Mathematica
Balkanica

New Series Vol. 26, 2012, Fasc. 1-2

Calculus of Variations with Classical and Fractional Derivatives

Tatiana Odzijewicz [1], *Delfim F. M. Torres* [2]

Presented at 6^{th} International Conference "TMSF' 2011"

We give a proper fractional extension of the classical calculus of variations. Necessary optimality conditions of Euler–Lagrange type for variational problems containing both classical and fractional derivatives are proved. The fundamental problem of the calculus of variations with mixed integer and fractional order derivatives as well as isoperimetric problems are considered.

MSC 2010: 49K05, 26A33

Key Words: variational analysis, optimality, Riemann–Liouville fractional operators, fractional differentiation, isoperimetric problems

1. Introduction

One of the classical problems of mathematics consists in finding a closed plane curve of a given length that encloses the greatest area: the *isoperimetric problem*. The legend says that the first person who solved the isoperimetric problem was Dido, the Queen of Carthage, who was offered as much land as she could surround with the skin of a bull. Dido's problem is nowadays part of the *calculus of variations* [23, 35].

Fractional calculus is a generalization of (integer) differential calculus, allowing to define derivatives (and integrals) of real or complex order [25, 30, 32]. The first application of fractional calculus belongs to Niels Henrik Abel (1802–1829) and goes back to 1823 [1]. Abel applied the fractional calculus to the solution of an integral equation which arises in the formulation of the *tautochrone problem*. This problem, sometimes also called the *isochrone problem*, is that of finding the shape of a frictionless wire lying in a vertical plane such that the time of a bead placed on the wire slides to the lowest point of the wire in the *same time* regardless of where the bead is placed. The cycloid is the isochrone as well as the *brachistochrone* curve: it gives the *shortest time* of slide and marks the born of the *calculus of variations*.

The study of fractional problems of the calculus of variations and respective Euler–Lagrange type equations is a subject of current strong research due to its many applications in science and engineering, including mechanics, chemistry, biology, economics, and control theory [27]. In 1996–1997 Riewe obtained a version of the Euler–Lagrange equations for fractional variational problems combining the conservative and nonconservative cases [33, 34]. Since then, numerous works on the fractional calculus of variations, fractional optimal control and its applications have been written—see, e.g., [4,7,8,11–14,20–22,26,28] and references therein. For the study of fractional isoperimetric problems, see [5].

In the pioneering paper [2], and others that followed, the fractional necessary optimality conditions are proved under the hypothesis that admissible functions y have continuous left and right fractional derivatives on the closed interval $[a, b]$. By considering that the admissible functions y have continuous left fractional derivatives on the whole interval, then necessarily $y(a) = 0$; by considering that the admissible functions y have continuous right fractional derivatives, then necessarily $y(b) = 0$. This fact has been independently remarked, in different contexts, at least in [5, 11, 13, 24].

In our work we want to be able to consider arbitrarily given boundary conditions $y(a) = y_a$ and $y(b) = y_b$ (and isoperimetric constraints). For that we consider variational functionals with integrands involving not only a fractional derivative of order $\alpha \in (0, 1)$ of the unknown function y, but also the classical derivative y'. More precisely, we consider dependence of the integrands on the independent variable t, unknown function y, and $y' + k\, {}_aD_t^\alpha y$ with k a real parameter. As a consequence, one gets a proper extension of the classical calculus of variations, in the sense that the classical theory is recovered with the particular situation $k = 0$. We remark that this is not the case with all the previous literature on the fractional variational calculus, where the classical theory is not included as a particular case and only as a limit, when $\alpha \to 1$.

The text is organized as follows. In Section 2 we briefly recall the necessary definitions and properties of the fractional calculus in the sense of Riemann–Liouville. Our results are stated, proved, and illustrated through an example, in Section 3. We end with Section 4 of conclusion.

2. Preliminaries

In this section some basic definitions and properties of fractional calculus are given. For more on the subject we refer the reader to the books [25, 30, 32] and historical survey [27].

Definition 1. (Left and right Riemann–Liouville derivatives) Let f be a function defined on $[a,b]$. The operator ${}_aD_t^\alpha$,

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} D^n \int_a^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau \,, $$

is called the left Riemann–Liouville fractional derivative of order α, and the operator ${}_tD_b^\alpha$,

$$ {}_tD_b^\alpha f(t) = \frac{-1}{\Gamma(n-\alpha)} D^n \int_t^b (\tau - t)^{n-\alpha-1} f(\tau) d\tau \,, $$

is called the right Riemann–Liouville fractional derivative of order α, where $\alpha \in \mathbb{R}^+$ is the order of the derivatives and the integer number n is such that $n-1 \leq \alpha < n$.

Definition 2. (Mittag–Leffler function) Let $\alpha, \beta > 0$. The Mittag–Leffler function is defined by

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \,. $$

Theorem 3. *(Integration by parts) If f, g and the fractional derivatives ${}_aD_t^\alpha g$ and ${}_tD_b^\alpha f$ are continuous at every point $t \in [a,b]$, then*

$$ \int_a^b f(t) {}_aD_t^\alpha g(t) dt = \int_a^b g(t) {}_tD_b^\alpha f(t) dt \qquad (1) $$

for any $0 < \alpha < 1$.

Remark 4. If $f(a) \neq 0$, then ${}_aD_t^\alpha f(t)|_{t=a} = \infty$. Similarly, if $f(b) \neq 0$, then ${}_tD_b^\alpha f(t)|_{t=b} = \infty$. Thus, if f possesses continuous left and right Riemann–Liouville fractional derivatives at every point $t \in [a,b]$, then $f(a) = f(b) = 0$. This explains why the usual term $f(t)g(t)|_a^b$ does not appear on the right-hand side of (1).

3. Main results

Following [24], we prove optimality conditions of Euler–Lagrange type for variational problems containing classical and fractional derivatives simultaneously. In Section 3.1 the fundamental variational problem is considered, while in Section 3.2 we study the isoperimetric problem. Our results cover fractional variational problems subject to arbitrarily given boundary conditions. This is in contrast with [2–4,15], where the necessary optimality conditions are valid for appropriate zero valued boundary conditions (cf. Remark 4). For a discussion on this matter see [11, 13, 24].

3.1. The Euler–Lagrange equation

Let $0 < \alpha < 1$. Consider the following problem: find a function $y \in C^1[a,b]$ for which the functional

$$\mathcal{J}(y) = \int_a^b F\left(t, y(t), y'(t) + k\, {}_aD_t^\alpha y(t)\right) dt \tag{2}$$

subject to given boundary conditions

$$y(a) = y_a, \quad y(b) = y_b, \tag{3}$$

has an extremum. We assume k is a fixed real number, $F \in C^2([a,b] \times \mathbb{R}^2; \mathbb{R})$, and $\partial_3 F$ (the partial derivative of $F(\cdot,\cdot,\cdot)$ with respect to its third argument) has a continuous right Riemann–Liouville fractional derivative of order α.

Definition 5. A function $y \in C^1[a,b]$ that satisfies the given boundary conditions (3) is said to be *admissible* for problem (2)–(3).

For simplicity of notation we introduce the operator $[\cdot]_k^\alpha$ defined by

$$[y]_k^\alpha(t) = \left(t, y(t), y'(t) + k\, {}_aD_t^\alpha y(t)\right) .$$

With this notation we can write (2) simply as

$$\mathcal{J}(y) = \int_a^b F[y]_k^\alpha(t) dt \,.$$

Theorem 6. *(The fractional Euler–Lagrange equation) If y is an extremizer (minimizer or maximizer) of problem (2)–(3), then y satisfies the Euler–Lagrange equation*

$$\partial_2 F[y]_k^\alpha(t) - \frac{d}{dt}\partial_3 F[y]_k^\alpha(t) + k\, {}_tD_b^\alpha \partial_3 F[y]_k^\alpha(t) = 0 \tag{4}$$

for all $t \in [a,b]$.

Proof. Suppose that y is a solution of (2)–(3). Note that admissible functions $\hat{y}$ can be written in the form $\hat{y}(t) = y(t) + \epsilon\eta(t)$, where $\eta \in C^1[a,b]$, $\eta(a) = \eta(b) = 0$, and $\epsilon \in \mathbb{R}$. Let

$$J(\epsilon) = \int_a^b F\left(t, y(t) + \epsilon\eta(t), \frac{d}{dt}\left(y(t) + \epsilon\eta(t)\right) + k_a D_t^\alpha \left(y(t) + \epsilon\eta(t)\right)\right) dt.$$

Since ${}_aD_t^\alpha$ is a linear operator, we know that

$${}_aD_t^\alpha \left(y(t) + \epsilon\eta(t)\right) = {}_aD_t^\alpha y(t) + \epsilon_a D_t^\alpha \eta(t).$$

On the other hand,

$$\left.\frac{dJ}{d\epsilon}\right|_{\epsilon=0} = \left.\int_a^b \frac{d}{d\epsilon} F[\hat{y}]_k^\alpha(t)dt\right|_{\epsilon=0}$$
$$= \int_a^b \Big(\partial_2 F[y]_k^\alpha(t)\cdot\eta(t) + \partial_3 F[y]_k^\alpha(t)\frac{d\eta(t)}{dt} + k\partial_3 F[y]_k^\alpha(t)\,{}_aD_t^\alpha\eta(t)\Big)dt. \tag{5}$$

Using integration by parts we get

$$\int_a^b \partial_3 F\frac{d\eta}{dt}dt = \partial_3 F\eta|_a^b - \int_a^b \left(\eta\frac{d}{dt}\partial_3 F\right)dt \tag{6}$$

and

$$\int_a^b \partial_3 F{}_aD_t^\alpha\eta dt = \int_a^b \eta\,{}_tD_b^\alpha\partial_3 F dt. \tag{7}$$

Substituting (6) and (7) into (5), and having in mind that $\eta(a) = \eta(b) = 0$, it follows that

$$\left.\frac{dJ}{d\epsilon}\right|_{\epsilon=0} = \int_a^b \eta(t)\Big(\partial_2 F[y]_k^\alpha(t) - \frac{d}{dt}\partial_3 F[y]_k^\alpha(t) + k\,{}_tD_b^\alpha\partial_3 F[y]_k^\alpha(t)\Big)dt.$$

A necessary optimality condition is given by $\frac{dJ}{d\epsilon}\big|_{\epsilon=0} = 0$. Hence,

$$\int_a^b \eta(t)\Big(\partial_2 F[y]_k^\alpha(t) - \frac{d}{dt}\partial_3 F[y]_k^\alpha(t) + k\,{}_tD_b^\alpha\partial_3 F[y]_k^\alpha(t)\Big)dt = 0. \tag{8}$$

We obtain equality (4) by applying the fundamental lemma of the calculus of variations to (8). ■

Example 7. Note that for $k = 0$ our necessary optimality condition (4) reduces to the classical Euler–Lagrange equation [23, 35].

3.2. The fractional isoperimetric problem

As before, let $0 < \alpha < 1$. We now consider the problem of extremizing a functional

$$\mathcal{J}(y) = \int_a^b F\left(t, y(t), y'(t) + k\,{}_aD_t^\alpha y(t)\right)dt \tag{9}$$

in the class $y \in C^1[a, b]$ when subject to given boundary conditions

$$y(a) = y_a\,, \quad y(b) = y_b, \tag{10}$$

and an isoperimetric constraint

$$\mathcal{I}(y) = \int_a^b G(t, y(t), y'(t) + k\, {}_aD_t^\alpha y(t))dt = \xi\,. \tag{11}$$

We assume that k and ξ are fixed real numbers, $F, G \in C^2([a,b] \times \mathbb{R}^2; \mathbb{R})$, and $\partial_3 F$ and $\partial_3 G$ have continuous right Riemann–Liouville fractional derivatives of order α.

Definition 8. A function $y \in C^1[a,b]$ that satisfies the given boundary conditions (10) and isoperimetric constraint (11) is said to be *admissible* for problem (9)–(11).

Definition 9. An admissible function y is an *extremal* for $\mathcal{I}$ if it satisfies the fractional Euler–Lagrange equation

$$\partial_2 G[y]_k^\alpha(t) - \frac{d}{dt}\partial_3 G[y]_k^\alpha(t) + k\, {}_tD_b^\alpha \partial_3 G[y]_k^\alpha(t) = 0$$

for all $t \in [a,b]$.

The next theorem gives a necessary optimality condition for the fractional isoperimetric problem (9)–(11).

Theorem 10. *Let y be an extremizer to the functional* (9) *subject to the boundary conditions* (10) *and the isoperimetric constraint* (11). *If y is not an extremal for $\mathcal{I}$, then there exists a constant λ such that*

$$\partial_2 H[y]_k^\alpha(t) - \frac{d}{dt}\partial_3 H[y]_k^\alpha(t) + k\, {}_tD_b^\alpha \partial_3 H[y]_k^\alpha(t) = 0 \tag{12}$$

for all $t \in [a,b]$, where $H(t,y,v) = F(t,y,v) - \lambda G(t,y,v)$.

P r o o f. We introduce the two parameter family

$$\hat{y} = y + \epsilon_1 \eta_1 + \epsilon_2 \eta_2, \tag{13}$$

in which η_1 and η_2 are such that $\eta_1, \eta_2 \in C^1[a,b]$ and they have continuous left and right fractional derivatives. We also require that

$$\eta_1(a) = \eta_1(b) = 0 = \eta_2(a) = \eta_2(b).$$

First we need to show that in the family (13) there are curves such that $\hat{y}$ satisfies (11). Substituting y by $\hat{y}$ in (11), $\mathcal{I}(\hat{y})$ becomes a function of two parameters ϵ_1, ϵ_2. Let

$$\hat{I}(\epsilon_1, \epsilon_2) = \int_a^b G(t, \hat{y}, \hat{y}' + k_a D_t^\alpha \hat{y})dt - \xi.$$

Then, $\hat{I}(0,0)=0$ and

$$\left.\frac{\partial \hat{I}}{\partial \epsilon_2}\right|_{(0,0)} = \int_a^b \eta_2 \left(\partial_2 G - \frac{d}{dt}\partial_3 G + k_t D_b^\alpha \partial_3 G\right) dt.$$

Since y is not an extremal for $\mathcal{I}$, by the fundamental lemma of the calculus of variations there is a function η_2 such that

$$\left.\frac{\partial \hat{I}}{\partial \epsilon_2}\right|_{(0,0)} \neq 0.$$

By the implicit function theorem, there exists a function $\epsilon_2(\cdot)$ defined in a neighborhood of zero, such that $\hat{I}(\epsilon_1, \epsilon_2(\epsilon_1)) = 0$. Let $\hat{J}(\epsilon_1, \epsilon_2) = \mathcal{J}(\hat{y})$. Then, by the Lagrange multiplier rule, there exists a real λ such that

$$\nabla(\hat{J}(0,0) - \lambda \hat{I}(0,0)) = \mathbf{0}.$$

Because

$$\left.\frac{\partial \hat{J}}{\partial \epsilon_1}\right|_{(0,0)} = \int_a^b \eta_1 \left(\partial_2 F - \frac{d}{dt}\partial_3 F + k_t D_b^\alpha \partial_3 F\right) dt$$

and

$$\left.\frac{\partial \hat{I}}{\partial \epsilon_1}\right|_{(0,0)} = \int_a^b \eta_1 \left(\partial_2 G - \frac{d}{dt}\partial_3 G + k_t D_b^\alpha \partial_3 G\right) dt,$$

one has

$$\int_a^b \eta_1 \left[\left(\partial_2 F - \frac{d}{dt}\partial_3 F + k_t D_b^\alpha \partial_3 F\right) - \lambda \left(\partial_2 G - \frac{d}{dt}\partial_3 G + k_t D_b^\alpha \partial_3 G\right)\right] dt = 0.$$

Since η_1 is an arbitrary function, (12) follows from the fundamental lemma of the calculus of variations. ■

3.3. An example

Let $\alpha \in (0,1)$ and $k, \xi \in \mathbb{R}$. Consider the following fractional isoperimetric problem:

$$\begin{gathered}\mathcal{J}(y) = \int_0^1 \left(y' + k\, {}_0D_t^\alpha y\right)^2 dt \longrightarrow \min \\ \mathcal{I}(y) = \int_0^1 \left(y' + k\, {}_0D_t^\alpha y\right) dt = \xi \\ y(0) = 0\,,\ y(1) = \int_0^1 E_{1-\alpha,1}\left(-k\,(1-\tau)^{1-\alpha}\right)\xi d\tau.\end{gathered} \tag{14}$$

In this case the augmented Lagrangian H of Theorem 10 is given by $H(t,y,v) = v^2 - \lambda v$. One can easily check that

$$y(t) = \int_0^t E_{1-\alpha,1}\left(-k\,(t-\tau)^{1-\alpha}\right)\xi d\tau \tag{15}$$

- is not an extremal for $\mathcal{I}$;
- satisfies $y' + k\, {}_0D_t^\alpha y = \xi$ (see, e.g., [25, p. 297, Theorem 5.5]).

Moreover, (15) satisfies (12) for $\lambda = 2\xi$, i.e.,

$$-\frac{d}{dt}\left(2\left(y' + k\, {}_0D_t^\alpha y\right) - 2\xi\right) + k\, {}_tD_1^\alpha \left(2\left(y' + k\, {}_0D_t^\alpha y\right) - 2\xi\right) = 0.$$

We conclude that (15) is the extremal for problem (14).

Example 11. Choose $k = 0$. In this case the isoperimetric constraint is trivially satisfied, (14) is reduced to the classical problem of the calculus of variations

$$\begin{gathered}\mathcal{J}(y) = \int_0^1 (y'(t))^2 dt \longrightarrow \min \\ y(0) = 0\,, \quad y(1) = \xi,\end{gathered} \tag{16}$$

and our general extremal (15) simplifies to the well-known minimizer $y(t) = \xi t$ of (16).

Example 12. When $\alpha \to 1$ the isoperimetric constraint is redundant with the boundary conditions, and the fractional problem (14) simplifies to the classical variational problem

$$\begin{gathered}\mathcal{J}(y) = (k+1)^2 \int_0^1 y'(t)^2 dt \longrightarrow \min \\ y(0) = 0\,, \quad y(1) = \frac{\xi}{k+1}.\end{gathered} \tag{17}$$

Our fractional extremal (15) gives $y(t) = \frac{\xi}{k+1}t$, which is exactly the minimizer of (17).

Example 13. Choose $k = \xi = 1$. When $\alpha \to 0$ one gets from (14) the classical isoperimetric problem

$$\begin{gathered} \mathcal{J}(y) = \int_0^1 \left(y'(t) + y(t)\right)^2 dt \longrightarrow \min \\ \mathcal{I}(y) = \int_0^1 y(t)dt = \frac{1}{e} \\ y(0) = 0\,, \quad y(1) = 1 - \frac{1}{\mathrm{e}}. \end{gathered} \tag{18}$$

Extremal (15) is then reduced to the classical extremal $y(t) = 1 - \mathrm{e}^{-t}$ of (18).

Example 14. Choose $k = 1$ and $\alpha = \frac{1}{2}$. Then (14) gives the following fractional isoperimetric problem:

$$\begin{gathered} \mathcal{J}(y) = \int_0^1 \left(y' + {}_0D_t^{\frac{1}{2}} y\right)^2 dt \longrightarrow \min \\ \mathcal{I}(y) = \int_0^1 \left(y' + {}_0D_t^{\frac{1}{2}} y\right) dt = \xi \\ y(0) = 0\,, \quad y(1) = -\xi \left(1 - \operatorname{erfc}(1) + \frac{2}{\sqrt{\pi}}\right), \end{gathered} \tag{19}$$

where erfc is the complementary error function. The extremal (15) for the particular fractional problem (19) is

$$y(t) = -\xi \left(1 - \mathrm{e}^t \operatorname{erfc}(\sqrt{t}) + \frac{2\sqrt{t}}{\sqrt{\pi}}\right).$$

4. Conclusion

Fractional variational calculus provides a very useful framework to deal with nonlocal dynamics in Mechanics and Physics [6, 16]. It has received considerable interest in recent years, with several researchers applying this field to develop fractional classical and quantum mechanics [10, 18, 29]. Motivated by the results and insights of [5,8,24], in this paper we generalize previous fractional Euler–Lagrange equations by proving optimality conditions for fractional problems of the calculus of variations where the highest derivative in the Lagrangian is of integer order. This approach avoids the difficulties with the given boundary conditions when in presence of the Riemann–Liouville derivatives [24]. For the case with the Caputo fractional derivatives ([32]) we refer the reader to [31].

We focus our attention to problems subject to integral constraints (fractional isoperimetric problems), which have recently found a broad class of important applications [9,17,19]. For $k = 0$ our results are reduced to the classical ones [35]. This is in contrast with the standard approach to fractional variational calculus, where the integer-order case is obtained only in the limit.

Acknowledgements. This work was first announced at the IFAC Workshop on Fractional Derivatives and Applications (IFAC FDA'2010), held in University of Extremadura, Badajoz, Spain, October 18-20, 2010; then subsequently at conference "TMSF' 2011". It was supported by *FEDER* funds through *COMPETE* — Operational Programme Factors of Competitiveness ("Programa Operacional Factores de Competitividade") and by Portuguese funds through the *Center for Research and Development in Mathematics and Applications* (CIDMA), University of Aveiro, and the Portuguese Foundation for Science and Technology ("FCT — Fundação para a Ciência e a Tecnologia"), within project PEst-C/MAT/UI4106/2011 with COMPETE number FCOMP-01-0124-FEDER-022690. Odzijewicz was also supported by FCT through the Ph.D. fellowship SFRH/BD/33865/2009.

References

[1] N.H. Abel, *Euvres Completes de Niels Henrik Abel*. Christiana: Imprimerie de Grondahl and Son, New York and London, Johnson Reprint Corporation (1965).

[2] O.P. Agrawal, Formulation of Euler-Lagrange equations for fractional variational problems. *J. Math. Anal. Appl.* **272**, No 1 (2002), 368–379.

[3] O.P. Agrawal, A general finite element formulation for fractional variational problems. *J. Math. Anal. Appl.* **337**, No 1 (2008), 1–12.

[4] O.P. Agrawal, D. Baleanu, A Hamiltonian formulation and a direct numerical scheme for fractional optimal control problems. *J. Vib. Control* **13**, No 9-10 (2007), 1269–1281.

[5] R. Almeida, R.A.C. Ferreira, D.F.M. Torres, Isoperimetric problems of the calculus of variations with fractional derivatives. *Acta Math. Sci. Ser. B Engl. Ed.* **32**, No 2 (2012), 619–630.

[6] R. Almeida, A.B. Malinowska, D.F.M. Torres, A fractional calculus of variations for multiple integrals with application to vibrating string. *J. Math. Phys.* **51**, No 3 (2010), 033503, 12pp.

[7] R. Almeida, S. Pooseh, D.F.M. Torres, Fractional variational problems depending on indefinite integrals. *Nonlinear Anal.* **75**, No 3 (2012), 1009–1025.

[8] R. Almeida, D.F.M. Torres, Calculus of variations with fractional derivatives and fractional integrals. *Appl. Math. Lett.* **22**, No 12 (2009), 1816–1820.

[9] R. Almeida, D.F.M. Torres, Hölderian variational problems subject to integral constraints. *J. Math. Anal. Appl.* **359**, No 2 (2009), 674–681.

[10] R. Almeida, D.F.M. Torres, Generalized Euler-Lagrange equations for variational problems with scale derivatives. *Lett. Math. Phys.* **92**, No 3 (2010), 221–229.

[11] R. Almeida, D.F.M. Torres, Leitmann's direct method for fractional optimization problems. *Appl. Math. Comput.* **217**, No 3 (2010), 956–962.

[12] R. Almeida, D.F.M. Torres, Fractional variational calculus for nondifferentiable functions. *Comput. Math. Appl.* **61**, No 10 (2011), 3097–3104.

[13] T.M. Atanacković, S. Konjik, S. Pilipović, Variational problems with fractional derivatives: Euler-Lagrange equations. *J. Phys. A* **41**, No 9 (2008), 095201, 12pp.

[14] D. Baleanu, New applications of fractional variational principles. *Rep. Math. Phys.* **61**, No 2 (2008), 199–206.

[15] D. Baleanu, O. Defterli, O.P. Agrawal, A central difference numerical scheme for fractional optimal control problems. *J. Vib. Control* **15**, No 4 (2009), 583–597.

[16] D. Baleanu, J.I. Trujillo, A new method of finding the fractional Euler-Lagrange and Hamilton equations within Caputo fractional derivatives. *Commun. Nonlinear Sci. Numer. Simul.* **15**, No 5 (2010), 1111–1115.

[17] V. Blåsjö, The isoperimetric problem. *Amer. Math. Monthly* **112**, No 6 (2005), 526–566.

[18] J. Cresson, G.S.F. Frederico, D.F.M. Torres, Constants of motion for non-differentiable quantum variational problems. *Topol. Methods Nonlinear Anal.* **33**, No 2 (2009), 217–231.

[19] J.P. Curtis, Complementary extremum principles for isoperimetric optimization problems. *Optim. Eng.* **5**, No 4 (2004), 417–430.

[20] R.A. El-Nabulsi, D.F.M. Torres, Necessary optimality conditions for fractional action-like integrals of variational calculus with Riemann-Liouville derivatives of order (α, β). *Math. Methods Appl. Sci.* **30**, No 15 (2007), 1931–1939.

[21] G.S.F. Frederico, D.F.M. Torres, A formulation of Noether's theorem for fractional problems of the calculus of variations. *J. Math. Anal. Appl.* **334**, No 2 (2007), 834–846.

[22] G.S.F. Frederico, D.F.M. Torres, Fractional conservation laws in optimal control theory. *Nonlinear Dynam.* **53**, No 3 (2008), 215–222.

[23] I.M. Gelfand, S.V. Fomin, *Calculus of Variations*. Revised English edition translated and edited by Richard A. Silverman, Prentice-Hall Inc., Englewood Cliffs, N.J. (1963).

[24] Z.D. Jelicic, N. Petrovacki, Optimality conditions and a solution scheme for fractional optimal control problems. *Struct. Multidiscip. Optim.* **38**, No 6 (2009), 571–581.

[25] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*. Elsevier Science B.V., Amsterdam (2006).

[26] M. Klimek, Lagrangean and Hamiltonian fractional sequential mechanics. *Czechoslovak J. Phys.* **52**, No 11 (2002), 1247–1253.

[27] J.T. Machado, V. Kiryakova, F. Mainardi, Recent history of fractional calculus. *Commun. Nonlinear Sci. Numer. Simul.* **16**, No 3 (2011), 1140–1153.

[28] A.B. Malinowska, D.F.M. Torres, Generalized natural boundary conditions for fractional variational problems in terms of the Caputo derivative. *Comput. Math. Appl.* **59**, No 9 (2010), 3110–3116.

[29] A.B. Malinowska, D.F.M. Torres, The Hahn quantum variational calculus. *J. Optim. Theory Appl.* **147**, No 3 (2010), 419–442.

[30] K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*. A Wiley-Interscience Publication, John Wiley & Sons Inc., New York (1993).

[31] T. Odzijewicz, A.B. Malinowska, D.F.M. Torres, Fractional variational calculus with classical and combined Caputo derivatives. *Nonlinear Anal.* **75**, No 3 (2012), 1507–1515.

[32] I. Podlubny, *Fractional Differential Equations*. Academic Press Inc., San Diego, CA (1999).

[33] F. Riewe, Nonconservative Lagrangian and Hamiltonian mechanics. *Phys. Rev. E (3)* **53**, No 2 (1996), 1890–1899.

[34] F. Riewe, Mechanics with fractional derivatives. *Phys. Rev. E (3)* **55**, No 3, part B (1997), 3581–3592.

[35] B. van Brunt, *The Calculus of Variations*. Springer-Verlag, New York (2004).

[1,2] *Center for Research and Development in Mathematics and Applications*
Department of Mathematics, University of Aveiro
3810-193 Aveiro, PORTUGAL

e-mail: [1] *tatianao@ua.pt* , [2] *delfim@ua.pt* *Received: October 13, 2011*