CALIBRATIONS AND LAGRANGIAN SUBMANIFOLDS IN THE SIX SPHERE

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Abstract. We study the first and second variations of volume for Lagrangian submanifolds in the six-dimensional sphere.

1. Introduction. Let (P, ω) be a symplectic manifold of dimension $2n$. An n-dimensional submanifold

$$f : \mathscr{L}^n \to P$$

is called *Lagrangian* if

$$f^*\omega \equiv 0 \,.$$

By Darboux's theorem P is locally symplectically equivalent to $T^*\mathbf{R}^n$. If g is any smooth function on $\mathbf{R}^n$ then

$$dg : \mathbf{R}^n \to T^*\mathbf{R}^n$$

defines a (possibly singular) Lagrangian submanifold in $T^*\mathbf{R}^n$. Hence the problem of constructing Lagrangian submanifolds in P is locally trivial. Further, by replacing g by $g + \zeta$ where ζ has compact support, one can produce an infinite-dimensional space of Lagrangian submanifolds which agree with $dg(\mathbf{R}^n)$ off a compact set. This situation can change drastically when P is no longer required to be symplectic, specifically when $d\omega = 0$ is no longer required to hold. The six-dimensional sphere is well-known to carry a nonintegrable almost complex structure with respect to which the standard metric is Hermitian but not Kähler. With regard to Lagrangian submanifolds in S^6 we have the following remarkable theorem of Ejiri [4].

THEOREM 1.1 (Ejiri). *Let $\mathscr{L}$ be a smooth Lagrangian submanifold in S^6. Then $\mathscr{L}$ is minimal.*

A consequence of Ejiri's theorem and the ellipticity of the equation for minimal submanifolds is that each component of the space of Lagrangian immersions in S^6 is finite dimensional. Its dimension is bounded by the nullity, i.e. the dimension of the kernel of the stability operator. We discuss below the relationship of Ejiri's theorem with the theory of calibrations as developed by Harvey and Lawson [6].

The second variation for minimal Lagrangian submanifolds in Einstein Kähler

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manifolds was developed by Oh and subsequently studied by many authors [9], [2], [11], [14]. In this case, Hamiltonian variations of the ambient manifold preserve the property that the submanifold is Lagrangian and the second variation of volume can be analyzed in the compact case using the spectrum of the Laplacian acting on 1-forms. In the case of Lagrangian submanifold in the nearly Kähler S^6 the Hamiltonian variations are orthogonal to those variations preserving the property of being Lagrangian (Lagrangian variations). The stability operator restricted to Hamiltonian variations can be analyzed using the spectrum of the Laplacian acting on functions while the second variation for variations orthogonal to the Hamiltonian variations can be studied using first order operators.

2. Calibrations. Let X^n be a smooth oriented n-manifold and let $\pi : G_k \to X$ denote the fibre bundle whose fibre over $p \in X$ is the Grassmannian of oriented k-planes in $T_p X$. If $\theta \in \Omega_k(X)$, the space of smooth k-forms on X, then θ defines a map $\theta : G_k \to \mathbf{R}$ in the obvious way. The comass of θ is defined by

$$\text{comass}(\theta) := \sup_{p \in X} \ \sup_{\pi \in G_K(X)_p} \ \langle \theta, \pi \rangle \ .$$

DEFINITION. A *calibration* θ on X is a smooth, closed differential form of comass 1.

The principal result concerning calibration is the following. If θ is as above, an oriented k-dimensional submanifold M in X is called a θ-submanifold if

$$\langle \vec{T}_p M, \theta \rangle \equiv 1 \ ,$$

where $\vec{T}_p M$ denotes the oriented tangent plane at p. If this is the case and M' is any other oriented k-submanifold in X such that $M \setminus M'$ is a boundary (and $\partial M = \partial M'$ if $\partial M \neq \varnothing$) then

$$\text{vol}(M) = \int_M \omega = \int_{M'} \omega \leq \text{vol}(M')$$

and hence M minimizes k-volume in its homology class.

We let $\mathbf{O}$ denote the octonians (Cayley numbers) which are an 8-dimensional non-commutative, non-associative, normed algebra over the reals. The identification

$$\mathbf{R}^7 \approx \text{Im} \, \mathbf{O}$$

endows $\mathbf{R}^7$ with the cross product

$$a \times b := \text{Im}(\bar{b} \cdot a) \qquad a, b \in \text{Im} \, \mathbf{O} \ .$$

We recall that for $\chi \in S^6(1) \subset \mathbf{R}^7$

$$J_\chi : T_\chi S^6 \to T_\chi S^6, \quad a \mapsto \chi \times a$$

defines a non-integrable almost complex structure on $S^6(1)$.

The primary examples of calibrations on $\mathbf{R}^7$ were found by Harvey and Lawson [6]. These are the *associative calibration* $\varphi \in \Lambda^3(\mathbf{R}^7)^*$ defined by

$$\varphi(a, b, c) := \langle a \times b, c \rangle,$$

where $\langle\,,\,\rangle$ denotes the usual inner product on $\mathbf{R}^7$ and the *coassociative calibration* $\psi \in \Lambda^4(\mathbf{R}^7)^*$ defined by

$$\tag{1} \psi := *\varphi,$$

where $*$ is the Hodge operator. Let $\omega_\chi(a, b) := \langle J_\chi a, b \rangle$; $a, b \in T_\chi S^6$ denote the Kähler form of S^6.

PROPOSITION 2.1. *For $\chi \in S^6$,*
(i) $\omega_\chi = \chi \,\lrcorner\, \varphi$
(ii) $d\omega = 3\varphi |_{S^6}$
(iii) $(\chi \,\lrcorner\, \psi)(a, b, c) = \langle a \times b, J_\chi c \rangle, \quad a, b, c \in T_\chi S^6$
(iv) $d(\chi \,\lrcorner\, \psi) = 4\omega \wedge \omega$.

PROOF. For $a, b \in T_\chi S^6$,

$$\omega_\chi(a, b) := \langle J_\chi a, b \rangle = \chi \,\lrcorner\, \varphi$$

proving (i). The statement (ii) follows from the formula (5.1) of [6]. Since $G_2 \subset SO(7)$, for all $\alpha \in G_2$

$$\alpha^* \psi = \alpha^* * \varphi = *\alpha^* \varphi = *\varphi = \psi,$$

from which it follows that

$$\alpha^*(\alpha\chi \,\lrcorner\, \psi) = \chi \,\lrcorner\, \psi, \qquad \alpha \in G_2, \quad \chi \in S^6.$$

Let θ denote the 3-form on S^6 defined by

$$\theta_\chi(a, b, c) := \langle a \times b, \chi \times c \rangle = \langle a \times b, J_\chi c \rangle; \qquad a, b, c \in T_\chi S^6.$$

Then since the action of G_2 preserves the cross product, we have

$$\alpha^* \theta_{\alpha\chi} = \theta_\chi.$$

Therefore both θ and $\chi \,\lrcorner\, \psi$ are G_2-invariant 3-forms on S^6 so the equality (iii) need only be checked at one point, say e_1, of S^6 using a multiplication table for $\mathbf{R}^7$. This is left to the reader. To prove (iv), one again uses formula (5.1) of [6] to obtain $d(\chi \,\lrcorner\, \psi) = 4\psi |_{S^6}$. At $\chi = e_1$ it is easily checked that $\omega \wedge \omega = \psi$ and the result follows from the G_2-invariance of ψ and $\omega \wedge \omega$.   q.e.d.

Recall that for any submanifold $f : M \to S^n$, the cone CM over M is the manifold

$$CM \approx M \times (0, 1) \to \mathbf{R}^{n+1}, \qquad p, t \mapsto tf(p).$$

PROPOSITION 2.2.   *A smooth 3-manifold $\chi: \mathscr{L} \to S^6$ is Lagrangian if and only if the cone, $C\mathscr{L} \to \mathbf{R}^7$, is a ψ-submanifold.*

PROOF.   If $\chi: \mathscr{L} \to S^6$ is Lagrangian, then as shown in [4], an orientation on $\mathscr{L}$ can be chosen such that for any pair of orthonormal vectors e_1, $e_2 \in T_\chi\mathscr{L}$, the set $\{e_1, e_2, e_3 := -J(e_1 \times e_2)\}$ is a positively oriented orthonormal basis for $T_\chi\mathscr{L}$. By (iii) of Proposition 2.1, this means that

(2) $$(\chi \lrcorner \psi)(e_1, e_2, e_3) = \langle e_1 \times e_2, -JJ(e_1 \times e_2)\rangle = 1$$

on any positively oriented orthonormal basis $\{e_1, e_2, e_3\}$ of $T_\chi\mathscr{L}$. However, if $\{e_1, e_2, e_3\}$ is a positively oriented orthonormal basis of $T_\chi\mathscr{L}$, then $\{\chi, e_1, e_2, e_3\}$ can be regarded as a positively oriented orthonormal basis of $T_{r\chi}C\mathscr{L}$ for all $r > 0$. It follows that $C\mathscr{L}$ is a ψ-submanifold.

Conversely, if $C\mathscr{L}$ is a ψ-submanifold then (2) holds for any positively oriented orthonormal basis $\{\chi, e_1, e_2, e_3\}$ of $T_{r\chi}C\mathscr{L}$. It follows that Je_3 is parallel to $e_1 \times e_2$ and hence Je_3 is orthogonal to e_1, e_2 and e_3. Therefore e_3 is orthogonal to $T_\chi\mathscr{L}$. By interchanging e_j, $j = 1, 2$ with e_3 one obtains that $\mathscr{L}$ is Lagrangian.   q.e.d.

COROLLARY 2.3.   *If $\chi: \mathscr{L} \to S^6$ is a Lagrangian immersion, then the volume element on $\mathscr{L}$ is given by*

$$dV_\mathscr{L} = \chi \lrcorner \psi.$$

3.   Second variation formula for Lagrangian submanifolds in S^6.   Let $\chi: \mathscr{L} \to S^6$ be a smooth immersion of a Lagrangian submanifold. Let $\xi \in \Gamma(\perp\mathscr{L})$ be the space of smooth sections of the normal bundle (with compact support if $\mathscr{L}$ is not compact). In order to consider the second variation of 3-volume, we introduce the following quantities. Let $\tilde{R}$ denote the curvature tensor of S^6 and define

$$\bar{R}(\xi) := \sum_{i=1}^{3} \tilde{R}(e_i, \xi)e_i$$

where $\{e_1, e_2, e_3\}$ is a local orthonormal frame field on $\mathscr{L}$. If

$$\mathrm{II}: T\mathscr{L} \times T\mathscr{L} \to \perp\mathscr{L}$$

is the second fundamental form, define

$$\tilde{\mathrm{II}} := \mathrm{II} \circ \mathrm{II}^t.$$

The second variation of volume is then given by

(3) $$\delta_\xi^2|\mathscr{L}| = \int_\mathscr{L} |\nabla^\perp\xi| + \langle\bar{R}(\xi), \xi\rangle - \langle\tilde{\mathrm{II}}\xi, \xi\rangle * 1.$$

Because $\mathscr{L}$ is Lagrangian we can write

$$\xi = JV, \qquad V \in T\mathscr{L}$$

and express the second variation in terms of V. Let Δ_1 denote the Laplacian acting on vector fields on $\mathscr{L}$, i.e.

$$\Delta_1 V := -((\delta d + d\delta)V^{\flat})^{\sharp} = \sum_{i=1}^{3} (\nabla_i \nabla_i - \nabla_{\nabla_i e_i})V - \rho V$$

where $\flat$ and $\sharp$ denote the usual duality operators on $\mathscr{L}$ and ρ is defined by $\langle \rho V, W \rangle = \mathrm{Ric}_{\mathscr{L}}(V, W)$. Define

$$\mathrm{curl}(V) := (*dV^{\flat})^{\sharp}.$$

PROPOSITION 3.1.

(4) $$\delta_{JV}^2 |\mathscr{L}| = -(V, \tilde{\mathscr{F}} V)$$

where

$$\tilde{\mathscr{F}} V := \Delta_1 V + 3V + 2\,\mathrm{curl}(V)$$

and $(\,,\,)$ *denotes the* L^2-*inner product of vector fields on* $\mathscr{L}$.

PROOF. For $A, B \in T_\chi S^6$ define $G(A, B) := A \times B - \langle A \times B, \chi \rangle \chi$. We have

$$\nabla^{\perp} JV = J\nabla V + G(\,\cdot\,, V)$$

and hence

$$|\nabla^{\perp} JV|^2 = |\nabla V|^2 + \sum_{1}^{3} (|G(e_i, V)|^2 + 2\langle J\nabla_i V, G(e_i, V)\rangle).$$

Using a multiplication table for $\mathbf{R}^7$, (see for example [4]), one sees that each Lagrangian plane in TS^6 is closed under the cross product and is isomorphic to $\mathbf{R}^3$ with the usual vector product. Choosing a basis for $T_p\mathscr{L}$ such that $V_p = |V_p|e_3$ gives

$$\sum_{1}^{3} |G(e_i, V)|^2 = |V_p|^2(|e_1 \times e_3|^2 + |e_2 \times e_3|^2) = 2|V_p|^2.$$

Using Corollary 2.3, we can write

$$\sum_{1}^{3} \langle J\nabla_i V, G(e_i, V)\rangle = - *dV^{\flat}(V) = -\langle \mathrm{curl}\, V, V\rangle$$

and we obtain

(5) $$|\nabla^{\perp} JV|^2 = |\nabla V|^2 + 2|V|^2 - 2\langle \mathrm{curl}\, V, V\rangle.$$

By definition of the adjoint

$$\langle \mathrm{II}^t JV, e_i \otimes e_j \rangle = \langle JV, \mathrm{II}(e_i, e_j) \rangle = \langle Je_j, \mathrm{II}(e_i, V) \rangle ,$$

using a well-known property of the second fundamental form of a Lagrangian submanifold, (see (3.5) of [3]). Therefore

$$\langle \tilde{\mathrm{II}}\xi, \xi \rangle = \langle \mathrm{II}^t JV, \mathrm{II}^t JV \rangle = \sum_{i,j=1,2,3} \langle Je_j, \mathrm{II}(e_i, V) \rangle^2 = \sum_{i=1,2,3} |\mathrm{II}(e_i, V)|^2 .$$

Since S^6 has constant curvature 1,

(6) $$\tilde{R}(A, B)C = \langle B, C \rangle A - \langle A, C \rangle B , \qquad A, B, C \in TS^6$$

from which

(7) $$\langle \tilde{R}\xi, \xi \rangle = -3|\xi|^2 = -3|V|^2$$

follows. From the Gauss equation

$$\langle \tilde{R}(A, B)C, E \rangle = \langle R_{\mathscr{L}}(A, B)C, E \rangle + \langle \mathrm{II}(A, C), \mathrm{II}(B, E) \rangle - \langle \mathrm{II}(A, E), \mathrm{II}(B, C) \rangle$$

and the minimality of $\mathscr{L}$, it follows that

(8) $$\langle \tilde{\mathrm{II}}\xi, \xi \rangle = \sum_{i=1,2,3} |\mathrm{II}(e_i, V)|^2 = 2|V|^2 - \mathrm{Ric}_{\mathscr{L}}(V, V) .$$

Combining (3), (5), (7) and (8) gives

$$\delta_\xi^2 |\mathscr{L}| = \int_{\mathscr{L}} (|\nabla V|^2 + \mathrm{Ric}_{\mathscr{L}}(V, V) - 2\langle \mathrm{curl}\, V, V \rangle - 3|V|^2) * 1$$

which is the same as (4).   q.e.d.

4. Lagrangian-Jacobi fields. Throughout this section $\chi : \mathscr{L} \to S^6$ will be a Lagrangian immersion of a closed manifold. The zeros of the stability operator on any minimal submanifold are called Jacobi fields. On a Lagrangian submanifold in S^6, any variation through Lagrangian submanifolds clearly preserves minimality. Since the property of being Lagrangian is a first order condition, one expects $\tilde{\mathscr{J}}$ to factor.

DEFINITION 4.1. $\xi \in \Gamma(\perp \mathscr{L})$ will be called a Lagrangian-Jacobi field if

(9) $$(L_\xi \omega)|_{\mathscr{L}} \equiv 0 .$$

Note that (9) is the condition that to first order, the flow of JV restricted to $\mathscr{L}$ gives a one-parameter family of Lagrangian submanifolds. More generally, it will be useful to consider the equation

(10) $$(L_{JV}\omega)|_{\mathscr{L}} = (3 - \lambda) * V^\flat, \qquad \lambda \in \mathbf{R} .$$

PROPOSITION 4.2.   Let $V \in \Gamma T\mathscr{L}$ solve (10) with $\lambda \neq 0$. Then
(i) $-dV^\flat + \lambda * V^\flat = 0$

(ii) $\delta V^{\flat} = 0$

(iii) $\Delta_1 V + \lambda^2 V = 0$

(iv) $(\tilde{\mathscr{J}} + \mu)V = 0$, where $\mu = \lambda^2 - 2\lambda - 3$

(v)   on $\mathscr{L}$, V solves the steady state Euler equations

$$\operatorname{div} V = 0, \qquad d(V \lrcorner \, dV^{\flat}) = 0.$$

REMARK.   A solution of the steady state Euler equations on $\mathscr{L}$ generates a geodesic in the group of volume preserving diffeomorphisms of $\mathscr{L}$. We refer the reader to [5] for details.

PROOF.   The equation (i) follows from

$$L_{JV}\omega = d(JV \lrcorner \, \omega) + JV \lrcorner \, d\omega = -dV^{\flat} + 3V \lrcorner \, (\chi \lrcorner \, \psi) = -dV^{\flat} + 3 * V^{\flat}$$

using Corollary 2.3. Equation (ii) follows immediately from (i) since $\delta_1 = -*d*$. To prove (iii), we have by (ii),

$$\Delta_1 V^{\flat} = -(d\delta + \delta d)V^{\flat} = -\delta dV^{\flat} = -*d*dV^{\flat} = -*d*\lambda * V^{\flat}$$
$$= -\lambda * dV^{\flat} = -\lambda^2 ** V^{\flat} = -\lambda^2 V^{\flat}.$$

Equation (iv) follows from (i), (iii) and the definition of $\tilde{\mathscr{J}}$. Finally, (v) follows from (i) and (ii).   q.e.d.

Recall the Hodge decomposition

(11) $\Gamma(T\mathscr{L}) = (dC^{\infty})^{\sharp} \oplus (\delta\Omega_2)^{\sharp} \oplus \mathscr{H}$

where $\mathscr{H}$ denotes the space of harmonic vector fields. Using the facts that the Laplacian commutes with both d and δ and that

$$\operatorname{curl}(\delta\tau)^{\sharp} = *d\delta\tau = (*d*)d*\tau = (\delta d*\tau)^{\sharp}$$

one sees that $\tilde{\mathscr{J}}$ preserves the Hodge decomposition. Consequently if $V \in \Gamma(T\Lambda)$ is decomposed

$$V = \nabla\psi + (\delta\tau)^{\sharp} + \eta$$

using (11) then the second variation can be expressed

(12) $\delta^2_{JV} |\mathscr{L}| = -(\nabla(\Delta_0\psi + 3\psi), \nabla\psi) - (\tilde{\mathscr{J}}(\delta\tau)^{\sharp}, (\delta\tau)^{\sharp}) - 3\|\eta\|^2_{L^2}.$

Variations of $\mathscr{L}$ generated by $\xi = J\nabla\psi$ with $\psi \in C^{\infty}(\mathscr{L})$ are called *Hamiltonian variations*. In the case of a Lagrangian submanifold in an Einstein-Kähler manifold, these variations preserve the property that the submanifold is Lagrangian. In the present case we have shown that if JV is the variation field of a deformation through Lagrangian submanifolds, then $\operatorname{div} V = 0$ holds so that

$$(JV, J\nabla\psi) = (V, \nabla\psi) = 0$$

using the Hodge decomposition. It follows that for Lagrangian submanifolds in S^6, the space of variations through Lagrangian submanifolds is L^2-orthogonal to the space of Hamiltonian variations.

Define $\tilde{\mathscr{J}}_1$ to be the restriction of $\tilde{\mathscr{J}}$ to $\{V \mid \delta V^b = 0\}$.

LEMMA 4.3.

$$\text{spectrum}(\tilde{\mathscr{J}}_1) \subset [-4, \infty) \, .$$

We give two proofs of the lemma below.

PROOF 1.   Let V be a vector field with $\delta V^b = 0$. Then

$$
\begin{aligned}
-(V, \tilde{\mathscr{J}}_1 V) &= -(V^b, [\delta + *][-d + 3*]V^b) \\
&= -([d + *]V^b, [-d + 3*]V^b) \\
&= \|dV^b\|_{L^2}^2 - 2(dV^b, *V^b) - 3\|V^b\|_{L^2}^2 \\
&\geq \|dV^b\|_{L^2}^2 - (\|dV^b\|_{L^2}^2 + \|V^b\|_{L^2}^2) - 3\|V^b\|_{L^2}^2 \\
&= -4\|V\|_{L^2}^2 \, .
\end{aligned}
$$

PROOF 2.   Let $C\mathscr{L}_\varepsilon$ denote the truncated cone over $\mathscr{L}$

$$\mathscr{L}_\varepsilon \approx [\varepsilon, 1] \times \mathscr{L} \to \mathbf{R}^7 \, , \qquad (t, p) \mapsto t\chi(p) \, .$$

Then for every $\varepsilon \in (0, 1)$, $C\mathscr{L}_\varepsilon$ minimizes the 4-volume with respect to its boundary. Following Simons [13], consider a normal variation of $C\mathscr{L}_\varepsilon$ of the form

$$v = g(t)\xi$$

where g has compact support in $(\varepsilon, 1)$ and ξ is a section of the normal bundle of $\mathscr{L}$ in S^6. If $\mathscr{J}_C$ denotes the stability operator of $C\mathscr{L}_\varepsilon$, then by results of [13],

$$\mathscr{J}_C v = -\Lambda v \, , \qquad \Lambda \in \mathbf{R} \, ,$$

if and only if

$$\Lambda = (3 + \sigma) + \tau \, ,$$

where $\sigma \in \text{spectrum}(\mathscr{J})$ and $\tau \in \{1 + (j\pi/\log \varepsilon)^2 \mid j = 0, 1, 2, \ldots\}$. Since $C\mathscr{L}_\varepsilon$ is minimizing and hence stable

$$0 \leq 3 + \sigma + 1 + \left(\frac{j\pi}{\log \varepsilon}\right)^2 \, , \qquad j = 0, 1, 2, \ldots$$

for every ε and hence $\sigma \geq -4$.   q.e.d.

LEMMA 4.4.   *For* $\mu \in (-4, \infty)$ *with* $\mu = (\lambda + 1)(\lambda - 3)$,

(13) $\ker(\tilde{\mathscr{J}}_1 + \mu)^b = \ker(-d + \lambda*) \oplus \ker(-d + (\lambda - 2)*) \, .$

PROOF.   For divergence free vector fields $(\tilde{\mathscr{J}}_1 + \mu)$ factors

(14)
$$(\tilde{\mathscr{J}}_1 + \mu)^b = [\delta + (2-\lambda)*][-d+\lambda*]$$

(15)
$$= [\delta - \lambda*][-d+(2-\lambda)*] \ .$$

This shows that the right hand side of (13) is contained in the left hand side. Now suppose that for some $\mu > -4$, $(\tilde{\mathscr{J}}_1 + \mu)V = 0$ and $(-d+\lambda*)V^b := \tau \neq 0$. Then

$$0 = [\delta + (\lambda - 2)*]\tau = -*[-d+(2-\lambda)*]*\tau$$

and therefore $*\tau \in \ker(-d+(2-\lambda)*)$. Furthermore $\lambda \neq 1$ since $\mu \neq -4$ and then

$$[-d+\lambda*]\frac{*\tau}{(2\lambda-2)} = \tau$$

follows.   q.e.d.

PROPOSITION 4.5.   *Let $\chi : \mathscr{L} \to S^6$ be a Lagrangian immersion of a closed manifold. Then the nullity of $\mathscr{L}$, $\mathrm{Nul}(L)$, and the Morse index satisfy*

$$\mathrm{Nul}(\mathscr{L}) = \dim \ker(\Delta_0 + 3) + \dim \ker(-d_1 + 3*) + \dim \ker(\delta_2 + *) \ .$$

$$\mathrm{Index}(\mathscr{L}) \geq \sum_{\lambda < 3} \dim \ker(\Delta_0 + \lambda) + \sum_{\substack{-1 < \lambda < 3 \\ \lambda \neq 1}} \dim \ker(-d_1 + \lambda*) + \dim H^1(\mathscr{L}) \ .$$

(Here d_1 and δ_2 are respectively the exterior derivative operator restricted to 1-forms and its adjoint.)

5. Circle bundles over complex curves.

In this section we apply the results above to a special class of examples first studied in [3]. We begin with a new proof of a result of [3].

THEOREM 5.1 (see [3]).   *Let $f : \Sigma \to \mathbf{P}^2(4)$ be a complex curve and let $\mathscr{L}_\Sigma \subset S^5$ be the S^1 bundle over Σ which is the pull-back of the Hopf fibration. Then after composing with the totally geodesic inclusion of S^5 in S^6 given by $\{x_4 = 0\}$, $\mathscr{L}_\Sigma$ is Lagrangian in S^6.*

$$
\begin{array}{ccc}
\mathscr{L}_\Sigma & \xrightarrow{\ \chi\ } & S^5 \ \hookrightarrow\ S^6 \\
\downarrow & & \downarrow \\
\Sigma & \xrightarrow{\ f\ } & \mathbf{P}^2(4) \ .
\end{array}
$$

PROOF.   Locally f is given by $f = [F]$ where $[F]$ is a holomorphic, non-zero, $\mathbf{C}^3$ valued function. Then $\mathscr{L}_\Sigma$ is locally given by $\{e^{i\theta}F/|F| \,|\, e^{i\theta} \in S^1\}$ and so the cone over $\mathscr{L}_\Sigma$ is locally given by $\{\lambda F \,|\, \lambda \in \mathbf{C}^*\}$. This is clearly a complex subvariety of $\mathbf{C}^3$. Let

$$\omega_3 = \sum_{j=1,2,3} dx_j \wedge dx_{j+4}$$

denote the Kähler form of C^3. Then the form $(1/2)\omega_3 \wedge \omega_3$ calibrates the cone over $\mathscr{L}_\Sigma$. Using (1) and a multiplication table for R^7, (see for example [4]), one sees that ψ is given by

$$\psi = dx_{4567} + dx_{2367} + dx_{2345} + dx_{1357} - dx_{41346} - dx_{1256} - dx_{1247}$$

where $dx_{ijkl} := dx_i \wedge dx_j \wedge dx_k \wedge dx_l$. A simple calculation then shows that

$$\frac{1}{2}\omega_3 \wedge \omega_3 = \psi|_{C^3}$$

and so the cone over $\mathscr{L}_\Sigma$ is coassociative. By Proposition (2.2), $\mathscr{L}_\Sigma$ is Lagrangian in S^6.

q.e.d.

Let $\pi : S^5 \to P^2(4)$ be the Hopf fibration and let α denote the almost complex structure of C^3. If χ denotes the position vector on $S^5 \subset C^3$, then the vector field $Y := \alpha\chi$ on S^5 is tangent to the fibres of π. Let $\eta := Y^\flat$. Then

$$d\eta = 2\pi^*\omega_{P^2} ,$$

where ω_{P^2} denotes the Kähler form of P^2.

The following lemma is well known. We include its proof for completeness.

LEMMA 5.2. *The k-th eigenvalues of the Laplacian on Σ and $\mathscr{L}_\Sigma$ satisfy*

$$\lambda_k(\mathscr{L}_\Sigma) \le \lambda_k(\Sigma) .$$

PROOF. Let $f : \Sigma \to R$ be a smooth function. We claim

(16) $\pi_* \nabla^{\mathscr{L}}(\pi^*f) = \nabla^\Sigma f .$

Note that

$$\nabla^{\mathscr{L}} f = (\pi^* df)^\sharp = (d\pi^* f)^\sharp$$

and that

$$0 = df(\pi_* Y) = \langle Y, \nabla^{\mathscr{L}} f \rangle_{\mathscr{L}} .$$

Let $A \in T_p\mathscr{L}$ with $A \perp Y$. Then

$$df(\pi_* A) = d(\pi^* f)(A) = \langle A, \nabla^{\mathscr{L}}\pi^* f \rangle_{\mathscr{L}} = \langle \pi_* A, \pi_* \nabla^{\mathscr{L}}\pi^* f \rangle_\Sigma$$

since π is a Riemannian submersion. Since π_* is surjective, this proves the claim.

Using (16), we find

$$d(\pi^* f) \wedge *_{\mathscr{L}} d(\pi^* f) = d(\pi^* f) \wedge [\nabla^{\mathscr{L}}\pi^* f] \lrcorner dV_{\mathscr{L}} .$$

Since $\mathscr{L}$ is an invariant submanifold of S^5, we have

(17) $dV_{\mathscr{L}} = \eta \wedge \dfrac{1}{2} d\eta = \eta \wedge \pi^*\omega_\Sigma$

and so

$$[\nabla^{\mathscr{L}}\pi^*f] \,\lrcorner\, dV_{\mathscr{L}} = -\eta \wedge [\nabla^{\mathscr{L}}\pi^*f] \,\lrcorner\, \pi^*\omega_{\Sigma} = -\eta \wedge \pi^*(\pi_*\nabla^{\mathscr{L}}(\pi^*f) \,\lrcorner\, \omega_{\Sigma})$$
$$= -\eta \wedge \pi^*(\nabla^{\Sigma}f \,\lrcorner\, \omega_{\Sigma}) = -\eta \wedge \pi^*(*_{\Sigma}df) \,.$$

Therefore

$$*_{\mathscr{L}}d(\pi^*f) = \pi^*(*_{\Sigma}df) \wedge \eta$$

and so

$$d(\pi^*f) \wedge *_{\mathscr{L}}d(\pi^*f) = [\pi^*df \wedge \pi^*(*_{\Sigma}d\eta)] \wedge \eta$$
$$= (df \wedge *_{\Sigma}df) \wedge \eta = \|\nabla^{\Sigma}f\|^2 \omega_{\Sigma} \wedge \eta \,.$$

It follows that

$$2\pi \int_{\Sigma} \|\nabla^{\Sigma}f\|^2 dA_{\Sigma} = \int_{\mathscr{L}} \|\nabla^{\mathscr{L}}f\|^2 dV_{\mathscr{L}} \,.$$

Similarly

$$2\pi \int_{\Sigma} f^2 dA_{\Sigma} = \int_{\mathscr{L}} f^2 dV_{\mathscr{L}}$$

and so

$$\frac{\int_{\Sigma} \|\nabla^{\Sigma}f\|^2 dA_{\Sigma}}{\int_{\Sigma} f^2 dA_{\Sigma}} = \frac{\int_{\mathscr{L}} \|\nabla^{\mathscr{L}}f\|^2 dV_{\mathscr{L}}}{\int_{\mathscr{L}} f^2 dV_{\mathscr{L}}} \,.$$

Finally, using the max-min characterization of the k-th eigenvalue

$$\lambda_k(\mathscr{L}) = \sup_{u_1,\dots,u_n} \inf_{\substack{f \perp u_j \\ j=1\dots n}} \frac{\int_{\mathscr{L}} \|\nabla^{\mathscr{L}}f\|^2 dV_{\mathscr{L}}}{\int_{\mathscr{L}} f^2 dV_{\mathscr{L}}}$$

gives the result. q.e.d.

PROPOSITION 5.3. *Let* $f : \Sigma \to \boldsymbol{P}^2(4)$ *be a compact, complex curve and let* φ *be an eigenfunction on* Σ *with eigenvalue* Λ,

$$\Delta_{\Sigma}\varphi + \Lambda\varphi = 0 \,.$$

Define

$$\lambda_{\pm} := 1 \pm \sqrt{\Lambda + 1} \,, \quad V_{\pm} := -\lambda_{\pm}\pi^*\varphi Y + (\pi^* * d\varphi)^{\sharp}$$

where $\pi: \mathscr{L}_\Sigma \to \Sigma$ *is the projection. Then on* $\mathscr{L}_\Sigma$,

$$dV_\pm^\flat = \lambda_\pm * V_\pm^\flat .$$

PROOF.   We have

$$\begin{aligned} dV_\pm^\flat &= d(-\lambda_\pm \pi^* \varphi \eta + (\pi^* * d\varphi)) \\ &= -\lambda_\pm (d\pi^* \varphi \wedge \eta + \varphi d\eta) - \Lambda(\pi^* \varphi \omega_\Sigma) \\ &= -\lambda_\pm d\pi^* \varphi \wedge \eta - (2\lambda_\pm + \Lambda)(\pi^* \varphi \omega_\Sigma) , \end{aligned}$$

using (17) and the fact that Σ is a complex submanifold of $\boldsymbol{P}^2$. On the other hand, by (17),

$$\begin{aligned} V_\pm \lrcorner dV_{\mathscr{L}} &= (V_\pm \lrcorner \eta) \pi^* \omega_\Sigma - \eta \wedge (V_\pm \lrcorner \pi^* \omega_\Sigma) \\ &= -\lambda_\pm \pi^* \varphi \omega_\Sigma - d\pi^* \varphi \wedge \eta . \end{aligned}$$

The result follows since $\lambda_\pm$ solves $x^2 - 2x - \Lambda = 0$.   q.e.d.

Note that if $\mu_\pm$ denotes the eigenvalue of $\tilde{\mathscr{J}}$ obtained from Proposition 4.2, then

$$\mu_\pm = (\lambda_\pm - 3)(\lambda_\pm + 1) = \Lambda - 3 .$$

LEMMA 5.4.   *The map* $\pi^*: H^1(\mathscr{L}_\Sigma, \boldsymbol{R}) \to H^1(\Sigma, \boldsymbol{R})$ *is injective.*

PROOF.   Let $[v] \in H^1(\Sigma, \boldsymbol{R})$ with $\pi^*[v] = 0$. Then $\pi^* v = df$ for some smooth function f on $\mathscr{L}_\Sigma$. However

$$df(Y) = \pi^* v(Y) = v(\pi_* Y) = 0$$

implies that f is S^1-invariant, i.e. there exists a function f_1 on Σ with $\pi^* f_1 = f$. It then follows that $df_1 = v$ and so $[v] = 0 \in H^1(\Sigma, \boldsymbol{R})$.   q.e.d.

THEOREM 5.5.

$$\text{Index}(\mathscr{L}_\Sigma) \geq 1 + 3 \sum_{0 < \Lambda < 3} \dim \ker(\Delta_\Sigma + \Lambda) + 2 \operatorname{genus}(\Sigma) .$$

PROOF.   The proof is achieved by combining the previous three results. Each eigenvalue Λ on Σ with $0 < \Lambda < 3$ contributes 1 to the index by Proposition 5.2 and contributes 2 to the index by Proposition 5.3.   q.e.d.

REMARK 5.6.   A result of Yau and Yang [15] states that for any compact, oriented 2-dimensional Riemannian manifold Σ of genus γ,

$$\Lambda_1 \leq \frac{8\pi(1 + \gamma)}{|\Sigma|} .$$

Note that when Σ is a complex curve in $\boldsymbol{P}^2$, $|\Sigma| = \pi d$ where d denotes the degree of the immersion. It is known that for a fixed value of γ there exist immersions of genus

γ surfaces into P^2 of arbitrarily high degree and hence Λ_1 can be made arbitrarily small.

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