ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 2, Number 1, Winter 1972

CALCULUS OF VARIATIONS IN COMPLEX VECTOR BUNDLES

ALFRED GRAY

1. **Introduction.** The theory of the calculus of variations, and in particular Jacobi vector fields, is a very important tool in the study of the topology of Riemannian manifolds. Recently Goldberg and Kobayashi [1] have introduced the notion of holomorphic bisectional curvature of a Kähler manifold. Actually this notion makes sense in an arbitrary almost Hermitian vector bundle (see §2). In fact in a slightly different form the notion of holomorphic bisectional curvature has been used by several authors [3], [5], [6] to define notions of positivity of holomorphic vector bundles which generalize that of holomorphic line bundles.

In this paper we define and study a calculus of variations in almost Hermitian vector bundles via holomorphic bisectional curvature. The basis of our discussion is the following observation. Let M be a Kähler manifold with almost complex structure J. If σ is a geodesic in M and Y is a Jacobi vector field along σ, then JY is not a Jacobi vector field. Moreover if $X = Y + JY$ then X satisfies the differential equation

$$2X'' = JR_{\sigma' J\sigma'}X, \tag{1.1}$$

where R denotes the curvature operator of M. Now (1.1) makes sense in an arbitrary almost Hermitian vector bundle with a given covariant derivative. This is the starting point for our theory of the calculus of variations. We obtain new versions of several well-known theorems in the calculus of variations of a Riemannian manifold, including Myer's theorem and the Rauch comparison theorem. (We do not discuss the Morse index theorem, because versions of it already are known in the context of vector bundles, e.g., [7].) A defect in our theory is that there seems to be no good substitute for the exponential map of Riemannian geometry, although we define the notion of conjugate point.

In §2 we describe analogues for an almost Hermitian bundle E of Jacobi vector fields and the index form. We develop some properties of the index form in §3 and prove a version of Myer's theorem for

Received by the editors August 21, 1970.

AMS 1970 *subject classifications.* Primary 53C55, 49F22; Secondary 32C10.

almost Hermitian vector bundles. Rauch's comparison theorem is generalized in §4.

Our study parallels the treatment of the calculus of variations for Riemannian manifolds given in [4].

2. **Fundamental definitions.** Let M be a C^∞ differentiable manifold; denote by $\mathcal{F}(M)$ the ring of C^∞ real valued functions on M, and by $\mathcal{X}(M)$ the Lie algebra of C^∞ vector fields on M. We assume that M is an *almost Hermitian* manifold, that is, the tangent bundle $\tau(M)$ has an almost complex structure J and a Riemannian metric $\langle\ ,\ \rangle$ such that $\langle JX, JY\rangle = \langle X, Y\rangle$ for all $X, Y \in \mathcal{X}(M)$.

Let E be another complex vector bundle over M. We use the same letter J to denote the almost complex structure of E. Denote by $\mathcal{S}(E)$ the C^∞ sections of E; $\mathcal{S}(E)$ is a module over $\mathcal{F}(M)$. We shall also assume that E is an *almost Hermitian vector bundle* in the sense that E has a Riemannian metric $\langle\ ,\ \rangle$ such that $\langle JA, JB\rangle = \langle A, B\rangle$ for $A, B \in \mathcal{S}(E)$. Note that the tangent bundle $\tau(M)$ is by assumption almost Hermitian in this sense, and $\mathcal{X}(M) = \mathcal{S}(\tau(M))$.

Denote by ∇ a covariant derivative for E which is compatible with J and $\langle\ ,\ \rangle$. In other words we require that $\nabla : \mathcal{X}(M) \times \mathcal{S}(E) \to \mathcal{S}(E)$ and has the following properties

$$\nabla_X(A + B) = \nabla_X A + \nabla_X B, \tag{2.1}$$

$$\nabla_X(fA) = f\nabla_X A + (Xf)A, \tag{2.2}$$

$$\nabla_{fX+gY} = f\nabla_X + g\nabla_Y, \tag{2.3}$$

$$X\langle A, B\rangle = \langle\nabla_X A, B\rangle + \langle A, \nabla_X B\rangle, \tag{2.4}$$

$$\nabla_X JA = J\nabla_X A, \tag{2.5}$$

for $X, Y \in \mathcal{X}(M)$, $A, B \in \mathcal{S}(E)$, and $f, g \in \mathcal{F}(M)$.

Such a covariant derivative always exists. The curvature operator of ∇ is given by the usual formula

$$R_{XY}A = \nabla_{[X,Y]}A - [\nabla_X, \nabla_Y]A. \tag{2.6}$$

It is linear in all variables with respect to elements of $\mathcal{F}(M)$; in addition, it has the following properties:

$$R_{XY} = -R_{YX}, \tag{2.7}$$

$$\langle R_{XY}A, B\rangle = -\langle R_{XY}B, A\rangle, \tag{2.8}$$

$$R_{XY}JA = JR_{XY}A, \tag{2.9}$$

for $X, Y \in \mathcal{X}(M)$ and $A, B \in \mathcal{S}(E)$.

Following [1] we define the *holomorphic bisectional curvature* B_{XA} of ∇ ($X \in \mathcal{X}(M)$, $A \in \mathcal{S}(E)$) by the formula

$$B_{XA}\|X\|^2 \|A\|^2 = \langle R_{XJX}A, JA\rangle \tag{2.10}$$

wherever X and A are nonzero. The holomorphic bisectional curvature plays almost the same role for the almost Hermitian vector bundle E that the ordinary sectional curvature plays for the tangent bundle of M. Strictly speaking, however, the holomorphic bisectional curvature is the sum of two sectional curvatures. The *Ricci curvature* k of E (with respect to ∇) is given by

$$k(X, Y) = \sum_{i=1}^{n} \langle R_{XJY}E_i, JE_i\rangle \tag{2.11}$$

where $\{E_1, \cdots, E_n, JE_1, \cdots, JE_n\}$ is a local orthonormal frame for E, and $X, Y \in \mathcal{X}(M)$. Thus whenever $X \neq 0$,

$$k(X, X) = \|X\|^2 \sum_{i=1}^{n} B_{XE_i}. \tag{2.12}$$

Let γ be a curve in M with velocity γ'. A *section* A *of* E *along* γ is a function which assigns to each point t in the domain of γ an element $A_{\gamma(t)} \in E_{\gamma(t)}$, where $E_{\gamma(t)}$ denotes the fiber of E over $\gamma(t)$. If A is a differentiable (i.e., C^∞) section along γ, we define another such section A' by $A'(t) = (\nabla_{\gamma'}A)_{\gamma(t)}$. (As usual, one must extend A and γ' to sections of E in a neighborhood of each point of γ, and then prove the independence of such extensions, in order to insure that $A'(t)$ is well defined.)

DEFINITION. Let σ be a unit speed geodesic in M, and let A be a piecewise differentiable section of E along σ.

(i) A is *parallel* if and only if wherever A is differentiable we have

$$A' = 0. \tag{2.13}$$

(ii) A is a *J-section* if and only if wherever A is differentiable we have

$$2A'' = JR_{\sigma' J\sigma'}A. \tag{2.14}$$

From the theory of ordinary differential equations we have the following result.

PROPOSITION (2.1). *Let A be a differentiable section of E along a geodesic σ.*

(i) *If A is parallel, then A is uniquely determined by its value at* *any one point of σ. In particular the dimension of the space of (continuous) parallel sections along σ is equal to the (real) fiber dimension of E.*

(ii) *If A is a J-section, then A is uniquely determined by the values of A and A′ at any one point of σ. In particular the dimension of the space of differentiable J-sections along σ is twice the (real) fiber dimension of E.*

In contrast to the situation with ordinary Jacobi vector fields, the following is true.

PROPOSITION (2.2). *If A is a J-section, so is JA.*

Next we describe the index form and the notion of conjugacy with respect to an almost Hermitian vector bundle.

DEFINITION. Let A and B be piecewise differentiable sections of E along a geodesic segment $\sigma : [a, b] \rightarrow M$. Put

$$I(A, B) = \int_a^b \{2\langle A', B'\rangle - \langle R_{\sigma' J\sigma'} A, JB\rangle\}(t)\, dt. \tag{2.15}$$

Then I is a symmetric bilinear form on the space of piecewise differentiable sections of E along σ which we call the *index form* of E along σ. When the limits a and b are important we write $I(A, B) = I_a{}^b(A, B)$.

For geometric interpretations of our results we shall find the following notion useful.

DEFINITION. Let $p, q \in M$, and assume $\sigma : [a, b] \rightarrow M$ is a geodesic with $\sigma(a) = p$ and $\sigma(b) = q$. Then p and q are *E-conjugate along* σ (with respect to the connection ∇ of E) if and only if there exists a nonzero differentiable J-section along σ which vanishes both at a and b. The *multiplicity* of E-conjugate points p and q is the dimension of the space of nonzero differentiable J-sections along σ which vanish at p and q.

The following fact, which may be deduced from the theory of ordinary differential equations, will be useful.

PROPOSITION (2.3). *Let $\sigma : [a, b] \rightarrow M$ be a geodesic segment. If $a \leqq c \leqq b$, then there exists a neighborhood U of c in $[a, b]$ such that no point of U is E-conjugate to any other point of U along σ.*

COROLLARY (2.4). *Let $\sigma : [a, b] \rightarrow M$ be a geodesic segment. If there exists an E-conjugate point of $\sigma(a)$ along σ, then there exists a first E-conjugate point to $\sigma(a)$.*

3. **Properties of the index form.** In this section we show that

analogues of some basic formulas for the index form for ordinary Jacobi fields also hold for the index form of J-sections. Denote by $\sigma: [a, b] \to M$ a unit speed geodesic segment.

THEOREM (3.1). *Let A and B be piecewise differentiable sections of E along σ. Assume that $[a, b]$ is subdivided as $a = t_0 < t_1 < \cdots < t_h = b$ so that A is differentiable on each subinterval $[t_{j-1}, t_j]$. Then*

$$
\begin{aligned}
I(A, B) = &- \int_a^b \langle 2A'' - JR_{\sigma' J\sigma'} A, B\rangle(t)\, dt \\
&+ 2 \sum_{j=1}^{h-1} \langle A'(t_j^-) - A'(t_j^+), B(t_j)\rangle \qquad (3.1)\\
&+ 2\langle A'(b), B(b)\rangle - 2\langle A'(a), B(a)\rangle,
\end{aligned}
$$

where $A'(t_j^-)$ and $A'(t_j^+)$ denote the left- and right-hand derivatives of A at t_j $(1 \leqq j \leqq h-1)$.

PROOF. From (2.15) and the fact that $\langle A', B\rangle' = \langle A'', B\rangle + \langle A', B'\rangle$ (wherever A and B are differentiable) we have

$$
(3.2) \qquad I(A, B) = \sum_{j=1}^{h} \int_{t_{j-1}}^{t_j} \{2\langle A', B\rangle' - \langle 2A'' - JR_{\sigma' J\sigma'} A, B\rangle\}(t)\, dt.
$$

Then (3.1) follows easily from (3.2).

We next obtain three consequences of Theorem (3.1).

COROLLARY (3.2). *Let A be a differentiable J-section along σ, and let B be any piecewise differentiable section of E along σ. Then*

$$
I(A, B) = 2\langle A'(b), B(b)\rangle - 2\langle A'(a), B(a)\rangle.
$$

COROLLARY (3.3). *Suppose the holomorphic bisectional curvature of E is nonpositive along σ (i.e., if A is a section of E along σ then $B_{\sigma' A} \leqq 0$ wherever $A \neq 0$). Then no two points of σ are $\bar{E}$-conjugate along σ.*

PROOF. Let A be a differentiable J-section along σ which (without loss of generality) vanishes at both $\sigma(a)$ and $\sigma(b)$. By Corollary (3.2) we have $I(A, A) = 0$. By hypothesis, $2\|A'\|^2 - \langle R_{\sigma' J\sigma'} A, JA\rangle \geqq 0$ on $[a, b]$, and so $\|A'\|^2 \equiv 0$ on $[a, b]$. Since A vanishes at $\sigma(a)$, this implies that $A = 0$ on $[a, b]$ by Proposition (2.1). Thus the corollary follows.

COROLLARY (3.4). *A is a differentiable J-section of E along σ if* *and only if* $I(A, B) = 0$ *for all piecewise differentiable sections* B *of* E *along* σ.

PROOF. If A is a differentiable J-section, it follows easily from (3.1) that $I(A, B) = 0$ for all piecewise differentiable sections B of E along σ.

Conversely assume that $I(A, B) = 0$ whenever B is a piecewise differentiable section of E along σ. Let f be a differentiable function such that $f(t_j) = 0$, $j = 1, \cdots, h$, and is positive everywhere else. Set $B = f(2A'' - JR_{\sigma' J\sigma'}A)$. Then from (3.1) we see that

$$0 = I(A, B) = -\int_a^b f\|2A'' - JR_{\sigma' J\sigma'}A\|^2(t)\, dt.$$

Hence A is a J-section which is differentiable on each subinterval $[t_{j-1}, t_j]$. Moreover let C_k be a piecewise differentiable section of E such that $C_k(t_k) = A'(t_k^-) - A'(t_k^+)$ and $C_k(t_j) = 0$ for $j \neq k$. Then from (3.1) we have

$$0 = 2\|A'(t_k^-) - A'(t_k^+)\|^2.$$

Hence $A'(t_k^-) = A'(t_k^+)$ for $k = 1, \cdots, h-1$. Since A is determined on each subinterval $[t_{j-1}, t_j]$ by its value at any point of the subinterval (Proposition (2.1)) it follows that A is a differentiable J-section of E along all of σ.

PROPOSITION (3.5). *Suppose* A *and* B *are* J*-sections along* σ. *Then*
(i) $\langle A, B'\rangle - \langle A', B\rangle$ *is constant along* σ;
(ii) *if* A *and* B *vanish at some point of* σ, *then* $\langle A, B'\rangle \equiv \langle A', B\rangle$.

PROOF. We have

$$\begin{aligned} 2\langle A, B'\rangle' &= 2\langle A', B'\rangle + 2\langle A, B''\rangle = 2\langle A', B'\rangle - \langle JA, R_{\sigma' J\sigma'}B\rangle \\ &= 2\langle A', B'\rangle - \langle JB, R_{\sigma' J\sigma'}A\rangle = 2\langle A', B\rangle'. \end{aligned}$$

Hence $(\langle A, B'\rangle - \langle A', B\rangle)' = 0$, and so the proposition follows.

THEOREM (3.6). *Let* $\sigma : [a, b] \to M$ *be a geodesic segment such that* $\sigma(a)$ *has no conjugate point along* σ. *Let* B *be a piecewise differentiable* J*-section along* σ *which vanishes at* $\sigma(a)$. *Suppose* A *is any piecewise differentiable section along* σ *such that* $A(a) = 0$ *and* $A(b) = B(b)$. *Then*

$$I(A, A) \geqq I(B, B). \tag{3.3}$$

Equality holds if and only if $A \equiv B$.

PROOF. Let $2n$ be the (real) fiber dimension of E. By Proposition

(2.1) there exist linearly independent J-sections $B_1, \cdots, B_{2n}$ along σ which vanish at $\sigma(a)$; these J-sections form a basis for the space of J-sections along σ which vanish at $\sigma(a)$. Hence there exist constants $a_1, \cdots, a_{2n}$ such that

$$B = \sum_{i=1}^{2n} a_i B_i.$$

Furthermore $B_1, \cdots, B_{2n}$ are linearly independent at each point of $\sigma(t)$, $a \leqq t \leqq b$. Therefore they form a basis for $E_{\sigma(t)}$. It follows that there exist piecewise differentiable functions $f_1, \cdots, f_{2n}$ defined on $[a, b]$ such that $A = \sum_{i=1}^{2n} f_i B_i$. We have

$$\|A'\|^2 = \|\sum f_i' B_i\|^2 + \|\sum f_i B_i'\|^2 + 2\langle \sum f_i' B_i, \sum f_i B_i' \rangle. \tag{3.4}$$

Since each B_i is a J-section,

$$\begin{aligned} -\langle R_{\sigma' J\sigma'} A, JA \rangle &= -\sum f_i \langle R_{\sigma' J \sigma'} B_i, JA \rangle \\ &= 2\sum f_i \langle B_i'', A \rangle = 2\langle \sum f_i B_i'', \sum f_i B_i \rangle. \end{aligned} \tag{3.5}$$

Furthermore

$$\begin{aligned} \langle \sum f_i B_i, \sum f_i B_i' \rangle' &= \langle \sum f_i' B_i, \sum f_i B_i' \rangle + \|\sum f_i B_i'\|^2 \\ &\quad + \langle \sum f_i B_i, \sum f_i' B_i' \rangle + \langle \sum f_i B_i, \sum f_i B_i'' \rangle. \end{aligned} \tag{3.6}$$

Combining equations (3.4), (3.5) and (3.6), we obtain

$$\begin{aligned} 2\|A'\|^2 - \langle R_{\sigma' J\sigma'} A, JA \rangle &= 2\|\sum f_i' B_i\|^2 + 2\|\sum f_i B_i'\|^2 \\ &\quad + 4\langle \sum f_i' B_i, \sum f_i B_i' \rangle + 2\langle \sum f_i B_i'', \sum f_i B_i \rangle \\ &= 2\|\sum f_i' B_i\|^2 + 2\langle \sum f_i B_i, \sum f_i B_i' \rangle' \\ &\quad + 2\langle \sum f_i' B_i, \sum f_i B_i' \rangle - 2\langle \sum f_i B_i, \sum f_i' B_i' \rangle. \end{aligned} \tag{3.7}$$

By Proposition (3.5) we have

$$\begin{aligned} &\langle \sum f_i' B_i, \sum f_i B_i' \rangle - \langle \sum f_i B_i, \sum f_i' B_i' \rangle \\ &\quad = \sum_{i,j} \{ f_i' f_j \langle B_i, B_j' \rangle - f_i f_j' \langle B_i, B_j' \rangle \} \\ &\quad = \sum_{i,j} f_i' f_j (\langle B_i, B_j' \rangle - \langle B_i', B_j \rangle) = 0. \end{aligned} \tag{3.8}$$

Thus from (3.7) and (3.8) we obtain

$$2\|A'\|^2 - \langle R_{\sigma' J\sigma'} A, JA \rangle = 2\|\sum f_i' B_i\|^2 + 2\langle \sum f_i B_i, \sum f_i B_i' \rangle'. \tag{3.9}$$

Hence

$$(3.10)\qquad \begin{aligned} I(A, A) &= \int_a^b \{2\|A'\|^2 - \langle R_{\sigma' J\sigma'} A, JA\rangle\}(t)\, dt \\ &= 2\int_a^b \|f_i' B_i\|^2(t)\, dt + 2\langle \textstyle\sum f_i B_i, \sum f_i B_i'\rangle(b). \end{aligned}$$

The same argument applied to B shows that

$$(3.11)\qquad I(B, B) = 2\langle \textstyle\sum a_i B_i, \sum a_i B_i'\rangle(b).$$

From the assumption that $A(b) = B(b)$ we obtain $a_i = f_i(b)$ for $i = 1, \cdots, n$. Hence from (3.10) and (3.11) we have

$$(3.12)\qquad I(A, A) - I(B, B) = 2\int_a^b \|f_i' B_i\|^2(t)\, dt \geqq 0,$$

proving (3.3). If $I(A, A) = I(B, B)$, then we must have $f_i' = 0$ on $[a, b]$ for $i = 1, \cdots, 2n$, by (3.12). Thus $A \equiv B$.

Corollary (3.7). *Let $\sigma : [a, b] \to M$ be a geodesic segment such that $\sigma(a)$ has no conjugate points along σ. If A is a piecewise differentiable section along σ such that $A(a) = A(b) = 0$, then $I(A, A) \geqq 0$.*

Proof. In Theorem (3.6) let $B = 0$.

Theorem (3.8). *Suppose the Ricci curvature of E (with respect to ∇) is positive definite with all eigenvalues $\geqq 2nh_0 > 0$, where $2n$ is the (real) fiber dimension of E. Then for every geodesic σ of M, the distance between any two consecutive conjugate points of σ is at most $\pi/\sqrt{h_0}$.*

Proof. Let $\sigma(c)$ be the first conjugate point of σ along $\sigma(a)$. Let $a < b < c$. Let $\{E_1, \cdots, E_n, JE_1, \cdots, JE_n\}$ be parallel sections of E along σ, and assume $E_1, \cdots, E_n, JE_1, \cdots, JE_n$ are orthonormal at each point of σ. Let f be a nonzero real valued function defined on $[a, b]$ with $f(a) = f(b) = 0$. By Corollary (3.7) we have

$$(3.13)\qquad \begin{aligned} 0 &\leqq \sum_{i=1}^n I(fE_i, fE_i) \\ &= \int_a^b \left\{2\sum_{i=1}^n \|f' E_i\|^2 - f^2 \sum_{i=1}^n \langle R_{\sigma' J\sigma'} E_i, JE_i\rangle\right\}(t)\, dt \\ &\leqq 2n\int_a^b (f'^2 - h_0 f^2)(t)\, dt. \end{aligned}$$

Let $f(t) = \sin(\pi(t - a)/(b - a))$. Then (3.13) reduces to

$$b - a \leqq \pi/\sqrt{h_0}.$$

Let $f(t) = \sin(\pi(t - a)/(b - a))$. Then (3.13) reduces to

$$b - a \leqq \pi/\sqrt{h_0}.$$

Since b is arbitrarily close to c, the theorem follows.

COROLLARY (3.9). *Suppose the holomorphic bisectional curvature of E satisfies* $B_{AC} \geqq 2h_0 > 0$. *Then for every geodesic* σ *of M, the distance between any two consecutive conjugate points of* σ *is at most* $\pi/\sqrt{h_0}$.

THEOREM (3.10). *Let* $\sigma : [a, b] \rightarrow M$ *be a geodesic segment in M. Then there is an E-conjugate point* $\sigma(c)$ *of a along* σ *with* $a < c < b$ *if and only if there is a piecewise differentiable section A of E along* σ *such that* $A(a) = A(b) = 0$ *and* $I(A, A) < 0$.

PROOF. If there exists a piecewise differentiable section A of E along σ such that $A(a) = A(b) = 0$ and $I(A, A) < 0$, then by Corollary (3.7), $\sigma(a)$ has an E-conjugate point $\sigma(c)$ with $a < c < b$.

Conversely let $\sigma(c)$ be a conjugate point of $\sigma(a)$ with $a < c < b$. By Proposition (2.3) there exists a neighborhood U of c such that no point of U is E-conjugate to any other point of U along σ. Let $\delta > 0$ be such that $c - \delta \in U$ and $c + \delta \in U$. Denote by τ the geodesic segment $\sigma \mid [c - \delta, c + \delta]$. Since $\tau(c - \delta)$ is not conjugate to $\tau(c + \delta)$, the linear mapping from the set of J-sections on τ into $E_{\tau(c-\delta)} \oplus E_{\tau(c+\delta)}$ is one-to-one and therefore onto, because both spaces have the same dimension. Hence there is a J-section with prescribed values at both ends of τ.

Let B be a J-section along $\sigma \mid [a, c]$ with $B(a) = B(c) = 0$, and let C be a J-section along $\tau = \sigma \mid [c - \delta, c + \delta]$ with $C(c - \delta) = B(c - \delta)$ and $C(c + \delta) = 0$. Define a piecewise differentiable section A of E along σ by

$$\begin{aligned} A(t) &= B(t), && a \leqq t \leqq c - \delta, \\ &= C(t), && c - \delta \leqq t \leqq c + \delta, \\ &= 0, && c + \delta \leqq t \leqq b. \end{aligned}$$

Also define a piecewise differentiable section D of E along τ by

$$\begin{aligned} D(t) &= B(t), && c - \delta \leqq t \leqq c, \\ &= 0. && c \leqq t \leqq c + \delta. \end{aligned}$$

By Corollary (3.2) we have

$$0 = I_a{}^c(B, B) = I_a{}^{c-\delta}(B, B) + I_{c-\delta}^{c}(B, B). \tag{3.14}$$

Also, from Theorem (3.6) it follows that

(3.15) $$I_{c-\delta}^{c+\delta}(C, C) < I_{c-\delta}^{c+\delta}(D, D).$$

From (3.14) and (3.15) we obtain

$$\begin{aligned} I_a^{\,b}(A, A) &= I_a^{\,c-\delta}(A, A) + I_{c-\delta}^{c+\delta}(A, A) + I_{c+\delta}^{b}(A, A) \\ &= I_a^{\,c-\delta}(B, B) + I_{c-\delta}^{c+\delta}(C, C) \\ &= -I_{c-\delta}^{c+\delta}(D, D) + I_{c-\delta}^{c+\delta}(C, C) < 0. \end{aligned}$$

This completes the proof.

4. **Comparison theorems.** In this section we show that the Rauch comparison theorem generalizes completely to J-sections of almost Hermitian vector bundles over almost Hermitian manifolds. Our main result is the following.

THEOREM (4.1). *Let E and F be almost Hermitian vector bundles over almost Hermitian manifolds M and N, respectively. Assume that E and F have the same fiber dimension. Let σ and τ be unit speed geodesics defined on $[a, b]$ in M and N, respectively. Also, let C be a J-section of E along σ and D a J-section of F along τ. Assume*

(i) $C(a) = D(a) = 0$;

(ii) $\|C'(a)\| = \|D'(a)\|$;

(iii) *$\sigma(a)$ and $\tau(a)$ have no E-conjugate (resp. F-conjugate) points along σ (resp. τ);*

(iv) *if $x \in E_{\sigma(t)}$ and $y \in F_{\tau(t)}$, the holomorphic bisectional curvatures of E and F satisfy*

(4.1) $$B_{\sigma'(t)x} \geqq B_{\tau'(t)y}.$$

Then we have $\|C(t)\| \leqq \|D(t)\|$ for $a \leqq t \leqq b$.

PROOF. Let $u(t) = \|C(t)\|^2$ and $v(t) = \|D(t)\|^2$. Then $u(t) \neq 0$ and $v(t) \neq 0$ for $a < t \leqq b$. Set

$$\mu(t) = u(t)^{-1} \int_a^t \{2\|C'\|^2 - \langle R_{\sigma' J\sigma'} C, JC\rangle\}(t)\, dt,$$

$$\nu(t) = v(t)^{-1} \int_a^t \{2\|D'\|^2 - \langle R_{\tau' J\tau'} D, JD\rangle\}(t)\, dt.$$

Then $u'(t) = u(t)\mu(t)$ and $v'(t) = v(t)\nu(t)$. Thus for every c with $a < c < b$ we have

$$u(t) = u(c) \exp \int_c^t \mu(t)\, dt \quad \text{and} \quad v(t) = v(c) \exp \int_c^t \nu(t)\, dt.$$

On the other hand by L'Hospital's rule, we have

$$\lim_{\epsilon\to a}\frac{u(\epsilon)}{v(\epsilon)}=\lim_{\epsilon\to a}\frac{\langle C, C'\rangle(\epsilon)}{\langle D, D'\rangle(\epsilon)}=\lim_{\epsilon\to a}\frac{(\|C'\|^2+\langle C, C'\rangle)(\epsilon)}{(\|D'\|^2+\langle D, D'\rangle)(\epsilon)}=1.$$

Therefore $u(t)/v(t)=\exp\int_a^t(\mu(t)-\nu(t))\,dt$ for all $a<t\leqq b$.

To complete the proof, it suffices to show that $\mu(t)\leqq\nu(t)$ whenever $a<t\leqq b$. Fix c with $a<c<b$. Set $\bar{C}=u(c)^{-1/2}C$ and $\bar{D}=v(c)^{-1/2}D$; then $\|\bar{C}(c)\|=\|\bar{D}(c)\|=1$, and C and D are J-sections. Let $f_c: F_{\tau(c)}\to E_{\sigma(c)}$ be any metric preserving linear isomorphism with $f_c(\bar{D}(c))=\bar{C}(c)$. Denote by $\sigma_s{}^r: E_{\sigma(r)}\to E_{\sigma(s)}$ (resp. $\tau_s{}^r: F_{\tau(r)}\to F_{\tau(s)}$) the parallel translation along σ (resp. τ) from $\sigma(r)$ to $\sigma(s)$ (resp. $\tau(r)$ to $\tau(s)$). Then set $f_t=\sigma_t{}^c\circ f\circ\tau_c{}^t$, and define a section A of E along σ by $A(t)=f_t(\bar{D}(t))$. We then have $A(0)=0=C(0)$ and $A(c)=f_c(\bar{D}(c))=\bar{C}(c)$. Also, it can be verified that $A'(t)=f_t(\bar{D}'(t))$ for $a<t<b$. Hence by Theorem (3.6), and assumption (iv) we have

$$\begin{aligned}I_a{}^c(\bar{C},\bar{C})&\leqq I_a{}^c(A,A)=\int_a^c\{2\|A'\|^2-B_{\sigma'A}\|A\|^2\}(t)\,dt\\&\leqq\int_a^c\{2\|\bar{D}'\|^2-B_{\tau'\bar{D}}\|\bar{D}\|^2\}(t)\,dt=I_a{}^c(\bar{D},\bar{D}).\end{aligned}$$

Thus

$$\begin{aligned}\mu(c)&=u(c)^{-1}I_a{}^c(C,C)=I_a{}^c(\bar{C},\bar{C})\leqq I_a{}^c(\bar{D},\bar{D})\\&=v(c)^{-1}I_a{}^c(D,D)=\nu(c).\end{aligned}$$

Since c is arbitrary, the theorem follows.

COROLLARY (4.2). *Let E and F be almost Hermitian vector bundles over M and N, respectively, and assume that E and F have the same fiber dimension. Let σ and τ be unit speed geodesics in M and N, respectively, both defined on $[a,b]$. Assume that if $x\in E_{\sigma(t)}$ and $y\in F_{\tau(t)}$, the holomorphic bisectional curvatures of E and F satisfy*

$$B_{\sigma'(t)x}\geqq B_{\tau'(t)y}.$$

Then if $\sigma(a)$ has no E-conjugate point along σ, $\tau(a)$ has no F-conjugate point along τ.

PROOF. Assume the contrary. By Corollary (2.4) there exists c such that $a<c\leqq b$ and $\tau(c)$ is the first F-conjugate point of $\tau(a)$ along τ. Let D be a nonzero J-section of F along τ such that $D(a)=D(c)=0$. By Proposition (2.1) there exists a J-section C of E along σ such that $C(a)=0$ and $\|C'(a)\|=\|D'(a)\|$. By Theorem (4.1) we have $\|C(t)\|\leqq\|D(t)\|$ for $a\leqq t<c$. Hence we have

$$\|C(c)\| = \lim_{t\to c} \|C(t)\| \leqq \lim_{t\to c} \|D(t)\| = 0,$$

and so $C(c) = 0$. This contradicts the assumption that $\sigma(a)$ has no E-conjugate points along σ.

COROLLARY (4.3). *Let E be an almost Hermitian vector bundle over an almost Hermitian manifold M and assume the holomorphic bisectional curvature of E satisfies*

$$0 < 2b_0 \leqq B_{XA} \leqq 2b_1,$$

where b_0 and b_1 are constants, for $X \in \mathcal{X}(M)$, $A \in \mathcal{S}(E)$. If $\sigma : [a, b] \to M$ is a unit speed geodesic such that $\sigma(b)$ is the first E-conjugate point of $\sigma(a)$ along σ, then

$$\pi/\sqrt{b_1} \leqq b - a \leqq \pi/\sqrt{b_0}.$$

REFERENCES

1. S. I. Goldberg and S. Kobayashi, *Holomorphic bisectional curvature,* J. Differential Geometry **1** (1967), 225–233. MR **37** #3485.

2. A. Gray, *Nearly Kähler manifolds,* J. Differential Geometry **4** (1970), 283–309.

3. P. Griffiths, *Hermitian differential geometry, Chern classes, and positive vector bundles,* Global Analysis (Papers in Honor of K. Kodaira), Univ. Tokyo Press, Tokyo, 1969, pp. 185–251. MR **41** #2717.

4. S. Kobayashi and R. Nomizu, *Foundations of differential geometry.* II, Interscience Tracts in Pure and Appl. Math., no. 15, vol. 2, Interscience, New York, 1969. MR **38** #6501.

5. S. Nakano, *On complex analytic vector bundles,* J. Math. Soc. Japan **7** (1955), 1–12. MR **17**, 409.

6. L. Nirenberg and D. C. Spencer, *On rigidity of holomorphic imbeddings,* Contributions to Function Theory (Internat. Colloq. Function Theory, Bombay, 1960), Tata Institute of Fundamental Research, Bombay, 1960, pp. 133–137. MR **23** #A3279.

7. S. Smale, *On the Morse index theorem,* J. Math. Mech. **14** (1965), 1049–1055. MR **31** #6251.

UNIVERSITY OF MARYLAND, COLLEGE PARK, MARYLAND 20742