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Calculations of Neutron Distributions by Statistical Method*

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Considering the displacement in the generalized random walk process of isotropic scattering neutrons as random variable, $\varphi_n(x)$, the probability density function after n steps is obtained through the use of the statistical characteristic function. The total neutron distribution $\varphi(x)=\sum_{n=0}^{\infty}\varphi_n(x)$ is expressed as a Fourier transform of the power series of this characteristic function. Using this statistical method, the total distributions $\varphi(x)$ are shown by means of concrete examples to agree with the solutions of the relevant linear Boltzmann equations which are already solved. The one dimensional space-angle dependent case is examined for a system containing random empty gaps. To evaluate $\varphi_n(x)$ for large n ($n\gg1$), the central limit theorem is used.

I. INTRODUCTION

The common practice in reactor theory is to use as basic equation the expression of the balance of neutron numbers in a small volume in phase space, *i.e.*, the Boltzmann equation.

The neutron distribution is on the other hand habitually calculated by the Multiple Collision Method. The present method is a statistical treatment of neutron random walk, which belongs to the latter category.

The random walk is regarded as a succession of "elementary events" composed of a collision and subsequent travers of free path. These events change the state of the system, and are assumed to be statistically independent of each other. To solve the random walk problem, we require the probability density $\varphi_n(\vec{x})$ of the resultant displacement vector in phase space, *i.e.* $\vec{x}=(\vec{r},\vec{\Omega},v)$, after n steps. Here $\vec{r}$, $\vec{\Omega}$ and v are respectively the displacement space vector, the direction vector and the velocity difference.

Many authors have adopted this approach: Chandrasekar[1] developed the three dimensional random flight problem using Markoff's method, and derived $\varphi_n(\vec{x})$ for large n as a Gaussian function. Grosjean[2] treated the random flight of one-velocity particles in the space of arbitrary dimensions and represented $\varphi_n(\vec{x})$ by the linear recurrence relation. This recurrence method is, however, too complicated for evaluating the total distribution $\varphi(\vec{x})$ in practical reactor problems. Wigner[3] studied the problem of multiple scattering of point particles under the condition that the transition is invariant under a group G, (the case of infinite homogeneous isotropic medium) and proposed a structure for G in the velocity dependent case. Guth & Inönü[4] applied this group theoretical treatment to time-dependent neutron slowing down problems, and compared the results with Waller's[5]. They further clarified the connection between the generalized random walk and the related linear Boltzmann equation[6].

For the case of space-energy dependent transport of neutrons in an infinite medium, Syros[7] expanded the neutron propagator, defined as the solution of the integral transformation of the transport equation (Laplace and Fourier transformations for lethargy and space variables respectively), in a series of ascending power of a certain function $\chi(\omega)$, and applied the central limit theorem to the Fourier inversion of $\{\chi(\omega)\}^n$ for $n\gg1$.

The present work is attempt to evaluate

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the density function $\varphi_n(\vec{x})$ in infinite homogeneous medium by means of a statistical method that provides for conversion of random modulation into displacement. The treatment is generalized to the case where each elementary event is governed by an independent transition probability, which should provide information on the statistical aspects of the random walk problem. The evaluation of $\varphi_n(\vec{x})$ for large n is greatly facilitated by making use of the central limit theorem[1].

We limit our consideration to the case where the transition probabilities can be represented as functions of displacements in phase space, which displacements are regarded as random variables. This permits us to consider the succession of elementary events as an additive process, and the problem is reduced to one of finding the probability density of the cumulated sum of these independent random variables.

For analytical representation of the density function $\varphi_n(\vec{x})$, the characteristic functions of the transition densities play an important role. As concrete examples, space-energy dependent and space-time dependent cases are treated in Chap. III, where the analytical representations of total distribution are shown to coincide with the results of Papmehl, who solved linear Boltzmann equations for cases of physical conditions identical to the present. In Chap. IV, the present statistical method is applied to a case of one-dimensional space-angle dependency in a system containing random empty gaps. In this case, the once scattering probability density $\varphi_1(x)$ is expressed by a combination of the orthodox exponential function and periodic functions that appear on account of the random modulation of free path. The effect of the oscillation term will be shown numerically, and for evaluating the total distribution, use is made of the central limit theorem.

II. Method

Tracing the process of neutron random walk, we consider the sequence of elementary events $\varepsilon_0, \varepsilon_1, \cdots \varepsilon_n$, of which the final i-th elementary event is described by the transition density $f_i(\vec{x})$ $(i=1,\cdots n)$, defined as the transition probability with displacement variables, where $\vec{x}$ is the displacement in the phase space which contains the time coordinate, *i. e.*, $\vec{x}=(\Delta\vec{r}, \Delta\vec{\Omega}, \Delta v, \Delta t)$.

The initial state of the neutron is determined by the random variable denoted by $\vec{X}_0$ with probability density $f_0(\vec{x})$ (source density). At the first step the state is changed as specified by the random variable $\vec{X}_1$ with density $f_1(\vec{x})$, at the second step by $\vec{X}_2$ with $f_2(\vec{x})$ and so forth. Each random variable $\vec{X}_i$ with probability density function $f_i(\vec{x})$ is statistically independent, so that the state of the neutron after n steps will be the sum of the independent random variables $\vec{S}_n=\vec{X}_0+\vec{X}_1+\cdots+\vec{X}_n$ which itself is also a random variable.

To find the neutron distribution after n steps, we only need to find the probability density function of this sum $\vec{S}_n$ of independent random variables.

We define the characteristic function of $f_i(\vec{x})$ by

$$\chi_i(\vec{\omega}) \equiv \langle e^{-i\vec{\omega}\vec{x}'}\rangle = \int_{-\infty}^{\infty} d\vec{x} e^{-i\vec{\omega}\vec{x}'} f_i(\vec{x}), \tag{1}$$

where $\vec{x}'$ is the transpose of the vector $\vec{x}$. Then the characteristic function for the random variable $\vec{S}_n$ is expressed by

$$\langle e^{-i\vec{\omega}\vec{s}_n'}\rangle = \langle e^{-i\vec{\omega}\vec{x}_0'}\rangle\langle e^{-i\vec{\omega}\vec{x}_1'}\rangle\cdots\langle e^{-i\vec{\omega}\vec{x}_n'}\rangle = \chi_0(\vec{\omega}_0)\chi_1(\vec{\omega})\cdots\chi_n(\vec{\omega}). \tag{2}$$

Hence, we have

$$\varphi_n(\vec{x}) = \left(\frac{1}{2\pi}\right)^k \int_{-\infty}^{\infty} d\vec{\omega} e^{i\vec{\omega}\vec{x}'}\chi_0(\vec{\omega})\cdots\chi_n(\vec{\omega}), \tag{3}$$

where k is the number of phase space dimensions.

For $n\gg 1$, $\varphi_n(\vec{x})$ can be approximately derived from the central limit theorem as a Gaussian function, provided the existence of the mean vectors $\vec{m}_i$ and covariance matrices C_i of $f_i(\vec{x})$ $(i=0, 1, 2, \cdots, n)$[10]:

$$\varphi_n(\vec{x}) = \left(\frac{1}{2\pi}\right)^k |C|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(\vec{x}-\vec{m})' C^{-1}(\vec{x}-\vec{m})\right\}, \tag{4}$$

where $\vec{m}$ is the sum of the mean vectors $\vec{m}_i$, and C is the sum of covariance matrices C_i $(i=0, 1, \cdots, n)$:

$$\vec{m}_i \equiv \langle\vec{x}\rangle$$

$$\vec{m} = \sum_{i=0}^{n} \vec{m}_i$$

$$C_i \equiv \langle (\vec{x}-\vec{m}_i)(\vec{x}-\vec{m}_i)' \rangle$$
$$C=\sum_{i=0}^{n} C_i$$

Habitually, the space and time displacements are expressed respectively as $\Delta\vec{r}=|\Delta\vec{r}|\vec{\Omega}$ and $\Delta t=|\Delta\vec{r}|/v$, where $|\Delta\vec{r}|$ is the free path length, $\vec{\Omega}$ the direction vector, and v the velocity at that elementary event. Hence the transition density at i-th elementary event will have the form

$$f_i(\vec{x};\vec{\Omega}_i,v_i)=\int_0^\infty d|\Delta\vec{r}|g_i(|\Delta\vec{r}|,\Delta\vec{\Omega},\Delta v;\ \vec{\Omega}_i,v_i) \cdot\delta(|\Delta\vec{r}|\vec{\Omega}_i-\Delta\vec{r})\delta(|\Delta\vec{r}|/v_i-\Delta t), \quad (5)$$

where $g_i(|\Delta\vec{r}|,\Delta\vec{\Omega},\Delta v;\ \vec{\Omega},v)$ is the density function of $|\Delta\vec{r}|,\Delta\vec{\Omega},\Delta v$, and the Dirac delta-functions are the projection of these displacements to space and time displacements. If the cross sections are space independent, the density g_i may be written in the form

$$g_i(|\Delta\vec{r}|,\Delta\vec{\Omega},\Delta v;\vec{\Omega},v) =\Sigma_s(\Delta\vec{\Omega},\Delta v;\vec{\Omega}_i,v_i)\exp(-\Sigma(v_i)|\Delta\vec{r}|),$$

and for $i=0$,

$$g_0(|\Delta\vec{r}|,\vec{\Omega},v)=S(\vec{\Omega},v)\exp(-\Sigma(v)|\Delta\vec{r}|),$$

where $S(\vec{\Omega},v)$ is the velocity vector source density, and $\Sigma_s(\Delta\vec{\Omega},\Delta v;\vec{\Omega},v)$ the probability of displacement $\Delta v,\Delta\vec{\Omega}$ at $v,\vec{\Omega}$, and Σ the total cross section—which may include absorption. Also,

$$v_i=v_0+\Delta v_1+\cdots+\Delta v_i$$
$$\vec{\Omega}_i=\vec{\Omega}_0+\Delta\vec{\Omega}_1+\cdots+\Delta\vec{\Omega}_i$$

where $v_0,\vec{\Omega}_0$ respectively are the initial velocity and direction, $\Delta v_i,\Delta\vec{\Omega}_i$ the velocity and direction displacements in i-th elementary event. Both displacements $\Delta\vec{\Omega}$ and Δv, which are random variables of the displacement density $\Sigma_s(\Delta\vec{\Omega},\Delta v;\vec{\Omega},v)$, depend on the kind of collision.

In the space-time-energy dependent case, the displacements of neutron random walk are, strictly speaking, not statistically independent, because the space and time displacements $\Delta\vec{r}$, Δt_i at the i-th elementary event depend on the sum of all the previous displacements: $\vec{\Omega}_i=\vec{\Omega}_0+\cdots+\Delta\vec{\Omega}_i$, $v_i=v_0+\cdots+\Delta v_i$. In this case, the characteristic function $\chi_i(\vec{\omega};\vec{\Omega},v)$ of the probability density Eq. (5) has the velocity vector as variable, and if we average it with an appropriately given weight function, and introduce these averaged characteristic functions into Eq. (3) we have an approximated representation for $\varphi_n(\vec{x})$.

In what follows, however, we shall consider the cases where the characteristic functions do not explicitely contain velocity variables, and where scattering is assumed to be isotropic. In such cases, excluding $\chi_0(\vec{\omega})$, all the remaining characteristic functions $\chi_i(\vec{\omega})$ $(i=1, 2,\cdots)$ becomes mutually identical and we then have the exact analytical form of the total neutron distribution:

$$\varphi(\vec{x})=\sum_{n=0}^{\infty}\varphi_n(\vec{x}) =\left(\frac{1}{2\pi}\right)^k\int_{-\infty}^{\infty}d\vec{\omega}\,\frac{\chi_0(\vec{\omega})}{1-\chi(\vec{\omega})}e^{i\vec{\omega}\vec{x}_i'}. \quad (6)$$

III. Calculation of Green's Function

1. Purely Energy Dependent Slowing Down

We consider only the lethargy dependent neutron slowing down process, and assume the scattering of neutrons to be elastic and isotropic in the L-system. The random variable of lethargy difference is given from the common transition density

$$f(\Delta u)=\begin{cases}\dfrac{1}{1-\alpha}e^{-\Delta u} & \text{for } 0\leq\Delta u\leq q_M\\ 0 & \text{otherwise}\end{cases} \quad (7)$$

where

$$\alpha=\left(\frac{A-1}{A+1}\right)^2$$
$$q_M=-\log\alpha,$$

and A is the mass of the target atom. From the definition (1) we obtain the characteristic function of $f(\Delta u)$:

$$\chi(\omega)=\frac{1}{1-\alpha}\cdot\frac{1-\exp\{-(1+i\omega)q_M\}}{1+i\omega}. \quad (8)$$

From expression (3),

$$\varphi_n(u)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega e^{i\omega u}\frac{\chi_0(\omega)}{(1-\alpha)^n}\left[\frac{1-\exp\{-(1+i\omega)q_M\}}{1+i\omega}\right]^n. \quad (9)$$

If the source density $f_0(u)$ is the delta-function — *i.e.* $f_0(u)=\delta(u)$ and the corresponding characteristic function $\chi_0(\omega)$ equals unity, then the integral of Eq. (9) is calculated exactly by the residue theorem as Waller[11] has shown:

$$\varphi_n(u)=\frac{e^{-u}}{(1-\alpha)^n}\sum_{j=0}^{\bar{j}}\binom{n}{j}(-1)^j(u-jq_M)^{n-1}, \quad (10)$$

where $\bar{j}$ is the maximum number that still keeps the term $u-jq_M$ positive. This result can also be obtained by the direct operator method, which is useful for cases where the cross sections are functions of lethargy. This approach will be the subject of a forthcoming paper. The total distribution becomes

$$\varphi(u)=\delta(u)+\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega e^{i\omega u} \cdot\frac{1-\exp\{-(1+i\omega)q_M\}}{(1-\alpha)(1+i\omega)-1+\exp\{-(1+i\omega)q_M\}}. \tag{11}$$

This is also a Green function of the relevant linear Boltzmann equation, and the integral is evaluated by the residue theorem.

2. Space-Energy Dependent Slowing Down

We assume an infinite plane symmetrical medium, constant cross sections and isotropic neutron scattering in the L-system. The random variable of free path length must be projected onto the x-axis parallel to the direction of the plane. For this, we use the random variable of direction cosine. Then the random variable of the x component, the lethargy difference and the direction cosine in each elementary event will be given from transition density:

$$f(\Delta x,\mu,\Delta u)=\int_0^{\infty}dz\frac{1}{1-\alpha}\cdot\frac{\Sigma_s}{2}e^{-z\Sigma}\ e^{-\Delta u}\delta(\mu z-\Delta x), \tag{12}$$

where z is the free path length, μ the direction cosine. In the case of isotropic scattering, the direction of free path at the i-th step is independent of the previous direction of free path, so that μ can always be taken for the direction cosine referred to the x-axis, and the sum of random variables of angle displacements need not be obtained. Fourier transformation of Eq. (12) with respect to space Δx and to lethargy Δu leads to the characteristic function with parameter μ:

$$\chi(\omega_1,\omega_2;\mu)=\frac{\Sigma_s[1-\exp\{-(1+i\omega_2)q_M\}]}{2(1-\alpha)(\Sigma+i\mu\omega_1)(1+i\omega_2)}, \tag{13}$$

integration of which with respect to μ leads to the characteristic function

$$\chi(\omega_1,\omega_2)=\frac{\Sigma_s}{2(1-\alpha)i\omega_1}\log\left(\frac{\Sigma+i\omega_1}{\Sigma-i\omega_1}\right) \cdot\frac{1-\exp\{-(1+i\omega_2)q_M\}}{1+i\omega_2}. \tag{14}$$

If a plane source is placed at the origin and we assume mono-directionality with lethargy zero, the initial density function corresponding to the elementary event ε_0 will be given by

$$f_0(\Delta x,\mu,u)=\int_0^{\infty}dze^{-z\Sigma}\delta(u)\delta(\mu-\mu_0)\delta(\Delta x-\mu z), \tag{15}$$

and the characteristic function of $f_0(\Delta x,\mu,u)$ becomes

$$\chi_0(\omega_1,\omega_2;\mu)=(\Sigma+i\omega_1\mu_0)^{-1}. \tag{16}$$

The angular dependence of neutron distribution is determined by the initial and final directions of neutron paths, hence we have a sequential series of characteristic functions $\chi_0(\omega_1,\omega_2;\mu_0)$, $\chi(\omega_1,\omega_2)$, ……, $\chi(\omega_1,\omega_2)$, $\chi(\omega_1,\omega_2;\mu)$, corresponding respectively to the elementary events $\varepsilon_0,\varepsilon_1,\cdots,\varepsilon_{n-1},\varepsilon_n$. We obtain the angular dependent distribution after n steps for $n\geq1$:

$$\begin{aligned}\varphi_n(x,\mu_0,\mu,u)&=\left(\frac{1}{2\pi}\right)^2\int_{-\infty}^{\infty}d\omega_1d\omega_2\, e^{i(\omega_1x+\omega_2u)}\chi_0(\omega_1,\omega_2;\mu_0)\chi^{n-1}(\omega_1,\omega_2)\chi(\omega_1,\omega_2;\mu)\\&=\left(\frac{1}{2\pi}\right)^2\left(\frac{\Sigma_s}{1-\alpha}\right)^n\int_{-\infty}^{\infty}d\omega_1\frac{e^{i\omega_1x}}{\Sigma+i\omega_1\mu_0}\left\{\frac{1}{2i\omega_1}\log\left(\frac{\Sigma+i\omega_1}{\Sigma-i\omega_1}\right)\right\}^{n-1}\\&\quad\cdot\frac{1}{2(\Sigma+i\omega_1\mu)}\int_{-\infty}^{\infty}d\omega_2e^{i\omega_2u}\left[\frac{1-\exp\{-(1+i\omega_2)q_M\}}{1+i\omega_2}\right]^n,\end{aligned} \tag{17}$$

and $\varphi_0(x,\mu_0,\mu,u)=f_0(x,\mu,\mu_0,u)$. (18)

From the expression (17) one will easily see that $\varphi_n(x,\mu_0,\mu,u)$ is separated into space and energy parts. We have then the corresponding total distribution

$$\begin{aligned}\varphi(x,\mu_0,\mu,u)&=\left(\frac{1}{2\pi}\right)^2\int_{-\infty}^{\infty}d\omega_1d\omega_2e^{i(\omega_1x+\omega_2u)}\chi_0(\omega_1,\omega_2;\mu_0)\\&\quad\cdot\left\{\delta(\mu-\mu_0)+\frac{\chi(\omega_1,\omega_2;\mu)}{1-\chi(\omega_1,\omega_2)}\right\}.\end{aligned} \tag{19}$$

Integrating $\varphi(x,\mu_0,\mu,u)$ by μ_0 and μ, we obtain the total space-energy dependent distribution

$$\varphi(x,u)=\left(\frac{1}{2\pi}\right)^2\int_{-\infty}^{\infty}d\omega_1d\omega_2e^{i(\omega_1x+\omega_2u)}\frac{\frac{1}{i\omega_1}\log\left(\frac{\Sigma+i\omega_1}{\Sigma-i\omega_1}\right)}{1-\chi(\omega_1,\omega_2)}. \tag{20}$$

Papmehl[8] derived the same result in the form of a Green function of the relevant linear Boltzmann equation. He evaluated the integrals of Eq. (20) for sufficiency large distances from source, and derived the age diffusion approximation.

3. Space-Time Dependent Case

We assume isotropic scattering in a plane symmetrical infinite medium, together with energy independence, and describe the process by space-time variables. The random variables of free path length and direction cosine in each elementary event are sampled with the transition density $\Sigma_s \exp(-z\Sigma)$ and 1/2, then projecting these variables onto space and time coordinates, we have transition density

$$f(\Delta x, \Delta t, \mu)=\frac{\Sigma_s}{2}\int_0^\infty dz e^{-z\Sigma}\delta\left(\frac{z}{v}-\Delta t\right)\delta(\mu z-\Delta x). \tag{21}$$

Similarly, the initial density function of the elementary event ε_0 is given by

$$f_0(\Delta x, \Delta t, \mu)=\int_0^\infty dz e^{-z\Sigma}\delta(\mu z-\Delta x)\delta(\mu-\mu_0)\delta\left(\frac{z}{v}-\Delta t\right), \tag{22}$$

where v is the neutron speed which we assume to be constant. The characteristic function of $f(\Delta x, \Delta t, \mu)$ with parameter μ is given by

$$\chi(\omega_1, \omega_2; \mu)=\frac{\Sigma_s}{2\{\Sigma+i(\omega_1\mu+\omega_2/v)\}}. \tag{23}$$

Integrating Eq. (23) with respect μ over $[-1, 1]$ we obtain the characteristic function

$$\chi(\omega_1, \omega_2)=\frac{\Sigma_s}{2i\omega_1}\log\left\{\frac{\Sigma+i(\omega_2/v+\omega_1)}{\Sigma+i(\omega_2/v-\omega_1)}\right\}. \tag{24}$$

We have also the characteristic function of $f_0(\Delta x, \Delta t, \mu)$:

$$\chi_0(\omega_1, \omega_2; \mu_0)=\{\Sigma+i(\omega_1\mu_0+\omega_2/v)\}^{-1}. \tag{25}$$

By a procedure quite similar to that used in deriving Eq. (17) in the previous example, the angular dependent probability density function after n steps is obtained for $n\geq 1$:

$$\begin{aligned}\varphi_n&(x, \mu_0, \mu, t)\\&=\left(\frac{1}{2\pi}\right)^2\int_{-\infty}^{\infty} d\omega_1 d\omega_2 e^{i(\omega_1 x+\omega_2 t)}\chi_0(\omega_1, \omega_2; \mu_0)\\&\qquad\cdot\chi^{n-1}(\omega_1, \omega_2)\chi(\omega_1, \omega_2; \mu).\end{aligned} \tag{26}$$

This time, however, the characteristic functions are not separated with respect to ω_1 and ω_2, so that separation of the variables x and t is not obtained in $\varphi_n(x, \mu, t)$, as it was in the preceding case. Introducing the characteristic functions (23), (24) and (25) into Eq. (19), we obtain the corresponding total neutron distribution:

$$\begin{aligned}\varphi(x, \mu_0, \mu, t)=&\left(\frac{1}{2\pi}\right)^2\int_{-\infty}^{\infty} d\omega_1 d\omega_2\\&\cdot e^{i(\omega_1 x+\omega_2 t)}\{\Sigma+i(\omega_1\mu_0+\omega_2/v)\}^{-1}\\&\cdot\left[\delta(\mu-\mu_0)+\frac{\Sigma_s/2}{\Sigma+i(\omega_1\mu+\omega_2/v)}\right.\\&\left.\cdot\left\{1-\frac{\Sigma_s}{2i\omega_1}\log\frac{\Sigma+i(\omega_2/v+\omega_1)}{\Sigma-i(\omega_2/v-\omega_1)}\right\}^{-1}\right].\end{aligned} \tag{27}$$

Integrating $\varphi(x, \mu_0, \mu, t)$ with respect to μ_0 and μ, the total space-time dependent distribution is derived:

$$\begin{aligned}\varphi(x, t)=&\left(\frac{1}{2\pi}\right)^2\int_{-\infty}^{\infty} d\omega_1 d\omega_2 e^{i(\omega_1 x+\omega_2 t)}\\&\cdot\frac{2\chi(\omega_1, \omega_2)}{\Sigma_s\{1-\chi(\omega_1, \omega_2)\}}.\end{aligned} \tag{28}$$

The integrand of Eq. (28) can be separated into real and imaginary parts, and the imaginary part is shown to be an odd function of ω_1 and ω_2, and consequently vanishes. The real part is an even function and can be calculated numerically. These results agree with the Green function of the space-time dependent Boltzmann equation, as treated by Papmehl[9].

VI. System Containing Random Gaps

In a system containing random empty holes, the length of the free path will be modified by the distribution, shape and size of the voids. We will here treat a one-dimensional space-angle dependent isotropic multiple collision process of neutrons in an infinite medium containing anisotropic random holes. We assume the shape of the holes to be a gap of width D perpendicular to the space coordinate. This random gap model is a one-dimensional approximation of a series of closely spaced (strongly correlated) spherical holes along a line perpendicular to the x direction, a configuration that is quite likely to occur in a boiling water reactor. Let $p_n(z, \mu)$ denote the probability of a neutron traversing n gaps in the direction μ, covering thereby a distance through the medium between the values of z and $z+dz$. Then, the system is characterized by the probability $p_n(z, \mu)$ as well as by the cross sections of atoms constituting the medium. The probability density of a neutron that has a free path length z

and a direction cosine μ passing through n gaps is $p_n(z,\mu)\Sigma_s \exp(-z\Sigma)/2$. Thus, the free path length z is lengthened by $nD/|\mu|$, the probability density being $p_n(z,\mu)\Sigma_s \exp(-z\Sigma)/2$, and we denote this lengthened free path by z'. The event of encountering n gaps in a free path is mutually exclusive for each n, hence the probability of a free path which has length z and direction cosine μ encountering any number of gaps will be the sum[12]

$$\sum_{n=0}^{\infty} p_n(z,\mu)\frac{\Sigma_s}{2}e^{-z\Sigma}.$$

Projecting the lengthened free path z' for each n onto the x coordinate, and integrating in the interval $(0,\infty)$ with respect to z and z', we obtain the transition density of the space difference, with parameter μ, for each elementary event except ε_0:

$$f(x,\mu)=\int_0^{\infty} dz dz' \delta(x-\mu z')\sum_{n=0}^{\infty}\frac{\Sigma_s}{2} \cdot e^{-z\Sigma}p_n(z,\mu)\delta\left(z'-z-n\frac{D}{|\mu|}\right). \tag{29}$$

The corresponding density function of the elementary event ε_0 will be given by

$$f_0(x,\mu)=\int_0^{\infty} dz dz' \delta(x-\mu z')\sum_{n=0}^{\infty}\delta(\mu-\mu_0) \cdot e^{-z\Sigma}p_n(z,\mu)\delta\left(z'-z-n\frac{D}{|\mu|}\right). \tag{30}$$

We assume the probability of the occurrence of gaps in a given interval along the x coordinate to be Poissonian in distribution, and that the average number of gaps in a unit length of x to be N. For the projected length $z|\mu|$, the mean number of holes will then be $z|\mu|N$, whence we have

$$p_n(z,\mu)=\frac{1}{n!}(z|\mu|N)^n \exp(-z|\mu|N). \tag{31}$$

Substituting the relation (31) into Eq. (29), we obtain the Fourier transform of $f(x,\mu)$ with respect to x, which is the characteristic function with parameter μ:

$$\chi(\omega;\mu)=\frac{\Sigma_s}{2}[\Sigma+|\mu|N\{1-\exp(-i\omega D\mu/|\mu|)\}+i\omega\mu]^{-1}. \tag{32}$$

Similarly the characteristic function of $f_0(x,\mu)$ is

$$\chi_0(\omega;\mu_0)=[\Sigma+|\mu_0|N \cdot\{1-\exp(-i\omega D\mu_0/|\mu_0|)\}+i\omega\mu_0]^{-1}. \tag{33}$$

Integration of $\chi(\omega,\mu)$ with respect to μ leads to the characteristic function

$$\chi(\omega)=\frac{\Sigma_s}{2}\left\{\frac{1}{g_+(\omega)}\log\frac{\Sigma+g_+(\omega)}{\Sigma} + \frac{1}{g_-(\omega)}\log\frac{\Sigma}{\Sigma-g_-(\omega)}\right\}, \tag{34}$$

where
$$g_+(\omega)=i\omega+N\{1-\exp(-i\omega D)\}$$
$$g_-(\omega)=i\omega-N\{1-\exp(i\omega D)\}. \tag{35}$$

From the definition expressed by Eq. (35) we easily obtain the relation $g_+(\omega)=-g_-(-\omega)$, so that we can rewrite the expression (34) in the form

$$\chi(\omega)=W(\omega)+W(-\omega), \tag{36}$$

where
$$W(\omega)=\frac{\Sigma_s}{2}\frac{1}{g_+(\omega)}\log\frac{\Sigma+g_+(\omega)}{\Sigma}. \tag{37}$$

A treatment similar to the preceding section leads to the angular dependent density after n steps:

$$\varphi_n(x,\mu_0,\mu)=\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{i\omega x}\chi_0(\omega;\mu)\chi^{n-1}(\omega)\chi(\omega;\mu). \tag{38}$$

and the total distribution:

$$\varphi(x,\mu_0,\mu)=\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{i\omega x}\chi_0(\omega;\mu) \cdot\left\{\delta(\mu-\mu_0)+\frac{\chi(\omega;\mu_0)}{1-\chi(\omega)}\right\}. \tag{39}$$

Integrating $\varphi(x,\mu_0,\mu)$ with respect to μ_0 and μ, we have

$$\varphi(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{i\omega x}\frac{\chi_0(\omega)}{1-\chi(\omega)}. \tag{40}$$

In a system devoid of hole, *i.e.*, $N=0$ or $D=0$, the expression (39) is reduced to

$$\varphi(x,\mu_0,\mu)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega e^{i\omega x}}{\Sigma+i\omega\mu} \cdot\left[\delta(\mu-\mu_0)+\frac{\Sigma_s}{2(\Sigma+i\mu_0\omega)}\left\{1-\frac{\Sigma_s}{2\omega i}\log\frac{\Sigma+\omega i}{\Sigma-\omega i}\right\}^{-1}\right]. \tag{41}$$

This expression agrees with the Green function for the space-angle dependent linear Boltzmann equation applicable to a corresponding physical condition[13].

To evaluate the angular dependent total distribution near the source, we first calculate the once scattered neutron distribution $\varphi_1(x,\mu_0,\mu)$. From Eq. (38),

$$\varphi_1(x,\mu_0,\mu)=\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{i\omega x}\chi_0(\omega;\mu_0)\chi(\omega;\mu). \tag{42}$$

We here assume that μ_0 is positive, and if $\mu>0$, the expression (42) becomes

$$\varphi_1(x,\mu_0,\mu)=\frac{\Sigma_s}{4\pi}\int_{-\infty}^{\infty}\frac{d\omega e^{i\omega x}}{\Sigma+g_+(\omega)\mu_0}\cdot\frac{1}{\Sigma+g_+(\omega)\mu}. \tag{43}$$

The integral of Eq. (43) is evaluated by the residue theorem, and if $x>0$, it is written

$$\varphi_1(x,\mu_0,\mu)=\frac{\Sigma_s}{2\Sigma}\cdot\frac{1}{\mu_0-\mu}\{I(x,\mu_0)-I(x,\mu)\}, \tag{44}$$

where

$$I(x,\mu)=\sum_j \exp(i\omega_j^* x)[1+ND\exp(-i\omega_j^* D)]^{-1}, \tag{45}$$

and where $\omega_j^*(\mu)$ is the j-th root of the transcendental equation $\Sigma/\mu+g_+(\omega)=0$. One of these roots is situated on the imaginary axis, and the other poles are located at symmetrical points referred to the same axis, all the points being in the upper half of the ω-plane. Denoting the pure imaginary root by $i\omega_0^*$, and the coordinates of other poles by

$$\omega_j^*=\pm Re\,\omega_j^*+iI_m\omega_j^*,$$

the expression (43) may be written in the form

$$I(x,\mu)=\frac{\exp(-\omega_0^* x)}{1+DN\exp(\omega_j^* D)}+2\sum_j\exp(-xI_m\omega_j^*)\cdot\frac{\cos(xRe\omega_j^*)+ND\exp(DI_m\omega_j^*)\cos\{(x+D)Re\omega_j^*\}}{1+2ND\cos(DRe\omega_j^*)\exp(I_m\omega_j^*)+(ND)^2\exp(2DI_m\omega_j^*)}. \tag{46}$$

The above expression depends mainly on the magnitude of the imaginary part of the poles $\omega_j^*(j=1,2,\cdots)$. If the gap width D is sufficiently small, the imaginary part of $\omega_j^*(j=1,2,\cdots)$ becomes large enough to let the second term of Eq. (46) become negligible. If $N=0$ or $D=0$, expression (44) is reduced to the familiar form of a system without void:

$$I(x,\mu)=\exp(-\Sigma x/\mu).$$

When $N\to\infty$ or $D\to\infty$, the corresponding roots ω_j^* are definite and from Eq. (44), $I(x,\mu)$ tends to zero.

When $\mu_0>0$ and $\mu>0$, there are no poles in the lower half of the ω-plane, and consequently

$$\varphi_1(x,\mu_0,\mu)=0 \qquad \text{for } x<0. \tag{47}$$

This relation is physically reasonable for once-scattered neutron distribution. For negative values of μ, we obtain the expression

$$\varphi_1(x,\mu_0,\mu)=\frac{\Sigma_s}{4\pi}\int_{-\infty}^{\infty}\frac{d\omega e^{i\omega x}}{\Sigma+g_+(\omega)\mu_0}\cdot\frac{1}{\Sigma+g_-(\omega)\mu}.$$

The roots of $\Sigma+g_-(\omega)\mu=0$ are equal to the roots of $\Sigma+g_+(\omega)\mu=0$, but with the sign of ω inverted. Then we have

$$\varphi_1(x,\mu_0,\mu)=\frac{\Sigma_s}{2\Sigma}\cdot\frac{1}{\mu_0-\mu}\begin{cases}I(x,\mu_0) & \text{for } x>0,\ \mu<0\\ I(x,\mu) & \text{for } x<0,\ \mu<0.\end{cases} \tag{48}$$

We now use the central limit theorem to evaluate the total distribution. In this case, the mean vector $\vec{m}_i$ and the covariance matrix C_i in the formula (4) are reduced to mean m_i and variance σ_i^2 respectively. As isotropic scattering is assumed, the angular dependent distribution depends only on the first and final directions taken by the neutron. For the process of random walk of neutrons the series of displacement densities $f_0(x,\mu)$, $f(x)$, $\cdots$, $f(x)$, $f(x,\mu)$ are the same as in Sec. II-2. The mean value and variance of $f_0(x,\mu)$ are obtained as

$$m_0=(1+ND)\mu_0/\Sigma^2$$

$$\sigma_0^2=\frac{1}{\Sigma^2}\left\{\frac{(1+ND)^2}{\Sigma}+\frac{ND^2}{|\mu_0|}\right\}\mu_0^2.$$

With respect to $f(x)$,

$$m=0$$

$$\sigma^2=\frac{1}{\Sigma^2}\left\{\frac{2}{3}\cdot\frac{(1+ND)^2}{\Sigma}+ND^2\right\},$$

and for those of the last step displacement density $f(x,\mu)$,

$$m(\mu)=(1+ND)\mu/\Sigma^2$$

$$\sigma^2(\mu)=\frac{1}{\Sigma^2}\left\{\frac{(1+ND)^2}{\Sigma}+\frac{ND^2}{|\mu|}\right\}\mu^2.$$

Then the distribution after n steps can be approximated for $n\gg1$ from Eq. (4) in the form

$$\varphi_n(x,\mu_0,\mu)=\frac{1}{\sqrt{2\pi\sigma_n^2}}\exp\left\{-\frac{(x-m_n)^2}{2\sigma_n^2}\right\}\frac{\Sigma_s^n}{2}, \tag{49}$$

where $m_n=(1+ND)(\mu_0+\mu)/\Sigma^2$

$$\sigma_n^2=\frac{1}{\Sigma^2}\left[\frac{(1+ND)^2}{\Sigma}\left\{\mu_0^2+\mu^2+\frac{2}{3}(n-1)\right\}+ND^2\left\{\frac{\mu_0^2}{|\mu_0|}+\frac{\mu^2}{|\mu|}+(n-1)\right\}\right].$$

The curves for total angular distribution computed with the above expression for $n\geq2$ are represented in **Fig. 1**.

For spatial density, the second term of the expression (45) indicates the existence of spatial oscillation, whose periods are proportional to the real part of the complex poles ω_j^*. The space dependency of once-scattering density $\varphi_1(x,\mu_0,\mu)$ is as shown in **Fig. 2**.

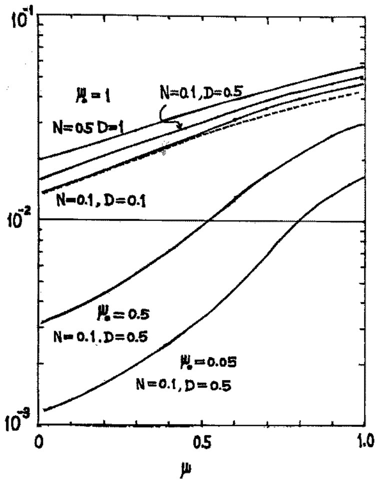

Fig. 1 Angular dependence of total distribution ($\Sigma=1$, $\Sigma_s=0.5$, $x=4$) with N, D and μ_0 as parameters (Broken line taken from Beach, *et al.*[14] for $N=D=0$)

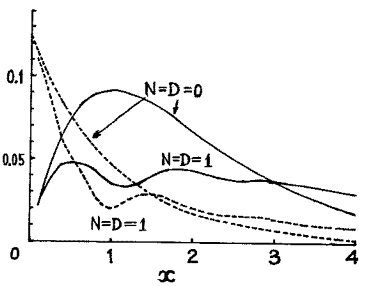

Fig. 2 Plane parallel ($\mu=0.99$) once-scattering density for monodirectional source ($\Sigma=1$, $\Sigma_s=0.5$, $\mu_0=1$; Broken lines are for $\mu=-1$.)

We now drive the asymptotic form of the total space dependent distribution (40) for isotropic source by means of the residue theorem. The poles of the integrand (40) are the roots of the transcendental equation

$$\chi(\omega)=1, \tag{50}$$

and the characteristic function $\chi(\omega)$ has the property

$$\bar{\chi}(\omega)=\chi(-\bar{\omega}). \tag{51}$$

Therefore, if ω is a root of Eq. (50), so is $-\bar{\omega}$. The poles and branch cuts in the upper half of the ω-plane present the form shown in **Fig. 3**. Hence, the integral of (40) can be expressed by

$$\varphi(x)=\frac{1}{2\pi}\Big[R_{es}(\omega_0)+\sum_j\{R_{es}(\omega_j)+R_{es}(-\bar{\omega}_j)\} + \sum_l\int_{C_l}\frac{d\omega e^{i\omega x}\chi_0(\omega)}{1-\chi(\omega)}\Big], \tag{52}$$

where $R_{es}(\omega_j)$ means the residue at the j-th pole ω_j:

$$R_{es}(\omega_j)=2\pi i e^{i\omega_j x}\Big/\Big(-\frac{\partial\chi}{\partial\omega}(\omega_j)\Big). \tag{53}$$

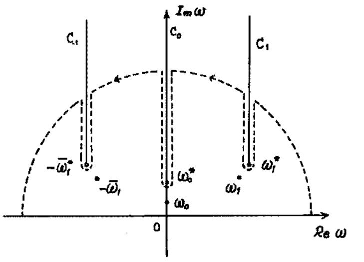

Fig. 3 Inversion contour (broken line) in ω-plane (ω_j, ω_j^* and C_j are poles, branch points and cut lines, respectively.)

The results of numerical calculations on the asymptotic term of Eq. (52) are shown in **Fig. 4**.

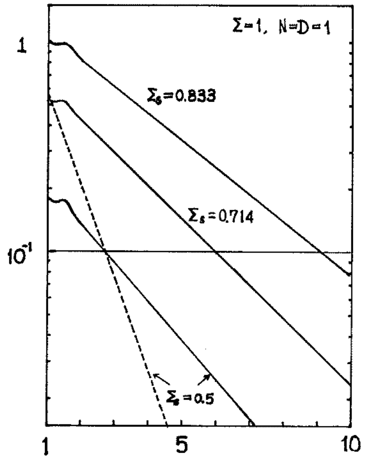

Fig. 4 Asymptotic space dependence of total density $\varphi(x)$ (Broken line taken from Beach, *et al.*[14] for $N=D=0$)

V. Discussion

As shown in Chap. II, if the displacements of neutron random walk at each step are statistically independent, the collision probabilities $\varphi_n(x)$ can be obtained exactly by Fourier inversion of the derived characteristic function. When n is small $(n=0,1)$, $\varphi_n(x)$ is dominant in the transient part of the total distribution, while for $n \gg 1$ —when the central limit theorem can be applied in the evaluation— $\varphi_n(x)$ contributes instead to the asymptotic part.

In the space-time-energy dependent case, the relations between displacements, $\Delta\vec{r}=|\Delta\vec{r}|\vec{\Omega}$ and $\Delta t=|\Delta\vec{r}|/v$ cause the displacements between each elementary event to be statistically interdependent, so that in this case the correlations between displacements must be taken into account in calculating $\varphi_n(x)$.

In the random gap system of Chap.IV, the behavior of the angular dependent probability density curves (Fig. 1) is similar to that of the density of a system devoid of holes, the only difference being in the vertical position of the curve, depending on the width of the gaps and on the numbers in which they occur. The total distribution thus determined has only one term that is exact, which is φ_1 for once-scattered neutron density; subsequent terms are approximated by Eq. (49), where the number of φ_n terms has been increased until the ratio $\varphi_{m+1}/\sum_1^m \varphi_n$ dropped below 10^{-3} (needing about 10 terms). In Fig. 2 one will see how after one collision, the presence of the random gaps carries the neutron further from the source as compared with the case of no gap, and how the spatial oscillations increase their prominence with the progression of N and D. When scattering is in the forward direction, the gap width D being unity, neutrons having traversed n gaps will not be found in the region $x<n$, but will all have reached $n \leqq x \leq \infty$, so that the neutrons in the interval $n \leq x \leq n+1$, will have passed at most only n gaps. Thus the density $\varphi_1(x,1,0.99)$ of once-scattered neutrons in the forward direction represents in the interval $0 \leq x \leq 1$ the probability of a neutron not encountering any gap. The probability of encountering a gap would therefore increase along with the free path length, while $\varphi_1(x,1,1)$ would decrease more rapidly as compared to the case of $D=0$. In the interval $1 \leq x \leq 2$, the curve of $\varphi_1(x,1,1)$ represents the probability of passing at most one gap, and in the interval $2 \leq x \leq 3$ at most two gaps and so on. Thus, $\varphi_1(x,1,1)$ with unit mean free path $(\Sigma=1)$ decreases exponentially with x.

In the backward direction, the behavior of the density $\varphi_1(x,1,-1)$ is explained as follows: From Eq. (48), $\varphi_1(x,1,\mu)$ for $x>0, \mu<0$ is inversely proportional to μ, hence $\varphi_1(x,1,-0)=2\varphi_1(x,1,-1)$, indicating that the upward (downward) direction density $\varphi_1(x,1,-0)$ behaves in the same manner as the backward direction density $\varphi_1(x,1,-1)$. For upward (downward) direction, the projection of the second step displacement onto the x coordinate is zero, so that $\varphi_1(x,1,-0)$ will be equivalent to the probability of progression of one step further from the forward direction neutron source with respect to space dependence. Thus the behavior of $\varphi_1(x,1,-0)$ can be explained by analogy with the preceding case.

In the space dependent total distribution represented by Eq. (40), the poles of the integrand correspond to the inverse diffusion length in the eigen-value problem. The first fundamental mode is real, and higher modes become complex, which causes spatial oscillation of the total distribution $\varphi(x)$. One may consider this spatial oscillation of $\varphi(x)$ to mean physically that random gaps affect the diffusion of neutrons just as if the gaps were at regular intervals. When $N=1$ and $D=1$, we have obtained the values $\omega_0=0.426i$ and $\omega_1=\pm 4.80+1.57i$ by the method of conformal mapping. The asymptotic curves obtained with these values are presented in Fig. 4, along with the case of no gaps. It shows how the presence of gaps distinctly affects the fundamental modes (the slopes of lines), as compared to the case of gap-free system. The figure also reveals that the oscillation term provides little influence at ample distances from sources.

The machine time on IBM 360 H 50 was a few seconds per curve.

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