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CALCULUS OF VARIATIONS ON TIME SCALES

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ABSTRACT. We introduce a version of the calculus of variations on time scales, which includes as special cases the classical calculus of variations and the discrete calculus of variations. Necessary conditions for weak local minima are established, among them the Euler condition, the Legendre condition, the strengthened Legendre condition, and the Jacobi condition.

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1. INTRODUCTION

In order to motivate the results presented in this paper, we start by recalling the well-known Legendre necessary condition from the classical calculus of variations as well as the corresponding result from the discrete theory.

Result 1.1 (Legendre's Necessary Condition)**.** *If $\hat{y}$ is a weak local minimum (see [7]) of the variational problem*

$$\mathcal{L}(y) = \int_a^b L(t, y(t), \dot{y}(t))\mathrm{d}t \to \min, \quad y(a) = \alpha, \; y(b) = \beta, \tag{1.1}$$

where $a, b \in \mathbb{R}$ with $a < b$; $\alpha, \beta \in \mathbb{R}^n$ with $n \in \mathbb{N}$, and $L : \mathbb{R}^{2n+1} \to \mathbb{R}$ is a C^2-function, then necessarily

$$P(t) \geq 0 \quad \textit{for all} \quad t \in [a, b], \tag{1.2}$$

where $P = L_{vv}(\cdot, \hat{y}, \dot{\hat{y}})$.

Note that the above P is the second derivative of L w.r.t. the third variable. It is an $n \times n$-matrix-valued function, and $P(t) \geq 0$ means that $P(t)$ is a positive semidefinite matrix.

Result 1.2 (Discrete Version of Legendre's Necessary Condition)**.** *If $\hat{y}$ is a local minimum (see [3]) of the discrete variational problem*

$$\mathcal{L}(y) = \sum_{t=a}^{b-1} L(t, y(t+1), \Delta y(t)) \to \min, \quad y(a) = \alpha, \; y(b) = \beta, \tag{1.3}$$

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where $a, b \in \mathbb{Z}$ *with* $a < b$; $\alpha, \beta \in \mathbb{R}^n$ *with* $n \in \mathbb{N}$, *and* $L : \mathbb{Z} \times \mathbb{R}^{2n} \to \mathbb{R}$ *is a* C^2*-function in the last two variables, then necessarily*

$$P(t) + R(t) + R^T(t) + Q(t) + P(t+1) \geq 0 \quad \text{for all} \quad t \in [a, b-2] \cap \mathbb{Z}, \tag{1.4}$$

where $P(t) = L_{vv}(t, \hat{y}(t+1), \Delta\hat{y}(t))$, $Q(t) = L_{xx}(t, \hat{y}(t+1), \Delta\hat{y}(t))$, *and* $R(t) = L_{xv}(t, \hat{y}(t+1), \Delta\hat{y}(t))$.

Note that above the subscript x denotes differentiation of L w.r.t. the second variable.

On the first view conditions (1.2) and (1.4) do not have much in common. E.g., neither of the two conditions implies the other one. In fact we may have in the discrete scalar case a local minimum at $\hat{y}$ such that $p(t) = L_{vv}(t, \hat{y}(t+1), \Delta\hat{y}(t)) < 0$ at every other time (for example, if $L(t, x, v) = (-1)^t v^2 + \frac{1}{2}x^2$ and $\alpha = \beta = 0$, then $\hat{y} = 0$ is a local minimum), which is of course impossible in the continuous case.

In this paper we will present (among other results) a version of Legendre's necessary condition that contains both Result 1.1 and Result 1.2 as special cases. A theory that allows to establish this kind of results has been introduced by Stefan Hilger in [8] and continued in e.g., [1, 2, 4–6, 9], namely the calculus on time scales. It is able to explain the nature of differences as e.g., occurring between Result 1.1 and Result 1.2, and allows an extension of the results to other time scales, possibly different from those that correspond to the continuous and to the discrete case. To illustrate this approach we finish this introduction by now stating Legendre's condition for an arbitrary time scale $\mathbb{T}$. In order to compare this result with Result 1.1 and Result 1.2 it is at this point sufficient to know that

- if $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$, $\mu(t) = 0$, $[a, b]^{\kappa^2} = [a, b]$,
$$f^\Delta = f, \quad \text{and} \quad \int_a^b f(t)\Delta t = \int_a^b f(t)\mathrm{d}t;$$
- if $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t + 1$, $\mu(t) = 1$, $[a, b]^{\kappa^2} = [a, b-2] \cap \mathbb{Z}$,
$$f^\Delta = \Delta f, \quad \text{and} \quad \int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t).$$

We abbreviate $f \circ \sigma$ by f^σ.

Result 1.3 (Legendre's Necessary Condition for Time Scales)**.** *If* $\hat{y}$ *is a weak local minimum* (*see Definition 3.1 below*) *of the variational problem*

$$\mathcal{L}(y) = \int_a^b L(t, y^\sigma(t), y^\Delta(t))\Delta t \to \min, \quad y(a) = \alpha, \ y(b) = \beta, \tag{1.5}$$

where $a, b \in \mathbb{T}$ *with* $a < b$; $\alpha, \beta \in \mathbb{R}^n$ *with* $n \in \mathbb{N}$, *and* $L : \mathbb{T} \times \mathbb{R}^{2n} \to \mathbb{R}$ *satisfies the assumption in Lemma 3.4 below, then necessarily*

$$P(t) + \mu(t)\left\{R(t) + R^T(t) + \mu(t)Q(t) + (\mu(\sigma(t)))^\dagger P(\sigma(t))\right\} \geq 0, \quad t \in [a, b]^{\kappa^2}, \tag{1.6}$$

where

$$P = L_{vv}(\cdot, \hat{y}^\sigma, \hat{y}^\Delta),\ Q = L_{xx}(\cdot, \hat{y}^\sigma, \hat{y}^\Delta),\ R = L_{xv}(\cdot, \hat{y}^\sigma, \hat{y}^\Delta), \tag{1.7}$$

and where $\alpha^\dagger = \frac{1}{\alpha}$ *if* $\alpha \in \mathbb{R} \setminus \{0\}$ *and* $0^\dagger = 0$.

It is interesting to compare (1.5) with (1.1) and (1.3), and (1.6) with (1.2) and (1.4).

This paper is organized as follows. In the next section we give a brief introduction into the time scales calculus. Section 3 introduces the variational problem (1.5) and its so-called first and second variation. In Sections 4 and 5 we then present versions of Euler's and Legendre's (see Result 1.3) necessary conditions, respectively. Finally, in Section 6, we discuss the strengthened Legendre condition as well as Jacobi's condition on time scales.

2. THE TIME SCALES CALCULUS

A closed subset $\mathbb{T}$ of $\mathbb{R}$ is called a *time scale*. The *jump operators* $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} :\ s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T} :\ s < t\}$$

(supplemented by $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$). A point $t \in \mathbb{T}$ is called *right-dense*, *right-scattered*, *left-dense*, and *left-scattered* if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$, and $\rho(t) < t$, respectively. Throughout we let $a, b \in \mathbb{T}$ with $a < b$. For an interval $[a,b] \cap \mathbb{T}$ we simply write $[a,b]$ when this is not ambiguous. We also define

$$[a,b]^\kappa = [a,b] \setminus (\rho(b), b] \quad \text{and} \quad [a,b]^{\kappa^2} = [a,b] \setminus (\rho(\rho(b)), b].$$

Next, the *graininess* μ is defined by $\mu(t) = \sigma(t) - t$. We say that a function f defined on $\mathbb{T}$ is *differentiable* at $t \in \mathbb{T}$ if for all $\varepsilon > 0$ there is a neighborhood U of t such that for some α the inequality

$$|f(\sigma(t)) - f(s) - \alpha(\sigma(t) - s)| < \varepsilon|\sigma(t) - s|$$

is true for all $s \in U$, and in this case we write $f^\Delta(t) = \alpha$. Note that in right-dense points $f^\Delta(t) = \lim_{s\to t} \frac{f(t)-f(s)}{t-s}$ provided this limit exists and in right-scattered points $f^\Delta(t) = \frac{f(\sigma(t))-f(t)}{\mu(t)}$ provided f is continuous at t. For differentiable f, the formula

$$f^\sigma = f + \mu f^\Delta \tag{2.1}$$

is very useful and easy to prove. If f and g are both differentiable, then so is fg with

$$(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta. \tag{2.2}$$

Next, a function f on $\mathbb{T}$ is called *rd-continuous* if it is continuous in right-dense points and if its left-sided limit exists in left-dense points. By $\mathrm{C_{rd}}$ we denote the set of all rd-continuous functions, while $\mathrm{C^1_{rd}}$ denotes the set of all differentiable functions with rd-continuous derivative. It is known that rd-continuous functions possess an *antiderivative*, i.e., there exists a function F with $F^\Delta = f$, and in this case an *integral* is defined by $\int_a^b f(t)\Delta t = F(b) - F(a)$. It satisfies

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t). \tag{2.3}$$

We need one more result that has not been available in the literature so far. But its proof is easy and we will sketch it below. To present this result we need another definition.

Definition 2.1. A function f defined on $[a,b]\times\mathbb{R}$ is called *continuous in the second variable, uniformly in the first variable*, if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|x_1 - x_2| < \delta$ implies $|f(t,x_1) - f(t,x_2)| < \varepsilon$ for all $t \in [a,b]$.

Lemma 2.2. *Suppose that $F(x) = \int_a^b f(t,x)\Delta t$ is well defined. If $f_x = \frac{\partial f}{\partial x}$ is continuous in x, uniformly in t, then $F'(x) = \int_a^b f_x(t,x)\Delta t$.*

Proof. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that $|f_x(t,x_1) - f_x(t,x_2)| < \varepsilon$ whenever $|x_1 - x_2| < \delta$ and $t \in [a,b]$. Let $h \in \mathbb{R}$ with $|h| < \delta$. Then

$$\begin{aligned}
&\left|\frac{F(x+h)-F(x)}{h} - \int_a^b f_x(t,x)\Delta t\right| = \left|\int_a^b \frac{f(t,x+h)-f(t,x)}{h}\Delta t - \int_a^b f_x(t,x)\Delta t\right| \\
&= \left|\int_a^b f_x(t,x+\theta h)\Delta t - \int_a^b f_x(t,x)\Delta t\right| \\
&\leq \int_a^b |f_x(t,x+\theta h) - f_x(t,x)|\Delta t \;<\; \varepsilon(b-a),
\end{aligned}$$

where $\theta = \theta(t,x) \in (0,1)$. □

3. THE VARIATIONAL PROBLEM

We now consider the variational problem (1.5).

Definition 3.1. For $f \in \mathrm{C}^1_{\mathrm{rd}}$ we define the norm

$$||f|| = \max_{t\in[a,b]^\kappa} |f^\sigma(t)| + \max_{t\in[a,b]^\kappa} |f^\Delta(t)|.$$

A function $\hat{y} \in \mathrm{C}^1_{\mathrm{rd}}$ with $\hat{y}(a) = \alpha$ and $\hat{y}(b) = \beta$ is called a (*weak*) *local minimum* of (1.5) provided there exists $\delta > 0$ such that $\mathcal{L}(\hat{y}) \leq \mathcal{L}(y)$ for all $y \in \mathrm{C}^1_{\mathrm{rd}}$ with $y(a) = \alpha$ and $y(b) = \beta$ and $||y - \hat{y}|| < \delta$. If $\mathcal{L}(\hat{y}) < \mathcal{L}(y)$ for all such $y \neq \hat{y}$, then $\hat{y}$ is said to be *proper*. Finally, an $\eta \in \mathrm{C}^1_{\mathrm{rd}}$ is called an *admissible variation* provided $\eta(a) = \eta(b) = 0$.

Now, for an admissible variation η, we define a function $\Phi : \mathbb{R} \to \mathbb{R}$ by

$$\Phi(\varepsilon) = \Phi(\varepsilon; y, \eta) = \mathcal{L}(y + \varepsilon\eta), \qquad \varepsilon \in \mathbb{R}.$$

The *first and second variation* of the variational problem (1.5) are defined by

$$\mathcal{L}_1(y,\eta) = \dot{\Phi}(0;y,\eta) \quad \text{and} \quad \mathcal{L}_2(y,\eta) = \ddot{\Phi}(0;y,\eta),$$

respectively.

The following two results are easy to prove (see e.g., [1, Theorems 1 and 2] and offer necessary and sufficient conditions for local minima of (1.5) in terms of its first and second variation.

Theorem 3.2 (Necessary Conditions)**.** *If $\hat{y}$ is a local minimum of* (1.5)*, then*

$$\mathcal{L}_1(\hat{y},\eta) = 0 \quad \textit{and} \quad \mathcal{L}_2(\hat{y},\eta) \geq 0 \quad \textit{for all admissible variations } \eta.$$

Theorem 3.3 (Sufficient Condition)**.** *Let $\hat{y} \in \mathrm{C}^1_{\mathrm{rd}}$ with $\hat{y}(a) = \alpha$ and $\hat{y}(b) = \beta$. If $\mathcal{L}_1(\hat{y},\eta) = 0$ and $\mathcal{L}_2(\hat{y},\eta) > 0$ for all nontrivial admissible variations η, then $\hat{y}$ is a proper weak local minimum of* (1.5)*.*

Functionals $\mathcal{L}_2$ that satisfy the condition from Theorem 3.2 and Theorem 3.3 are called *positive semidefinite* and *positive definite*, respectively.

In view of the above two results it will be important to find another representation of the first and second variation. This is done in the following lemma.

Lemma 3.4. *Put $f(t,\varepsilon) = L(t, y^\sigma(t) + \varepsilon\eta^\sigma(t), y^\Delta(t) + \varepsilon\eta^\Delta(t))$. If f_ε and $f_{\varepsilon\varepsilon}$ are continuous in ε, uniformly in t, then*

$$\begin{aligned}
\mathcal{L}_1(y,\eta) &= \int_a^b \left\{L_x(t,y^\sigma(t),y^\Delta(t))\eta^\sigma(t) + L_v(t,y^\sigma(t),y^\Delta(t))\eta^\Delta(t)\right\}\Delta t,\\
\mathcal{L}_2(y,\eta) &= \int_a^b \left\{(\eta^\sigma)^T Q\eta^\sigma + 2(\eta^\sigma)^T R\eta^\Delta + (\eta^\Delta)^T P\eta^\Delta\right\}(t)\Delta t,
\end{aligned}$$

where P, Q, and R are defined by (1.7) (*with $\hat{y}$ replaced by y*)*.*

Proof. This is a direct consequence of Lemma 2.2. □

In the sequel we always assume without mentioning it further that L satisfies the assumption of Lemma 3.4 for all y and η.

4. EULER'S CONDITION

We start with the following easy extension of the fundamental lemma of variational analysis to time scales.

Lemma 4.1 (Dubois–Reymond)**.** *Let $g \in \mathrm{C}_{\mathrm{rd}}$, $g : [a,b] \to \mathbb{R}^n$. Then*

$$\int_a^b g^T(t)\eta^\Delta(t)\Delta t = 0 \textit{ for all } \eta \in \mathrm{C}^1_{\mathrm{rd}} \textit{ with } \eta(a) = \eta(b) = 0 \tag{4.1}$$

holds if and only if

$$g(t) \equiv c \textit{ on } [a,b]^\kappa \textit{ for some } c \in \mathbb{R}^n. \tag{4.2}$$

Proof. First suppose that (4.2) holds. Then, by the definition of the integral,

$$\begin{aligned}\int_a^b g^T(t)\eta^\Delta(t)\Delta t &= \int_a^b c^T\eta^\Delta(t)\Delta t \\ &= c^T\int_a^b \eta^\Delta(t)\Delta t \\ &= c^T\left(\eta(b)-\eta(a)\right) = 0\end{aligned}$$

if $\eta \in \mathrm{C}^1_{\mathrm{rd}}$ with $\eta(a)=\eta(b)=0$, and hence (4.1) holds. Next, assume (4.1), and put

$$G(t) = \int_a^t g(\tau)\Delta\tau, \quad t\in[a,b] \quad \text{and} \quad c = \frac{G(b)}{b-a}.$$

Define $\eta(t) = G(t)-(t-a)c$ for $t\in[a,b]$. Then $\eta^\Delta(t) = g(t)-c$ so that $\eta\in\mathrm{C}^1_{\mathrm{rd}}$. Also,

$$\eta(a) = G(a) = 0$$

and

$$\eta(b) = G(b)-(b-a)c = G(b)-(b-a)\frac{G(b)}{b-a} = 0.$$

Hence, by (4.1),

$$\begin{aligned}0 &= \int_a^b g^T(t)\eta^\Delta(t)\Delta t = \int_a^b g^T(t)\left\{g(t)-c\right\}\Delta t \\ &= \int_a^b \left\{g(t)-c\right\}^T\left\{g(t)-c\right\}\Delta t + \int_a^b c^T\left\{g(t)-c\right\}\Delta t \\ &= \int_a^b \left|g(t)-c\right|^2\Delta t + c^TG(b) - c^Tc(b-a) \\ &= \int_a^b \left|g(t)-c\right|^2\Delta t.\end{aligned}$$

Therefore (see e.g., [8]) $g(t)-c=0$ for all $t\in[a,b]^\kappa$ so that (4.2) holds. □

Now, using the fundamental lemma from above, we can derive Euler's necessary condition.

Theorem 4.2 (Euler's Necessary Condition)**.** *If* $\hat{y}$ *is a local minimum of* (1.5), *then the* Euler–Lagrange equation

$$L_v^\Delta(t,\hat{y}^\sigma(t),\hat{y}^\Delta(t)) = L_x(t,\hat{y}^\sigma(t),\hat{y}^\Delta(t)), \quad t\in[a,b]^\kappa \tag{4.3}$$

holds.

Proof. Let $\hat{y}$ be a local minimum of (1.5) and suppose that η is an admissible variation. Then we have by Theorem 3.2

$$\begin{aligned}
0 &= \mathcal{L}_1(\hat{y},\eta) = \dot{\Phi}(0) \\
&= \int_a^b \left\{L_x(t,\hat{y}^\sigma(t),\hat{y}^\Delta(t))\eta^\sigma(t) + L_v(t,\hat{y}^\sigma(t),\hat{y}^\Delta(t))\eta^\Delta(t)\right\}\Delta t \\
&= \int_a^b \Bigg\{\left[\int_a^t L_x(\tau,\hat{y}^\sigma(\tau),\hat{y}^\Delta(\tau))\Delta\tau\eta(t)\right]^\Delta - \left[\int_a^t L_x(\tau,\hat{y}^\sigma(\tau),\hat{y}^\Delta(\tau))\Delta\tau\right]\eta^\Delta(t) \\
&\qquad + L_v(t,\hat{y}^\sigma(t),\hat{y}^\Delta(t))\eta^\Delta(t)\Bigg\}\Delta t \\
&= \int_a^b \left\{L_v(t,\hat{y}^\sigma(t),\hat{y}^\Delta(t)) - \int_a^t L_x(\tau,\hat{y}^\sigma(\tau),\hat{y}^\Delta(\tau))\Delta\tau\right\}\eta^\Delta(t)\Delta t.
\end{aligned}$$

Therefore, by Lemma 4.1, we have

$$L_v(t,\hat{y}^\sigma(t),\hat{y}^\Delta(t)) = \int_a^t L_x(\tau,\hat{y}^\sigma(\tau),\hat{y}^\Delta(\tau))\Delta\tau, \quad t\in[a,b],$$

and hence (4.3) follows. □

Example 4.3. Find the solution of the problem

$$\int_a^b \sqrt{1+(y^\Delta(t))^2}\Delta t \to \min, \quad y(a)=0, \quad y(b)=1. \tag{4.4}$$

Writing (4.4) in the form of (1.5), we have $\alpha=0$, $\beta=1$, $n=1$, and

$$L(t,x,v)=\sqrt{1+v^2}.$$

Next,

$$L_x(t,x,v)=0 \quad \text{and} \quad L_v(t,x,v)=\frac{v}{\sqrt{1+v^2}}.$$

Suppose $\hat{y}$ is a local minimum of (4.4). Then, by Theorem 4.2, the Euler equation (4.3) must hold, i.e.,

$$L_v^\Delta(t,\hat{y}^\sigma(t),\hat{y}^\Delta(t))=0, \quad t\in[a,b]^\kappa. \tag{4.5}$$

Now (4.5) implies that there exists a constant $c\in\mathbb{R}$ such that

$$L_v(t,\hat{y}^\sigma(t),\hat{y}^\Delta(t))\equiv c, \quad t\in[a,b],$$

i.e.,

$$\hat{y}^\Delta(t)=c\sqrt{1+(\hat{y}^\Delta(t))^2}, \quad t\in[a,b] \tag{4.6}$$

holds. It is easy to solve equation (4.6) to obtain

$$\hat{y}(t)=\frac{t-a}{b-a} \quad \text{for all} \quad t\in[a,b],$$

i.e., $\hat{y}$ is the line connecting the points $(a,0)$ and $(b,1)$, as expected.

5. LEGENDRE'S CONDITION

In this section we prove Result 1.3 stated in the introduction.

Proof of Result 1.3. Let $s \in [a,b]^{\kappa^2}$. We consider two cases. First, we suppose that $s < \sigma(s) < b$. Let $\gamma \in \mathbb{R}^n$ be arbitrary and define $\eta : [a,b] \to \mathbb{R}^n$ by

$$\eta(t) = \begin{cases} \gamma\sqrt{\mu(s)} & \text{if } t = \sigma(s), \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $\eta(a) = \eta(b) = 0$ and that η is an admissible variation. We have

$$\eta^\Delta(s) = \frac{\eta(\sigma(s)) - \eta(s)}{\mu(s)} = \frac{\gamma\sqrt{\mu(s)}}{\mu(s)} = \frac{\gamma}{\sqrt{\mu(s)}}$$

and

$$\eta^\Delta(\sigma(s)) = \frac{\eta(\sigma(\sigma(s))) - \eta(\sigma(s))}{\mu(\sigma(s))} = -\frac{\gamma\sqrt{\mu(s)}}{\mu(\sigma(s))}.$$

Furthermore, $\eta^\Delta(t) = 0$ for all $t \in [a,b] \setminus \{s, \sigma(s)\}$. Thus,

$$\begin{aligned} &\int_a^{\sigma(s)} \left\{(\eta^\sigma)^T Q \eta^\sigma + 2(\eta^\sigma)^T R \eta^\Delta + (\eta^\Delta)^T P \eta^\Delta\right\}(t)\Delta t \\ &= \mu(s)\left\{(\eta^\sigma)^T Q \eta^\sigma + 2(\eta^\sigma)^T R \eta^\Delta + (\eta^\Delta)^T P \eta^\Delta\right\}(s) \\ &= \gamma^T\left\{P(s) + \mu(s)\left[R(s) + R^T(s) + \mu(s)Q(s)\right]\right\}\gamma. \end{aligned}$$

The integral

$$\int_{\sigma(s)}^b \left\{(\eta^\sigma)^T Q \eta^\sigma + 2(\eta^\sigma)^T R \eta^\Delta + (\eta^\Delta)^T P \eta^\Delta\right\}(t)\Delta t$$

is equal to zero if $\sigma(s)$ is right-dense, and it is equal to

$$\mu(\sigma(s))(\eta^\Delta(\sigma(s)))^T P(\sigma(s))\eta^\Delta(\sigma(s)) = \frac{\mu(s)}{\mu(\sigma(s))}\gamma^T P(\sigma(s))\gamma$$

if $\sigma(s)$ is right-scattered. In either case, by adding the integrals from a to $\sigma(s)$ and from $\sigma(s)$ to b and by using Theorem 3.2, the matrix in (1.6) for $t = s$ is indeed positive semidefinite.

Second, suppose that $s \in (a,b)$ is right-dense, i.e., $\sigma(s) = s$. (If the point $s = a$ is right-dense, one can use continuity of the matrices occurring in (1.7) in order to show (1.6) for $t = a$, and similarly for $t = b$ is b is left-dense.) We assume that s is also left-scattered (the case that s is also left-dense can be treated similarly). Then there exists a strictly decreasing sequence $\{s_k : k \in \mathbb{N}\} \subset [a,b]$ with $\lim_{k\to\infty} s_k = s$. Let $\gamma \in \mathbb{R}^n$ and define for $k \in \mathbb{N}$

$$\eta_k(t) = \begin{cases} \gamma\frac{s_k - t}{\sqrt{s_k - s}} & \text{if } t \in [s, s_k], \\ 0 & \text{otherwise.} \end{cases}$$

Again it follows that η_k is an admissible variation, and we have

$$\begin{aligned}
&\int_a^s \left\{(\eta^\sigma)^T Q\eta^\sigma + 2(\eta^\sigma)^T R\eta^\Delta + (\eta^\Delta)^T P\eta^\Delta\right\}(t)\Delta t \\
&= \mu(\rho(s))\left\{(\eta^\sigma)^T Q\eta^\sigma + 2(\eta^\sigma)^T R\eta^\Delta + (\eta^\Delta)^T P\eta^\Delta\right\}(\rho(s))\Delta t \\
&= \mu(\rho(s))\left\{(s_k - s)\gamma^T Q(\rho(s))\gamma + 2\frac{s_k - s}{\mu(\rho(s))}\gamma^T R(\rho(s))\gamma + \frac{s_k - s}{(\mu(\rho(s)))^2}\gamma^T P(\rho(s))\gamma\right\} \\
&= (s_k - s)\gamma^T\left\{2R(\rho(s)) + \mu(\rho(s))Q(\rho(s)) + \frac{1}{\mu(\rho(s))}P(\rho(s))\right\}\gamma
\end{aligned}$$

and

$$\begin{aligned}
&\int_s^b \left\{(\eta^\sigma)^T Q\eta^\sigma + 2(\eta^\sigma)^T R\eta^\Delta + (\eta^\Delta)^T P\eta^\Delta\right\}(t)\Delta t \\
&\qquad = \int_s^{s_k}\left\{\frac{(s_k - \sigma(t))^2}{s_k - s}\gamma^T Q(t)\gamma - 2\frac{s_k - \sigma(t)}{s_k - s}\gamma^T R(t)\gamma + \frac{1}{s_k - s}\gamma^T P(t)\gamma\right\}\Delta t.
\end{aligned}$$

By Theorem 3.2, we have

$$\mathcal{L}_2(\hat{y}, \eta_k) \geq 0,$$

and letting $k \to \infty$, we obtain

$$\gamma^T P(s)\gamma \geq 0.$$

Again, the matrix in (1.6) is (note $\mu(s) = 0$) positive semidefinite. □

6. JACOBI'S CONDITION

We need the following result from [1, 10].

Theorem 6.1. *Suppose A, B, and C are $n \times n$-matrix-valued rd-continuous functions on $\mathbb{T}$ such that $I - \mu(t)A(t)$ is invertible, $B(t)$ is invertible and symmetric, and $C(t)$ is symmetric for each $t \in \mathbb{T}$. Then $\int_a^b \left\{(x^\sigma)^T C x^\sigma + u^T B u\right\}(t)\Delta t > 0$ for all pairs (x, u) with $x(a) = x(b) = 0$, $x \neq 0$, and $x^\Delta = Ax^\sigma + Bu$, if and only if the linear Hamiltonian dynamic system*

$$x^\Delta = A(t)x^\sigma + B(t)u, \quad u^\Delta = C(t)x^\sigma - A^T(t)u \tag{6.1}$$

is disconjugate on $[a, b]$, i.e., the $n \times n$-matrix-valued solution (X, U) of (6.1) with $X(a) = 0$ and $U(a) = I$ satisfies

$$X^\sigma \text{ invertible, } X(X^\sigma)^{-1}(I - \mu A)^{-1}B \geq 0 \quad \text{on} \quad [a, b]^\kappa. \tag{6.2}$$

We now give a condition (the so-called "strengthened Legendre condition") that ensures that we can rewrite $\mathcal{L}_2$ as a quadratic functional of the form given in Theorem 6.1.

Lemma 6.2. *Suppose the strengthened Legendre condition*

$$P(t) \text{ and } P(t)+\mu(t)R(t) \text{ are invertible for all } t \in [a,b]^\kappa \tag{6.3}$$

holds. Then

$$\mathcal{L}_2(\hat{y},\eta) = \int_a^b \left\{(\eta^\sigma)^T C\eta^\sigma + \xi^T B\xi\right\}(t)\Delta t,$$

where $\xi = P\eta^\Delta + R^T\eta^\sigma$, $B = P^{-1}$, *and* $C = Q - RP^{-1}R^T$.

Proof. We have

$$\begin{aligned}
\mathcal{L}_2(\hat{y},\eta) &= \int_a^b \left\{(\eta^\sigma)^T Q\eta^\sigma + 2(\eta^\sigma)^T R\eta^\Delta + (\eta^\Delta)^T P\eta^\Delta\right\}(t)\Delta t \\
&= \int_a^b \left\{(\eta^\sigma)^T [C+RBR^T]\eta^\sigma + 2(\eta^\sigma)^T RBP\eta^\Delta + (P\eta^\Delta)^T B(P\eta^\Delta)\right\}(t)\Delta t \\
&= \int_a^b \left\{(\eta^\sigma)^T C\eta^\sigma + [P\eta^\Delta + R^T\eta^\sigma]^T B[P\eta^\Delta + R^T\eta^\sigma]\right\}(t)\Delta t \\
&= \int_a^b \left\{(\eta^\sigma)^T C\eta^\sigma + \xi^T B\xi\right\}(t)\Delta t.
\end{aligned}$$

This proves our result. □

The question of positive definiteness of $\mathcal{L}_2$, which arises when trying to apply Theorem 3.3, is now satisfactorily answered by the following theorem. For this we put

$$A = -P^{-1}R^T, \qquad B = P^{-1}, \qquad C = Q - RP^{-1}R^T$$

and note that $\xi = P\eta^\Delta + R^T\eta^\sigma$ is equivalent to $\eta^\Delta = A\eta^\sigma + B\xi$.

Theorem 6.3 (Jacobi's Condition)**.** *Assume that the strengthened Legendre condition* (6.3) *holds. Then* $\mathcal{L}_2$ *is positive definite if and only if the linear Hamiltonian system*

$$\eta^\Delta = A(t)\eta^\sigma + B(t)\xi, \qquad \xi^\Delta = C(t)\eta^\sigma - A^T(t)\xi$$

is disconjugate on $[a,b]$.

Proof. Note that (6.3) implies that

$$I - \mu A = I + \mu P^{-1}R^T = P^{-1}(P+\mu R^T) = \left\{(P+\mu R)P^{-1}\right\}^T \text{ is invertible.}$$

Hence our theorem is a special case of Theorem 6.1. □

Remark 6.4. If R is symmetric and differentiable, then we can write

$$\begin{aligned}
(\eta^T R\eta)^\Delta &= (\eta^\sigma)^T R^\Delta\eta^\sigma + (\eta^\sigma)^T R\eta^\Delta + (\eta^\Delta)^T R\eta \\
&= (\eta^\sigma)^T R^\Delta\eta^\sigma + 2(\eta^\sigma)^T R\eta^\Delta - \mu(\eta^\Delta)^T R\eta^\Delta
\end{aligned}$$

and hence

$$\begin{aligned}\mathcal{L}_2(y,\eta) &= \int_a^b \left\{(\eta^\sigma)^T Q\eta^\sigma + (\eta^T R\eta)^\Delta - (\eta^\sigma)^T R^\Delta \eta^\sigma\right. \\ &\qquad \left. +\mu(\eta^\Delta)^T R\eta^\Delta + (\eta^\Delta)^T P\eta^\Delta\right\}(t)\Delta t \\ &= \int_a^b \left\{(\eta^\sigma)^T \tilde{Q}\eta^\sigma + (\eta^\Delta)^T \tilde{P}\eta^\Delta\right\}(t)\Delta t,\end{aligned}$$

where $\tilde{Q} = Q - R^\Delta$ and $\tilde{P} = P + \mu R$. We hence have the following result.

Theorem 6.5. *Let R be symmetric and invertible. If the alternative strengthened Legendre condition*

$$P(t) + \mu(t)R(t) \quad \textit{is invertible for all} \quad t \in [a,b]^\kappa$$

holds, then $\mathcal{L}_2$ is positive definite if and only if the linear Hamiltonian system

$$\eta^\Delta = \tilde{P}^{-1}(t)\xi, \quad \xi^\Delta = \tilde{Q}(t)\eta^\sigma$$

is disconjugate on $[a,b]$.

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