{1}{2} - \sqrt{\frac{1}{4} + \frac{1}{27}}} .)$$

The maximum relative error in any transmission zero by our method is $O(10^{-16})$. However, relative errors are $O(10^{-12})$ by Davison's method.

Example 3: This high-order example is also chosen from Davison and Wang [25]:

$n = 100, r = m = 1$ and

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \cdot & & \cdot & \cdot & \cdot & \\ \cdot & & \cdot & \cdot & \cdot & \\ \cdot & & \cdot & \cdot & \cdot & \\ \cdot & & \cdot & \cdot & \cdot & \\ \cdot & & \cdot & \cdot & \cdot & \\ \cdot & & \cdot & \cdot & \cdot & \\ 0 & \dots & \dots & \dots & \dots & 0 & 1 \\ 0 & \dots & \dots & \dots & \dots & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{pmatrix},$$

$$C = (1, c, 0, \dots, 0), \quad D = d.$$

It can be verified analytically that $\det \begin{pmatrix} A-\lambda I & B \\ C & D \end{pmatrix} = d\lambda^{100} + c\lambda + 1$.

Three cases are considered and the analytical solutions are summarized below:

$$\text{CASE 1: } c = 0, d = 0; \quad T = \emptyset$$

$$\text{CASE 2: } c = 1, d = 0; \quad T = \{-1\}$$

$$\text{CASE 3: } c = 0, d = -1; \quad T = \{\lambda: \lambda^{100} = 1\}.$$

We now summarize errors and CPU times for these three cases for our method and the Davison and Wang method:

		Avg. Rel. Error	CPU Time
Case 1	L&M	0	7.5 sec.
	D&W	0	>19.7 sec.
Case 2	L&M	0	7.5 sec.
	D&W	$O(10^{-15})$	>16.1 sec.
Case 3	L&M	$O(10^{-14})$	20.0 sec.
	D&W	$O(10^{-13})$	>17.0 sec.

The times given for Davison's method are fairly generous lower bounds inasmuch as the time required for an initial test for degeneracy is not included and we are assuming only two values of ρ were used. Moreover, for a prescribed accuracy it is still not completely certain what ρ to choose. The next example illustrates a potentially more severe problem with the "high-gain" approach.

Example 4: Consider the following controllable, observable system with

$$A = \begin{pmatrix} 1+\tau & 1-\tau \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$C = (1 \quad 1), \quad D = 0,$$

where $\tau \neq 0$ is a parameter to be specified later. It is immediately verified that

$$\det \begin{pmatrix} A-\lambda I & B \\ C & 0 \end{pmatrix} = 2(\lambda - \tau)$$

so there is one transmission zero at τ .

Let us now compute the transmission zeros for this system using the Davison and Wang algorithm. This involves determining the "finite" eigenvalues of $A + \rho BKC$. Without any loss of generality, we may set $K = 1$. Suppose we are computing in double precision on an IBM machine with machine precision on the order of 10^{-16} . Then according to the algorithm ([2] or [25]) we choose

$$\rho = 10^{15} \frac{\|A\|}{\|BC\|} = 0(10^{15}) \quad .$$

Now, suppose τ is such that $\text{fl}(\tau + x) = x$ where $\text{fl}(\cdot)$ denotes floating point evaluation and rounding to the above-mentioned precision. In this case, $\tau = 0.01$ will do, for example. Then

$$A + \rho BC = \begin{pmatrix} 1+\tau+\rho & 1-\tau+\rho \\ \rho & \rho \end{pmatrix}$$

but

$$\text{fl}(A + \rho BC) = \begin{pmatrix} 1+\rho & 1+\rho \\ \rho & \rho \end{pmatrix} \quad .$$

Any reliable eigenvalue routine will now give an "infinite" eigenvalue at $1 + 2\rho$ and a finite eigenvalue at 0 for the transmission zero--obviously an extremely bad approximation to the true transmission zero at 0.01. The QZ approach, on the other hand, determines the zero at 0.01 to approximately

14 decimal places which is consistent with its condition number of approximately 150.0 as the following analysis shows.

Following [10] we can explicitly compute by hand the condition number of the zero at $\lambda = \tau$ for this particular example. For the generalized eigenvalue problem $Lz = \lambda Mz$, compute the normalized (Euclidean norm = 1) right and left generalized eigenvectors x and y , respectively. From $(L - \lambda M)x = 0$ we find

$$x^T = (2+\tau^2)^{-\frac{1}{2}} (1 \quad -1 \quad -\tau)^T$$

while from $y^T(L - \lambda M) = 0$ we find

$$y^T = 3^{-\frac{1}{2}} (1 \quad -1 \quad -1)^T.$$

Then the condition number of $\lambda = \tau$ is given by $(a^2 + b^2)^{-\frac{1}{2}}$ where $a = y^T Lx$, $b = y^T Mx$. We thus have $\text{cond}(\lambda) = \frac{3(2+\tau^2)}{4\tau}$ which is approximately 150 when $\tau = 0.01$. Thus one might expect to lose approximately 2 decimal digits of accuracy with a stable algorithm such as QZ.

However, there is a significant degradation of accuracy in using the high-gain approach for any value of ρ . Clearly, increasing ρ will not help matters, whereas if ρ is decreased, one has but to decrease τ (noting, however, that $\text{cond}(\lambda) = O(\tau^{-1})$) for the same qualitative results to obtain. Even for $\tau = 0.01$ though, decreasing ρ (to 10^7) still gives, at best, only 7 decimal places of accuracy -- a very disappointing result in view of the relatively well-conditioned nature of the problem.

Moreover, it would be easy to expand this example to include another transmission zero which needs a different value of ρ (say, the

more usual 10^{15}) to be accurately determined. Then even using many values of ρ we are still in a quandary as to the accuracy of any particular zero, and, as we have seen, a particular zero may still be determined much less accurately than its condition would indicate.

Clearly, if one wants to have a reliable algorithm, the choice of ρ cannot depend on the particular values of the transmission zeros.

Of course, the above remarks derive entirely from the effect of finite word length (i.e., the statement above that $f\ell(\tau+x) = x$). Given that constraint, the ρ parameter of the high-gain approach has the potential for destroying critical information in the A matrix. That is precisely how this numerical counterexample was constructed.

Because of the potential, in well-conditioned problems, for unwarranted inaccuracy and because of the difficulty in satisfactorily determining ρ a priori (aside from the inefficiency in using more than 2 values) we must categorize the high-gain approach as somewhat unsatisfactory.

On the basis of Examples 2, 3, and 4 and many others, we would recommend our algorithm over that of Davison and Wang as it is generally faster, more accurate (in the sense of minimizing the smallest singular value of $\begin{pmatrix} A - \lambda I & B \\ C & D \end{pmatrix}$), and certainly more reliable in the sense of computing the zeros as accurately as their condition warrants. The QZ approach does, however, require at least twice as much storage as the high-gain approach. In most situations, though, with only a hundred or so states, neither storage nor CPU time is usually too critical. For larger systems sparse matrix techniques will probably be necessary. Finally, while the QZ approach is good mathematical software, we would definitely recommend Davison's development for the purposes of system-theoretic insight as the QZ algorithm has moderately little to offer, so far, in that respect.

Example 5: This example simply illustrates the dangers of having to rely on computation of quantities such as the rank of CB. Consider the controllable, observable system with

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1+\mu & 1 \\ 1-\mu & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 2 & 1+\mu & 1-\mu \\ 2 & 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

where μ is some number less than the square root of the particular machine precision of the digital computer on which it is assumed we are doing our floating point arithmetic (e.g., $\mu = 10^{-4}$ in single precision computations on an IBM 360/370). In other words $(1\pm\mu)^2$ will be computed (rounded) as $1\pm 2\mu$. Then CB will be computed as $\begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}$ which, by any reliable rank algorithm, will be determined to have rank 1. One might then be forced to conclude - erroneously, of course - that the system has at most $3-2-1=0$ transmission zeros (see Section 4.3). In reality, as is easily verified, the system does have one transmission zero at -2. A similar fate awaits floating point computations with $C(sI-A)^{-1}B$. The QZ approach, however, does not suffer from these difficulties, and the transmission zero at -2 is computed with ease. The point is that the QZ approach avoids unnecessary computations (multiplications) which introduce additional roundoff problems needlessly. Moreover, this example exhibits a phenomenon which is not merely pathological but rather is of sometimes critical concern in the context of noisy or uncertain data.

Example 6: Example 5 is obviously near-degenerate. In fact, one can check that $\det \begin{pmatrix} A-\lambda I & B \\ C & D \end{pmatrix} = -8\mu^2(2+\lambda)$ so that the degeneracy becomes

more acute as μ decreases. To illustrate the consistent behavior of the QZ approach in the face of near-degeneracy we computed the transmission zero at -2 for various values of μ with the following results:

μ	1	10^{-4}	10^{-8}	10^{-12}
Rel. error in tr. zero	$O(10^{-16})$	$O(10^{-13})$	$O(10^{-9})$	$O(10^{-5})$

Note that $\mu = 10^{-8}$ is approximately the square root of machine precision ϵ while $\mu = 10^{-12}$ is considerably less than $\sqrt{\epsilon}$. Yet the transmission zero is still computed with reasonable accuracy. What is most important, moreover, is that fact that while -2 is being determined by a ratio of two numbers both $O(1)$, near-degeneracy is being automatically detected by another (α_i, β_i) pair with $\alpha_i \sim O(\mu)$, $\beta_i \sim O(10^{-16})$. Finally, setting $\mu=0$ results in no transmission zero being computed but degeneracy being signaled by an (α_i, β_i) pair with $\alpha_i \sim O(10^{-16})$ and $\beta_i \sim O(10^{-16})$.

Example 7:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C = (1 \quad 0 \quad 0), \quad D = (0 \quad 0).$$

$$E_1 = (5 \quad -20 \quad 8), \quad F_1 = (-3 \quad 8): \quad T_1 = \{4.5, 3.0\}$$

$$E_2 = (4 \quad -50 \quad -3), \quad F_2 = (9 \quad -5): \quad T_2 = \{3.0, -8.0\}.$$

(All computations were accurate to 16 significant figures.)

Thus this system has one transmission zero at 3. Note that 3 is an unobservable mode. Since (C,A) is not observable, 3 may or may not be considered a transmission zero by other definitions. In any case, it is the only number which reduces the rank of $\begin{pmatrix} A-\lambda I & B \\ C & D \end{pmatrix}$. We also checked that

$I = \emptyset$ and $O = \{2.0, 3.0\}$. Taking $C = (1 \ 1 \ 1)$ instead and different E_i, F_i we computed $T_1 = \{0.224, 0.962\}$, $T_2 = \{0.298, 0.460\}$ (rounded) so this system has no transmission zeros.

Example 8: We recomputed the transmission zeros of Example 1 of [2], noting the typographical error that $a_{8,5}$ of A should be +0.001 and not -0.001 as published. This 9th order system with $m=r=2$ describes a boiler system. We computed 6 finite transmission zeros (CB has rank deficiency 1). Each of our numbers agrees exactly with the corresponding 7 significant figures published. For reference purposes, we reproduce below the transmission zeros computed by the QZ algorithm:

```

-26.39513729221503
- 2.957771983411796 + 0.3352672040386323 j
  0.7486064352926953
  0.09403261463223497
- 0.009546070736281448 .

```

Based on extended precision results, the relative error in these values is on the order of 10^{-12} or 10^{-13} . This is precisely all one can reasonably expect from a stable algorithm since $\|A\|$ for this example is on the order of 10^4 (and $\|A\|$ appears in the backward error bound). Note that substitution of these various values of λ into $\begin{pmatrix} A - \lambda I & B \\ C & 0 \end{pmatrix}$ yields a smallest singular value of the order of the machine precision used in the original calculations.

7. TIMING ESTIMATES

We ran tests on randomly generated system matrices with $m=r=10$ and $n=30, 50, 70$. The CPU times were 1.6, 4.2, and 10.2 seconds respectively. We summarize this data by saying that CPU time is approximately $20 q^3 \mu \text{ sec}$ where $q = n+m$ ($m=r$). If QZVEC is also used to compute the eigenvectors the time is approximately $33 q^3 \mu \text{ sec}$. Because of the special diagonal form of M these times are somewhat less than the EISPACK

times of approximately $30 q^3 \mu \text{ sec}$ (without eigenvectors) and $54 q^3 \mu \text{ sec}$ (with eigenvectors).

We emphasize that these estimates are very approximate but $20 q^3 \mu \text{ sec}$ is a good ballpark figure to keep in mind when using this algorithm to compute transmission zeros. It should also be noted that for moderate-sized systems (say $n \leq 50$) peripheral costs (such as a card reader or line printer) are usually more than CPU costs.

One additional consideration that bears emphasis is that because the QZ approach does no "preprocessing" of the system data A, B, C, D any sparsity initially present will be retained. Good QZ code can sometimes take advantage of this sparsity (cf., Example 3). This certainly will be a fruitful area for future research.

8. GENERALIZED EIGENVECTORS AND PRINCIPAL VECTORS

For $\lambda \in T$ the corresponding generalized eigenvector is a vector z satisfying (2). MacFarlane and Karcanias give these eigenvectors (which they call invariant-zero directions) a nice system-theoretic interpretation in a series of theorems in [3]. The typical sort of result is that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

if and only if the input $u(t) = e^{\lambda t} \eta$ yields state trajectory $x(t) = e^{\lambda t} \xi$ and output $y(t) \equiv 0, t \geq 0$. In other words, transmission is blocked at the frequency λ .

The ξ 's can also be used to construct a basis for V^* or V^*/R^* (in the notation of [13]). We would argue that this fact will become one of the

main motivations for using the QZ approach since computation of these subspaces is of fundamental importance in questions of synthesis and design for linear multivariable control systems. We refer the interested reader to [26] for details.

In case $\lambda \in T$ is of multiplicity p with fewer than p corresponding eigenvectors we must, in theory, resort to generalized principal vectors to get a spanning set for V^* , say. A generalized principal vector of grade k will be defined by an equation of the form

$$(L - \lambda M)z_k = Mz_{k-1}$$

where, of course, the generalized principal vectors of grade 1 are simply the generalized eigenvectors.

It is known (see, for example [27]) that the complete solution of the eigenvalue-eigenvector problem for non-normal matrices poses severe numerical and practical problems for defective or near-defective matrices. In fact, in the presence of rounding error, it cannot even be determined if a matrix is defective. Rather than compute principal vectors satisfying the "chain conditions" one must instead be content to compute other well-defined bases of the appropriate invariant subspaces. By analogy then, other numerical algorithms will have to be brought to bear on the computation of bases for V^* or V^*/R^* in the defective case.

9. CONCLUSIONS

We have presented, in a moderately tutorial style, application of QZ techniques to the problem of computing transmission zeros for a linear time-invariant control system. We strongly advocate such techniques because

of their basis in sound numerical analysis, their direct applicability, their wide availability, and their consequent overall reliability.

10. REFERENCES

- [1] Davison, E.J. and S.H. Wang, "Remark on Multiple Transmission Zeros of a System", *Automatica*, 12(1976), 195.
- [2] Davison, E.J. and S.H. Wang, "Properties and Calculation of Transmission Zeros of Linear Multivariable Systems", *Automatica*, 10 (1974), 643-658.
- [3] MacFarlane, A.G.J. and N. Karcanias, "Poles and Zeros of Linear Multivariable Systems: A Survey of the Algebraic, Geometric, and Complex-variable Theory", *Int. J. Control*, 24(1976), 33-74.
- [4] Francis, B.A. and W.M. Wonham, "The Role of Transmission Zeros in Linear Multivariable Regulators", *Int. J. Control*, 22(1975), 657-681.
- [5] Kouvaritakis, B. and A.G.J. MacFarlane, "Geometric Approach to Analysis and Synthesis of System Zeros, Part 1, Square Systems; and Part 2. Non-square Systems", *Int. J. Control*, 23(1976), 149-166 and 167-181.
- [6] Desoer, C.A. and J.D. Schulman, "Zeros and Poles of Matrix Transfer Functions and Their Dynamical Interpretation", *IEEE Trans. Circ. Sys.*, CAS-21(1974), 3-8.
- [7] Moler, C.B. and G.W. Stewart, "An Algorithm for Generalized Matrix Eigenvalue Problems", *SIAM J. Numer. Anal.*, 10(1973), 241-256.
- [8] Sinswat, V., R.V. Patel, and F. Fallside, "A Method of Computing Invariant Zeros and Transmission Zeros of Invertible Systems", *Int. J. Control*, 23(1976), 183-196.
- [9] Stewart, G.W., "On the Sensitivity of the Eigenvalue Problem $Ax = \lambda Bx$ ", *SIAM J. Numer. Anal.*, 9(1972), 669-686.
- [10] Stewart, G.W., "Gershgorin Theory for the Generalized Eigenvalue Problem $Ax = \lambda Bx$ ", *Math. Comp.*, 29(1975), 600-606.
- [11] Patel, R.V., "On Computing the Invariant Zeros of Multivariable Systems", *Int. J. Control*, 24(1976), 145-146.

- [12] Kaufman, I., "On Poles and Zeros of Linear Systems", IEEE Trans. Circ. Theory, 20(1973), 93-101.
- [13] Wonham, W.M., Linear Multivariable Control, Lecture Notes in Economics and Mathematical Systems, Vol. 101, Springer-Verlag, Berlin, 1974.
- [14] Gantmacher, F.R., The Theory of Matrices, Vol. II, Chelsea, New York, 1959, Ch. XII.
- [15] Thompson, G.L. and R.L. Weil, "Reducing the Rank of $(A-\lambda B)$ ", Proc. A.M.S., 26(1970), 548-554.
- [16] Thompson, G.L. and R.L. Weil, "The Roots of Matrix Pencils $(A\lambda = \lambda B)$: Existence, Calculations, and Relations to Game Theory", Lin. Alg. and Its Applics., 5(1972), 207-226.
- [17] Lancaster, P., Lambda-matrices and Vibrating Systems, Pergamon, Oxford, 1966.
- [18] Garbow, B.S., et al., Matrix Eigensystems Routines--EISPACK Guide Extension, Lecture Notes in Computer Science, Vol. 51, Springer-Verlag, Berlin, 1977.
- [19] Ward, R.C., "The Combination Shift QZ Algorithm", SIAM J. Numer. Anal., 12(1975), 835-853.
- [20] Kaufman, L., "The LZ Algorithm to Solve the Generalized Eigenvalue Problem", SIAM J. Numer. Anal., 11(1974), 997-1024.
- [21] Van Loan, C.F., "A General Matrix Eigenvalue Algorithm", SIAM J. Numer. Anal., 12(1975), 819-834.
- [22] Karcanias, N., Ph.D. Thesis, University of Manchester, 1975.
- [23] Kouvaritakis, B. and U. Shaked, "Asymptotic Behavior of Root-Loci of Linear Multivariable Systems", Int. J. Control, 23(1976), 297-340.
- [24] Miller, R.J. and R.D. Hackney, F100 Multivariable Control System Engine Models/Design Criteria, Pratt and Whitney Aircraft Group, United Technologies Corporation, West Palm Beach, Florida, Technical Report No. AFAPL-TR-76-74, Nov. 1976.
- [25] Davison, E.J. and S.H. Wang, An Algorithm for the Calculation of Transmission Zeros of the System (C, A, B, D) Using High Gain Output Feedback, Systems Control Report No. 7619, Department of Electrical Engineering, University of Toronto, Dec. 1976.

- [26] Moore, B.C. and A.J. Laub, Computation of Supremal (A,B)-Invariant and Controllability Subspaces, Systems Control Report 7706, Systems Control Group, Department of Electrical Engineering, University of Toronto, March 1977.
- [27] Golub, G.H., and J.H. Wilkinson, "Ill-Conditioned Eigensystems and the Computation of the Jordan Canonical Form," SIAM Review, 18(1976), 578-619.