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CALCULATION OF TRANSONIC FLOWS USING AN EXTENDED INTEGRAL EQUATION METHOD

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Abstract

An extended integral equation method for transonic flows is developed. In the extended integral equation method velocities in the flow field are calculated in addition to values on the aerofoil surface, in contrast with the less accurate 'standard' integral equation method in which only surface velocities are calculated. The results obtained for aerofoils in subcritical flow and in supercritical flow when shock waves are present compare satisfactorily with the results of recent finite difference methods.

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Introduction

At the present time the most common method of calculating the inviscid pressure distribution around aerofoils at transonic speeds is by using finite difference techniques. For subcritical flows one of the most accurate methods is that of Sells (1). For supercritical flows with shock waves the initial work of Murman and Cole (2) has been followed by many developments, notably those by Murman and Krupp (3), Garabedian and Korn (4) and by Jameson (5). It is pointed out by Murman (6) , however, that the difference scheme used in these methods is incorrect at the shock location, leading to incorrect shock jump relations; corrected results are presented in Ref.(6).

One of the earliest attempts to solve the transonic flow problem was the integral equation method first developed for non-lifting flows by Oswatitsch (7) with subsequent extensions by Spreiter and Alksne (8). Later developments to include lifting flows are by Norstrud (9) and Nixon and Hancock (10) although it should be noted that the formulation used by Norstrud for lifting flows is incorrect (11).

In the integral equation method for two-dimensional flows the non-linear partial differential equation for the perturbation velocity potential is written in integral form using Green's Theorem. The resulting integral equation for the velocity potential can be differentiated to give the perturbation velocity in terms of a line integral over the aerofoil chord, involving only linear terms and a surface integral over the flow field involving only non-linear terms. The line integral can be easily evaluated using standard methods but the evaluation of the field integral requires a knowledge of the velocity distribution over the entire flow field. In the standard integral equation method the field integral is reduced to a line integral by representing the transverse variation of the velocity by a fairly arbitrary approximation function involving only the velocity on the aerofoil surface. The resulting integral equation for the surface velocity can be solved without much difficulty. This approximate evaluation of the field integral, although accurate for shock-free flows, is not sufficiently accurate to give satisfactory results when shock waves are present.

Many problems in subsonic 'linear' aerodynamics, both in steady flow and oscillatory flows, can be formulated in terms of integral equations and there exists a considerable expertise in the solution of these equations. Similarly many problems in transonic aerodynamics, again both for steady and oscillatory flows, can be formulated as integral equations; examples of steady flow problems are given by Norstrud (9) and an example of oscillatory flows is given by Nixon (12). The transonic integral equation contains a line integral similar to one appearing in subsonic aerofoil theory and the available expertise in the evaluation of this integral can be incorporated into any solution of the transonic integral equation. The main advantage of the integral equation methods over finite difference methods is that, in principle, the numerical solution of the integral equation is easier than the solution of the differential equation and is typified by rapid convergence. The main disadvantage of existing solutions of the transonic integral equation is the unsatisfactory results due to the rather primitive and inaccurate approximation of the field integral component of the transonic integral equation. If the accuracy of the evaluation of the field integral can be improved then it is suggested that integral equation methods will become a useful technique in a wide range of problems in transonic aerodynamics.

In the extended integral equation method presented in this paper an alternative means of evaluating the field integral is developed. The flow field is divided into a number of streamwise strips and the transverse variation of the perturbation velocities across each of these strips is approximated by an interpolation function in terms of values on the strip edges. The field integral is then reduced to a line integral which is in turn evaluated by quadrature. The fundamental integral equation is thus approximated by a set of non-linear algebraic equations.

The pressure distribution around both lifting and non-lifting thick aerofoils in subcritical flow is calculated and compares favourably with the results of other exact methods. The pressure distribution around aerofoils in supercritical flow with shock waves is also calculated and there is good agreement with the results of recent finite difference methods. The computing time for a lifting supercritical example is about 80-100 secs. on an ICL 1904S computer.

Basic Equations

A two dimensional cartesian co-ordinate system is chosen with the origin at the leading edge of the aerofoil; the x-axis is in the freestream direction and the z-axis is normal to the freestream; x and z are non dimensionalized with respect to the aerofoil chord.

In order to simplify the governing equations for transonic flow it is frequently assumed that the flow is both isentropic and irrotational even when shock waves are present. The perturbed flow may thus be represented by a perturbation velocity potential, $\phi(x, z)$, defined by

$$\frac{\partial\phi}{\partial x} = u \quad , \quad \frac{\partial\phi}{\partial z} = w \qquad (1)$$

where u and w are the non-dimensional perturbation velocities in the x and z directions respectively, relative to the freestream velocity. The full potential equation for steady flow is of third order in ϕ but for transonic flows a number of the higher order terms may be neglected for most practical applications.

The variables

$$\bar{\Phi}(\bar{x},\bar{z}) = \frac{k}{\beta^2}\phi(x,z)\,,\ \bar{U}(\bar{x},\bar{z}) = \frac{k}{\beta^2}U(x,z)\,,\ \bar{W}(\bar{x},\bar{z}) = \frac{k}{\beta^3}W(x,z)$$
$$\bar{x} = x \quad , \quad \bar{z} = \beta z \qquad (2)$$

are introduced where, if M_∞ is the freestream Mach number and γ is the ratio of specific heats

$$\beta = (1 - M_\infty^2)^{\frac{1}{2}}$$

and

$$k = k(\gamma, M_\infty) \qquad (3)$$

A second order potential equation for transonic flow can then be written as

$$\bar{\Phi}_{\bar{x}\bar{x}} + \bar{\Phi}_{\bar{z}\bar{z}} = g_{\bar{x}} \qquad (4)$$

where

$$g(\bar{x},\bar{z}) = \frac{\bar{U}^2(\bar{x},\bar{z})}{2} + \frac{\beta^2 M_\infty^2}{k}\bar{W}^2(\bar{x},\bar{z}) \qquad (5)$$

If the second term on the right-hand side of Eq. (5) is neglected then Eq. (4) reduces to the transonic small disturbance equation.

The boundary conditions are:

(a) that the perturbation velocity potential and its derivatives vanish at an infinite distance upstream of the aerofoil,

(b) that the flow direction of the aerofoil surface is tangential to the aerofoil surface,

(c) that for a subsonic trailing edge the Kutta condition of finite pressure at the trailing edge must be satisfied.

If $z = z_u(x)$ and $z = z_l(x)$ are the equations of the upper and lower surfaces of the aerofoil respectively then the tangency boundary condition can be written in the variables of Eq. (2) as

$$\bar W(\bar x, \bar z_u) = Z_{u\bar x}(\bar x)\{1 + \frac{\beta^2}{k}\bar U(\bar x, \bar z_u)\}$$
$$W(\bar x, \bar z_l) = Z_{l\bar x}(\bar x)\{1 + \frac{\beta^2}{k}\bar U(\bar x, \bar z_l)\} \qquad (6)$$

where

$$Z_u(\bar x) = \beta Z_u(x), \quad Z_l(\bar x) = \beta z_l(x)$$

$$Z_u(\bar x) = \frac{k}{\beta^3} z_u(x), \quad Z_l(\bar x) = \frac{k}{\beta^3} z_l(x)$$

It can be shown that Eq. (4) is elliptic (subsonic) when $\bar U < 1 - \left(\frac{M_\infty^2 \bar\varphi}{k}\right)^2$, hyperbolic (supersonic) when $\bar U > 1 - \left(\frac{M_\infty^2 \bar\varphi}{k}\right)^2$, sonic conditions exist when $\bar U = 1 - \left(\frac{M_\infty^2 \bar\varphi}{k}\right)^2$ When Eq. (4) is hyperbolic in character then a solution of Eq. (4) may have discontinuities in the velocity components $\bar u, \bar w$. These discontinuities are the shock waves of Eq.(4). The shock jump relations can be found fairly easily from Eq.(4) since it is written in conservation form: thus

$$\left[\Phi_{\bar x} - g\right]_-^+ \cos(n, \bar x_s) + \left[\Phi_{\bar z}\right]_-^+ \cos(n, \bar z_s) = 0 \qquad (7a)$$

where $[\quad]_-^+$ denotes the jump across the shock wave and $\cos(n, \bar x_s)$, $\cos(n, \bar z_s)$ are the direction cosines of the shock wave. It should also be noted that

$$\left[\Phi\right]_-^+ = 0 \qquad (7b)$$

Treatment of Boundary Conditions

There are two common methods of treating the boundary conditions in aerofoil theory, namely thin aerofoil theory and thick aerofoil theory. Thin aerofoil theory introduces an approximation to the exact problem by using the assumption that the disturbances from the freestream are of the same order of magnitude as some power of the thickness/chord ratio of the aerofoil section, which is usually small compared with unity. The boundary conditions and and the potential equation are then expanded as a series in powers of the thickness/chord ratio giving an infinite set of differential equations with their associated boundary conditions . Generally only the first few terms of these series are retained. The tangency boundary condition for each equation is now satisfied on the plane $z = \pm 0$ rather than on the aerofoil surface. The accuracy of thin aerofoil theory can be improved by progressively including more terms in the series. This method of solution gives rise to singularities due to the assumption of small disturbances breaking down in the neighbourhood of the leading edge and these singularities must be removed in the course of the solution.

In thick aerofoil theory no assumption regarding the magnitude of the flow disturbances is made and consequently thick aerofoil theory is generally more difficult than thin aerofoil theory and the numerical procedures are more complex although, of course, the 'exact' result is computed.

The considerable experience available in using both the 'standard' [8,9,10] and extended [13] integral equation method is based on the formulation of the problem using the thin aerofoil boundary conditions, particularly that the boundary conditions are satisfied on the plane $z = \pm 0$ rather than on the actual aerofoil surface. In transonic flows when the non-linear potential equation, Eq.(4), is applicable, the series solution normaly used when the non-linear terms in Eq.(4) can be neglected is impractical and some alternative must be sought. The principle requirement is that the tangency boundary condition must be satisfied on the plane $z = \pm 0$.

If it is assumed that the external flow (that is, external to the aerofoil) can be continued analytically inside the aerofoil to the plane $z = +0$, $z = -0$, for the upper and lower half planes respectively, with the corresponding boundary conditions on the plane $z = \pm 0$, then existing expertise can be used in formulating the integral equation for a modified problem, the domain of which now includes both the real external flow and a fictitious internal flow. Since it is assumed that the interior flow is an analytic continuation of the external flow then the value of $\bar{w}(\bar{x},\pm 0)$ can be obtained from the physical boundary conditions Eq. (6) using a Taylors series expansion; thus for example $\bar{w}(\bar{x},+0)$ is given by

$$\bar{w}(\bar{x},+0) = \bar{\phi}_{\bar{z}}(\bar{x},+0) = \bar{\phi}_{\bar{z}}(\bar{x},\bar{z}_u) - \bar{z}_u(\bar{x})\bar{\phi}_{\bar{z}\bar{z}}(\bar{x},\bar{z}_u) + \cdots\cdots \qquad (8)$$

If only the first two terms of the series in Eq. (8) are taken then on using Eqs. (4,6) the boundary conditions for the modified flow are

$$\begin{aligned} w(\bar{x},+0) = \bar{\phi}_{\bar{z}}(\bar{x},+0) &= Z_{u\bar{x}}(\bar{x})\{1+\beta_k^2 \bar{U}(\bar{x},\bar{z}_u)\} \\ &\quad + \bar{z}_u(\bar{x})\{\bar{U}_{\bar{x}}(\bar{x},\bar{z}_u) - g_{\bar{x}}(\bar{x},\bar{z}_u)\} \\ w(\bar{x},-0) = \bar{\phi}_{\bar{z}}(\bar{x},-0) &= Z_{L\bar{x}}(\bar{x})\{1+\beta_k^2 \bar{U}(\bar{x},\bar{z}_L)\} \qquad (9) \\ &\quad + \bar{z}_L(\bar{x})\{\bar{U}_{\bar{x}}(\bar{x},\bar{z}_L) - g_{\bar{x}}(\bar{x},\bar{z}_L)\} \end{aligned}$$

The boundary conditions, Eq. (9), are used in the subsequent analysis. Since there is no assumption of small disturbances except possibly in the number of terms retained in the Taylors series expansion, no singularities due to approximations in the boundary conditions need arise. Once the solution to the modified problem is known the solution to the physical problem is found by considering only external flow.

The Integral Equations

The partial differential equation for the velocity potential Eq. (4), with the associated shock relations Eq. (7), can be written in integral form using Green's Theorem; thus

$$\begin{aligned} \bar{\Phi}(\bar{x},\bar{z}) = \frac{1}{2\pi}\int_0 \{\psi(\bar{x},\bar{\xi};\bar{z},0)\Delta\bar{\phi}_{\bar{\zeta}}(\bar{\xi}) - \psi_{\bar{\zeta}}(\bar{x},\bar{\xi};\bar{z},0)\Delta\bar{\phi}(\bar{\xi})\}d\bar{\xi} \\ - \frac{1}{2\pi}\iint_{S'} \psi_{\bar{\xi}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})\, g(\bar{\xi},\bar{\zeta})\, d\bar{\xi}\, d\bar{\zeta} \qquad (10) \end{aligned}$$

where ($\bar{\xi},\bar{\zeta}$) are co-ordinates equivalent to $(\bar{x},\bar{z})$,

$$\psi(\bar{x},\bar{\xi};\bar{z},\bar{\zeta}) = \ln\{(\bar{x}-\bar{\xi})^2+(\bar{z}-\bar{\zeta})^2\}^{\frac{1}{2}} \qquad (11)$$

and the operator ' Δ ' is defined for $f(\bar{x},\bar{z})$ as

$$\Delta f(\bar{x}) = f(\bar{x},+0) - f(\bar{x},-0) \qquad (12)$$

The surface integral in Eq. (10) is defined for $\bar{z} > 0$ as

$$\begin{aligned} \iint_{S'} F\, d\bar{\xi}\, d\bar{\zeta} = \lim_{\substack{\delta\to 0\\ \epsilon\to 0}} \Big\{ &\int_0^{\bar{z}-\delta}\Big(\int_{-\infty}^{\bar{x}_{su}-\epsilon} F\, d\bar{\xi} + \int_{\bar{x}_{su}+\epsilon}^{\infty} F\, d\bar{\xi}\Big)d\bar{\zeta} + \int_{\bar{z}+\delta}^{\infty}\Big(\int_{-\infty}^{\bar{x}_{su}-\epsilon} F\, d\bar{\xi} + \int_{\bar{x}_{su}+\epsilon}^{\infty} F\, d\bar{\xi}\Big)d\bar{\zeta} + \int_{\bar{z}-\delta}^{\bar{z}+\delta}\Big(\int_{-\infty}^{\bar{x}-\{\delta^2-(\bar{z}-\bar{\zeta})^2\}^{\frac{1}{2}}} F\, d\bar{\xi} \\ &+ \int_{\bar{x}+\{\delta^2-(\bar{z}-\bar{\zeta})^2\}^{\frac{1}{2}}}^{\infty} F\, d\bar{\xi}\Big)d\bar{\zeta} + \int_{-\infty}^{0}\Big(\int_{-\infty}^{\bar{x}_{sL}-\epsilon} F\, d\bar{\xi} + \int_{\bar{x}_{sL}+\epsilon}^{\infty} F\, d\bar{\xi}\Big)d\bar{\zeta} \Big\} \qquad (13) \end{aligned}$$

where $\bar{x}_{su}$, $\bar{x}_{sL}$ denote the location of the shock waves on the upper and lower half planes respectively.

Eq. (10) can be differentiated with respect to $\bar{X}$ to give the velocity component $\bar{U}(\bar{X},\bar{Z})$ thus

$$\bar{U}(\bar{X},\bar{Z}) = \frac{1}{2\pi}\int_0^1\{\psi_{\bar{X}}(\bar{X},\bar{\xi};\bar{Z},0)\Delta\bar{\phi}_{\bar{\xi}}(\bar{\xi})\}d\bar{\xi} - \frac{1}{2\pi}\int_0^1\psi_{\bar{\xi}\bar{X}}(\bar{X},\bar{\xi};\bar{Z},0)\Delta\bar{\phi}(\bar{\xi})d\bar{\xi} + \frac{g(\bar{X},\bar{Z})}{2} - \frac{1}{2\pi}\iint_{S'}\psi_{\bar{\xi}\bar{X}}(\bar{X},\bar{\xi};\bar{Z},\bar{\zeta})g(\bar{\xi},\bar{\zeta})d\bar{\xi}d\bar{\zeta} \qquad (14)$$

The term $\frac{g(\bar{X},\bar{Z})}{2}$ arises from differentiation of the limits around the singular point $(\bar{X},\bar{Z})$ in the field integral. If the second integral on the right-hand side of Eq. (14) is integrated by parts then

$$\frac{1}{2\pi}\int_0^1\psi_{\bar{\xi}\bar{X}}(\bar{X},\bar{\xi};\bar{Z},0)\Delta\bar{\phi}(\bar{\xi})d\bar{\xi} = \frac{1}{2\pi}\int_0^1\psi_{\bar{X}}(\bar{X},\bar{\xi};\bar{Z},0)\Delta\bar{U}(\bar{\xi})d\bar{\xi} \qquad (15)$$

It may be shown that

$$\frac{-1}{2\pi}\iint_{S'}\psi_{\bar{\xi}\bar{X}}(\bar{X},\bar{\xi};\bar{Z},\bar{\zeta})g(\bar{\xi},\bar{\zeta})d\bar{\xi}d\bar{\zeta} = \frac{g(\bar{X},\bar{Z})}{2} + I_s(\bar{X},\bar{Z},\bar{X}_s) \qquad (16)$$

where

$$I_s(\bar{X},\bar{Z},\bar{X}_s) = \frac{-1}{2\pi}\iint_S\psi_{\bar{\xi}\bar{X}}(\bar{X},\bar{\xi};\bar{Z},\bar{\zeta})\,g(\bar{\xi},\bar{\zeta})d\bar{\zeta}d\bar{\xi} \qquad (17)$$

and the surface integral in Eq. (17) is defined by

$$\iint_S F d\bar{\xi}d\bar{\zeta} = \lim_{\substack{\delta\to 0\\ \epsilon\to 0}}\Big\{\int_{-\infty}^{\bar{X}-\delta}\Big(\int_0^{\infty}F d\bar{\zeta}\Big)d\bar{\xi} + \int_{\bar{X}+\delta}^{\bar{X}_{su}-\epsilon}\Big(\int_0^{\infty}F d\bar{\zeta}\Big)d\bar{\xi} + \int_{\bar{X}_{su}+\epsilon}^{\infty}\Big(\int_0^{\infty}F d\bar{\zeta}\Big)d\bar{\xi} + \int_{-\infty}^{\bar{X}_{sl}-\epsilon}\Big(\int_{-\infty}^{0}F d\bar{\zeta}\Big)d\bar{\xi} + \int_{\bar{X}_{sl}+\epsilon}^{\infty}\Big(\int_{-\infty}^{0}F d\bar{\zeta}\Big)d\bar{\xi}\Big\}$$

On substition of Eqs. (15, 16) into Eq. (14)

$$\bar{U}(\bar{X},\bar{Z}) - g(\bar{X},\bar{Z}) = I_L(\bar{X},\bar{Z}) + I_s(\bar{X},\bar{Z},\bar{X}_s) \qquad (19)$$

where

$$I_L(\bar{X},\bar{Z}) = \frac{1}{2\pi}\int_0^1\psi_{\bar{X}}(\bar{X},\bar{\xi};\bar{Z},0)\Delta\bar{W}(\bar{\xi})d\bar{\xi} - \frac{1}{2\pi}\int_0^1\psi_{\bar{Z}}(\bar{X},\bar{\xi};\bar{Z},0)\Delta\bar{U}(\bar{\xi})d\bar{\xi} \qquad (20)$$

If the limit as $\bar{Z}\to +0$ is taken it can be shown that Eq. (19) reduces to

$$\Big\{\bar{U}(\bar{X},+0) - \frac{\Delta\bar{U}(\bar{X})}{2}\Big\} - \Big\{g(\bar{X},+0) - \frac{\Delta g(\bar{X})}{2}\Big\} = \frac{1}{2\pi}\int_0^1\psi_{\bar{X}}(\bar{X},\bar{\xi};0,0)\Delta\bar{W}(\bar{\xi})d\bar{\xi} - \frac{1}{4\pi}\iint_S\psi_{\bar{\xi}\bar{X}}(\bar{X},\bar{\xi};0,\bar{\zeta}) \times \{g(\bar{\xi},\bar{\zeta}) + g(\bar{\xi},-\bar{\zeta})\}d\bar{\xi}d\bar{\zeta} \qquad (21)$$

which is symmetric with respect to the $\bar{Z}$-axis. A similar result occurs if the limit as $\bar{Z}\to -0$ is taken. The additional equation to give the anti-symmetric component of $\bar{U}(\bar{X},\pm 0)$, $\Delta\bar{U}(\bar{X})$, can be obtained by differentiating Eq. (10) with respect to $\bar{z}$ and taking the limit as $\bar{z} \to +0$. Thus, after some manipulation

$$\bar{\omega}_T(\bar{x}) = -\frac{1}{2\pi}\int_0^1 \frac{(\Delta \bar{U}(\xi) - \Delta g(\xi))}{(\bar{x}-\xi)} d\xi + I_e(\bar{x}, \bar{x}_0) \qquad (22)$$

where

$$\bar{\omega}_T(\bar{x}) = \frac{1}{2}\{\bar{\omega}(\bar{x}, +0) + \bar{\omega}(\bar{x}, -0)\} \qquad (23)$$

and

$$I_e(\bar{x}, \bar{x}_0) = \frac{1}{2\pi}\iint_S \psi_{\bar{z}\bar{z}}(\bar{x}, \xi; +0, \zeta)\{g(\xi, \zeta) - g'(\xi, \pm 0)\} d\xi d\zeta \qquad (24)$$

where

$$g'(\xi \pm 0) = \begin{cases} g(\xi, +0) & \zeta > 0 \\ g(\xi, -0) & \zeta < 0 \end{cases}$$

Eq. (22) can be inverted to give

$$\Delta \bar{U}(\bar{x}) - \Delta g(\bar{x}) = \frac{2}{\pi}\left(\frac{1-\bar{x}}{\bar{x}}\right)^{\frac{1}{2}} \int_0^1 \frac{\bar{\omega}_T(\xi)}{(\bar{x}-\xi)} \cdot \left(\frac{\xi}{1-\xi}\right)^{\frac{1}{2}} d\xi$$
$$- \frac{2}{\pi}\left(\frac{1-\bar{x}}{\bar{x}}\right)^{\frac{1}{2}} \int_0^1 \frac{I_e(\xi, \bar{x}_0)}{(\bar{x}-\xi)} \cdot \left(\frac{\xi}{1-\xi}\right)^{\frac{1}{2}} d\xi \qquad (25)$$

Combination of Eq.(25) and Eq.(21) then gives the required $\bar{U}(\bar{x}, \pm 0)$.

If the integral $I_L(\bar{x} \pm 0)$ is redefined as

$$I_L(\bar{x}, \pm 0) = \frac{1}{2\pi}\int_0^1 \frac{\Delta \bar{\omega}(\xi)}{(\bar{x}-\xi)} d\xi \pm \frac{1}{\pi}\left(\frac{1-\bar{x}}{\bar{x}}\right)^{\frac{1}{2}} \int_0^1 \frac{\bar{\omega}_T(\xi)}{(\bar{x}-\xi)}\left(\frac{\xi}{1-\xi}\right)^{\frac{1}{2}} d\xi \qquad (26)$$

and if the integral $I_s(\bar{x}, \pm 0, \bar{x}_0)$ is redefined as

$$I_s(\bar{x}, \pm 0, \bar{x}_0) = \frac{1}{4\pi}\iint_S \psi_{\bar{x}}(\bar{x}, \xi; 0, \zeta)\{g(\xi, \zeta) + g(\xi, -\zeta)\} d\xi d\zeta$$
$$\mp \frac{1}{\pi}\left(\frac{1-\bar{x}}{\bar{x}}\right)^{\frac{1}{2}} \int_0^1 \frac{I_e(\xi, \bar{x}_0)}{(\bar{x}-\xi)} \cdot \left(\frac{\xi}{1-\xi}\right)^{\frac{1}{2}} d\xi \qquad (27)$$

then Eq. (19) can be used for all $\bar{z}$.

In order to simplify the present analysis it is assumed that any shock waves in the flow are normal to the freestream. Using this assumption the shock relations, Eq. (7a) become

$$\left[\bar{U} - \frac{\bar{U}^2}{2}\right]_-^+ = 0 \qquad (28)$$

If shock waves are present in the flow then, with $g(\bar{x}, \bar{z})$ given by Eq. (5), it can be shown that in order to ensure finite acceleration through the line $(\bar{U} = 1)$, Eq. (19) must be solved subject to the conditions

$$\left\{ \frac{\beta^2 M_\infty^2}{k} \bar{U}^2(\bar{x},\bar{z}) + \mathcal{I}_L(\bar{x},\bar{z}) + \mathcal{I}_s(\bar{x},\bar{z},\bar{x}_s) \right\} \Big|_{\bar{x}=\bar{x}_o(\bar{z})} \doteq \frac{1}{2} \tag{29a}$$

$$\left\{ \frac{\partial}{\partial \bar{x}} \left\{ \frac{\beta^2 M_\infty^2}{k} \bar{U}^2(\bar{x},\bar{z}) + \mathcal{I}_L(\bar{x},\bar{z}) + \mathcal{I}_s(\bar{x},\bar{z},\bar{x}_s) \right\} \right\} \Big|_{\bar{x}=\bar{x}_o(\bar{z})} = 0 \tag{29b}$$

$$\left\{ \frac{\partial}{\partial \bar{z}} \left\{ \frac{\beta^2 M_\infty^2}{k} \bar{U}^2(\bar{x},\bar{z}) + \mathcal{I}_L(\bar{x},\bar{z}) + \mathcal{I}_s(\bar{x},\bar{z},\bar{x}_s) \right\} \right\} \Big|_{\bar{x}=\bar{x}_o(\bar{z})} = 0 \tag{29c}$$

where $\bar{x} = \bar{x}_o(\bar{z})$ denotes the $(\bar{U}=1)$ line. It can be shown that Eq.(29c) is always satisfied; The remaining equations Eq.(29a, 29b) are sufficient to give location of the $(\bar{U}=1)$ line $\bar{x}_o(\bar{z})$ and the shock locations $\bar{x}_{su}$, $\bar{x}_{sl}$.

The perturbation velocity component $\bar{U}(\bar{x},\bar{z})$ can then be found from Eq.(19) with Eq.(9) and Eq.(29). Having found $\bar{U}(\bar{x},\bar{z})$ the velocity component $\bar{w}(\bar{x},\bar{z})$ can be found using the irrotationality relation

$$\bar{w}(\bar{x},\bar{z}) = \frac{\partial}{\partial \bar{z}} \int_{-\infty}^{\bar{x}} \bar{U}(\xi,\bar{z})\, d\xi \tag{30}$$

Evaluation of the Integrals

In order to solve Eq.(19) the line integral $\mathcal{I}_L(\bar{x},\bar{z})$ and the field integrals in $\mathcal{I}_s(\bar{x},\bar{z},\bar{x}_s)$ need to be evaluated. The line integral involves only the value of $\bar{U}(\bar{x},\bar{z})$ on the aerofoil surface through the boundary conditions Eq.(9) and can be evaluated using standard techniques in terms of specific values of $\bar{U}(\bar{x},\bar{z})$ on the aerofoil surface. The evaluation of the field integrals in $\mathcal{I}_s(\bar{x},\bar{z},\bar{x}_s)$ is more complex since they involve $g(\xi,\zeta)$ over the field An approximate, although sufficiently accurate, evaluation of $\mathcal{I}_s(\bar{x},\bar{z},\bar{x}_s)$ is possible if $g(\xi,\zeta)$ is known at specific points in the flow field and an interpolation function used to express $g(\xi,\zeta)$ over the rest of the flow field. The practical requirement is an accurate estimation of the velocity on the aerofoil surface and hence an acceptable solution to Eq.(19) need not necessarily be very accurate in the far field provided the surface velocities are calculated to sufficient accuracy.

Let the flow field be divided into 2N strips as shown in Fig.1. In principle the outermost strips in each half plane extend to infinity but in practice they need only extend to a suitable finite location in the far field. It is assumed that the function $g(\xi,\zeta)$ is known on each strip edge $(\zeta = \zeta_j ,\ j = 1, 2N)$ and the variation of

$g(\bar{\xi},\bar{\zeta})$ in each strip is represented by some interpolation function; in the present work linear interpolation is used. Thus in the j^{th} strip

$$g(\bar{\xi},\bar{\zeta}) = g(\bar{\xi},\bar{\zeta}_j) + \frac{[g(\bar{\xi},\bar{\zeta}_{j+1}) - g(\bar{\xi},\bar{\zeta}_j)](\bar{\zeta}-\bar{\zeta}_j)}{(\bar{\zeta}_{j+1}-\bar{\zeta}_j)} \qquad \bar{\zeta}_j \leq \bar{\zeta} \leq \bar{\zeta}_{j+1} \tag{31}$$

Using Eq. (31) the integrations with respect to $\bar{\zeta}$ in the field integrals can be performed. The subsequent line integrals in the $\bar{\xi}$ direction can be evaluated using standard methods in terms of $g(\bar{x}_i, \bar{z}_j)$ where the $\bar{x}_i$ ($i = 1, m$) are specified values of $\bar{x}$. On putting $\bar{x} = \bar{x}_i$ and $\bar{z} = \bar{\zeta}_j = \bar{z}_j$ the integral equation Eq. (19) reduces to a set of non-linear algebraic equations for $\bar{u}(\bar{x}_i, \bar{z}_j)$ if $\bar{w}(\bar{x}_i, \bar{z}_j)$ is assumed known; thus

$$\bar{u}(\bar{x}_i,\bar{z}_j) - g(\bar{x}_i,\bar{z}_j) = I_L(\bar{x}_i,\bar{z}_j) + I_F(\bar{x}_i,\bar{z}_j,\bar{x}_s) \tag{32}$$

which may be solved by iteration.

Once the $\bar{u}(\bar{x}_i, \bar{z}_j)$ are known the $\bar{w}(\bar{x}_i, \bar{z}_j)$ can be found from Eq. (30) and the surface velocities can be found using interpolation.

Since high accuracy is required only at the aerofoil surface the function $g(\bar{x}, \bar{z})$ on a certain number of outer strip edges may be adequately estimated in terms of values on the inner strip edges using some suitable approximation function without significantly affecting the accuracy of the calculation of the surface velocities. In the present work it is assumed that

$$g(\bar{\xi},\bar{\zeta}) = \frac{g(\bar{\xi},\bar{\zeta}_{NN})}{\left\{1+\dfrac{|\bar{\zeta}-\bar{\zeta}_{NN}|}{b(\bar{\xi},\bar{\zeta}_{NN})}\right\}^2}, \quad |\bar{\zeta}| > |\bar{\zeta}_{NN}| \tag{33}$$

where

$$b(\bar{\xi},\bar{\zeta}_{NN}) = \left|\frac{2(\bar{\zeta}_{NN}-\bar{\zeta}_{NN-1})\,g(\bar{\xi},\bar{\zeta}_{NN})}{\{g(\bar{\xi},\bar{\zeta}_{NN})-g(\bar{\xi},\bar{\zeta}_{NN-1})\}}\right|$$

and $\bar{\zeta}_{NN}$ is some value of $\bar{\zeta}$ in each half plane.

Calculation Procedure for Supercritical Flow with Shock Waves

When no shock waves are present Eq. (32) can be solved directly. When shock waves are present Eq. (32) must be solved subject to the conditions Eq.(29). It is difficult to satisfy Eq. (29) at each stage of the iteration by altering the shock location $\bar{x}_s$ since the location of the discontinuity in $\bar{U}(\bar{x},\bar{z})$ then changes at each iteration. An alternative method of satisfying Eq. (29) is as follows.

Eq.(32) can be written in the form

$$\boxed{\bar{U}(\bar{x}_i,\bar{z}_j) - g(\bar{x}_i,\bar{z}_j) = I_L(\bar{x}_i,\bar{z}_j) + \epsilon(\bar{z}_j) I_s(\bar{x}_i,\bar{z}_j,\bar{x}_s)} \tag{34}$$

where the $\epsilon(\bar{z}_j)$ is a parameter which is constant along each strip edge. An initial estimate is made of the shock locations $\bar{x}_{su}$, $\bar{x}_{sL}$, the function $g(\bar{x}_i, \bar{z}_j)$ and $\bar{U}(\bar{x}_i, \bar{z}_j)$. The integrals $I_L(\bar{x}_i, \bar{z}_j)$ and $I_s(\bar{x}_i, \bar{z}_j, \bar{x}_s)$ can then be evaluated. The parameter $\epsilon(\bar{z})$ and the location of the $(\bar{U} = 1)$ point on the strip edge $(\bar{z} = \bar{z}_j)$, $\bar{x}_o(\bar{z})$ are evaluated by enforcing the condition of finite streamwise acceleration along each strip edge, that is satisfying Eq. (29a, 29b) along each strip edge. New values of $\bar{U}(\bar{x}_i, \bar{z}_j)$, $g(\bar{x}_i, \bar{z}_j)$ are then computed and the process repeated until convergence of the $\epsilon(\bar{z}_j)$. The shock locations are then changed and the procedure outlined above repeated. The shock locations are correct when the $\epsilon(\bar{z}_j)$ are unity. Since high accuracy is required only on the aerofoil surface the accuracy of the solution in the flow field can decrease as $|\bar{\zeta}| \to \infty$ provided the surface solution is not affected to any significant degree. In practice this is taken to mean that the converged value of $\epsilon(\bar{z}_j)$ may be allowed to differ from unity on the outermost strip edges.

Results

The pressure distributions over several aerofoils have been computed for both subcritical and supercritical flows with shock waves. For the subcritical flows the function $g(\bar{x},\bar{z})$ is defined by Eq.(5) and the transonic parameter k, Eq.(3), is defined after Hancock[14] to be

$$k = \{3 + (\gamma-2)M_\infty^2\} M_\infty^2 \tag{35}$$

For supercritical flows with shock waves the function $g(\bar{x},\bar{z})$ is taken to be

$$\boxed{g(\bar{x},\bar{z}) = \frac{\bar{U}^2(\bar{x},\bar{z})}{2}} \tag{36}$$

in order to simplify the computation, and the parameter k is taken to be

$$k = (\gamma+1) M_\infty^{3/2} \tag{37}$$

In all cases the exact pressure-velocity relation is used.

Three examples of subcritical flow calculations are shown in Figs. 2-5 and are compared to exact results. The results of the standard integral equation method [15] are shown for comparison. In Fig. 2 the flow around a NACA 0012 aerofoil at 0° incidence and $M_{\infty} = 0.72$ is shown and it can be seen that the present results compare well with the 'exact' results of Sells [16]. In Fig. 3 the pressure distribution around a NLR 0.1 - 0.75 - 1.25 aerofoil at 0° incidence and $M_{\infty} = 0.745$ is shown. The agreement with the 'exact' result [16] is fairly good except in the vicinity of the leading edge. In Fig. 4. the pressure distribution around a NACA 0012 aerofoil at 2° incidence and $M_{\infty} = 0.63$ is shown. The agreement with the 'exact' result[16] is good again except in the neighbourhood of the leading edge. A similar agreement with the 'exact' result is shown in Fig. 5. for the pressure distribution around a 14% thick NPL 3111 aerofoil at 1.2° incidence and $M_{\infty} = 0.667$.

Three examples of supercritical flow are shown in Figs. 6-8. In Fig.6 the pressure distribution around a NACA oo12 aerofoil at zero incidence and $M_{\infty} = 0.816$ is shown and the present results agree fairly well with the finite difference results calculated using the method of Garabedian and Korn[4] using the non-conservative difference scheme. In Fig.7 the pressure distribution around a NACA 0012 aerofoil 2° incidence and $M_{\infty} = 0.75$ is compared to the finite difference results calculated by Bauer and Korn[17] using both'non-conservative' and 'conservative' difference schemes. As in the non-lifting example the present results agree fairly well with the 'non-conservative' results. In Fig.8 the pressure distribution around a NACA 64A410 aerofoil at 0° incidence and $M_{\infty} = 0.72$ is shown. As in the previous example the present results agree fairly well with the'non-conservative' finite difference results[18]. This tendency of the present calculations to agree with the 'non-conservstive' rather than the correct 'conservative' results may be attributed to the use of the transonic small perturbation equation for the integral equation calculations as opposed to the use of the full potential equation by Bauer and Korn[17], and Jameson [18].

For the subcritical examples 30 streamwise elements and 5 strips in each half plane are used. The velocity on the outer strip edges in each half plane is estimated using Eq.(33). Convergence is obtained after about 10 iterations.

In the supercritical examples 30 streamwise elements and 8 strips in each half plane are used. The velocity on the 3 outer strip edges is estimated using Eq.(33). For each shock location convergence of the $\varepsilon(z_i)$ is obtained after about 10 iterations. A typical computing time for the lifting supercritical example is 80 - 100 secs. on an ICL 1904S. Further improvements in the numerical scheme are feasible and it is probable that the computing time can be considerably decreased.

Conclusions

An extended integral equation method has been developed for the transonic flow around lifting and non-lifting aerofoils. The number of iterative steps to convergence is small and this is reflected in the comparatively low computing time. Improvements in the numerical solution of the integral equation are possible which should decrease the overall computing time. The results obtained for both lifting and non-lifting aerofoils in subcritical flow agree satisfactorily with 'exact' results. The extended integral equation method gives considerably improved results over the earlier standard integral equation method. The results for supercritical flows with shock waves are in fairly good agreement with existing finite difference results, although there appears to be better agreement with the incorrect 'non-conservative' finite difference results than with the correct 'conservative' results. This is attributed to the present use of the transonic small disturbance potential equation as opposed to the full potential equation used in the finite difference calculations.

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Figure Captions

Fig. 1. Arrangement of strips

Fig. 2. Pressure distribution around a NACA 0012 aerofoil at 0° incidence; $M_{\infty} = 0.72$

Fig. 3. Pressure distribution around an NLR 0.1 - 0.75 - 1.25 aerofoil at 0° incidence; $M_{\infty} = 0.745$

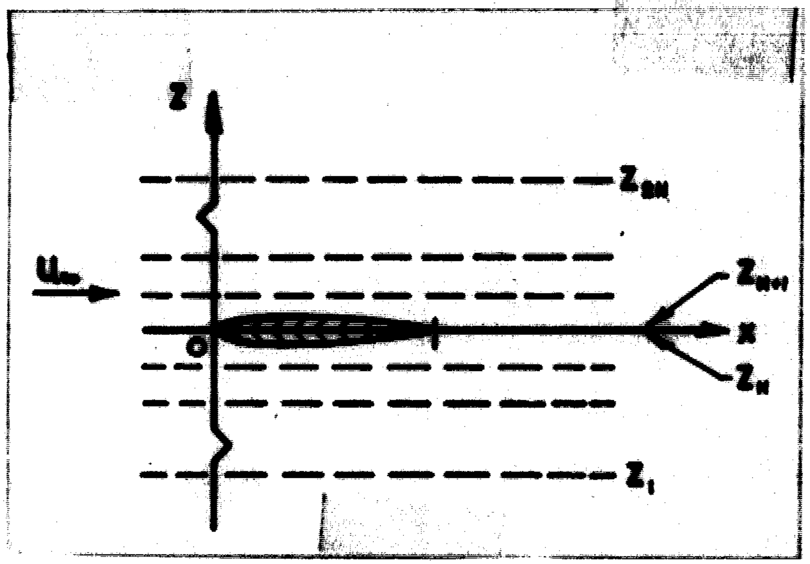
Fig. 4. Pressure distribution around a NACA 0012 aerofoil at 2° incidence; $M_{\infty} = 0.63$

Fig. 5. Pressure distribution around an NPL 3111 aerofoil at 1.2° incidence; $M_{\infty} = 0.667$

Fig. 6. Pressure distribution around a NACA 0012 aerofoil at 0° incidence; $M_{\infty} = 0.816$

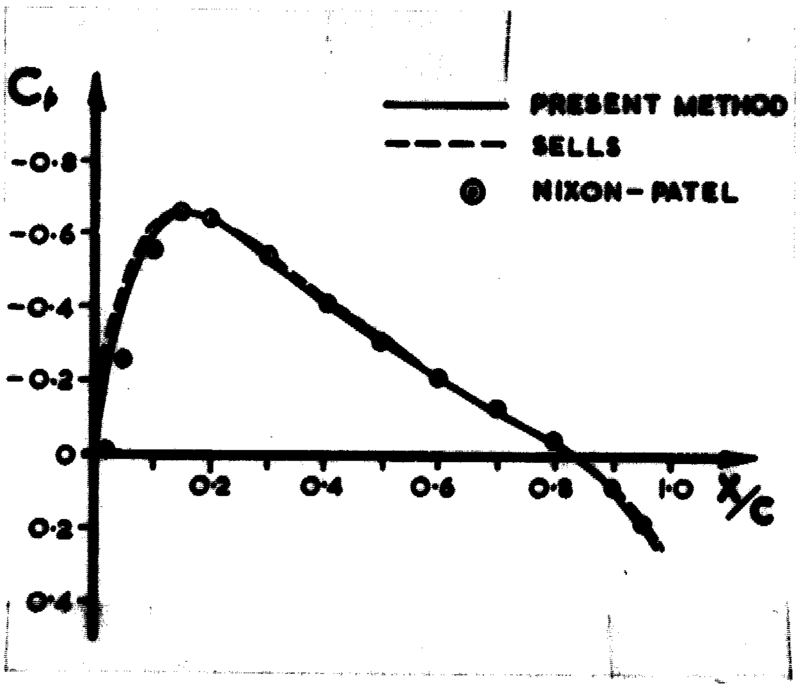
Fig. 7. Pressure distribution around a NACA 0012 aerofoil at 2° incidence; $M_{\infty} = 0.75$

Fig. 8. Pressure distribution around a NACA 64A410 aerofoil at 0° incidence; $M_{\infty} = 0.72$



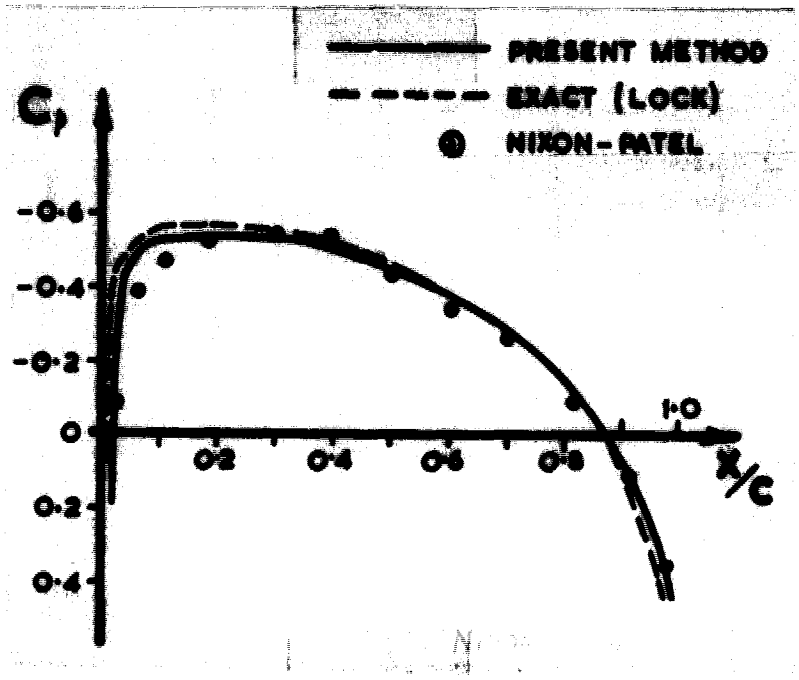
Arrangement of strips.

Fig. 1



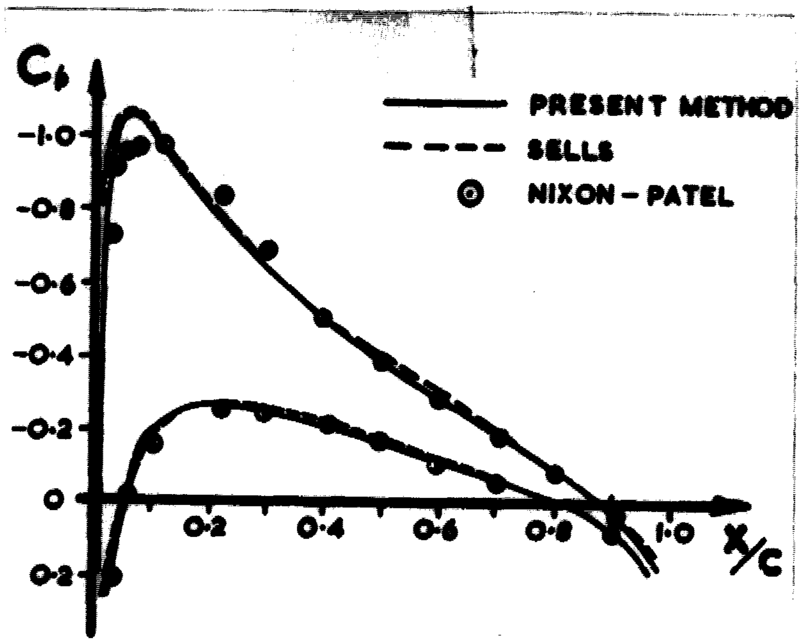
Pressure distribution around a NACA 0012 aerofoil at 0^0 incidence; M_∞ = 0.72 .

Fig. 2



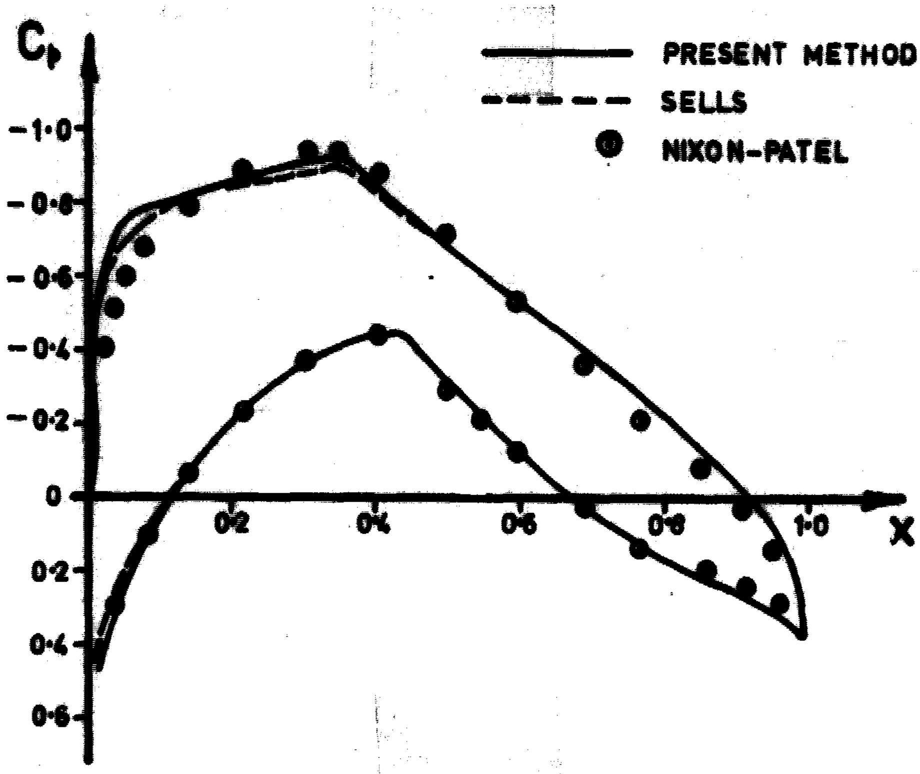
Pressure distribution around an NLR 0.1 - 0.75 - 1.25 aerofoil at 0^0 incidence; M_∞ = 0.745

Fig. 3



Pressure distribution around a NACA 0012 aerofoil at 2^0 incidence; M_∞ = 0.63

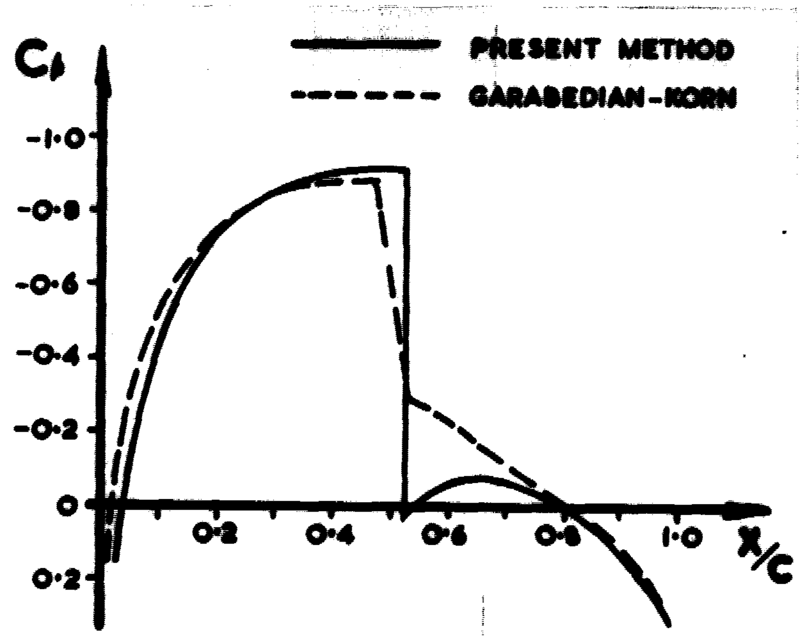
Fig. 4



Pressure distribution around an NPL 3111 aerofoil at

1.2^0 incidence; $M_\infty = 0.667$

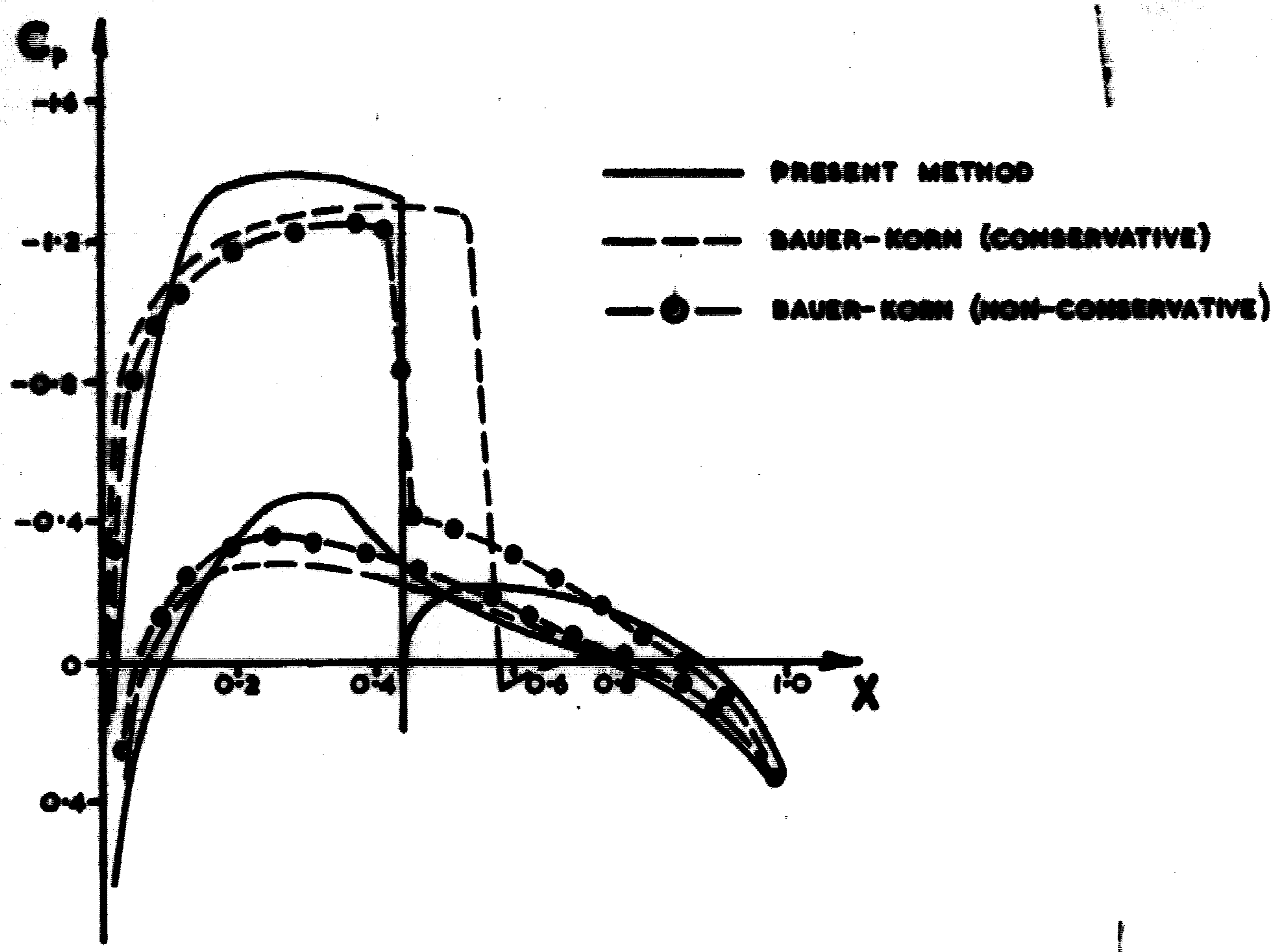
Fig. 5



Pressure distribution around a NACA 0012 aerofoil at

0^0 incidence; $M_\infty = 0.816$

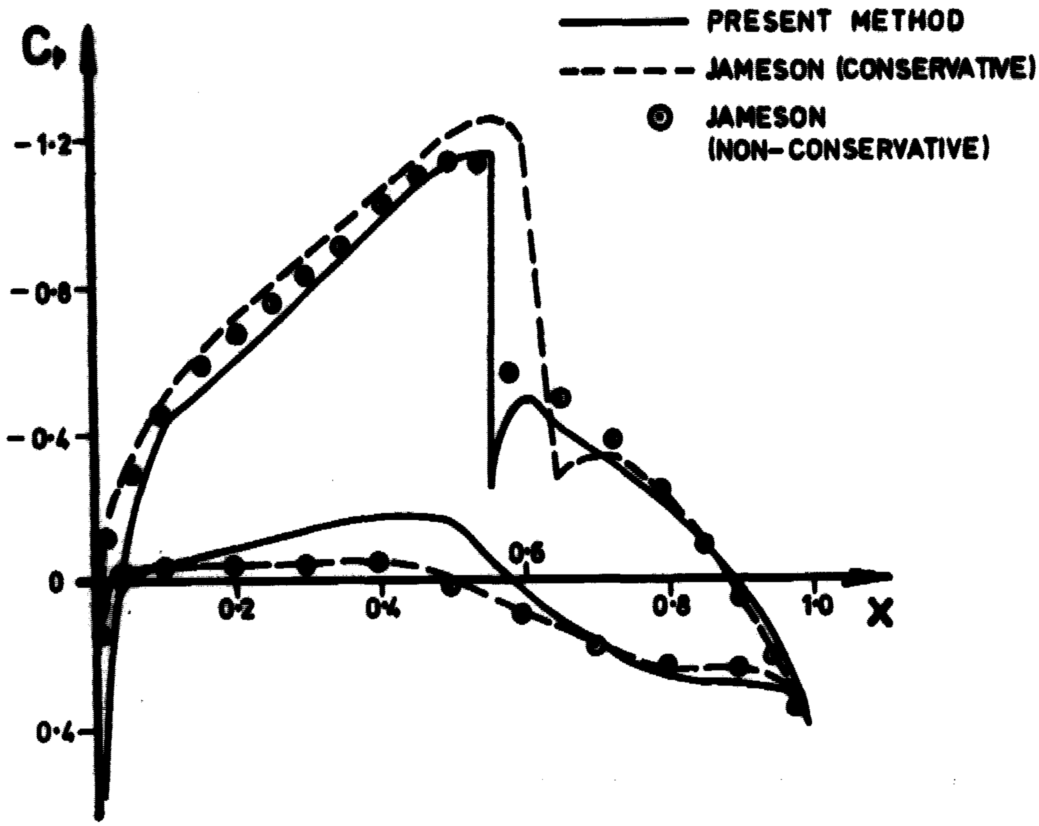
Fig. 6



Pressure distribution around a NACA 0012 aerofoil at

2^{0} incidence; $M_{\infty} = 0.75$

Fig. 7



Pressure distribution around a NACA 64A410 aerofoil

at 0^{0} incidence; $M_{\infty} = 0.72$

Fig. 8