

Calculi for Intuitionistic Normal Modal Logic

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Abstract

This paper provides a call-by-name and a call-by-value term calculus, both of which have a Curry-Howard correspondence to the box fragment of the intuitionistic modal logic **IK**. The strong normalizability and the confluence of the calculi are shown. Moreover, we define a CPS transformation from the call-by-value calculus to the call-by-name calculus, and show its soundness and completeness.

1 Introduction

It is well-known that the intuitionistic propositional logic exactly corresponds to the simply typed λ -calculus: formulae as types and proofs as terms. Such a correspondence is called a Curry-Howard correspondence after [12]. A Curry-Howard correspondence enables us to study an equality on proofs of a logic computationally. Though Curry-Howard correspondences for higher-order and predicate logics were provided in [3], we investigate only propositional logics in this paper. The aim of this study is to give a proper calculus that have a Curry-Howard correspondence with a modal logic.

Modal logics have a long history and are now widely studied both theoretically and practically. Especially, studies about Kripke semantics [14] of modal logics are quite active. Curry-Howard correspondences of modal logics are, however, less studied except for linear logics [9]. (In fact, exponentials of linear logics are a kind of **S4** modality.) Since **K** is known to be the simplest modal logic, first we focus the intuitionistic modal logic **IK**. A difficulty of a calculus for **K** is lack of acknowledged models. Because a model of the modality in call-by-name **S4** is acknowledged as a monoidal comonad, a model of the modality in **K** should be a generalization of a monoidal comonad. This paper defines a call-by-name calculus, which is called the $\lambda\Box$ -calculus, based on a categorical model proposed by Bellin et al. in [4]. Another difficulty is a problem about natural deductions of modal logics pointed out in [24]. A solution of the problem in **IS4** is found in [2], but it cannot be applied to **IK**. The formulation of [4] and this paper is a natural deduction style, and solves this problem.

On the other hand, studies on Curry-Howard correspondences for modal logics, especially **IS4**, are applied to staged computations and information flow analysis (*e.g.*, [6], [17]) in the field of programming languages. Since our $\lambda\Box$ -calculus can

$$\begin{array}{c}
\frac{}{\Gamma \vdash c^\tau : \tau} \quad \frac{}{\Gamma, x : \tau, \Gamma' \vdash x : \tau} \\
\frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x^\sigma. M : \sigma \supset \tau} \quad \frac{\Gamma \vdash M : \sigma \supset \tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau} \\
\frac{x_1 : \sigma_1, \dots, x_n : \sigma_n \vdash M : \tau \quad \Gamma \vdash N_1 : \square \sigma_1 \quad \dots \quad \Gamma \vdash N_n : \square \sigma_n}{\Gamma \vdash \mathbf{box} \langle x_1^{\sigma_1}, \dots, x_n^{\sigma_n} \rangle \mathbf{be} \langle N_1, \dots, N_n \rangle \mathbf{in} M : \square \tau}
\end{array}$$

Figure 1: Typing rules of $\lambda\square$ -calculus

be extended easily to **IT**, **IK4**, **IS4**, and so on, this work is expected to contribute such programming language matters.

This paper provides not only a call-by-name but also a call-by-value calculus. A call-by-value calculus is usually defined by a CPS transform, which is originally introduced by [8] and [23]; for example, a call-by-value control operator is defined by a CPS transform in [7]. In [25], Sabry and Felleisen showed that the λ_c -calculus [19] is sound and complete for CPS semantics. We give the call-by-value $\lambda\square$ -calculus as an extension of the λ_c -calculus. Moreover, we define a CPS transformation from the call-by-value $\lambda\square$ -calculus to the call-by-name $\lambda\square$ -calculus. The soundness and completeness for the CPS semantics are shown along the line of [25].

2 Call-by-Name Calculus

First, we remark special notations used in this paper. We use a notation “ \vec{M} ” for a sequence of meta-variables “ M_1, \dots, M_n ” including the empty sequence. Hence, an expression “ \vec{M}, \vec{N} ” stands for the concatenation of \vec{M} and \vec{N} . For a unary operator $\Phi(-)$, we write “ $\Phi(\vec{M})$ ” for the sequence “ $\Phi(M_1), \dots, \Phi(M_n)$ ”. We use also “ $\vec{N}(\lambda \vec{x}. M)$ ” as an abbreviation for “ $N_1(\lambda x_1. \dots N_n(\lambda x_n. M) \dots)$ ”.

A hole in a context is represented by “ $-$ ” in this paper. For a context C , “ $C[M]$ ” denotes the result of filling holes in C with M as usual.

Definition 1. Types σ and terms M of the call-by-name $\lambda\square$ -calculus are defined as follows:

$$\begin{array}{l}
\sigma ::= p \mid \sigma \supset \sigma \mid \square \sigma, \\
M ::= c \mid x \mid \lambda x^\sigma. M \mid MM \mid \mathbf{box} \langle x^\sigma, \dots, x^\sigma \rangle \mathbf{be} \langle M, \dots, M \rangle \mathbf{in} M,
\end{array}$$

where p , c , and x range over type constants, constants, and variables, respectively. Free variables of $\mathbf{box} \langle \vec{x} \rangle \mathbf{be} \langle \vec{N} \rangle \mathbf{in} M$ are free variables of \vec{N} . The typing rules are given in Figure 1. The reduction rules are given in Figure 2. Define \mathbf{n} as the set $\{\beta_\supset, \eta_\supset, \text{id}_\square, \beta_\square\}$.

We remark that all free variables of M are included by $\{\vec{x}\}$ if $\mathbf{box} \langle \vec{x} \rangle \mathbf{be} \langle \vec{N} \rangle \mathbf{in} M$ is typable.

The $\lambda\square$ -calculus has essentially the same syntax as [4]. Hence, one can see that our calculus corresponds to the intuitionistic modal logic. The calculus can be

$$\begin{array}{l}
(\lambda x. M)N \longrightarrow_{\beta_{\supset}} M[N/x] \\
\lambda x. Mx \longrightarrow_{\eta_{\supset}} M \qquad x \notin \text{FV}(M) \\
\mathbf{box} \langle x \rangle \mathbf{be} \langle M \rangle \mathbf{in} x \longrightarrow_{\text{id}_{\square}} M \\
\mathbf{box} \langle \vec{w}, x, \vec{z} \rangle \mathbf{be} \langle \vec{P}, \mathbf{box} \langle \vec{y} \rangle \mathbf{be} \langle \vec{L} \rangle \mathbf{in} N, \vec{Q} \rangle \mathbf{in} M \\
\longrightarrow_{\beta_{\square}} \mathbf{box} \langle \vec{w}, \vec{y}, \vec{z} \rangle \mathbf{be} \langle \vec{P}, \vec{L}, \vec{Q} \rangle \mathbf{in} M[N/x] \qquad |\vec{w}| = |\vec{P}|
\end{array}$$

Figure 2: Call-by-name reductions of $\lambda\square$ -calculus

regarded as a natural deduction by forgetting terms. Our logic is equivalent to the $\supset\square$ -fragment of the usual intuitionistic modal logic \mathbf{IK} with respect to provability. Let \mathbf{IK} be an intuitionistic Hilbert system with the axiom $\square(\sigma \supset \tau) \supset \square\sigma \supset \square\tau$ and the box inference rule. The axiom is validated in our calculus as the term

$$\vdash \lambda f. \lambda x. \mathbf{box} \langle f', x' \rangle \mathbf{be} \langle f, x \rangle \mathbf{in} f'x' : \square(\sigma \supset \tau) \supset \square\sigma \supset \square\tau.$$

The box rule is simulated as

$$\frac{\vdash M : \tau}{\vdash \mathbf{box} \langle \rangle \mathbf{be} \langle \rangle \mathbf{in} M : \square\tau}.$$

Conversely, the typing rule of the $\lambda\square$ -calculus is simulated by \mathbf{IK} :

$$\frac{\frac{\frac{\frac{\sigma_1, \dots, \sigma_n \vdash \tau}{\vdash \sigma_1 \supset \dots \supset \sigma_n \supset \tau}}{\vdash \square(\sigma_1 \supset \dots \supset \sigma_n \supset \tau)}}{\Gamma \vdash \square(\sigma_1 \supset \dots \supset \sigma_n \supset \tau)}}{\Gamma \vdash \square\sigma_1 \supset \square(\sigma_2 \supset \dots \supset \sigma_n \supset \tau)} \quad \Gamma \vdash \square\sigma_1}{\Gamma \vdash \square(\sigma_2 \supset \dots \supset \sigma_n \supset \tau)} \\
\vdots \\
\frac{\Gamma \vdash \square(\sigma_n \supset \tau)}{\Gamma \vdash \square\sigma_n \supset \square\tau} \quad \Gamma \vdash \square\sigma_n}{\Gamma \vdash \square\tau}.$$

According to this encoding, it is not trivial whether an exchange rule commutes with a box operation. Hence, we distinguish two terms, $\mathbf{box} \langle x, y \rangle \mathbf{be} \langle N, L \rangle \mathbf{in} M$ and $\mathbf{box} \langle y, x \rangle \mathbf{be} \langle L, N \rangle \mathbf{in} M$, in the $\lambda\square$ -calculus, although it is common to consider proofs up to exchanges. Commutativity with exchanges requires another axiom, *symmetry*, given in Section 6.

Remark 1. The typing rules of our calculus are the same as those of Bellin et al.'s [4], but reductions are essentially different. In [4], they address natural deduction style formulation and categorical semantics, but not a term calculus itself, so their calculus has room for improvement. Differences between Bellin et al.'s calculus and our $\lambda\square$ -calculus are the following.

- The first reduction of their calculus corresponds to a special case of $\longrightarrow_{\beta_{\square}}$.
- The direction of the first reduction is opposite to $\longrightarrow_{\beta_{\square}}$: our reduction merges adjacent two boxes into one box, while their reduction splits a box into two boxes.
- The second reduction of their calculus cannot be applied to any typable term.
- Their calculus does not have a reduction corresponding to $\longrightarrow_{\text{id}_{\square}}$.

Though Bellin et al.'s calculus does not have the semantic completeness, it has syntax for a diamond property. Intuitionistic characterization of a diamond property is not obvious, but the author has observed a diamond property in the classical modal logic \mathbf{K} in [13].

For a set of labels X , we write \longrightarrow_X as a reduction whose label is a member of X , and $=_X$ as the reflexive transitive symmetric closure of \longrightarrow_X . We also use \equiv for the α -equivalence.

We can easily check the subject reduction theorem for this calculus.

Proposition 1. *If $\Gamma \vdash M : \tau$ and $M \longrightarrow_{\mathbf{n}} N$ hold, then $\Gamma \vdash N : \tau$ holds.*

Other important properties, the strong normalizability and the confluency, also hold.

Proposition 2. *The call-by-name λ_{\square} -calculus is strongly normalizable with respect to $\longrightarrow_{\mathbf{n}}$.*

Proof. Define the transformation $[-]$ to the simply typed λ -calculus by

$$[\mathbf{box} \langle \vec{x} \rangle \mathbf{be} \langle \vec{N} \rangle \mathbf{in} M] = \lambda k. [\vec{N}](\lambda \vec{x}. k[M]).$$

Then, $M : \tau$ implies $[M] : [\tau]$ if we define the type transformation $[-]$ by $[\square\tau] = ([\tau] \supset p) \supset p$. One can see that $[\mathbf{box} \langle x \rangle \mathbf{be} \langle M \rangle \mathbf{in} x] \longrightarrow_{\eta_{\supset}}^+ [M]$ and

$$\begin{aligned} & [\mathbf{box} \langle \vec{w}, x, \vec{z} \rangle \mathbf{be} \langle \vec{P}, \mathbf{box} \langle \vec{y} \rangle \mathbf{be} \langle \vec{L} \rangle \mathbf{in} N, \vec{Q} \rangle \mathbf{in} M] \\ & \equiv \lambda k. [\vec{P}](\lambda \vec{w}. (\lambda h. [\vec{L}](\lambda \vec{y}. h[N]))(\lambda x. [\vec{Q}](\lambda \vec{z}. k[M]))) \\ & \longrightarrow_{\beta_{\supset}} \lambda k. [\vec{P}](\lambda \vec{w}. [\vec{L}](\lambda \vec{y}. (\lambda x. [\vec{Q}](\lambda \vec{z}. k[M]))[N])) \\ & \longrightarrow_{\beta_{\supset}} \lambda k. [\vec{P}](\lambda \vec{w}. [\vec{L}](\lambda \vec{y}. [\vec{Q}](\lambda \vec{z}. k[M][[N]/x]))) \\ & \equiv \lambda k. [\vec{P}](\lambda \vec{w}. [\vec{L}](\lambda \vec{y}. [\vec{Q}](\lambda \vec{z}. k[M[N/x]]))) \\ & \equiv [\mathbf{box} \langle \vec{w}, \vec{y}, \vec{z} \rangle \mathbf{be} \langle \vec{P}, \vec{L}, \vec{Q} \rangle \mathbf{in} M[N/x]] \end{aligned}$$

hold. Because the simply typed λ -calculus is SN w.r.t. $\longrightarrow_{\beta_{\supset}, \eta_{\supset}}$ (e.g., q.v. [10]), the call-by-name λ_{\square} -calculus is SN. \square

We note here that the strong normalization theorem was proved via a different calculus in [1].

Proposition 3. $\longrightarrow_{\mathbf{n}}$ is confluent.

Proof. By Newman's lemma [20], it is sufficient to check the local confluency. The call-by-name λ_{\square} -calculus has essentially four kinds of critical pairs other

than pairs of the λ -calculus:

$$\begin{array}{l}
\longmapsto_{\text{id}_\square} \text{box } \langle \vec{y} \rangle \text{ be } \langle \vec{L} \rangle \text{ in } N \\
\text{box } \langle x \rangle \text{ be } \langle \text{box } \langle \vec{y} \rangle \text{ be } \langle \vec{L} \rangle \text{ in } N \rangle \text{ in } x \\
\longmapsto_{\beta_\square} \text{box } \langle \vec{y} \rangle \text{ be } \langle \vec{L} \rangle \text{ in } N, \\
\longmapsto_{\text{id}_\square} \text{box } \langle x \rangle \text{ be } \langle N \rangle \text{ in } M \\
\text{box } \langle x \rangle \text{ be } \langle \text{box } \langle y \rangle \text{ be } \langle N \rangle \text{ in } y \rangle \text{ in } M \\
\longmapsto_{\beta_\square} \text{box } \langle y \rangle \text{ be } \langle N \rangle \text{ in } M[y/x], \\
\longmapsto_{\beta_\square} \text{box } \langle y \rangle \text{ be } \langle \text{box } \langle \vec{z} \rangle \text{ be } \langle \vec{P} \rangle \text{ in } L \rangle \text{ in } M[N/x] \\
\text{box } \langle x \rangle \text{ be } \langle \text{box } \langle y \rangle \text{ be } \langle \text{box } \langle \vec{z} \rangle \text{ be } \langle \vec{P} \rangle \text{ in } L \rangle \text{ in } N \rangle \text{ in } M \\
\longmapsto_{\beta_\square} \text{box } \langle x \rangle \text{ be } \langle \text{box } \langle \vec{z} \rangle \text{ be } \langle \vec{P} \rangle \text{ in } N[L/y] \rangle \text{ in } M, \\
\longmapsto_{\beta_\square} \text{box } \langle \vec{y}, x' \rangle \text{ be } \langle \vec{L}, \text{box } \langle \vec{y}' \rangle \text{ be } \langle \vec{L}' \rangle \text{ in } N' \rangle \text{ in } M[N/x] \\
\text{box } \langle x, x' \rangle \text{ be } \langle \text{box } \langle \vec{y} \rangle \text{ be } \langle \vec{L} \rangle \text{ in } N, \text{box } \langle \vec{y}' \rangle \text{ be } \langle \vec{L}' \rangle \text{ in } N' \rangle \text{ in } M \\
\longmapsto_{\beta_\square} \text{box } \langle x, \vec{y}' \rangle \text{ be } \langle \text{box } \langle \vec{y} \rangle \text{ be } \langle \vec{L} \rangle \text{ in } N, \vec{L}' \rangle \text{ in } M[N'/x'].
\end{array}$$

It is easily shown that all the pairs are joinable. \square

Last, we mention the subformula property of this calculus.

Theorem 4. *A normal form in the call-by-name $\lambda\square$ -calculus has the subformula property.*

Proof. By induction on construction of terms. If $\text{box } \langle \vec{x} \rangle \text{ be } \langle \vec{N} \rangle \text{ in } M$ is a normal form, then M is a normal form and each N_i has a form $yL_1 \cdots L_m$. Therefore, the subformula property holds in this case by the induction hypothesis. Other cases are just the same as the simply typed λ -calculus. \square

A characterization of the $\lambda\square$ -calculus by a standard translation into the predicate logic is given by Abe in [1]. Since our motivation arises from logics and categorical semantics, computational meaning of the calculus still remains to be studied. We believe the following discussions are helpful.

Because the logic **IK** is weaker than the logic **IS4**, the $\lambda\square$ -calculus is expected to be a subcalculus of a calculus for **IS4**. A method for extending the $\lambda\square$ -calculus to **IS4** is discussed in Section 6. Through this approach, computational analyses of **IS4** calculi might be applied to the $\lambda\square$ -calculus.

Another approach to understand computational meaning of the $\lambda\square$ -calculus is to investigate a relation to monads. It is remarkable that the transformation $[-]$ mentioned in the proof of Proposition 2 preserves the equality. It means that \square in the $\lambda\square$ -calculus can be interpreted as a continuation monad in the λ -calculus. In fact, such a transformation exists for any strong monad because a strong monad is a lax monoidal endofunctor. Hence, we can conclude that the $\lambda\square$ -calculus includes an abstract setting of strong monads. In [16], McBride and Paterson have studied a structure abstracting a strong monad. It must be strongly related to our calculus though their formulation has a tensorial strength with respect to cartesian products.

$$\begin{array}{l}
V, W : \text{value} \\
C : \text{simple evaluation context} \\
E : \text{evaluation context} \\
(\lambda x. x)M \longrightarrow_{\text{id}_{\supset}} M \\
(\lambda x. M)V \longrightarrow_{\beta_{\supset}^{\vee}} M[V/x] \\
\lambda x. Vx \longrightarrow_{\eta_{\supset}^{\vee}} V \quad x \notin \text{FV}(V) \\
C[(\lambda x. M)N] \longrightarrow_{\text{lift}} (\lambda x. C[M])N \\
C[yM] \longrightarrow_{\text{flat}} (\lambda x. C[x])(yM) \quad C \neq V- \\
(\lambda x. E[yx])M \longrightarrow_{\beta_{\Omega}} E[yM] \quad x \notin \text{FV}(E[y]) \\
\mathbf{box} \langle x \rangle \mathbf{be} \langle M \rangle \mathbf{in} x \longrightarrow_{\text{id}_{\square}} M \\
\mathbf{box} \langle \vec{w}, x, \vec{z} \rangle \mathbf{be} \langle \vec{W} \rangle, \mathbf{box} \langle \vec{y} \rangle \mathbf{be} \langle \vec{N} \rangle \mathbf{in} V, \vec{P} \rangle \mathbf{in} M \\
\longrightarrow_{\beta_{\square}^{\vee}} \mathbf{box} \langle \vec{w}, \vec{y}, \vec{z} \rangle \mathbf{be} \langle \vec{W}, \vec{N}, \vec{P} \rangle \mathbf{in} M[V/x] \quad |\vec{w}| = |\vec{W}|
\end{array}$$

Figure 3: Call-by-value reductions of $\lambda\square$ -calculus

3 Call-by-Value Calculus

Definition 2. Types σ , terms M , values V , simple evaluation contexts C , and evaluation contexts E of the call-by-value $\lambda\square$ -calculus are defined as follows:

$$\begin{array}{l}
\sigma ::= p \mid \sigma \supset \sigma \mid \square\sigma, \\
M ::= c \mid x \mid \lambda x^{\sigma}. M \mid MM \mid \mathbf{box} \langle x^{\sigma}, \dots, x^{\sigma} \rangle \mathbf{be} \langle M, \dots, M \rangle \mathbf{in} M, \\
V ::= c \mid x \mid \lambda x^{\sigma}. M \mid \mathbf{box} \langle x^{\sigma}, \dots, x^{\sigma} \rangle \mathbf{be} \langle V, \dots, V \rangle \mathbf{in} M, \\
C ::= -M \mid V- \mid \mathbf{box} \langle x^{\sigma}, \dots, x^{\sigma} \rangle \mathbf{be} \langle V, \dots, V, -, M, \dots, M \rangle \mathbf{in} M, \\
E ::= - \mid C[E].
\end{array}$$

The typing rules are just the same as the call-by-name. The reduction rules are given in Figure 3. Define \mathbf{v} as the set $\{\text{id}_{\supset}, \beta_{\supset}^{\vee}, \eta_{\supset}^{\vee}, \text{lift}, \text{flat}, \beta_{\Omega}, \text{id}_{\square}, \beta_{\square}^{\vee}\}$.

Proposition 5. *If $\Gamma \vdash M : \tau$ and $M \longrightarrow_{\mathbf{v}} N$ hold, then $\Gamma \vdash N : \tau$ holds.*

Since the definition of terms and the typing rules are the same as those of the call-by-name calculus, also the call-by-value $\lambda\square$ -calculus corresponds to **IK**. In order to define CPS semantics, however, we restrict terms as follows:

$$M ::= c \mid x \mid \lambda x^{\sigma}. M \mid MM \mid \mathbf{box} \langle x^{\sigma}, \dots, x^{\sigma} \rangle \mathbf{be} \langle M, \dots, M \rangle \mathbf{in} V.$$

These terms are closed under call-by-value reductions because values are closed under substitutions. Hence, we can say that the full call-by-value calculus is a conservative extension of the restricted version. In the rest of this section (and the first half of the next section), we focus on this restricted calculus.

Our call-by-value $\lambda\square$ -calculus is an extension of Sabry and Felleisen's calculus in [25]. As mentioned in [25], it is equivalent to the λ_c -calculus [19], which is ac-

$$\begin{array}{c}
\bar{p} = p \\
\overline{\sigma \supset \tau} = (\bar{\tau} \supset \mathbf{R}) \supset \bar{\sigma} \supset \mathbf{R} \\
\Box \bar{\sigma} = \Box \bar{\sigma} \\
\bar{x} = x \\
\bar{c} = c \\
\overline{\lambda x. M} = \lambda k. \lambda x. \llbracket M \rrbracket k \\
\hline
\overline{\mathbf{box} \langle \bar{x} \rangle \mathbf{be} \langle \bar{U} \rangle \mathbf{in} V} = \mathbf{box} \langle \bar{x} \rangle \mathbf{be} \langle \bar{U} \rangle \mathbf{in} \bar{V} \\
\llbracket x \rrbracket = \lambda k. k\bar{x} \\
\llbracket c \rrbracket = \lambda k. k\bar{c} \\
\llbracket \lambda x. M \rrbracket = \lambda k. k(\overline{\lambda x. M}) \\
\llbracket MN \rrbracket = \lambda k. \llbracket M \rrbracket (\lambda y. \llbracket N \rrbracket (yk)) \\
\llbracket \mathbf{box} \langle \bar{x} \rangle \mathbf{be} \langle \bar{M} \rangle \mathbf{in} V \rrbracket = \lambda k. \llbracket \bar{M} \rrbracket (\lambda \bar{y}. k(\overline{\mathbf{box} \langle \bar{x} \rangle \mathbf{be} \langle \bar{y} \rangle \mathbf{in} V}))
\end{array}$$

Figure 4: CPS transformation with \Box

known as a call-by-value language, with respect to equalities. We give CPS semantics of the call-by-value $\lambda\Box$ -calculus and show the soundness and completeness along the line of [25].

Definition 3. The CPS transformation $\llbracket - \rrbracket$ from the call-by-value $\lambda\Box$ -calculus to the call-by-name $\lambda\Box$ -calculus is defined by Fig 4. We write $\Phi(M, K)$ for the administrative normal form of $\llbracket M \rrbracket K$.

Proposition 6. *If $x_1 : \sigma_1, \dots, x_n : \sigma_n \vdash M : \tau$ holds, $x_1 : \bar{\sigma}_1, \dots, x_n : \bar{\sigma}_n \vdash \llbracket M \rrbracket : (\bar{\tau} \supset \mathbf{R}) \supset \mathbf{R}$ holds.*

Definition 4. The CPS language is defined as a subcalculus of the call-by-name $\lambda\Box$ -calculus:

$$\begin{array}{l}
V ::= c \mid x \mid \lambda k. K \mid \mathbf{box} \langle x, \dots, x \rangle \mathbf{be} \langle V, \dots, V \rangle \mathbf{in} V, \\
K ::= k \mid \lambda x. A \mid VK, \\
A ::= KV \mid (\lambda k. A)K.
\end{array}$$

The transformation $\Psi(-)$ from the CPS language to the call-by-value $\lambda\Box$ -calculus is defined by Fig 5.

Proposition 7. *The CPS language is closed under $\longrightarrow_{\mathbf{n}}$.*

The following lemma is the core of the soundness and completeness. An outline of the proof is just the same as [25]'s.

Lemma 8. *1. $M \longrightarrow_{\text{lift, flat}} N$ implies $\Phi(M, k) \equiv \Phi(N, k)$.
2. $M \longrightarrow_{\text{id}_{\supset}, \beta_{\supset}^{\vee}, \beta_{\supset}^{\vee}, \beta_{\Omega}, \text{id}_{\Box}, \beta_{\Box}^{\vee}} N$ implies $\Phi(M, k) \longrightarrow_{\mathbf{n}}^+ \Phi(N, k)$.
3. $M \longrightarrow_{\mathbf{n}} N$ implies $\Psi(M) \longrightarrow_{\vee}^* \Psi(N)$.*

$$\begin{aligned}
\Psi(c) &= c \\
\Psi(x) &= x \\
\Psi(\lambda k. k) &= \lambda x. x \\
\Psi(\lambda k. \lambda x. A) &= \lambda x. \Psi(A) \\
\Psi(\lambda k. VK) &= \lambda x. \Psi(VKx) \\
\Psi(\text{box } \langle \vec{x} \rangle \text{ be } \langle \vec{U} \rangle \text{ in } V) &= \text{box } \langle \vec{x} \rangle \text{ be } \langle \Psi(\vec{U}) \rangle \text{ in } \Psi(V) \\
\Psi(k) &= - \\
\Psi(\lambda x. A) &= (\lambda x. \Psi(A)) - \\
\Psi(cK) &= \Psi(K)[c-] \\
\Psi(xK) &= \Psi(K)[x-] \\
\Psi((\lambda k. H)K) &= \Psi(H[K/k]) \\
\Psi(KV) &= \Psi(K)[\Psi(V)] \\
\Psi((\lambda k. A)K) &= \Psi(A[K/k])
\end{aligned}$$

Figure 5: Inverse of CPS transformation

4. $M \xrightarrow{\text{lift, flat}^*} \Psi(\Phi(M, k))$.

Theorem 9. For $\lambda\Box$ -terms M and N , $M =_{\text{v}} N$ holds if and only if $\llbracket M \rrbracket =_{\text{n}} \llbracket N \rrbracket$ holds.

The lemma helps us to prove the strongly normalizing property and the confluency of the call-by-value $\lambda\Box$ -calculus too.

Proposition 10. The call-by-value $\lambda\Box$ -calculus is strongly normalizable with respect to $\longrightarrow_{\text{v}}$.

Proof. There is no infinite sequence of $\longrightarrow_{\text{lift}}$ and $\longrightarrow_{\text{flat}}$. Therefore, if there is an infinite reduction sequence in the call-by-value $\lambda\Box$ -calculus, there is an infinite reduction sequence in the call-by-name calculus via $\Phi(-, k)$. \square

Proposition 11. $\longrightarrow_{\text{v}}$ is confluent.

Proof. Although the confluency can be shown directly, we prove it using the lemma and the confluency of the call-by-name $\lambda\Box$ -calculus. Assume $M \xrightarrow{\text{v}^*} N_1$ and $M \xrightarrow{\text{v}^*} N_2$. Since $\Phi(M, k) \xrightarrow{\text{n}^*} \Phi(N_j, k)$, there is a term L such that $\Phi(N_j, k) \xrightarrow{\text{n}^*} L$. $\Psi(L)$ is an evidence of confluence. \square

4 Other Formulations of Call-by-Value

Although it has been shown that the call-by-value $\lambda\Box$ -calculus has expected properties, we can propose another call-by-value axiomatization following [18].

$$\begin{array}{l}
V, W : \text{value} \\
A : \text{non-value} \\
C : \text{simple evaluation context} \\
\text{let } x \text{ be } M \text{ in } x \longrightarrow_{\text{id}_{\text{let}}} M \\
\text{let } x \text{ be } V \text{ in } M \longrightarrow_{\beta_{\text{let}}^v} M[V/x] \\
(\lambda x. M)V \longrightarrow_{\beta_{\lambda}^v} M[V/x] \\
\lambda x. Vx \longrightarrow_{\eta_{\lambda}^v} V \quad x \notin \text{FV}(V) \\
\text{let } x \text{ be } (\text{let } y \text{ be } L \text{ in } N) \text{ in } M \\
\quad \longrightarrow_{\text{comp}} \text{let } y \text{ be } L \text{ in let } x \text{ be } N \text{ in } M \quad y \notin \text{FV}(M) \\
C[A] \longrightarrow_{\text{let}} \text{let } x \text{ be } A \text{ in } C[x] \\
\text{box } \langle x \rangle \text{ be } \langle M \rangle \text{ in } x \longrightarrow_{\text{id}_{\square}} M \\
\text{box } \langle \vec{w}, x, \vec{z} \rangle \text{ be } \langle \vec{W}, \text{box } \langle \vec{y} \rangle \text{ be } \langle \vec{N} \rangle \text{ in } V, \vec{P} \rangle \text{ in } M \\
\quad \longrightarrow_{\beta_{\square}^v} \text{box } \langle \vec{w}, \vec{y}, \vec{z} \rangle \text{ be } \langle \vec{W}, \vec{N}, \vec{P} \rangle \text{ in } M[V/x] \quad |\vec{w}| = |\vec{W}|
\end{array}$$

Figure 6: Computational reductions of λ_{\square} -calculus

Definition 5. Define the computational λ_{\square} -calculus by adding the new syntax $\text{let } x \text{ be } N \text{ in } M$ to the syntax of the call-by-value λ_{\square} -calculus. The reduction rules of the computational λ_{\square} -calculus are given in Figure 6. Define \mathbf{c} as the set $\{\text{id}_{\text{let}}, \beta_{\text{let}}^v, \beta_{\lambda}^v, \eta_{\lambda}^v, \text{comp}, \text{let}, \text{id}_{\square}, \beta_{\square}^v\}$.

Proposition 12. *If $\Gamma \vdash M : \tau$ and $M \longrightarrow_{\mathbf{c}} N$ hold, then $\Gamma \vdash N : \tau$ holds.*

It is easily seen that the computational λ_{\square} -calculus is equivalent to the previous call-by-value λ_{\square} -calculus with respect to equalities.

Proposition 13. *For λ_{\square} -terms M and N , $M =_v N$ holds if and only if $M =_{\mathbf{c}} N$ holds.*

We can show the strong normalization theorem of the computational λ_{\square} -calculus via the strong normalizability of the $\lambda_{\mathbf{c}}$ -calculus.

Proposition 14. *The computational λ_{\square} -calculus is strongly normalizable with respect to $\longrightarrow_{\mathbf{c}}$.*

Proof. Define $\llbracket - \rrbracket$ into the typed $\lambda_{\mathbf{c}}$ -calculus by

$$\begin{aligned}
\llbracket \text{box } \langle \vec{x} \rangle \text{ be } \langle \vec{V} \rangle \text{ in } M \rrbracket &= \llbracket M \rrbracket [\llbracket \vec{V} \rrbracket / \vec{x}], \\
\llbracket \text{box } \langle \vec{w}, x, \vec{z} \rangle \text{ be } \langle \vec{V}, A, \vec{N} \rangle \text{ in } M \rrbracket \\
&= \text{let } y \text{ be } \llbracket A \rrbracket \text{ in } \llbracket \text{box } \langle \vec{w}, x, \vec{z} \rangle \text{ be } \langle \vec{V}, y, \vec{N} \rangle \text{ in } M \rrbracket.
\end{aligned}$$

One can see that $\llbracket V \rrbracket$ is a value when V is a value, remembering that boxed terms are restricted to the form $\text{box } \langle \vec{x} \rangle \text{ be } \langle \vec{M} \rangle \text{ in } V$. Let $\longrightarrow_{\text{let}_{\square}}$ be the special case

of $\longrightarrow_{\text{let}}$:

$$\begin{aligned} & \text{box } \langle \vec{w}, x, \vec{z} \rangle \text{ be } \langle \vec{W}, A, \vec{P} \rangle \text{ in } M \\ & \longrightarrow_{\text{let}_{\square}} \text{let } y \text{ be } A \text{ in box } \langle \vec{w}, x, \vec{z} \rangle \text{ be } \langle \vec{W}, y, \vec{P} \rangle \text{ in } M \quad |\vec{w}| = |\vec{W}|. \end{aligned}$$

It can be checked that $M \longrightarrow_{\text{id}_{\square}, \beta_{\square}^{\vee}, \text{let}_{\square}} N$ implies $[M] \longrightarrow_{\text{c}}^* [N]$, otherwise, $M \longrightarrow_{\text{c}} N$ implies $[M] \longrightarrow_{\text{c}} [N]$. Because we know the λ_{c} -calculus is SN (it was proved by Hasegawa in [11]), it is sufficient to show there is no infinite sequence that consists of $\longrightarrow_{\text{id}_{\square}}$, $\longrightarrow_{\beta_{\square}^{\vee}}$, and $\longrightarrow_{\text{let}_{\square}}$.

We extend the transformation $[-]$, which is defined in the proof of Proposition 2, to the computational λ_{\square} -calculus by

$$[\text{let } x \text{ be } N \text{ in } M] = [M][[N]/x].$$

Then, $M \longrightarrow_{\text{id}_{\square}, \beta_{\square}^{\vee}} N$ implies $[M] \longrightarrow_{\beta_{\square}, \eta_{\square}}^+ [N]$, and $M \longrightarrow_{\text{let}_{\square}} N$ implies $[M] \equiv [N]$. Suppose the existence of an infinite sequence of $\longrightarrow_{\text{id}_{\square}}$, $\longrightarrow_{\beta_{\square}^{\vee}}$, and $\longrightarrow_{\text{let}_{\square}}$. Since there is no infinite reduction sequence in the simply typed λ -calculus, neither $\longrightarrow_{\text{id}_{\square}}$ nor $\longrightarrow_{\beta_{\square}^{\vee}}$ appears infinitely in the sequence. The assumption contradicts the fact that there is no infinite sequence of $\longrightarrow_{\text{let}_{\square}}$. \square

Proposition 15. $\longrightarrow_{\text{c}}$ is confluent.

Proof. According to Newman's lemma [20], we consider the local confluency. Because the λ_{c} -calculus and the call-by-name λ_{\square} -calculus are confluent, the following critical pairs are essential:

$$\begin{array}{l} \begin{array}{l} \longrightarrow_{\text{let}} \text{let } y \text{ be } M \text{ in box } \langle x \rangle \text{ be } \langle y \rangle \text{ in } x \\ \text{box } \langle x \rangle \text{ be } \langle M \rangle \text{ in } x \\ \searrow_{\text{id}_{\square}} M, \end{array} \\ \begin{array}{l} \longrightarrow_{\text{let}} \text{let } z \text{ be } (\text{box } \langle \vec{y} \rangle \text{ be } \langle \vec{N} \rangle \text{ in } V) \text{ in box } \langle x \rangle \text{ be } \langle z \rangle \text{ in } M \\ \text{box } \langle x \rangle \text{ be } \langle \text{box } \langle \vec{y} \rangle \text{ be } \langle \vec{N} \rangle \text{ in } V \rangle \text{ in } M \\ \searrow_{\beta_{\square}^{\vee}} \text{box } \langle \vec{y} \rangle \text{ be } \langle \vec{N} \rangle \text{ in } M[V/x]. \end{array} \end{array}$$

Confluence of the former pair is easily shown. For the latter case, let $\text{let } \vec{w} \text{ be } \vec{N}' \text{ in box } \langle \vec{y} \rangle \text{ be } \langle \vec{W} \rangle \text{ in } x$ be the $\longrightarrow_{\text{let}}$ -normal form of $\text{box } \langle \vec{y} \rangle \text{ be } \langle \vec{N} \rangle \text{ in } x$.

$$\begin{aligned} & \text{let } z \text{ be } (\text{box } \langle \vec{y} \rangle \text{ be } \langle \vec{N} \rangle \text{ in } V) \text{ in box } \langle x \rangle \text{ be } \langle z \rangle \text{ in } M \\ & \longrightarrow_{\text{let}}^* \text{let } z \text{ be } (\text{let } \vec{w} \text{ be } \vec{N}' \text{ in box } \langle \vec{y} \rangle \text{ be } \langle \vec{W} \rangle \text{ in } V) \text{ in box } \langle x \rangle \text{ be } \langle z \rangle \text{ in } M \\ & \longrightarrow_{\text{comp}}^* \text{let } \vec{w}, z \text{ be } \vec{N}', (\text{box } \langle \vec{y} \rangle \text{ be } \langle \vec{W} \rangle \text{ in } V) \text{ in box } \langle x \rangle \text{ be } \langle z \rangle \text{ in } M \\ & \longrightarrow_{\beta_{\text{let}}^{\vee}} \text{let } \vec{w} \text{ be } \vec{N}' \text{ in box } \langle x \rangle \text{ be } \langle \text{box } \langle \vec{y} \rangle \text{ be } \langle \vec{W} \rangle \text{ in } V \rangle \text{ in } M \\ & \longrightarrow_{\beta_{\square}^{\vee}} \text{let } \vec{w} \text{ be } \vec{N}' \text{ in box } \langle \vec{y} \rangle \text{ be } \langle \vec{W} \rangle \text{ in } M[V/x]. \end{aligned}$$

On the other hand, the lower term goes to the same term by $\longrightarrow_{\text{let}}^*$. \square

We have restricted forms of terms in the call-by-value calculi for CPS completeness. Leaving completeness on one side, now we can present another CPS transformation on full terms:

$$\begin{aligned}\Box\sigma' &= \Box((\sigma' \supset \mathbf{R}) \supset \mathbf{R}), \\ \llbracket \text{box } \langle \vec{x} \rangle \text{ be } \langle \vec{N} \rangle \text{ in } M \rrbracket' & \\ &= \lambda k. \llbracket \vec{N} \rrbracket'(\lambda \vec{y}. k(\text{box } \langle \vec{z} \rangle \text{ be } \langle \vec{y} \rangle \text{ in } \lambda h. \vec{z}(\lambda \vec{x}. \llbracket M \rrbracket' h))),\end{aligned}$$

where a non-overridden part of the definition is just the same as Figure 4. We remark that the definition of $\llbracket - \rrbracket'$ does not require a value transformation like $\overline{-}$. Also this transformation preserves the equality.

Theorem 16. *For $\lambda\Box$ -terms M and N , $M =_v N$ implies $\llbracket M \rrbracket' =_n \llbracket N \rrbracket'$.*

Unfortunately, it can be seen that this modified CPS transformation does not reflect the equality. For example,

$$\begin{aligned}\llbracket \text{box } \langle x \rangle \text{ be } \langle \text{box } \langle y \rangle \text{ be } \langle L \rangle \text{ in } N \rangle \text{ in } M \rrbracket' & \\ =_n \llbracket \text{box } \langle y \rangle \text{ be } \langle L \rangle \text{ in } (\lambda x. M)N \rrbracket', &\end{aligned}$$

but $\text{box } \langle x \rangle \text{ be } \langle \text{box } \langle y \rangle \text{ be } \langle L \rangle \text{ in } N \rangle \text{ in } M \neq_v \text{box } \langle y \rangle \text{ be } \langle L \rangle \text{ in } (\lambda x. M)N$ unless N is a value. It is still open to find an axiomatization complete for $\llbracket - \rrbracket'$.

5 Semantics

Since Kripke semantics [14] concern only provability, they are not suitable for our study. It is proposed by Bellin et al. in [4] that a model of **IK** is a cartesian closed category with a lax monoidal endofunctor with respect to cartesian products. (Fundamental properties of monoidal functors are found in [15].) Indeed, it is shown in [13] that the call-by-name $\lambda\Box$ -calculus with conjunctions is sound and complete for the class of such models. The completeness without conjunctions is expected to be proved in a way similar to the case of the simply typed λ -calculus. Bellin et al.'s calculus has the same syntax as ours, but it is not complete for the semantics.

Semantics for the call-by-value calculus is more complex than the call-by-name semantics. We show construction of a call-by-value model as follows.

Let a cartesian closed category \mathcal{C} have a strong monad $\langle T, \eta, \mu \rangle$ and a monoidal endofunctor $\langle \Box, \mathfrak{m}_1, \mathfrak{m} \rangle$. We focus on the Kleisli category \mathcal{C}_T , which is a model of the λ_c -calculus. For a morphism $f \in \mathcal{C}(B, A)$, there exists a morphism $\eta \circ \Box f \in \mathcal{C}_T(\Box B, \Box A)$. This explains a construction

$$\frac{x : \sigma \vdash V : \tau}{y : \Box\sigma \vdash \text{box } \langle x \rangle \text{ be } \langle y \rangle \text{ in } V : \Box\tau.}$$

The fact that the above construction is functorial justifies the following equations:

$$\begin{aligned}\text{box } \langle x \rangle \text{ be } \langle M \rangle \text{ in } x &=_v M, \\ \text{box } \langle x \rangle \text{ be } \langle \text{box } \langle y \rangle \text{ be } \langle M \rangle \text{ in } W \rangle \text{ in } V &=_v \text{box } \langle y \rangle \text{ be } \langle M \rangle \text{ in } V[W/x].\end{aligned}$$

The natural transformation $\{\mathfrak{m}_{A,B} \in \mathcal{C}(\Box A \times \Box B, \Box(A \times B))\}$ induces a type-indexed family $\{\eta \circ \mathfrak{m}_{A,B} \in \mathcal{C}_T(\Box A \times \Box B, \Box(A \times B))\}$. This family is not a natural transformation but natural in values. It explains an equation

$$\mathbf{box} \langle x, z \rangle \mathbf{be} \langle \mathbf{box} \langle y \rangle \mathbf{be} \langle N \rangle \mathbf{in} V, P \rangle \mathbf{in} M =_{\mathbf{v}} \mathbf{box} \langle y, z \rangle \mathbf{be} \langle N, P \rangle \mathbf{in} M[V/x].$$

If a monad T is a continuation monad, that is, $TX = \mathbf{R}^{\mathbf{R}^X}$, the categorical semantics coincides with the CPS semantics.

6 Extensions

In this section, we show an extension of the call-by-name calculus to **IS4**. A call-by-value axiomatization still remains future work.

We introduce type-indexed families of constants $\{\varepsilon_\sigma : \Box\sigma \supset \sigma\}$ and $\{\delta_\sigma : \Box\sigma \supset \Box\Box\sigma\}$ with the following axioms:

$$\begin{aligned} \varepsilon(\mathbf{box} \langle \vec{x} \rangle \mathbf{be} \langle \vec{N} \rangle \mathbf{in} M) &=_{\mathbf{nat}_\varepsilon} M[\varepsilon \vec{N} / \vec{x}], \\ \delta(\mathbf{box} \langle \vec{x} \rangle \mathbf{be} \langle \vec{N} \rangle \mathbf{in} M) &=_{\mathbf{nat}_\delta} \mathbf{box} \langle \vec{y} \rangle \mathbf{be} \langle \delta \vec{N} \rangle \mathbf{in} \mathbf{box} \langle \vec{x} \rangle \mathbf{be} \langle \vec{y} \rangle \mathbf{in} M, \\ \delta(\delta M) &=_{\mathbf{mon}} \mathbf{box} \langle x \rangle \mathbf{be} \langle \delta M \rangle \mathbf{in} \delta x, \\ \varepsilon(\delta M) &=_{\mathbf{mon}} \mathbf{box} \langle x \rangle \mathbf{be} \langle \delta M \rangle \mathbf{in} \varepsilon x =_{\mathbf{mon}} M. \end{aligned}$$

(It is trivial that this calculus corresponds to **IS4**.) We only consider equalities because it is not obvious in some equations which side is a result of a computation. Naturally, it is possible to give calculi for **IT** and **IK4** as fragments of this **IS4** calculus.

Bierman and de Paiva introduced the $\lambda^{\mathbf{S4}}$ -calculus in [5]. We show our calculus can emulate the \Box -fragment of their calculus. Let

$$\begin{aligned} \mathbf{box} M \mathbf{with} \vec{N} \mathbf{for} \vec{x} &\equiv \mathbf{box} \langle \vec{x} \rangle \mathbf{be} \langle \delta \vec{N} \rangle \mathbf{in} M, \\ \mathbf{unbox} M &\equiv \varepsilon M. \end{aligned}$$

The following equation holds in our calculus:

$$\mathbf{unbox}(\mathbf{box} M \mathbf{with} \vec{N} \mathbf{for} \vec{x}) = M[\vec{N} / \vec{x}].$$

On the other hand, Bierman and de Paiva's calculus does not emulate our calculus because ours is complete for the class of cartesian closed categories with monoidal comonads but theirs is not.

It is also possible to compare our calculus to a dual context version of **IS4** like Barber and Plotkin's DILL [2]. A dual context calculus for **IS4** is proposed by Pfenning and Davies in [22]. (Their calculus has a diamond modality too, but we just ignore it here.) The dual context calculus requires new syntax $\Box M$ and

$\text{let } \square x \text{ be } N \text{ in } M$ instead of $\text{box } \langle \vec{x} \rangle \text{ be } \langle \vec{N} \rangle \text{ in } M$. The typing rules consist of

$$\frac{\frac{\frac{\Delta, a : \tau, \Delta' \mid \Gamma \vdash a : \tau,}{\Delta \mid \vdash M : \tau}}{\Delta \mid \Gamma \vdash \square M : \square \tau},}{\Delta, a : \sigma \mid \Gamma \vdash M : \tau \quad \Delta \mid \Gamma \vdash N : \square \sigma} \frac{}{\Delta \mid \Gamma \vdash \text{let } \square a^\sigma \text{ be } N \text{ in } M : \tau},$$

and the usual rule of the simply typed λ -calculus with respect to right-hand contexts. We use a, b, \dots for variables of left-hand contexts to distinguish them from those of right-hand contexts. The equality is defined by

$$\begin{aligned} C &: \text{context,} \\ \text{let } \square a \text{ be } \square N \text{ in } M &= M[N/a], \\ \text{let } \square a \text{ be } M \text{ in } \square a &= M, \\ C[\text{let } \square a \text{ be } N \text{ in } M] &= \text{let } \square a \text{ be } N \text{ in } C[M] \quad a \notin \text{FV}(C), \end{aligned}$$

where C is a context such that its hole does not appear under a box.

In the $\lambda\square$ -calculus, we call the following equality the strongness condition:

$$\begin{aligned} \text{box } \langle \vec{w}, x, \vec{z} \rangle \text{ be } \langle \vec{P}, N, \vec{Q} \rangle \text{ in } M \\ &=_{\text{st}} \text{box } \langle \vec{w}, \vec{z} \rangle \text{ be } \langle \vec{P}, \vec{Q} \rangle \text{ in } M \quad |\vec{w}| = |\vec{P}|, \\ \text{box } \langle \vec{w}, x, y, \vec{z} \rangle \text{ be } \langle \vec{P}, N, N, \vec{Q} \rangle \text{ in } M \\ &=_{\text{st}} \text{box } \langle \vec{w}, x, \vec{z} \rangle \text{ be } \langle \vec{P}, N, \vec{Q} \rangle \text{ in } M[x/y] \quad |\vec{w}| = |\vec{P}|. \end{aligned}$$

The following equality is called the symmetricity.

$$\begin{aligned} \text{box } \langle \vec{w}, x, y, \vec{z} \rangle \text{ be } \langle \vec{P}, N, L, \vec{Q} \rangle \text{ in } M \\ &=_{\text{sym}} \text{box } \langle \vec{w}, y, x, \vec{z} \rangle \text{ be } \langle \vec{P}, L, N, \vec{Q} \rangle \text{ in } M \quad |\vec{w}| = |\vec{P}|. \end{aligned}$$

The terms “strong” and “symmetric” follow the terms “strong monoidal functor” and “symmetric monoidal functor” in the category theory. We show that the dual context calculus is equivalent to the $\lambda\square$ -calculus with the symmetricity and the strongness condition. Define the transformation $\langle - \rangle$ from the dual context calculus into the $\lambda\square$ -calculus by

$$\begin{aligned} \langle a \rangle &= \varepsilon a, \\ \langle \square M \rangle &= \text{box } \langle \vec{b} \rangle \text{ be } \langle \delta \vec{b} \rangle \text{ in } \langle M \rangle \quad \text{where } \{ \vec{b} \} = \text{FV}(M), \\ \langle \text{let } \square a \text{ be } N \text{ in } M \rangle &= (\lambda a. \langle M \rangle) \langle N \rangle. \end{aligned}$$

For a derivable judgment $a_1 : \rho_1, \dots \mid x_1 : \sigma_1, \dots \vdash M : \tau$, the judgment

$$a_1 : \square \rho_1, \dots, x_1 : \sigma_1, \dots \vdash \langle M \rangle : \tau$$

is derivable in the $\lambda\Box$ -calculus, and $M = N$ implies $\langle M \rangle = \langle N \rangle$ under the strongness condition and the symmetricity. Its inverse $\langle - \rangle$ can be defined by

$$\begin{aligned} \langle \varepsilon \rangle &= \lambda y. \text{let } \Box a \text{ be } y \text{ in } a, \\ \langle \delta \rangle &= \lambda y. \text{let } \Box a \text{ be } y \text{ in } \Box \Box a, \\ \langle \text{box } \langle \vec{x} \rangle \text{ be } \langle \vec{N} \rangle \text{ in } M \rangle &= \text{let } \Box \vec{a} \text{ be } \langle \vec{N} \rangle \text{ in } \Box (\langle M \rangle [\vec{a} / \vec{x}]). \end{aligned}$$

One can see that $M = N$ implies $\langle M \rangle = \langle N \rangle$. While $\langle \langle - \rangle \rangle$ is the identity up to the equality, $\langle \langle - \rangle \rangle$ is not the identity itself. The reason is that the dual context calculus is redundant in some sense: for example, two judgments,

$$\begin{aligned} &| x : \Box \sigma \vdash x : \Box \sigma, \\ a : \sigma \quad &| \vdash \Box a : \Box \sigma, \end{aligned}$$

have the same semantics.

Another possible extension of our calculus is a calculus corresponding to the classical modal logic \mathbf{K} . In [21], Parigot has extended the simply typed λ -calculus to the $\lambda\mu$ -calculus, which corresponds to the classical logic. We can extend the call-by-name $\lambda\Box$ -calculus with μ -operator in a straightforward way. A call-by-value version and analyses of the relation between call-by-name and call-by-value are found in [13].

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