Acta Math., 170 (1993), 83–120

Calibrations and spinors

by

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Introduction

On a Riemannian manifold a calibration ϕ is simply a smooth p-form closed under exterior differentiation which is less than or equal to the volume form induced on each oriented p-dimensional submanifold. Each calibration determines a geometry of distinguished submanifolds, namely those submanifolds for which ϕ is exactly the induced volume form. The fundamental result of the theory of calibrations says that each closed submanifold, distinguished by a calibration, is automatically homologically volume minimizing. See the papers Harvey–Lawson [4] and Harvey [2], [3] for more details.

The purpose of this paper is first, to establish a general procedure for constructing (constant coefficient) calibrations by squaring spinors, and second, to use this general procedure to explicitly calculate new calibrations in sixteen variables—and hence new geometries of submanifolds.

Part I describes the basic construction of calibrations from spinors. We first review the general facts about Clifford algebras $\mathrm{Cl}(V)$ and focus on the case when $\dim V = 8m$. The spin representations of $\mathrm{Spin}(V)\subset \mathrm{Cl}(V)$ on the space of spinors have a natural periodicity of order eight. We show then how to construct calibrations ϕ from products of spinors (Theorem 2.2) and how to determine the corresponding sets of calibrated planes, i.e. the contact sets $G(\phi)$ (Corollary 2.4). In the last section of Part I we examine the eight dimensional case. Here the group Spin(8) is transitive on the set of unit positive spinors and hence the calibrations produced from squares of spinors are all SO(8) equivalent. As elements of $\bigwedge(\mathbf{R}^8)^*$, these are equal to $1+\Phi+\lambda$, where λ is the unit volume element and where Φ is the Cayley calibration introduced in [4]. Therefore the first possibility for new calibrations in dimension $8m$ occurs in dimension sixteen.

The detailed analysis of the sixteen dimensional case is the subject of Part II. It is based on the complete classification of Spin(16) orbits in the space of spinors (Proposition 2.3). The main result of Part II is Theorem 4.1 in which we compute the largest contact sets that are obtainable from squares of spinors. There are eight classes of maximal contact sets naturally corresponding to the eight vertices of the Dynkin diagram of the reflection group of type E_8. Each contact set $G(\phi)$ is a union of compact symmetric spaces having up to eight components.

In Section 1 of Part III we construct a differential system for each geometry obtained by squaring an $8m$-dimensional spinor. This differential system is the analogue of the Cauchy–Riemann operator which distinguishes holomorphic curves among the real surfaces in $\mathbf{C}^2$ (expressed as graphs of functions).

In general it is not possible to classify all orbits of the Spin group in the space of spinors as we have done in dimension sixteen. However, one can identify some orbits of interest. One such orbit in the space of complexified spinors is the orbit of the highest weight which is classically known as the set of pure spinors. In Section III.2 we give a brief discussion of pure spinors and the corresponding calibrations.

Given a constant coefficient calibration ϕ on $\mathbf{R}^p$ one can always construct calibrations α, β on $\mathbf{R}^{p-1}$ by choosing a unit vector $e\in\mathbf{R}^p$ and writing

$$\phi = e^* \wedge \alpha + \beta,$$

where $\alpha, \beta \in \bigwedge(\mathbf{R}^{p-1})^*$ and $\mathbf{R}^p = \mathbf{R}e \oplus \mathbf{R}^{p-1}$ is the orthogonal direct sum. In this way the calibrations in sixteen dimensions obtained in Part II can be used to construct calibrations in lower dimensions. In particular we show in Section III.3 that all calibrations resulting from squares of 15-spinors can be obtained in this manner. This is especially significant since there is no reasonable classification of Spin(15) orbits in the space of 15-spinors.

I. Spinors in dimension $8m$

I.1. Spinors and pinors

Assume that V is a real vector space of dimension n which is equipped with a real positive definite bilinear form $\langle\ ,\ \rangle$. The associated Clifford algebra will be denoted $\mathrm{Cl}(V)$.

First, we recall some of the basic facts about the Clifford algebra in any dimension. More details are provided in Harvey [3]. As a vector space, $\mathrm{Cl}(V)$ is canonically isomorphic to $\bigwedge V$, the exterior algebra over V. Let $\langle\ ,\ \rangle$ also denote the canonical extension of the inner product to $\bigwedge V \cong \mathrm{Cl}(V)$. Each orthogonal transformation $f \in \mathrm{O}(V)$ has a canonical extension to an automorphism of the Clifford algebra. Utilizing $f \equiv -I \in \mathrm{O}(V)$, let $a \mapsto \tilde{a}$ denote the corresponding *canonical automorphism* of $\mathrm{Cl}(V)$. In terms of $\bigwedge V$, $\tilde{a}=a$ if a is of even degree, while $\tilde{a}=-a$ if a is of odd degree. The operation of reversing the order in a simple tensor $v_1 \otimes \ldots \otimes v_k \mapsto v_k \otimes \ldots \otimes v_1$ induces an anti-automorphism of the Clifford algebra, which will be denoted by $a \mapsto \check{a}$ and referred to as the *check anti-automorphism.* The composition of this anti-automorphism with the canonical automorphism yields a second anti-automorphism, which will be denoted by $a \mapsto \hat{a}$ and referred to as the *hat anti-automorphism.* If a is of degree $p \equiv 0$ or $1 \mod 4$ then $\check{a}=a$, while if a is a degree $p \equiv 2$ or $3 \mod 4$ then $\check{a}=-a$. These involutions map $a \in \bigwedge^p V$ to $\pm a$ as described in the following table.

$p \mod 4$	0	1	2	3
$\tilde{a}=$	$+a$	$-a$	$+a$	$-a$
$\check{a}=$	$+a$	$+a$	$-a$	$-a$
$\hat{a}=$	$+a$	$-a$	$-a$	$+a$

A choice of orientation on V uniquely determines a *unit volume element* $\lambda = e_1 \ldots e_n$, where $e_1, \ldots, e_n$ is an oriented orthonormal basis for V. If the dimension n is even, then conjugation by λ gives the canonical automorphism, i.e. $\tilde{a}=\lambda a \lambda^{-1}$. Clifford multiplication by an element $a \in \mathrm{Cl}(V)$ has adjoint given by $\hat{a}$. That is,

$$\langle ab, c\rangle = \langle b, \hat{a}c\rangle \quad \text{and} \quad \langle ba, c\rangle = \langle b, c\hat{a}\rangle$$

for all $a, b, c \in \mathrm{Cl}(V)$.

In the remainder of this section and the next section assume that the dimension $n = 8m$ is a multiple of 8 and that V is oriented. An irreducible representation of the Clifford algebra $\mathrm{Cl}(V)$ on a real vector space $\mathbf{P}$ is unique. In fact, any two such representations on, say, $\mathbf{P}_1$ and $\mathbf{P}_2$ determine an intertwining operation $f: \mathbf{P}_1 \to \mathbf{P}_2$ which is unique up to replacing f by cf, for some constant $c \in \mathbf{R}^*$. That is, $\mathbf{P}$ is projectively canonical. The space $\mathbf{P}$ is called the space of *pinors* and is of dimension 16^m, with $\mathrm{Cl}(V) \cong \mathrm{End}_{\mathbf{R}}(\mathbf{P})$. This isomorphism $\mathrm{Cl}(V) \cong \mathrm{End}_{\mathbf{R}}(\mathbf{P})$ distinguishes an n-dimensional subspace of $\mathrm{End}_{\mathbf{R}}(\mathbf{P})$

corresponding to the subspace $V \subset \mathrm{Cl}(V)$, of *vectors*. This subspace of $\mathrm{End}_{\mathbf{R}}(\mathbf{P})$ will also be denoted by V. The basic identity

$$v^2 = -|v|^2, \quad \text{for all } v \in V$$

is valid.

Conversely, given any n-dimensional subspace

$$V \subset \mathrm{End}_{\mathbf{R}}(\mathbf{P}) \quad (\text{with } \dim_{\mathbf{R}} \mathbf{P} \equiv 16^m)$$

with the property that $v^2 = -|v|^2$ for each $v \in V$, the inclusion map of V into $\mathrm{End}_{\mathbf{R}}(\mathbf{P})$ extends to an isomorphism $\mathrm{Cl}(V) \cong \mathrm{End}_{\mathbf{R}}(\mathbf{P})$. In this case an orthonormal basis for V is called a *set of γ matrices.*

The volume element λ squares to 1 with eigenspaces $\mathbf{S}^{\pm}$ of equal dimension $\frac{1}{2}16^m$. Thus the pinor space $\mathbf{P}$ is decomposed into $\mathbf{P} \equiv \mathbf{S}^+ \oplus \mathbf{S}^-$, with $\mathbf{S}^+$ the space of *positive spinors* and $\mathbf{S}^-$ the space of *negative spinors.* Also, $\mathrm{Cl}(V)^{\mathrm{even}} \cong \mathrm{End}_{\mathbf{R}}(\mathbf{S}^+) \oplus \mathrm{End}_{\mathbf{R}}(\mathbf{S}^-)$ is the even part of the Clifford algebra.

The pinor space $\mathbf{P}$ can be equipped with a positive definite inner product $\varepsilon(x, y)$ with the property that $\hat{a}$ is the adjoint a^t of $a \in \mathrm{End}_{\mathbf{R}}(\mathbf{P})$ with respect to this inner product ε, i.e. $\varepsilon(ax, y) = \varepsilon(x, \hat{a}y)$, for all $x, y \in \mathbf{P}$. Up to a conformal factor $c > 0$, ε is unique. The notation $\hat{\varepsilon}(x, y)$ rather than $\varepsilon(x, y)$ will also be used. Since $\lambda\lambda^t = 1$, the eigenspaces $\mathbf{S}^+$ and $\mathbf{S}^-$ are orthogonal.

Note that each Clifford element $A \in \mathrm{Cl}(V) \cong \mathrm{End}_{\mathbf{R}}(\mathbf{S}^+ \oplus \mathbf{S}^-)$ blocks as

$$A \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

With respect to this blocking,

$$\tilde{A} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \quad \text{and} \quad \check{A} = \begin{pmatrix} a^t & -c^t \\ -b^t & d^t \end{pmatrix}, \quad \text{since } \hat{A} = A^t.$$

Now $\mathrm{End}_{\mathbf{R}}(\mathbf{P})$ also comes equipped with a natural inner product, namely,

$$\langle a, b \rangle = 16^{-m} \,\mathrm{trace}(ab^t), \quad \text{for all } a, b \in \mathrm{End}_{\mathbf{R}}(\mathbf{P}) \cong \mathrm{Cl}(V).$$

The isomorphism $\mathrm{Cl}(V) \cong \mathrm{End}_{\mathbf{R}}(\mathbf{P})$ is, in fact, an isometry. Therefore, the left hand side of this formula can be considered to be the inner product on $\bigwedge V \cong \mathrm{Cl}(V)$.

Obviously, each form $\phi \in \bigwedge^p V^* \cong \bigwedge^p V$ is closed under exterior differentiation. If

$$\phi(\xi) \leqslant 1, \quad \text{for all } \xi \in G(p, V) \subset \bigwedge^p V$$

then ϕ is said to be a *calibration*. Here $G(p,V)\subset\bigwedge^p V$ denotes the Grassmannian of unit oriented p-planes ξ in V, i.e. $\xi\equiv e_1\wedge\ldots\wedge e_p$ where $e_1,\ldots,e_p$ is an oriented orthonormal basis for ξ. A p-plane $\xi\in G(p,V)$ with $\phi(\xi)=1$ is said to be a *ϕ-plane*. The *contact set* or *face*

$$G(\phi)\equiv\{\xi\in G(p,V):\phi(\xi)=1\}$$

is called the *ϕ-Grassmannian*. A calibration is *non-trivial* if $G(\phi)$ is non-empty.

I.2. Constructing calibrations from spinors

Consider the inner product space $\mathbf{P},\varepsilon$, and let $\hat{a}$ denote the adjoint of $a\in\mathrm{End}_{\mathbf{R}}(\mathbf{P})$. Since ε induces an identification of $\mathbf{P}$ with $\mathbf{P}^*$ and $\mathrm{End}_{\mathbf{R}}(P)=\mathbf{P}\otimes\mathbf{P}^*$, the product $x\circ y$ of two pinors $x,y\in\mathbf{P}$ is an element of $\mathrm{End}_{\mathbf{R}}(\mathbf{P})\cong\mathrm{Cl}(V)$ defined by:

$$(x\circ y)(z)=\varepsilon(y,z)x,\quad \text{for all}\ \ x,y,z\in\mathbf{P}.$$

That is, $x\circ y$ is just the tensor product of x and y^*.

Some of the basic properties are listed here for convenient reference.

PROPOSITION 2.1. *For all $a,b\in\mathrm{Cl}(V)$ and for all $x,y,z,w\in\mathbf{P}$:*
(1) $(ax)\circ y=a(x\circ y)$ *and* $x\circ(ay)=(x\circ y)\hat{a}$,
(2) $(x\circ y)(z\circ w)=\varepsilon(y,z)x\circ w$,
(3) $\widehat{x\circ y}=y\circ x$,
(4) $\langle a,b\rangle\equiv 16^{-m}\,\mathrm{trace}\,a\hat{b}$,
(5) $\mathrm{trace}\,x\circ y=\varepsilon(x,y)$,
(6) $\langle x\circ y,a\rangle=16^{-m}\varepsilon(x,ay)$,
(7) $\langle x\circ y,z\circ w\rangle=16^{-m}\varepsilon(x,z)\varepsilon(y,w)$.

Proof. (1), (2) and (3) are immediate from the definition of $x\circ y$, while (4) was discussed in the first section. To prove (5), let $x_1,\ldots,x_N$ denote an orthonormal basis for $\mathbf{P}$. Then

$$\begin{aligned}\mathrm{trace}\,x\circ y&=\sum^{N}\varepsilon((x\circ y)(x_i),x_i)=\sum^{N}\varepsilon(\varepsilon(y,x_i)x,x_i)\\&=\sum^{N}\varepsilon(y,x_i)\varepsilon(x,x_i)=\varepsilon(x,y).\end{aligned}$$

Now (6) follows from (4) and (5), since $(x\circ y)\hat{a}=x\circ(ay)$. Finally, (7) is a special case of (6). □

Spinor multiplication can be used to construct calibrations. If $x,y\in\mathbf{P}$ then $x\circ y\in\mathrm{End}_{\mathbf{R}}(\mathbf{P})$. However, because of the string of isomorphisms

$$\mathrm{End}_{\mathbf{R}}(\mathbf{P})\cong\mathrm{Cl}(V)\cong\bigwedge V\cong\bigwedge V^*,$$

the product $x\circ y$ can be considered as a differential form $\phi=x\circ y\in\bigwedge V^*$. Given a differential form $\phi\in\bigwedge V^*$, the degree p part of ϕ will be denoted by ϕ_p.

THEOREM 2.2. *Suppose V, $\langle\ ,\ \rangle$ is an oriented inner product space of dimension $8m$. Suppose $x\in\mathbf{S}^+$ is a unit positive spinor. Let*

$$\phi\equiv 16^m x\circ x\in \operatorname{End}_{\mathbf{R}}(\mathbf{S}^+)\subset \operatorname{End}(\mathbf{P})\cong \operatorname{Cl}(V)\cong\bigwedge V^*.$$

Then

(i) *$\phi_k=0$ unless $k=4p$ is a multiple of 4.*

(ii) *Each ϕ_k is a calibration, i.e.,*

$$\phi(\xi)\leqslant 1,\quad \textit{for all}\ \ \xi\in G(k,V).$$

(iii) *A k-plane $\xi\in G(k,V)$ is a ϕ-plane if and only if $\xi x=x$. Here ξ acts on x by considering $\xi\in\operatorname{End}_{\mathbf{R}}(\mathbf{P})$ via the isomorphisms*

$$G(k,V)\subset\bigwedge\nolimits^k V\subset\bigwedge V\cong\operatorname{Cl}(V)\cong\operatorname{End}_{\mathbf{R}}(\mathbf{P}).$$

(iv) *The isotropy subgroup of* $\mathrm{O}(V)$ *that fixes ϕ is isomorphic to the subgroup K_x of* $\operatorname{Spin}(V)$ *that fixes the spinor x.*

Proof. Note that

$$\phi(a)=\varepsilon(ax,x),\quad \text{for all}\ \ a\in\bigwedge V\cong\operatorname{Cl}(V)\cong\operatorname{End}_{\mathbf{R}}(\mathbf{P}),$$

by (6) of Proposition 2.1.

Proof of (i). If a is odd then $ax\in\mathbf{S}^-$ and hence $\phi(a)=\varepsilon(ax,x)=0$. Therefore $\phi\in\bigwedge^{\mathrm{even}}V$. By (3) above $\hat\phi=\phi$ which proves (i).

Proof of (ii). The Grassmannian $G(k,V)\subset\operatorname{Spin}(V)$ for k even. Therefore $\hat\xi\xi=1$. That is, ξ is an isometry of $\mathbf{P}$, $\hat\varepsilon$, for each $\xi\in G(k,V)$. Consequently,

$$\phi(\xi)=\varepsilon(\xi x,x)\leqslant|\xi x|\,|x|=1.$$

Here $|x|^2\equiv\varepsilon(x,x)$ denotes the norm of a spinor.

Proof of (iii). Equality occurs in this Cauchy–Schwarz inequality if and only if $\xi x=x$.

Proof of (iv). Let $\chi\colon\operatorname{Pin}(V)\to\mathrm{O}(V)$ denote the double cover defined by $\chi_a(v)=\tilde a v a^{-1}$, for each $a\in\operatorname{Pin}(V)$. If $a\in\operatorname{Pin}(V)$ then $a\hat a=1$ so that $a^{-1}=\hat a$. The orthogonal map $\chi_a\in\mathrm{O}(V)$ acts on $\bigwedge V^*\cong\bigwedge V\cong\operatorname{Cl}(V)$ by $\chi_a(\phi)=\tilde a\phi a^{-1}$. Now this equals

$$16^m\tilde a(x\circ x)a^{-1}=16^m\tilde a(x\circ x)\hat a=16^m(\tilde a x)\circ(ax).$$

If $\chi_a(\phi)=\phi$ then $(\tilde{a}x)\circ(ax)=x\circ x$. By the definition of spinor multiplication, this implies that $\tilde{a}x=cx$ for some scalar $c\in\mathbf{R}$. Since $|\tilde{a}x|=|x|$ the constant $c=\pm1$. The element $a\in\mathrm{Pin}(V)$ must be either even or odd. If a is odd $ax=\pm x$ is impossible since $ax\in\mathbf{S}^-$. This proves that χ_a with $a\in\mathrm{Pin}(V)$ fixes ϕ if and only if $a\in\mathrm{Spin}(V)$ and $ax=\pm x$. Since under the vector representation χ both a and $-a$ have the same image,

$$\chi\colon K_x\to\{g\in\mathrm{O}(V):g^*\phi=\phi\}$$

is surjective. Finally, $-1\notin K_x$ since -1 does not fix x. Therefore,

$$\chi\colon K_x\cong\{g\in\mathrm{O}(V):g^*\phi=\phi\}$$

is an isomorphism. □

Remark 2.3. The top and bottom components of ϕ are easy to compute. Since $\phi(\xi)=\varepsilon(\xi x,x)$ both $\phi(1)$ and $\phi(\lambda)$ equal 1. Therefore, $\phi_0=1$ and $\phi_n=\lambda$. Thus the first interesting case is $\phi_4\in\bigwedge^4V^*$.

COROLLARY 2.4. *The contact set $G(\phi_{4p})$ is the intersection of the isotropy subgroup $K_x\equiv\{a\in\mathrm{Spin}(V)\colon ax=x\}$ with the $4p$-Grassmannian $G(4p,V)$ inside the Clifford algebra $\mathrm{Cl}(V)\cong\bigwedge V$. Consequently the components of $G(\phi_{4p})$ are in one-to-one correspondence with K_x^0 conjugacy classes of elements of order two in K_x. In particular every component is a symmetric space.*

Proof. Elements of order two in $\mathrm{Spin}(V)$ are precisely the union of the various $G(4p,V)$. Elements of order two in any compact Lie group naturally are a union of symmetric spaces. □

Remark 2.5. A $4p$-plane $\xi\in G(4p,V)$ is a ϕ_{4p}-plane if and only if orthogonal reflection along ξ, denoted $\mathrm{Ref}_\xi\in\mathrm{O}(V)$, fixes ϕ. In particular, ϕ_{4p} is a non-trivial calibration if and only if $\mathrm{Ref}_\xi\,\phi=\phi$ for some k-plane $\xi\in G(k,V)$.

I.3. Triality and the octonians

Let $\mathbf{O}$ denote the normed algebra of octonians or Cayley numbers and let $e_1=1$, $e_2=i$, $e_3=j,\dots,e_8=ke$ be the standard basis. A concrete model for Cl(8) is available using the octonians. Let $V\equiv\mathbf{O}$ considered as an oriented eight dimensional real inner product space. With $\mathbf{P}\equiv\mathbf{S}^+\oplus\mathbf{S}^-$, also let $\mathbf{S}^+\equiv\mathbf{O}^+$, a copy of the octonians marked to distinguish it from $\mathbf{S}^-\equiv\mathbf{O}^-$. Then V can be identified with

$$\left\{\begin{pmatrix}0 & R_u\\ -R_{\bar{u}} & 0\end{pmatrix}\in\mathrm{End}_{\mathbf{R}}(\mathbf{P}):u\in\mathbf{O}\right\}.$$

Here $R_u \in \mathrm{End}_{\mathbf{R}}(\mathbf{O})$ denotes right multiplication by the octonian u. Since $u^2 = -|u|^2$, there is an induced isomorphism $\mathrm{Cl}(V) \cong \mathrm{End}_{\mathbf{R}}(\mathbf{P})$. (Here u^2 is a product in $\mathrm{End}_{\mathbf{R}}(\mathbf{P})$, not $\mathbf{O}$!) We have

$$\mathrm{Spin}(8) \equiv \{u_1 \dots u_{2k} \in \mathrm{Cl}(V) : u_1, \dots, u_{2k} \in V \text{ are of unit length}\}.$$

The *vector representation* of Spin(8) is given by: $\chi_a(u) \equiv aua^{-1}$, for all $u \in V$. Since $\mathrm{Spin}(8) \subset \mathrm{Cl}(V)^{\mathrm{even}} \cong \mathrm{End}_{\mathbf{R}}(\mathbf{S}^+) \oplus \mathrm{End}_{\mathbf{R}}(\mathbf{S}^-)$, each element $g \in \mathrm{Spin}(8)$ is of the form

$$g \equiv \begin{pmatrix} g_+ & 0 \\ 0 & g_- \end{pmatrix}.$$

In fact g_+, g_- have determinant equal to 1. Let $g_0 = \chi_g$. The notation $g \equiv (g_+, g_-) \in \mathrm{Spin}(8)$ as well as the notation $g \equiv (g_0, g_+, g_-) \in \mathrm{Spin}(8)$ will be used. The representation $\varrho_+(g) \equiv g_+$ is called the *positive spinor representation*, while $\varrho_-(g) \equiv g_-$ is the *negative spinor representation.* Each pair of these three 8-dimensional representations χ, ϱ_+ and ϱ_- of Spin(8) are distinct, because $\ker \chi = \{(I,I), (-I,-I)\}, \ker \varrho_+ = \{(I,I), (I,-I)\}$ and $\ker \varrho_- = \{(I,I), (-I,I)\}$ are distinct.

Let c denote conjugation on $\mathbf{O}$, and given $h \in \mathrm{End}_{\mathbf{R}}(\mathbf{O})$, let $h' = c \cdot h \cdot c$.

THEOREM 3.1 (Triality, [3]). *Suppose (g_0, g_+, g_-) is a triple of orthogonal maps on $\mathbf{O}$. Then $(g_+, g_-) \in \mathrm{Spin}(8)$ with g_0 the vector representation of (g_+, g_-) if and only if $g_+(xy) = g_-(x) g_0(y)$, for all $x, y \in \mathbf{O}$. Furthermore $\alpha(g_0, g_+, g_-) \equiv (g'_-, g'_+, g'_0)$ and $\beta(g_0, g_+, g_-) \equiv (g'_0, g'_-, g'_+)$ are outer automorphisms of* Spin(8) *satisfying $\alpha^2 = \beta^2 = (\alpha\beta)^3 = 1$ and therefore they generate a group isomorphic to the symmetric group on three letters.*

The Clifford algebra Cl(7) is a subalgebra of Cl(8) generated by $\mathrm{Im}\,\mathbf{O} \subset V \subset \mathrm{Cl}(8)$. Since the generators of Cl(7) are of the form

$$\begin{pmatrix} 0 & R_u \\ R_u & 0 \end{pmatrix} u \in \mathrm{Im}\,\mathbf{O}$$

we have

$$\mathrm{Cl}(7)^{\mathrm{even}} = \left\{ \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} : g \in \mathrm{End}(\mathbf{O}) \right\}.$$

It now follows from the Triality Theorem 3.1 that

$$\mathrm{Spin}(7) = \{(g_0, g_+, g_-) \in \mathrm{Spin}(8) : g_+ = g_-\} = \{(g_0, g_+, g_-) : g_0 s_1 = s_1\}.$$

It is now elementary to compute the elements of order two in this Spin(7). Any such element must be an element of some $4p$-Grassmannian. First, $G(0, \mathrm{Im}\,\mathbf{O}) = \{\pm I\} \in \mathrm{Cl}(8)$

is just the center of Spin(7). This leaves $G(4, \mathrm{Im}\,\mathbf{O})$ which consists of noncentral elements of order two. On $\mathbf{O}^+$ and $\mathbf{O}^-$ these elements are reflections along Cayley 4-planes (see below).

We shall denote the standard basis on $\mathbf{O}^+, \mathbf{O}^-$ by $s_1 = i$, $s_2 = j$, ..., $s_8 = ke$ to distinguish it from the standard basis for the vectors, i.e. for $\mathbf{O} \subset \mathrm{Cl}(8)$.

For $s_1 \in \mathbf{O}^+$ consider $\phi = 16 s_1 \circ s_1 \in \bigwedge V^* \cong \mathrm{End}(\mathbf{P})$. By Remark 2.3

$$\phi = 1 + \Phi + \lambda$$

with $\Phi \in \bigwedge^4 V^*$ and λ the unit volume element.

PROPOSITION 3.2. (i) *Φ is a calibration called the Cayley calibration (see [4]). The set of planes it calibrates, denoted $\mathrm{CAY}(\mathbf{O}) \subset G(4, \mathbf{O})$, is isometric to the Grassmannian $G(4, \mathbf{R}^7)$.*

(ii) *$\Phi(u_1 \wedge u_2 \wedge u_3 \wedge u_4) = \varepsilon(u_1, u_2 \times u_3 \times u_4)$, for all $\xi = u_1 \wedge u_2 \wedge u_3 \wedge u_4 \in G(4, \mathbf{O})$, where the triple cross product is defined by*

$$u_2 \times u_3 \times u_4 \equiv \tfrac{1}{2}(u_2(\bar{u}_3 u_4) - u_4(\bar{u}_3 u_2)).$$

Proof. (i) By triality the isotropy of Spin(8) at s_1 is conjugate by an *outer* automorphism to the above computed Spin(7).

$$K_{s_1} = \{(g_+, g_0, g'_-) : (g_0, g_+, g_-) \in \mathrm{Spin}(7)\}.$$

Now by Corollary 2.4 the planes calibrated by Φ are elements of order two in K_{s_1} which we have already computed.

(ii) The right hand side defines a 4-form on $\mathbf{O}$, the *Cayley calibration.* We may assume that u_1, u_2, u_3, u_4 are orthonormal. Recall $\Phi(\xi) = \varepsilon(\xi 1_+, 1_+)$.

The Clifford product

$$\xi = u_1 \cdot u_2 \cdot u_3 \cdot u_4 \equiv \begin{pmatrix} R_{u_1} R_{\bar{u}_2} R_{u_3} R_{\bar{u}_4} & 0 \\ 0 & R_{\bar{u}_1} R_{u_2} R_{\bar{u}_3} R_{u_4} \end{pmatrix}.$$

Therefore,

$$\begin{aligned} \Phi(\xi) &= \varepsilon(1_+, ((\bar{u}_4 u_3)\bar{u}_2)u_1) = \varepsilon(\bar{u}_1, (\bar{u}_4 u_3)\bar{u}_2) \\ &= \varepsilon(u_1, u_2(\bar{u}_3 u_4)) = \varepsilon(u_1, u_2 \times u_3 \times u_4). \end{aligned}$$

□

With calculations in the next part of this paper in mind we now identify some additional subgroups of Spin(8). First we put on $\mathbf{O}$ the quaternionic structure given by the following three complex structures

$$I = R_i, \quad J = R_j, \quad K = R_i \circ R_j \qquad (\neq \pm R_k!).$$

We can lift these elements of SO(8) to Spin(8) as follows:

$$I=(R_i, r_{12}, -L_i), \quad J=(R_j, r_{13}, -L_j), \quad K=(R_i\circ R_j, r_{23}, L_i\circ L_j),$$

where r_{ij} is the reflection along the (i,j)-plane in $\mathbf{O}^+$ and L_i is the left multiplication on $\mathbf{O}^-$ by $i\in\mathbf{O}$. The elements I,J,K also give a quaternionic structure on $\mathbf{O}^-$. The proof of the following lemma is now a straightforward calculation.

LEMMA 3.3. (i) *The subgroup of* Spin(8) *that commutes with* I *is*

$$\mathrm{U}(4)\cup\mathrm{U}(4)(I,-I,-I),$$

where U(4) *acts on* $\mathbf{O}$ *and* $\mathbf{O}^-$ *by unitary matrices in the complex structure determined by* I*, and on* $\mathbf{O}^+$ *by elements of* $\mathrm{SO}(2)\times\mathrm{SO}(6)$*. The center of* U(4) *gives rotations in the* $(1,2)$*-plane while elements of* SU(4) *fix this plane.*

(ii) *The subgroup of* Spin(8) *that commutes with* I,J,K *is* $\mathrm{Sp}(2)\cup\mathrm{Sp}(2)(I,-I,-I)$ *where* Sp(2) *acts on* $\mathbf{O}$ *and* $\mathbf{O}^-$ *by quaternionic unitary maps in the quaternionic structure* (I,J,K)*, and on* $\mathbf{O}^+$ *by elements of* SO(5)*. These fix the* $(1,2,3)$*-plane of* $\mathbf{O}^+$*.*

(iii) *The subgroup* $\mathrm{Sp}(1)\subset\mathrm{Spin}(8)$ *that on* $\mathbf{O}$ *and* $\mathbf{O}^-$ *consists of multiplications by unit quaternions acts on* $\mathbf{O}^+$ *by rotations in the* $(1,2,3)$*-plane.*

II. Dimension 16 and E_8

II.1. A model for the Clifford algebra Cl(16)

In this section we build a model for Cl(16) from the model of Cl(8) described in Section 3 of Part I. Our detailed knowledge of Cl(8) will make computations in Cl(16) in subsequent sections much easier.

Let $\nu=e_1e_2\dots e_8\in\mathrm{Cl}(8)$ be the volume element and define $\varepsilon_i,\varepsilon_i'\in\mathrm{Cl}(8)\otimes\mathrm{Cl}(8)$ as follows:

$$E_i\equiv\varepsilon_i\equiv e_i\otimes\nu, \quad E_{8+i}\equiv\varepsilon_i'\equiv 1\otimes e_i, \quad i=1,2,\dots,8.$$

A short computation shows that $E_i^2=-I$ and that $E_iE_j=-E_iE_j$ for $i\neq j$. So from the fundamental lemma of Clifford algebras we have

$$\begin{aligned}\mathrm{Cl}(16)&\cong\mathrm{Cl}(8)\otimes\mathrm{Cl}(8)\cong\mathrm{End}(\mathbf{O}^+\oplus\mathbf{O}^-)\otimes\mathrm{End}(\mathbf{O}^+\oplus\mathbf{O}^-)\\&\cong\mathrm{End}((\mathbf{O}^+\otimes\mathbf{O}^+)\oplus(\mathbf{O}^-\otimes\mathbf{O}^-)\oplus(\mathbf{O}^+\otimes\mathbf{O}^-)\oplus(\mathbf{O}^-\otimes\mathbf{O}^+))\end{aligned}$$

where

$$V(16)\equiv\mathrm{span}_{\mathbf{R}}\{E_j:j=1,2,\dots,16\}=V_8\oplus V_8'\equiv\mathrm{span}_{\mathbf{R}}\{\varepsilon_i\}\oplus\mathrm{span}_{\mathbf{R}}\{\varepsilon_i'\}.$$

The volume element $\lambda \equiv E_1 E_2 \dots E_{16}$ multiplies out to be

$$\lambda = \nu \otimes \nu \in \mathrm{Cl}(8) \otimes \mathrm{Cl}(8).$$

Therefore, λ is $+I$ on $(\mathbf{O}^+ \otimes \mathbf{O}^+) \oplus (\mathbf{O}^- \otimes \mathbf{O}^-) = \mathbf{S}^+(16)$ and $\lambda = -I$ on $(\mathbf{O}^+ \otimes \mathbf{O}^-) \oplus (\mathbf{O}^- \otimes \mathbf{O}^+) = \mathbf{S}^-(16)$.

If $\{s_1, s_2, \dots, s_8\}$ is the standard basis of $\mathbf{O}^+$ we set

$$S_i = s_i \otimes s_i \in \mathbf{S}^+(16), \quad i = 1, 2, \dots, 8,$$

and

$$D = \mathrm{span}_{\mathbf{R}}\{S_i : i = 1, 2, \dots, 8\} \subset \mathbf{S}^+(16).$$

Let $\mathrm{Spin}(8) \subset \mathrm{Spin}(16)$ (resp. $\mathrm{Spin}(8)' \subset \mathrm{Spin}(16)$) be the subgroup generated by $\varepsilon_i \varepsilon_j$, $i \neq j$ (resp. $\varepsilon'_i \varepsilon'_j$, $i \neq j$). The corresponding Clifford algebras $\mathrm{Cl}(8)$ and $\mathrm{Cl}(8)'$ are given by

$$\mathrm{Cl}(8) \equiv \mathrm{Cl}(8) \otimes 1 \subset \mathrm{Cl}(16)$$

and by

$$\mathrm{Cl}(8)' \equiv 1 \otimes \mathrm{Cl}(8) \subset \mathrm{Cl}(16).$$

Note that

$$\mathrm{Spin}(8) \cap \mathrm{Spin}(8)' = \{\pm I\},$$

and let $(\mathrm{Spin}(8) \times \mathrm{Spin}(8))/\mathbf{Z}_2 \subset \mathrm{Spin}(16)$ denote the group generated by $\mathrm{Spin}(8)$ and $\mathrm{Spin}(8)'$. This group is covered twice by $\mathrm{Spin}(8) \times \mathrm{Spin}(8)'$. Now $(\mathrm{Spin}(8) \times \mathrm{Spin}(8))/\mathbf{Z}_2$, and therefore $\mathrm{Spin}(8) \times \mathrm{Spin}(8)'$, leaves $\mathbf{O}^+ \otimes \mathbf{O}^+$ invariant:

$$(g, h) x \otimes y = g_+ x \otimes h_+ y, \quad x, y \in \mathbf{O}^+, \quad (g, h) \in \mathrm{Spin}(8) \times \mathrm{Spin}(8)'.$$

The $\mathrm{Spin}(8) \times \mathrm{Spin}(8)'$ action on $\mathbf{O}^+ \otimes \mathbf{O}^+$ is seen to be the lift of the $\mathrm{SO}(8) \times \mathrm{SO}(8)$ action on 8×8 real matrices. In this model D becomes the set of diagonal matrices which is a cross-section of all $\mathrm{SO}(8) \times \mathrm{SO}(8)$ orbits.

II.2. The orbit structure of Spin(16) on $\mathbf{S}^+(16)$

Consider the symmetric space $E_8/\mathrm{Spin}(16)$. It is well known [5], [1] that the isotropy representation of $\mathrm{Spin}(16)$ is equivalent to the $\mathbf{S}^+(16)$ representation. In $\mathbf{S}^+(16)$ there is an eight dimensional cross-section of all $\mathrm{Spin}(16)$ orbits and each $\mathrm{Spin}(16)$ orbit meets this cross-section in a W orbit. Here the W is a finite reflection group of type E_8 acting on the cross-section. We proceed to identify one such cross-section and the accompanying W concretely.

For $\lambda \in \mathbf{R}^8$ we let $S(\lambda) \in D$ denote

$$S(\lambda) = \sum_{i=1}^{8} \lambda_i S_i = \sum_{i=1}^{8} \lambda_i s_i \otimes s_i,$$

and identify λ with $S(\lambda)$ whenever convenient.

LEMMA 2.1. *$D\subset \mathbf{S}^+(16)$ is a cross-section of all* Spin(16) *orbits. Every* Spin(16) *orbit intersects D in an orbit of a finite reflection group W. This group is generated by reflections through hyperplanes $\lambda_i=\pm\lambda_j$, $i\neq j$, and the hyperplane $\sum\lambda_i=0$. It is a reflection group of type E_8.*

Proof. Let so(16) be the Lie algebra of Spin(16). According to [1] any Spin(16) orbit meets the orthogonal complement of $\mathrm{so}(16)\cdot x$, $x\in S^+(16)$. We choose $x=S_1=s_1\otimes s_1$. Evidently $\varepsilon_i\varepsilon_j'(s_1\otimes s_1)=\varepsilon_i s_1\otimes\varepsilon_j' s_1\in\mathbf{O}^-\otimes\mathbf{O}^-$, so that in fact $\mathbf{O}^-\otimes\mathbf{O}^-\subset\mathrm{so}(16)\cdot x$. Thus every Spin(16) orbit meets $\mathbf{O}^+\otimes\mathbf{O}^+$. Now using $\mathrm{Spin}(8)\times\mathrm{Spin}(8)'$ we may move any element of $\mathbf{O}^+\otimes\mathbf{O}^+$ into D, and hence D is a cross-section of all Spin(16) orbits. All such cross-sections are Spin(16) conjugate, (see [1]). Since W is a reflection group of type E_8 (see [5]) it suffices to find in W generators for E_8. First observe that the reflections through planes $\lambda_i=\pm\lambda_j$ are induced by elements of $\mathrm{Spin}(8)\times\mathrm{Spin}(8)'$. Next consider the circle in Spin(16) we shall call T^1:

$$T^1=\left\{t(\theta)=\prod_{i=1}^{8}(\cos\theta+\sin\theta\,\varepsilon_i\varepsilon_i'):\theta\in\mathbf{R}\right\}.$$

The generator of T^1 in the Lie algebra of Spin(16) is $X=\sum_{i=1}^{8}\varepsilon_i\varepsilon_i'$.

LEMMA 2.2. *$T^1\subset\mathrm{Spin}(16)$ fixes the hyperplane $H=\{S(\lambda):\sum_{i=1}^{\infty}\lambda_i=0\}$ of D. The element $t(\pi/8)$ maps D into itself and is the reflection through H. The element $\xi=t(\pi/4)$ fixes D pointwise and acts on $V(16)=V_8\oplus V_8'$ by*

$$\xi=\begin{pmatrix}0 & -I\\ I & 0\end{pmatrix}.$$

Proof. Application of the generator X of T^1 to $s_j\otimes s_j$, $j=1,2,\dots,8$ always yields $w\equiv\sum_{i=1}^{8}s_i\otimes s_i\in\mathbf{O}^-\otimes\mathbf{O}^-\subset\mathbf{S}^+(16)$, and hence X annihilates elements of H. Now

$$X\cdot w=8\sum_{i=1}s_i\otimes s_i\in D\subset\mathbf{O}^+\otimes\mathbf{O}^+,$$

which shows that T^1 leaves $D\oplus\mathbf{R}w$ stable, fixing $H\subset D$ pointwise. Next, since $\|Xw\|=8\|w\|$ the subgroup of T^1 that fixes $D\oplus\mathbf{R}w$ pointwise is generated by $t(\pi/4)$, and therefore $t(\pi/8)$ must be the reflection through H. The action of $\cos\theta+\sin\theta\,\varepsilon_j\varepsilon_j'$ on $V(16)$ is easily computed to be the rotation in the $(\varepsilon_j,\varepsilon_j')$-plane by the angle 2θ. This then implies that $\xi=t(\pi/4)$ blocks with respect to $V(16)=V_8\oplus V_8'$ as advertised. □

A Weyl chamber C (a fundamental region for W) is a closure of a connected component of the complement of all reflection hyperplanes in D. We choose

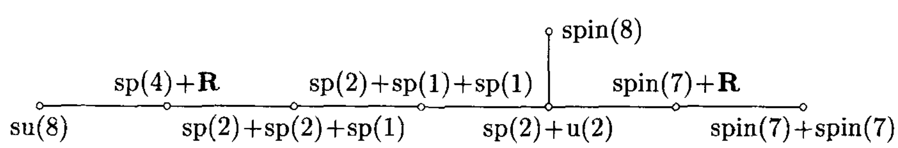
$$C=\{S(\lambda)\in D:\lambda_1\geqslant\lambda_2\geqslant\lambda_3\geqslant\lambda_4\geqslant\lambda_5\geqslant\lambda_6\geqslant|\lambda_7|,\text{ and }\textstyle\sum\lambda_i\leqslant 0\}.$$

Observe that C is bounded by the hyperplanes $\lambda_i=\lambda_{i+1}$, $i=1,2,\dots,6$, $\lambda_6=-\lambda_7$, and $\sum\lambda_i=0$. We then have

PROPOSITION 2.3. *Every orbit of* Spin(16) *on* $\mathbf{S}^+(16)$ *has a unique representative in* $C \subset D$.

II.3. Maximal isotropy subgroups in Spin(16)

It is well known [5] that the isotropy subgroup of Spin(16) at $S(\lambda) \in D$ depends only on the set of reflection hyperplanes $S(\lambda)$ lies in. The origin lies in all the hyperplanes, and the isotropy there is all of Spin(16). The maximal isotropy subgroups (the ones that are not included in larger *proper* isotropy subgroups of Spin(16)) correspond to the eight edges of the fundamental cone C. The 8 reflections through hyperplanes bounding C generate W and their geometric relationships to each other are encoded in the Dynkin diagram of W. A point on one of the edges of C lies in all but one of these hyperplanes. Thus to each vertex of the Dynkin diagram of E_8 corresponds a maximal isotropy subgroup. To compute the Lie algebras of these subgroups from the structure theory of symmetric spaces is easy: First we strike from the Dynkin diagram the vertex corresponding to the chosen edge of C. The resulting diagram corresponds to a compact connected and 1-connected Lie group G. Now G is transitive on precisely one simply connected symmetric space G/K of rank *equal* to the rank of G. The Lie algebra of K is then the Lie algebra of the maximal isotropy subgroup at the chosen edge. (In fact, there is a homomorphism $\varphi: G \to E_8$ with finite kernel that takes K onto the identity component of the maximal isotropy.) Consulting the list of symmetric spaces [5] we may label the Dynkin diagram by the maximal isotropy Lie subalgebras:



We now proceed to determine all the components of the maximal isotropy subgroups. We first let

$$Z(D) = \{g \in \mathrm{Spin}(16) : gx = x \text{ for all } x \in D\}.$$

We shall need the following lemma valid for any polar representation:

LEMMA 3.1. *Let G be a compact group and $G|V$ be a polar representation on a real inner product space V. If $\mathfrak{a} \subset V$ is a Cartan subspace for this action, let*

$$Z(\mathfrak{a}) = \{g \in G : g \cdot x = x, \ \text{all } x \in \mathfrak{a}\}.$$

Then for any $v \in \mathfrak{a}$ every component of the isotropy G_v at v intersects $Z(\mathfrak{a})$. That is, G_v is generated by its identity component and $Z(\mathfrak{a})$.

Proof. According to Theorem 3 in [1] the action of G_v^0, the identity component of G_v, on the orthogonal complement of $\mathfrak{g}\cdot v$ is polar with $\mathfrak{a}$ as a Cartan subspace. Let W be the reflection group induced by G on $\mathfrak{a}$. If v lies on only one reflection hyperplane of W then G_v^0 leaves this hyperplane fixed and contains an element that induces the reflection through this hyperplane (see [1]). More generally, if v lies on k reflection hyperplanes the G_v^0 induces on $\mathfrak{a}$ the reflection group generated by reflections through these k hyperplanes. But by a well known property of reflection groups this group is precisely the isotropy subgroup of W at v.

Now let $g\in G_v$ and consider $g\cdot\mathfrak{a}\subset(\mathfrak{a}v)^{\perp}$. Evidently $g\cdot\mathfrak{a}$ is a Cartan subspace, so there is an $h\in G_v^0$ such that $hg\cdot\mathfrak{a}\subset\mathfrak{a}$. By the preceding paragraph we may choose h so that hg not only fixes v but in fact all $x\in\mathfrak{a}$, that is $h\in Z(\mathfrak{a})$. □

To apply the above lemma to the computation of maximal isotropy subgroups we need to compute $Z(D)\subset\mathrm{Spin}(16)$.

The isomorphism between V_8 and V_8' given by $\varepsilon_j\to\varepsilon_j'$ defines an isomorphism between Spin(8) and Spin(8)′. The graph of this isomorphism is a subgroup of $\mathrm{Spin}(8)\times\mathrm{Spin}(8)'$ that we shall denote by diag Spin(8). The image of this subgroup in

$$(\mathrm{Spin}(8)\times\mathrm{Spin}(8))/\mathbf{Z}_2\subset\mathrm{Spin}(16)$$

we shall denote by diag SO(8) (it is covered twice by diag Spin(8)).

In $\mathrm{Spin}(8)\subset\mathrm{Cl}(8)$ we now consider the group $\hat{A}$ of even oriented axis planes: $\hat{A}$ is generated by $e_ie_j\in\mathrm{Cl}(8)$. We let

$$A=\tau(\hat{A}),$$

where $\tau\colon\mathrm{Spin}(8)\to\mathrm{Spin}(8)$ is the automorphism $\tau(g_0,g_+,g_-)=(g_+,g_0,g_-')$. Note that A is generated by elements that on $\mathbf{S}^+(8)$ change a pair of signs of the standard basis, whereas on $V(8)$ these elements give complex structures (see discussion preceding Lemma I.3.3). The group A is nonabelian, it is a central extension of $(\mathbf{Z}_2)^7$ by $\mathbf{Z}_2$. We shall also let A denote the corresponding group in diag Spin(8). For a multiindex I set $a_I\in A$ to be

$$a_I=\tau(e_I)=\tau\big(e_{i_1}e_{i_2}\dots e_{i_k}\big),\quad e_I\in\hat{A}.$$

Note that under the covering map $\pi\colon\mathrm{diag\,Spin}_8\to\mathrm{diag\,SO}_8$ we have

$$\pi(a_I)=\pi(a_{*I}),$$

where $*$ is the Hodge star operator on the set of multiindices of a standard basis for $\mathbf{R}^8$. Finally, let $\eta\in\mathrm{Spin}(8)\times\mathrm{Spin}(8)$ be the image of $((I,I,I),(-I,I,-I))\in\mathrm{Spin}(8)\times\mathrm{Spin}(8)'$, and note that on $V(16)$ η fixes ε_i and changes the sign of ε_i'.

LEMMA 3.2.

$$Z(D)\cap((\mathrm{Spin}(8)\times\mathrm{Spin}(8))/\mathbf{Z}_2)=\pi(A)\cup\pi(A)\eta.$$

Remark. Clearly η commutes with $\pi(A)$. The reason for introducing A is that multiplication in A amounts to manipulating indices, which is easy.

Proof. For $(g,h)\in\mathrm{Spin}(8)\times\mathrm{Spin}(8)'$ that fixes D pointwise and all $i=1,2,\ldots,8$ we must have

$$g_+s_i\otimes h_+s_i=s_i\otimes s_i.$$

Hence $g_+=h_+$ change even numbers of signs of the standard basis $\{s_i\}$. If $g=h$ then clearly $(g,g)\in A$. The only other possibility is $(g_0,g_+,g_-)=(-h_0,h_+,-h_-)$ in which case $\pi(g,h)\in\pi(A)\eta$. □

Next set $\xi\in T^1\subset\mathrm{Spin}(16)$ to be $\xi=t(\pi/4)$ (see Lemma 2.2). We have $\xi\in Z(D)$ and we note that on $V(16)$ ξ sends ε_i to ε_i' and ε_i' to $-\varepsilon_i$.

PROPOSITION 3.3. *$Z(D)$ acts faithfully on $V(16)$. It is a group of 512 elements generated by $\pi(A)$, η, and ξ.*

Proof. The kernel of the Spin(16) representation on $V(16)$ is $\{\pm I\}\subset\mathrm{Cl}(16)$. Since $-I$ cannot be in any isotropy subgroup of $x\in\mathbf{S}^+(16)$ we see that $Z(D)$ is faithfully represented by 16×16 orthogonal matrices.

Let $G\subseteq\mathrm{Spin}(16)$ be the full isotropy subgroup at $S_1\in D$. The identity component G_0 is $\mathrm{Spin}(7)\times\mathrm{Spin}(7)\subset(\mathrm{Spin}(8)\times\mathrm{Spin}(8))/\mathbf{Z}_2$ (see Proposition I.3.2.). In fact, the degree four calibration determined by S_1 is $\omega+\omega'$, where ω,ω' are the Cayley calibrations determined by $s_1\in\mathbf{O}^+$ on V_8 and V_8' respectively. Since $G_0\subset G$ is normal any $g\in G$ must map a G_0 invariant subspace of $V(16)$ into another G_0 invariant subspace. Thus either g or $g\xi$ acts on $V(16)$ by

$$\begin{pmatrix} a & 0\\ 0 & b\end{pmatrix},\quad a,b\in\mathrm{O}(8).$$

But since this element must fix $\omega+\omega'$ we in fact must have $a,b\in\mathrm{Spin}(7)$. Thus g or $g\xi\in(\mathrm{Spin}(8)\times\mathrm{Spin}(8))/\mathbf{Z}_2$ and by Lemma 3.2 the result follows. □

Remark 3.4. As noted before on $V(16)$ we have

$$\xi=\begin{pmatrix} 0 & -I\\ I & 0\end{pmatrix},\quad \eta=\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix}.$$

Furthermore, $a_{ij}\in Z(D)$ act on $V(16)$ by

$$a_{ij}=\begin{pmatrix} J_{ij} & 0\\ 0 & J_{ij}'\end{pmatrix},$$

where J_{ij} and J'_{ij} are complex structures on V_8 and V'_8: Identifying V_8 with $\mathbf{O}$ by $\varepsilon_j \to e_j$ reveals that J_{ij} is given by

$$J_{ij}(x) = -(xe_j)\bar{e}_i, \quad x \in \mathbf{O}.$$

II.4. Calibrations on $\mathbf{R}^{16}$ constructed by squaring spinors

Since each positive spinor is Spin(16) equivalent to a spinor

$$S(\lambda) \equiv \sum_{j=1}^{8} \lambda_j S_j = \sum_{j=1}^{8} \lambda_j s_j \otimes s_j \tag{4.1}$$

in the diagonal $D \subset \mathbf{S}^+(16)$, we need only consider unit spinors of this form.

THEOREM 4.1. *Consider the exterior form*

$$\phi \equiv 16^2 S(\lambda) \circ S(\lambda), \quad \text{on } V(16), \tag{4.2}$$

where $\lambda \equiv (\lambda_1, ..., \lambda_8)$ *is of unit length. The isotropy subgroup of* SO(16) *at* ϕ *lifts isomorphically (under the vector representation* χ: Spin(16)$\to$SO(16)) *to the isotropy subgroup* $K(\lambda)$ *of* Spin(16) *at the spinor* $S(\lambda)$. *The nonzero components of the form* ϕ *are of degree* 0, 4, 8, 12, *and* 16.

Each ϕ_{4k} *is a calibration on* $V(16)$ *fixed by the group* $\chi(K(\lambda)) \subset \mathrm{SO}(16)$. *The contact set is given in terms of* $K(\lambda)$ *by:*

$$G(\phi_{4k}) = K(\lambda) \cap G(4k, V(16)). \tag{4.3}$$

The isotropy subgroups $K(\lambda)$ *and hence the contact sets* $G(\phi_{4k})$ *are partially ordered under inclusion. For each spinor* $S(\lambda)$ *fixed by a maximal isotropy subgroup both the isotropy* $K(\lambda)$ *and the contact sets* $G(\phi_{4k})$, $k=1,2$, *are described below, case by case with each case labelled by the identity component of* $K(\lambda)$.

Remark. First, $\phi_0 = 1$ and ϕ_{16} is the unit volume element; no matter which spinor $S(\lambda)$ is selected. Second, $\phi_{12} = *\phi_4$ and $G(\phi_{12}) = *G(\phi_4)$. Thus we need only describe $G(\phi_4)$ and $G(\phi_8)$, for each choice of spinor $S(\lambda)$. Also each ϕ_8 is self dual so that the contact sets $G(\phi_8)$ are $*$ invariant.

Conventions. The subspaces $V_8, V'_8 \subset V(16)$ are isometric under the map $\varepsilon_j \to \varepsilon'_j$. If $\alpha \subset V_8$ is an oriented k-plane we shall denote by α' the corresponding k-plane in V'_8. Also, if $g \in \mathrm{SO}(V_8)$ then by the graph of g we mean the oriented 8-plane spanned by

$(\varepsilon_j, g(\varepsilon_j)')\in V_8\oplus V_8'$. By *the* quaternionic structure on $V(16)$, (resp. V_8 or V_8') we mean the following action by $\mathbf{H}$:

$$(a+bi+cj+dk)\cdot v = av+ba_{12}(v)+ca_{13}(v)+da_{23}(v),$$

where $v\in V(16)$ (resp. $v\in V_8$ or V_8'), and a_{12}, a_{13}, and $a_{23}\in Z(D)$ are complex structures described at the end of Section 3. Identifying V_8 with $\mathbf{O}$ by $\varepsilon_j\to e_j$ this quaternionic structure agrees with the one of Section 3, Part I.

Given a quaternionic structure $\{I,J,K\}$ on $\mathbf{R}^{4n}$ (with the standard inner product) let $\mathrm{SU}(I)\subset\mathrm{SO}(4n)$ be the special unitary group determined by I. Choose a $2n$ dimensional subspace $\mathbf{R}^{2n}\subset\mathbf{R}^{4n}$ such that $\mathbf{R}^{2n}$ is J stable and $\mathbf{R}^{2n}\oplus I\mathbf{R}^{2n}=\mathbf{R}^{4n}$, orthogonal direct sum. Let c be the conjugation $c(u+Iv)=u-Iv$, $u,v\in\mathbf{R}^{2n}$. We shall denote the set of special Lagrangian planes determined by the ordered pair (I,J) by $\mathrm{SLAG}(I,J)$. These are the planes that are $\mathrm{SU}(I)$ equivalent to $\mathbf{R}^{2n}$. The corresponding conjugations are $\mathrm{SU}(I)$ conjugate to c, they lie in the coset $\mathrm{SU}(I)J$, and exhaust all reflections in that coset.

Finally we note that the subset of J complex planes in $\mathrm{SLAG}(I,J)$ corresponds to the set of reflections in the coset $\mathrm{Sp}(I,J,K)\cdot J$. These reflections form the symmetric space $\mathrm{Sp}(n)/\mathrm{U}(n)$.

The Spin(8) *geometry.* Let

$$S(\lambda)\equiv\frac{1}{\sqrt{8}}(s_1\otimes s_1-s_2\otimes s_2-\ldots-s_8\otimes s_8),$$

i.e. choose $\lambda\equiv(1,-1,-1,-1,-1,-1,-1,-1)$. The isotropy subgroup of Spin(16) at $S(\lambda)$ is:

$$K(\lambda)=\mathrm{Spin}(8)\cup\mathrm{Spin}(8)\xi. \tag{4.4}$$

The calibration ϕ_4 is trivial, that is there are no calibrated 4-planes in this geometry. Calibrated 8-planes in this geometry are of four distinct types:

$$V_8\cup V_8', \tag{4.5}$$

$$\begin{gathered}\{\alpha\wedge\sigma(\alpha') : \alpha\in G(4,V_8)\}\cong G(4,\mathbf{R}^8),\\ \text{where } \sigma\colon G(4,V_8')\to G(4,V_8') \text{ is a reflection through } \mathrm{CAY}(V_8'),\end{gathered} \tag{4.6}$$

$$\{\alpha : \alpha \text{ is a graph of } g_+g_-^{-1}, (g_+,g_-)\in\mathrm{Spin}(8)\}\cong S^7, \tag{4.7}$$

$$\begin{gathered}\{\alpha : \alpha \text{ is a graph of } g_+rg_-^{-1}, (g_+,g_-)\in\mathrm{Spin}(8)\}\cong G(3,R^6),\\ \text{where } r \text{ is a reflection through any Cayley plane.}\end{gathered} \tag{4.8}$$

The Spin(7)×Spin(7) *geometry.* Let

$$S(\lambda)\equiv S_1=s_1\otimes s_1,$$

i.e. choose $\lambda\equiv(1,0,0,0,0,0,0,0)$. The isotropy subgroup of Spin(16) at $S(\lambda)$ is:

$$K(\lambda)\equiv(\mathrm{Spin}(7)\times\mathrm{Spin}(7))\cup(\mathrm{Spin}(7)\times\mathrm{Spin}(7))\xi. \tag{4.9}$$

Calibrated 4-planes in this geometry are:

$$G(\phi_4)=\mathrm{CAY}(V_8)\cup\mathrm{CAY}(V_8'). \tag{4.10}$$

Calibrated 8-planes in this geometry are of three distinct types:

$$V_8\cup V_8', \tag{4.11}$$

$$\{\alpha\wedge\beta:\alpha\in\mathrm{CAY}(V_8)\ \text{ and }\beta\in\mathrm{CAY}(V_8')\}, \tag{4.12}$$

$$\{\mathrm{graph}\,g:g\in\mathrm{Spin}(7)\}. \tag{4.13}$$

The Spin(7)×T^1 *geometry.* Let

$$S(\lambda)\equiv\frac{1}{\sqrt{56}}(7s_1\otimes s_1-s_2\otimes s_2-\ldots-s_8\otimes s_8),$$

i.e. choose

$$\lambda=\frac{1}{\sqrt{56}}(7,-1,-1,-1,-1,-1,-1,-1).$$

The isotropy subgroup of Spin(16) at $S(\lambda)$ is

$$K(\lambda)=\big(\mathrm{diag}\,\mathrm{Spin}(7)\times T^1\big)/\mathbf{Z}_2\cup\big(\mathrm{diag}\,\mathrm{Spin}(7)\times T^1\big)/\mathbf{Z}_2\cdot\eta. \tag{4.14}$$

There are no calibrated 4-planes in this geometry. The calibrated 8-planes in this geometry form five disjoint components:

$$\{\alpha\wedge\alpha':\alpha\in\mathrm{CAY}(V_8),\alpha'\in\mathrm{CAY}(V_8')\}\cong G(3,\mathbf{R}^7), \tag{4.15}$$

$$\{\mathrm{graph}\,J:J\in\mathrm{Spin}(7)\text{ is a complex structure Spin(7) conjugate to }a_{12}\}\cong S^6, \tag{4.16}$$

$$\{\mathrm{graph}\,J:J\in\mathrm{Spin}(7)\text{ is a complex structure Spin(7) conjugate to }a_{23}\}\cong G(2,\mathbf{R}^7), \tag{4.17}$$

$$\{t(\alpha):t\in T^1,\ \alpha\text{ is the graph of the identity}\}\cong S^1, \tag{4.18}$$

$$\{t(\alpha):t\in T^1,\ \alpha\text{ is a graph of Cayley reflection in Spin(7)}\}\cong S^1\times G(3,\mathbf{R}^7). \tag{4.19}$$

The SU(8) *geometry.* Let

$$S(\lambda)\equiv\frac{1}{\sqrt{2}}(s_1\otimes s_1-s_2\otimes s_2),$$

i.e. choose $\lambda=(1/\sqrt{2})(1,-1,0,0,0,0,0,0)$. The isotropy subgroup of Spin(16) at $S(\lambda)$ is

$$K(\lambda)=\mathrm{SU}(8)\cup\mathrm{SU}(8)\cdot a_{23}. \tag{4.20}$$

The calibrated 4-planes in this geometry are:

$$\{\alpha:\alpha \text{ is a complex 2-plane in the } a_{12} \text{ complex structure on } V(16)\}\cong G(2,\mathbf{C}^8). \tag{4.21}$$

The calibrated 8-planes form two components:

$$\{\alpha:\alpha \text{ is a complex 4-plane in the } a_{12} \text{ complex structure on } V(16)\}\cong G(4,\mathbf{C}^8), \tag{4.22}$$

$$\{\alpha:\alpha\in\mathrm{SLAG}(a_{12},a_{23})\}\cong\mathrm{SU}(8)/\mathrm{SO}(8). \tag{4.23}$$

The $\mathrm{Sp}(4)\times\mathrm{SO}(2)$ *geometry.* Let

$$S(\lambda)\equiv\frac{1}{\sqrt{6}}(2s_1\otimes s_1-s_2\otimes s_2-s_3\otimes s_3),$$

i.e. choose $\lambda=(1/\sqrt{6})(2,-1,-1,0,0,0,0,0)$. The isotropy subgroup of Spin(16) at $S(\lambda)$ is

$$K(\lambda)=\mathrm{Sp}(4)\times\mathrm{SO}(2)/\mathbf{Z}_2\cup(\mathrm{Sp}(4)\times\mathrm{SO}(2))/\mathbf{Z}_2\cdot a_{12}. \tag{4.24}$$

The calibrated 4-planes in this geometry are:

$$\{\alpha:\alpha \text{ is a quaternionic 1-plane}\}\cong\mathrm{Sp}(4)/(\mathrm{Sp}(3)\times\mathrm{Sp}(1)). \tag{4.25}$$

The calibrated 8-planes in this geometry form three components:

$$\{\alpha:\alpha \text{ is a quaternionic 2-plane}\}\cong\mathrm{Sp}(4)/(\mathrm{Sp}(2)\times\mathrm{Sp}(2)), \tag{4.26}$$

$$\begin{gathered}\{\alpha:\alpha \text{ is both a complex 4-plane in structure } a_{23}\\ \text{and } \alpha\in\mathrm{SLAG}(a_{12},a_{23})\}\cong\mathrm{Sp}(4)/\mathrm{U}(4),\end{gathered} \tag{4.27}$$

$$\begin{gathered}\{g(\alpha):g\in\mathrm{SO}(2),\ \alpha \text{ is both a complex 4-plane in the structure } a_{12}\\ \text{and } \alpha\in\mathrm{SLAG}(a_{23},a_{12})\}\cong(S^1\times\mathrm{Sp}(4)/\mathrm{U}(4))/\mathbf{Z}_2.\end{gathered} \tag{4.28}$$

The Sp(2)×Sp(2)×Sp(1) *geometry.* Let

$$S(\lambda)\equiv\frac{1}{\sqrt{3}}(s_1\otimes s_1+s_2\otimes s_2+s_3\otimes s_3),$$

i.e. choose $\lambda=(1/\sqrt{3})(1,1,1,0,0,0,0,0)$. The isotropy subgroup of Spin(16) at $S(\lambda)$ is

$$K(\lambda)=(\mathrm{Sp}(2)\times\mathrm{Sp}(2)\times\mathrm{diag}\,\mathrm{Sp}(1))/\mathbf{Z}_2\cup(\mathrm{Sp}(2)\times\mathrm{Sp}(2)\times\mathrm{diag}\,\mathrm{Sp}(1))/\mathbf{Z}_2\cdot\xi. \tag{4.29}$$

The calibrated 4-planes in this geometry are:

$$\{\alpha:\alpha \text{ is a quaternionic 1-plane in } V_8 \text{ or in } V_8'\}\cong S^4\cup S^4. \tag{4.30}$$

The calibrated 8-planes in this geometry are of five distinct types:

$$V_8\cup V_8', \tag{4.31}$$

$$\{\alpha\wedge\beta:\alpha \text{ (resp. } \beta) \text{ is a quaternionic 1-plane in } V_8 \text{ (resp. } V_8')\}\cong S^4\times S^4, \tag{4.32}$$

$$\begin{aligned}&\{g(\alpha\wedge\beta): g\in\mathrm{diag}\,\mathrm{Sp}(1), \alpha\subset V_8 \text{ (resp. } \beta\subset V_8') \text{ is a complex 2-plane in the}\\ &\quad\text{structure } a_{12} \textit{ and } \alpha,\beta\in\mathrm{SLAG}(a_{23},a_{12})\}\cong(S^2\times\mathrm{Sp}(2)/\,\mathrm{U}(2)\times\mathrm{Sp}(2)/\,\mathrm{U}(2))/\mathbf{Z}_2,\end{aligned} \tag{4.33}$$

$$\{\alpha:\alpha \text{ is a graph of } g\in\mathrm{Sp}(2)\}\cong\mathrm{Sp}(2), \tag{4.34}$$

$$\{\alpha:\alpha \text{ is a graph of } Jg,\ g\in\mathrm{Sp}(2), J \text{ unit imaginary quaternion}\}\cong(S^2\times\mathrm{Sp}(2))/\mathbf{Z}_2. \tag{4.35}$$

The Sp(1)×Sp(1)×Sp(2) *geometry.* Let

$$S(\lambda)\equiv\frac{1}{\sqrt{5}}(s_4\otimes s_4+s_5\otimes s_5+s_6\otimes s_6+s_7\otimes s_7+s_8\otimes s_8),$$

i.e. choose $\lambda=(1/\sqrt{5})(0,0,0,1,1,1,1,1)$. The isotropy subgroup of Spin(16) at $S(\lambda)$ is

$$K(\lambda)=(\mathrm{diag}\,\mathrm{Sp}(2)\times\mathrm{Sp}(1)\times\mathrm{Sp}(1))/\mathbf{Z}_2\cdot\xi. \tag{4.36}$$

There are no calibrated 4-planes in this geometry. The calibrated 8-planes in this geometry are of seven distinct types:

$$V_8\cup V_8', \tag{4.37}$$

$$\{\alpha\wedge\alpha':\alpha\subset V_8 \text{ is a quaternionic 1-plane}\}\cong S^4, \tag{4.38}$$

$$\{\alpha\wedge{*\alpha'}:\alpha\subset V_8 \text{ as above, } * \text{ is the Hodge operator on } V_8'\}\cong S^4, \tag{4.39}$$

$$\begin{aligned}\{h(\alpha\wedge\alpha'):h\in \mathrm{Sp}(1)\times\mathrm{Sp}(1),\ \alpha\subset V_8 \text{ is a complex 2-plane in the structure } a_{23},\\ \textit{and } \alpha\in\mathrm{SLAG}(a_{12},a_{23})\}\cong(S^2\times S^2\times\mathrm{Sp}(2)/\,\mathrm{U}(2))/\mathbf{Z}_2,\end{aligned}\tag{4.40}$$

$$\{\alpha:\alpha \text{ is a graph of } h\in\mathrm{Sp}(1)\}\cong S^3,\tag{4.41}$$

$$\begin{aligned}\{\alpha:\alpha \text{ is a graph of } hg,\ h\in\mathrm{Sp}(1),\ g\in\mathrm{Sp}(2) \text{ is a reflection}\\ \text{through a quaternionic 1-plane}\}\cong(S^3\times S^4)/\mathbf{Z}_2,\end{aligned}\tag{4.42}$$

$$\begin{aligned}\{\alpha:\alpha \text{ is a graph of } hg,\ h\in\mathrm{Sp}(1),\ g \text{ is a complex structure in } \mathrm{Sp}(2)\}\\ \cong(S^3\times\mathrm{Sp}(2)/\,\mathrm{U}(2))/\mathbf{Z}_2.\end{aligned}\tag{4.43}$$

The $\mathrm{Sp}(2)\times\mathrm{U}(2)$ *geometry.* Let

$$S(\lambda)\equiv\frac{1}{\sqrt{30}}(5s_3\otimes s_3-s_4\otimes s_4-s_5\otimes s_5-s_6\otimes s_6-s_7\otimes s_7-s_8\otimes s_8),$$

i.e. choose $\lambda=(1/\sqrt{30}\,)(0,0,5,-1,-1,-1,-1,-1)$. The isotropy subgroup of Spin(16) at $S(\lambda)$ is

$$K(\lambda)=(\mathrm{diag}\,\mathrm{Sp}(2)\times\mathrm{U}(2))/\mathbf{Z}_2\cup(\mathrm{diag}\,\mathrm{Sp}(2)\times\mathrm{U}(2))/\mathbf{Z}_2\cdot a_{13}.\tag{4.44}$$

There are no calibrated 4-planes in this geometry. The calibrated 8-planes in this geometry form eight components:

$$\{\alpha\wedge\alpha':\alpha \text{ is a quaternionic 1-plane in } V_8\}\cong S^4,\tag{4.45}$$

$$\{h(\alpha):h\in\mathrm{U}(2),\alpha=V_8\}\cong S^2,\tag{4.46}$$

$$\{h(\alpha\wedge\alpha'):h\in\mathrm{U}(2),\alpha\subset V_8 \text{ is a quaternionic 1-plane}\}\cong(S^2\times S^4)/\mathbf{Z}_2,\tag{4.47}$$

$$\begin{aligned}\{\alpha\wedge\alpha':\alpha\subset V_8 \text{ is a complex 2-plane in the structure } a_{12} \text{ and } \alpha\in\mathrm{SLAG}(a_{13},a_{12})\}\\ \cong\mathrm{Sp}(2)/\,\mathrm{U}(2)\cong G(2,\mathbf{R}^5),\end{aligned}\tag{4.48}$$

$$\begin{aligned}\{h(\alpha\wedge{*\alpha'}):\ h\in\mathrm{U}(2),\alpha\subset V_8 \text{ is a complex 2-plane in the structure } a_{12},\\ \alpha\in\mathrm{SLAG}(a_{13},a_{12}), \text{ and } * \text{ is the Hodge operator on } V_8'\}\cong(S^2\times G(2,\mathbf{R}^5))/\mathbf{Z}_2,\end{aligned}\tag{4.49}$$

$$\{\alpha:\alpha \text{ is a graph of } e^{i\theta}J\}\cong S^1,\tag{4.50}$$

$$\begin{aligned}\{\alpha:\alpha \text{ is a graph of } e^{i\theta}Jg,\ g\in\mathrm{diag}\,\mathrm{Sp}(2) \text{ so that } g|V_8 \text{ is a reflection}\\ \text{through a quaternionic 1-plane}\}\cong(S^1\times S^4)/\mathbf{Z}_2,\end{aligned}\tag{4.51}$$

$$\begin{aligned}\{h(\alpha\wedge\alpha'):h\in\mathrm{U}(2),\ \alpha\subset V_8 \text{ is a complex 2-plane in the structure } a_{13},\\ \alpha\in\mathrm{SLAG}(a_{12},a_{13})\}\cong[G(2,\mathbf{R}^5)\times\mathrm{U}(2)/\,\mathrm{O}(2)]/\mathbf{Z}_2.\end{aligned}\tag{4.52}$$

Proof of Theorem 4.1. For each conjugacy class of maximal isotropy subgroups we select $S(\lambda)\in D$ and compute explicitly the maximal isotropy subgroup $K(\lambda)$ at $S(\lambda)$

that represents the class. In all cases the isotropy $K(\lambda)$ has two components. Using the maximal torus theorem or other classical methods we determine the elements of order two in $K(\lambda)$ and then apply Corollary I.2.4 to identify all components of the contact sets $G(\phi_4)$ and $G(\phi_8)$.

The Spin(8) *geometry.* Let $\mathrm{Spin}(8)\subset\mathrm{Spin}(16)$ be the subgroup that acts on $V(16)$ as follows

$$(g_0, g_+, g_-)\cdot(x, y) = (g_+x, g_-y), \tag{4.53}$$

where $x\in V_8$, $y\in V_8'$. We claim that this Spin(8) fixes $S(\lambda)$ with $\lambda=(1,-1,-1,-1,-1,-1,-1,-1)$. First $\mathrm{Spin}(8)\subset(\mathrm{Spin}(8)\times\mathrm{Spin}(8))/\mathbf{Z}_2$ and it acts on $\mathbf{O}^+\otimes\mathbf{O}^+$ by

$$(g_0, g_+, g_-)u\otimes v = g_0u\otimes g_0'v. \tag{4.54}$$

To see this observe that by triality both (g_+, g_0, g_-') and (g_-, g_0', g_+') are in Spin(8) and from the definition (4.53) of this Spin(8) the second factors of these elements are relevant for the action on $\mathbf{O}^+\otimes\mathbf{O}^+$. It is then clear from (4.54) and the definition of $g_0'\equiv c\circ g_0\circ c$ that Spin(8) fixes $S(\lambda)$.

A short calculation shows that of the generators of $Z(D)$ the only one not in Spin(8) is ξ, and so we have the full isotropy subgroup

$$K(\lambda) = \mathrm{Spin}(8)\cup\mathrm{Spin}(8)\xi.$$

Next we determine the elements of order two in Spin(8). The center has four elements of order two. Besides the two elements that are $\pm I$ on $V(16)$ we get reflections through V_8 on V_8' giving us (4.5).

Besides the center, Spin(8) has only one conjugacy class of elements of order 2, and that is the Grassmannian $G(4,\mathbf{R}^8)\subset\mathrm{Spin}(8)$. The outer automorphism $(g_0, g_+, g_-)\rightarrow(g_0', g_-, g_+)$ of Spin(8) induces an involution σ on $G(4,\mathbf{R}^8)$ whose fixed point set is $G(4,\mathbf{R}^8)\cap\{(g_0,g_+,g_-): g_+=g_-\}=G(4,\mathbf{R}^8)\cap\mathrm{Spin}(7)$, the set of Cayley reflections (see Section 3, Part I). This establishes (4.6).

Finally we determine the elements of order two in $\mathrm{Spin}(8)\xi$. First we note that $K(\lambda)\subset\mathrm{End}(V(16))\cong\mathrm{Cl}(8)$ is really the group Pin(8). Indeed,

$$\xi\in\mathrm{End}(V(16)) = \mathrm{End}(V_8\oplus V_8')\cong\mathrm{End}(\mathbf{O}^+\oplus\mathbf{O}^-)$$

is the element

$$\xi = \begin{pmatrix} 0 & -R_1 \\ R_1 & 0 \end{pmatrix},$$

where R_1 is multiplication by $1\in\mathbf{O}$ (see Section I.3). The group Pin(8) is generated by Spin(8) and ξ. (We note that $Z(D)\subset\mathrm{Pin}(8)\subseteq\mathrm{Cl}(8)$ is just the group of all oriented axis planes in Cl(8).) Now among the odd dimensional Grassmannians in Pin(8) only $G(3,\mathbf{R}^8)$ and $G(7,\mathbf{R}^8)$ consist of elements of order 2. They are both conjugacy classes of Spin(8) whence (4.7), (4.8) follow once we find representatives. If $\{u_2,u_3,u_4\}$ is an orthonormal set in $\operatorname{Im}\mathbf{O}$, then $u_2\wedge u_3\wedge u_4\in G(3,\operatorname{Im}\mathbf{O})\subset G(3,\mathbf{O})$ is represented on $\mathbf{O}^+\oplus\mathbf{O}^-$ by

$$\begin{pmatrix} 0 & R_{u_2}\circ R_{u_3}\circ R_{u_4} \\ R_{u_2}\circ R_{u_3}\circ R_{u_4} & 0 \end{pmatrix}.$$

Now observe that $r=R_{u_2}\circ R_{u_3}\circ R_{u_4}$ is a reflection through a Cayley plane since $r^2=I$ and $(g_+,g_-)=(r,r)$ lies in

$$\operatorname{Spin}(7)=\{(g_+,g_-)\in\operatorname{Spin}(8):g_+=g_-\}.$$

The Spin(7)×Spin(7) *geometry.* The $\operatorname{Spin}(7)\times\operatorname{Spin}(7)\subset(\operatorname{Spin}(8)\times\operatorname{Spin}(8))/\mathbf{Z}_2$ that fixes $s_1\otimes s_1$ contains all the generators of $Z(D)$ except for ξ. Hence $K(\lambda)$ has two components (4.9). We proceed to find the elements of order 2 in $K(\lambda)$. In Spin(7) these are the center $\{\pm I\}$, and the reflections through Cayley planes (see Part I). This determines all elements of order two in Spin(7)×Spin(7) and establishes (4.10), (4.11), and (4.12).

The reflections in $\operatorname{Spin}(7)\times\operatorname{Spin}(7)\xi$ on $V(16)$ look like

$$\begin{pmatrix} 0 & g \\ g^{-1} & 0 \end{pmatrix},\quad g\in\operatorname{Spin}(7).$$

This set gives the planes in (4.13).

The $\operatorname{Spin}(7)\times T^1$ *geometry.* First, diag SO(8) fixes any $S(\lambda)$ with λ a multiple of $\mu=(1,1,1,1,1,1,1,1)$. Therefore $S(\lambda)=aS(\mu)+bS_1$ $(a,b\in\mathbf{R})$ is fixed by $\operatorname{diag}\operatorname{SO}(8)\cap(\operatorname{Spin}(7)\times\operatorname{Spin}(7))\equiv\operatorname{diag}\operatorname{Spin}(7)$. Since for $\lambda=(7,-1,-1,-1,-1,-1,-1,-1)$ we have $\sum\lambda_i=0$, Lemma 2.2 shows that $S(\lambda)$ is also fixed by T^1. On $V(16)$ the intersection of T^1 with diag Spin(7) is $\{\pm I\}$. Finally of the generators of $Z(D)$ the only one that is not in diag Spin(7) or in T^1 in η. Thus we have established (4.14).

We proceed to find elements of order two in $\operatorname{diag}\operatorname{Spin}(7)\times T^1/\mathbf{Z}_2$. Let $g\in\operatorname{diag}\operatorname{Spin}(7)$ and $t\in T^1$. If $t=I$ then g must be an element of order two in diag Spin(7). The center of diag Spin(7) contributes 0 and 16-dimensional planes and the "Cayley reflections" in diag Spin(7) give (4.15).

If $t^2=-I$ (i.e. $t=\xi$) then we must have $g^2=-I$ on $V(16)$ be a complex structure. Now

$$\operatorname{diag}\operatorname{Spin}(7)\subset\operatorname{Spin}(8)\subset\operatorname{End}(V_8\oplus V_8')\cong\operatorname{Cl}(8)\cong\textstyle\bigwedge\mathbf{O}$$

are the even elements in $\operatorname{diag}\operatorname{Pin}(7)\subset\operatorname{Pin}(8)$. Here diag Pin(7) is generated by the unit sphere in $\bigwedge^1(\operatorname{Im}\mathbf{O})\subset\operatorname{Cl}(8)$. The elements of diag Spin(7) that square to $-I\in\operatorname{Cl}(8)$ are the Grassmannians $G(2,\operatorname{Im}\mathbf{O})$ and $G(6,\operatorname{Im}\mathbf{O})$. These Grassmannians then contribute the following reflections on $V(16)$ (here we identify $V_8\oplus V_8'$ with $\mathbf{O}^+\oplus\mathbf{O}^-$)

$$\left\{\begin{pmatrix} 0 & R_u \\ -R_u & 0 \end{pmatrix} : |u|=1,\ u\in\operatorname{Im}\mathbf{O}\right\}$$

gives (4.16) and

$$\left\{\begin{pmatrix} 0 & R_u\circ R_v \\ -R_u\circ R_v & 0 \end{pmatrix} : u,v\in\operatorname{Im}\mathbf{O} \text{ orthonormal}\right\}$$

gives (4.17).

Finally we consider reflections in $(\mathrm{Spin}(7)\times T^1)/\mathbf{Z}_2\cdot\eta$. Noting that $(t\eta)^2=I$ for all $t\in T^1$ gives us (4.18). Since any $g\in\operatorname{diag}\mathrm{Spin}(7)$ commutes with $T^1\eta$, the only other reflections in this component are $gt\eta$ with $g\in\operatorname{diag}\mathrm{Spin}(7)$ a Cayley reflection, hence (4.19).

The SU(8) *geometry.* The connected isotropy at $\lambda=(1,-1,0,0,0,0,0,0)$ must be SU(8) because up to conjugation there is only one monomorphism from SU(8) to SO(16). We claim that this SU(8) is determined by the complex structure a_{12}. First, the subgroup of $\operatorname{diag}\mathrm{SO}_8$ that commutes with a_{12} is $\operatorname{diag}\mathrm{SU}(4)$ which fixes $s_1\otimes s_1$ and $s_2\otimes s_2$, and therefore also $S(\lambda)$. This follows from Lemma I.3.3. Let J be the complex structure that determines the connected isotropy SU(8). Since $\operatorname{diag}\mathrm{SU}(4)\subset\mathrm{SU}(8)$ leaves V_8, V_8' invariant we must have $J=\pm a_{12}$ separately on V_8 and V_8'. Let us arrange $J=a_{12}$ on V_8 by changing sign of J if necessary. If now $J=-a_{12}$ on V_8' then (by explicit calculation) the circle $\exp tJ\subset\mathrm{SO}(16)$ would fix $S(\lambda)$ showing that $\mathrm{U}(8)\subset K(\lambda)$, a contradiction. Finally we compute $Z(D)\cap\mathrm{SU}(8)$ by checking which elements in $Z(D)$ commute with a_{12}. Besides ξ and η we get

$$\{a_I\in A : I\cap\{1,2\}=\varnothing \text{ or } \{1,2\}\}.$$

Thus $Z(D)$ is generated by $Z(D)\cap\mathrm{SU}(8)$ and a_{23}, whence (4.20).

Now the reflections in SU(8) are reflections through even dimensional complex planes giving us (4.21) and (4.22). The reflections in $\mathrm{SU}(8)a_{23}$ form one SU(8) conjugacy class of conjugations on $\mathbf{C}^8\cong(\mathbf{R}^{16},a_{12})$. These are reflections through special Lagrangian planes denoted by $\mathrm{SLAG}(a_{12},a_{23})$ listed under (4.23).

The $\mathrm{Sp}(4)\times\mathrm{SO}(2)$ *geometry.* The SU(8) determined by the complex structure a_{13} fixes S_1-S_3. The intersection of this SU(8) with the one that fixes S_1-S_2 (discussed in the preceding case) gives an Sp(4) that certainly fixes $2S_1-S_2-S_3=(S_1-S_2)+(S_1-S_3)$. This Sp(4) is the set of quaternionic linear maps on $V(16)\cong\mathbf{H}^4$ where the quaternionic structure is given by a_{12}, a_{13}, and a_{23}. Let $\operatorname{diag}\operatorname{Sp}(1)\subset\operatorname{Spin}(16)$ be the group that on $V(16)$ consists of multiplication by unit quaternions. Let $\operatorname{SO}(2)\subset\operatorname{diag}\operatorname{Sp}(1)$ be the 1 parameter subgroup containing a_{23} (the "circle through a_{23}"). This circle acts diagonally on $\mathbf{O}^+\otimes\mathbf{O}^+\subset\mathbf{S}^+(16)$ by rotations in the $(2,3)$-planes of each $\mathbf{O}^+$ factor. Hence this $\operatorname{SO}(2)$ fixes $2S_1-S_2-S_3$. We note that $\operatorname{SO}(2)\cap\operatorname{Sp}(4)=\pm I$ on $V(16)$. Finally, $Z(D)\cap \operatorname{Sp}(4)$ is the set of elements of $Z(D)$ that commute both with a_{12} and a_{13}. Since $a_{23}\in \operatorname{SO}(2)$ a short calculation reveals that $Z(D)$ is generated by $Z(D)\cap(\operatorname{Sp}(4)\times\operatorname{SO}(2))/\mathbf{Z}_2$ and a_{12}, hence (4.24).

To determine reflections in $\operatorname{Sp}(4)\times\operatorname{SO}(2)/\mathbf{Z}_2$ let $g\in\operatorname{Sp}(4)$ and $t\in\operatorname{SO}(2)$. If gt is of order 2 we must have $g^2=t^2=\pm I$. If $t=I$ then the conjugacy classes of reflections in $\operatorname{Sp}(4)$ give (4.25) and (4.26). If $t^2=-I$ we may choose $t=a_{23}$. The set of complex structures g in $\operatorname{Sp}(4)$ is isometric to $\operatorname{Sp}(4)/\operatorname{U}(4)$. Observe that the reflection ga_{23} lies in $\operatorname{SU}(8)\cdot a_{23}$ (see $\operatorname{SU}(8)$ geometry) and that ga_{23} certainly commutes with the complex structure a_{23}, and hence (4.27).

To find the reflections in the second component we consider the equation $(gta_{12})^2=I$. Since $(ta_{12})^2=-I$ for all $t\in\operatorname{SO}(2)$ we must have $g\in\operatorname{Sp}(4)$ be a complex structure. Finally note that the reflection ga_{12} commutes with a_{12} and lies in the component $\operatorname{SU}(8)'a_{12}$ where $\operatorname{SU}(8)'$ is determined by the complex structure a_{23}. Thus the calibrated plane corresponding to ga_{12} is both complex and special Lagrangian (4.28).

The $\operatorname{Sp}(2)\times\operatorname{Sp}(2)\times\operatorname{Sp}(1)$ *geometry.* First we note that the group $\operatorname{diag}\operatorname{Sp}(1)$ introduced in the previous geometry acts on $\mathbf{O}^+\otimes\mathbf{O}^+\subset\mathbf{S}^+(16)$ by rotations in the (s_1,s_2,s_3)-plane in each $\mathbf{O}^+$ factor (Lemma I.3.3). Therefore $\operatorname{diag}\operatorname{Sp}(1)$ fixes $S(\lambda)$ with $\lambda=(1,1,1,0,0,0,0,0)$. Similarly the $\operatorname{Sp}(2)\times\operatorname{Sp}(2)\subset(\operatorname{Spin}(8)\times\operatorname{Spin}(8))/\mathbf{Z}_2$ determined by the complex structures a_{12}, a_{13}, and a_{23} on V_8, V_8' is seen to fix $S(\lambda)$. On $V(16)$ the intersection of $\operatorname{diag}\operatorname{Sp}(1)$ and $\operatorname{Sp}(2)\times\operatorname{Sp}(2)$ is $\{\pm I\}$. Finally, the only generator of $Z(D)$ not in the above determined identity component is ξ, and hence (4.29).

To classify the reflections let $(g_1,g_2)\in\operatorname{Sp}(2)\times\operatorname{Sp}(2)$ and $h\in\operatorname{diag}\operatorname{Sp}(1)$. In the identity component we must have $g_1^2g_2^2h^2=I$. If $h^2=I$ then each g_i must be a reflection (classified as in the $\operatorname{Sp}(4)$ geometry) and we get (4.30), (4.31), and (4.32). If $h^2=-I$ on $V(16)$ then both g_1 and g_2 must be complex structures in $\operatorname{Sp}(2)$. Taking $h=a_{12}$ and arguing analogously as in the $\operatorname{Sp}(4)$ geometry we obtain (4.33).

As for the reflections in the second component $h^2(g_1g_2\xi)^2=I$ on $V(16)$ implies that either $h=I$ or $h\in\operatorname{diag}\operatorname{Sp}(1)$ is a multiplication by an imaginary quaternion. Observe that on $V(16)$

$$g_1g_2\xi=\begin{pmatrix}0 & g_1\\ -g_2 & 0\end{pmatrix}.$$

Hence if $h=I$ we must have $g_2=-g_1^{-1}$, (4.34), and if $h^2=-I$ we must have $g_2=g_1^{-1}$, (4.35).

The $\mathrm{Sp}(1)\times\mathrm{Sp}(1)\times\mathrm{Sp}(2)$ *geometry.* The quaternionic structure $\{a_{12}, a_{13}, a_{23}\}$ on V_8 and V_8' defines a $\mathrm{Sp}(1)\times\mathrm{Sp}(1)$ action (multiplication by unit quaternions) on $V(16)$. This $\mathrm{Sp}(1)\times\mathrm{Sp}(1)$ as well as $\operatorname{diag}\mathrm{Sp}(2)\subset\mathrm{Sp}(2)\times\mathrm{Sp}(2)$ of the previous geometry fix $S(\lambda)$ with $\lambda=(0,0,0,1,1,1,1,1)$. The intersection of these groups is $\{\pm I\}$ on $V(16)$. As in the previous geometry the only generator of $Z(D)$ not in $(\mathrm{Sp}(1)\times\mathrm{Sp}(1)\times\operatorname{diag}\mathrm{Sp}(2))/\mathbf{Z}_2$ is ξ, hence (4.36).

To compute the reflections in the identity component let $g\in\operatorname{diag}\mathrm{Sp}(2)$ and $(h_1,h_2)\in\mathrm{Sp}(1)\times\mathrm{Sp}(1)$, and consider the equations $g^2h_i^2=I$ on V_8 and V_8'. If $g=I$ then $h=\pm I$ and we obtain (4.37). If $g^2=I$ but $g\neq I$ then either $h_1=h_2=\pm I$, (4.38), or $h_1=-h_2=\pm I$, (4.39).

If $g\in\operatorname{diag}\mathrm{Sp}(2)$ is a complex structure on $V(16)$ then both h_1 (resp. h_2) must be complex structures on V_8 (resp. V_8'), that is they must be unit imaginary quaternions (4.40).

To compute the reflections in the other component note that on $V(16)$

$$h_1h_2g\xi=\begin{pmatrix}0 & -h_1g\\ h_2g & 0\end{pmatrix}.$$

Again, if $g^2=I$ then $h_1=-h_2^{-1}$ and we get (4.41) in case $g=I$, or we get (4.42) in case $g\neq I$. Similarly, if $g^2=-I$ we must have $h_1=h_2^{-1}$ and the corresponding set of reflections is (4.43).

The $\mathrm{Sp}(2)\times\mathrm{U}(2)$ *geometry.* First note that $\operatorname{diag}\mathrm{Sp}(2)$ discussed in the previous geometry fixes $S(\lambda)$ with $\lambda=(0,0,5,-1,-1,-1,-1,-1)$. Next, the complex structure a_{12} defines multiplication by unit complex numbers on V_8 and V_8' separately. This gives the action of the 2-torus of diagonal matrices in $\mathrm{U}(2)$ on $V(16)$. This 2-torus operates trivially on $S_3, S_4, \dots, S_8$, and hence fixes $S(\lambda)$ as well. Finally, by Lemma 2.2 the $T^1\in\mathrm{Spin}(16)$ that operates on $V_8\oplus V_8'$ by

$$\begin{pmatrix}\cos\theta\, I & -\sin\theta\, I\\ \sin\theta\, I & \cos\theta\, I\end{pmatrix}$$

fixes $S(\lambda)$. This T^1 and the previously discussed 2-torus generate $\mathrm{U}(2)\subset\mathrm{Spin}(16)$ that commutes with $\operatorname{diag}\mathrm{Sp}(2)$. Observe that $\operatorname{diag}\mathrm{Sp}(2)\cap\mathrm{U}(2)=\{\pm I\}$ on $V(16)$. To compute the other components of the isotropy at $S(\lambda)$ note that $a_{12},\xi,\eta\in\mathrm{U}(2)$ and that

$$Z(D)\cap\operatorname{diag}\mathrm{Sp}(2)=\{a_I : I\cap\{1,2,3\}=\varnothing \ \text{ or } \{1,2,3\}\}.$$

Adding a_{13} to $Z(D)\cap(\operatorname{diag}\mathrm{Sp}(2)\times\mathrm{U}(2)/\mathbf{Z}_2)$ generates all of $Z(D)$, and hence (4.44).

Now let $g\in\operatorname{diag}\mathrm{Sp}(2)$ and $h\in\mathrm{U}(2)$. If $h^2g^2=I$ on $V(16)$ either $g^2=h^2=I$ or $g^2=h^2=-I$. In the former case if $h=I$ then $g=\pm I$ or $g\in\mathrm{Sp}(2)$ is a reflection through a quaternionic 1-plane (4.45). If $g=I$ and $h\in \mathrm{U}(2)$ is nontrivial reflection then (in the standard representation) h is a reflection through a complex line in $\mathbf{C}^2$, hence (4.46). Finally, if both g, h are nontrivial reflections we get (4.47).

Next, if $g\in \operatorname{diag}\mathrm{Sp}(2)$ is a complex structure then $h\in \mathrm{U}(2)$ is either *the* complex structure defining U(2) (in the standard representation) yielding (4.48) or h is obtained from the above complex structure by a sign change on a complex line. There is a $\mathbf{CP}^1$ of such complex structures and thus we get (4.49).

Now let us consider the reflections in the other component of $K(\lambda)$. Either we must have $g^2=(ha_{13})^2=I$ or $g^2=(ha_{13})^2=-I$. The elements g arising from these equations have already been discussed. By a short computation it is seen that $(he_{13})^2=-h\bar{h}$, where $\bar{h}\in \mathrm{U}(2)$ is the complex conjugate of h in the standard representation on $\mathbf{C}^2$. Let c be a conjugation on $\mathbf{C}^2$. If $h\in \mathrm{U}(2)$ then $(hc)^2=h\bar{h}$. Therefore the U(2) conjugacy class of conjugations in $\mathrm{U}(2)\cdot c$ is isometric to the complex structures in $\mathrm{U}(2)\cdot a_{13}\subset \mathrm{Spin}(16)$. This is the symmetric space $\mathrm{U}(2)/\mathrm{O}(2)$ ($\cong S^1\times \mathbf{RP}^2$) with a_{13} as a representative. The complex structures in $\mathrm{U}(2)\cdot c$ is the circle of multiplications by unit imaginary quaternions in $\mathbf{H}\cong\mathbf{C}^2$ orthogonal to the quaternion i that determines the U(2). Therefore there is a circle of reflections in $\mathrm{U}(2)a_{13}$, the U(2) conjugacy class of ξa_{13}:

$$\left\{\begin{pmatrix} 0 & e^{i\theta}J \\ -e^{i\theta}J & 0 \end{pmatrix} : \begin{pmatrix} 0 & e^{i\theta} \\ -e^{i\theta} & 0 \end{pmatrix} \in \mathrm{U}(2),\ \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \equiv a_{13}\right\}.$$

Putting these facts about conjugacy classes of $h\in \mathrm{U}(2)a_{13}$ and $g\in \operatorname{diag}\mathrm{Sp}(2)$ together yields (4.50), (4.51), and (4.52). □

III. Additional results in higher dimension

III.1. A differential system for each geometry

A calibration ϕ on a Riemannian manifold determines a geometry of distinguished submanifolds, sometimes called ϕ-submanifolds. Each such submanifold is homologically volume minimizing. Frequently there is a useful system of differential equations (more special than the minimal submanifold equations) whose solutions are exactly the ϕ-submanifolds. For example, if $\phi\equiv(1/p!)\omega^p$ where ω is the Kähler form on a Kähler manifold then the ϕ-submanifolds are just the p-dimensional complex submanifolds and these submanifolds are distinguished by the Cauchy–Riemann equations.

Remark. If ϕ is a calibration and a submanifold M is given then the differential equation $\phi(\overrightarrow{M})=1$ distinguishes the ϕ-submanifolds. However, this differential equation is not very useful. Note that in the example: ϕ the standard Kähler form on $\mathbf{C}^2$, if $M\equiv\operatorname{graph} f$ then $\phi(\overrightarrow{M})=1$ is not the Cauchy–Riemann equation for $f:\mathbf{C}\to\mathbf{C}$.

The problem of finding the ϕ version of the Cauchy–Riemann equations can be solved for the calibrations constructed in this paper.

In Harvey–Lawson [4], a partial solution to the general problem was described as follows. Suppose one can complete the inequality

$$\phi(\xi) \leqslant 1, \quad \text{for all } \xi \in G(p, TX),$$

to an equality,

$$\phi(\xi)^2 + \Psi_1(\xi)^2 + \ldots + \Psi_N(\xi)^2 = 1, \quad \text{for all } \xi \in G(p, TX),$$

where $\Psi_1, \ldots, \Psi_N$ are d-closed p-forms on X. Then M is a ϕ submanifold (with the appropriate orientation) if and only if M is an integral submanifold for the differential system generated by $\Psi_1, \ldots, \Psi_N$.

Now suppose ϕ is constructed from squaring a spinor as in Theorem I.2.2.

THEOREM 1.1. *Suppose V, $\langle\ ,\ \rangle$ is an oriented inner product space of dimension $8m$. Suppose $x \in \mathbf{S}^+$ is a unit positive spinor. Let*

$$\phi \equiv 16^m x \circ x \in \mathrm{End}_{\mathbf{R}}(\mathbf{S}^+) \subset \mathrm{End}_{\mathbf{R}}(\mathbf{P}) \cong \mathrm{Cl}(V) \cong \bigwedge V^*.$$

Consider a component calibration $\phi_{4p} \in \bigwedge^{4p} V^*$. There exist forms $\Psi_1, \ldots, \Psi_N \in \bigwedge^{4p} V^*$ such that: A submanifold M of V is a ϕ_{4p}-submanifold if and only if

$$\Psi_1|_M = \ldots = \Psi_N|_M = 0. \tag{1.1}$$

Proof. Let $x_0 \equiv x$ and complete to an orthonormal basis $x_0, x_1, \ldots, x_N$ for $\mathbf{S}^+$. Let Ψ_j denote the degree $4p$ component of

$$16^m x_j \circ x_0 \in \mathrm{End}_{\mathbf{R}}(\mathbf{S}^+) \subset \mathrm{End}_{\mathbf{R}}(\mathbf{P}) \cong \mathrm{Cl}(V) \cong \bigwedge V^*.$$

Given $\xi \in G(4p, V)$, first note that ξx_0 is of unit length in $\mathbf{S}^+$. By Proposition I.2.1 (6) (with $a \equiv \xi$),

$$\sum_{j=0}^{N} \Psi_j(\xi)^2 = \sum_{j=0}^{N} 16^{2m} \langle x_j \circ x_0, \xi\rangle^2 = \sum_{j=0}^{N} \varepsilon(x_j, \xi x_0)^2 = \varepsilon(\xi x_0, \xi x_0) = 1,$$

completing the proof. □

III.2. Pure spinors and calibrations

For simplicity we assume that V, $\langle\ ,\ \rangle$ is an oriented inner product space of dimension $8m=2p$. Other even dimensions will be discussed in a remark at the end of this section. The first objective is to develop a *model* for the *pinor representation*

$$\mathrm{Cl}(8m)\cong\mathrm{End}_{\mathbf{R}}(\mathbf{S}^+\oplus\mathbf{S}^-):$$

and for its complexification

$$\mathrm{Cl}_{\mathbf{C}}(8m)\cong\mathrm{End}_{\mathbf{C}}(\mathbf{S}^+_{\mathbf{C}}\oplus\mathbf{S}^-_{\mathbf{C}}).$$

A model for Cl$(8m)$. Let $e_1,\ldots,e_p$ denote the standard orthonormal basis for $\mathbf{R}^p$ with the standard inner product $(\ ,\ )$. Also, let $(\ ,\ )$ denote the induced inner product on the isomorphic vector spaces

$$\mathrm{Cl}(p)\cong\bigwedge\mathbf{R}^p. \tag{2.1}$$

This canonical vector space isomorphism will be considered as an identification, with $\mathbf{R}^p\subset\mathrm{Cl}(p)$ identified with $\bigwedge^1\mathbf{R}^p$. Let $E_u(x)\equiv u\wedge x$ denote exterior multiplication of $x\in\bigwedge\mathbf{R}^p$ by $u\in\mathbf{R}^p$ on the left. Let $I_u(x)\equiv u\llcorner x$ denote interior multiplication of $x\in\bigwedge\mathbf{R}^p$ by $u\in\mathbf{R}^p$ on the left. The canonical automorphism on $\mathrm{Cl}(p)\cong\bigwedge\mathbf{R}^p$ is $+1$ on $\bigwedge^{\mathrm{even}}\mathbf{R}^p$ and -1 on $\bigwedge^{\mathrm{odd}}\mathbf{R}^p$ and denoted by $x\mapsto\tilde{x}$. Right and left Clifford multiplication can be expressed by:

$$u\cdot x=u\wedge x-u\llcorner x,\quad \text{for all } u\in\mathbf{R}^p \text{ and } x\in\bigwedge\mathbf{R}^p$$

$$\tilde{x}\cdot u=u\wedge x+u\llcorner x,\quad \text{for all } u\in\mathbf{R}^p \text{ and } x\in\bigwedge\mathbf{R}^p.$$

Let $\sigma\equiv e_1\cdot\ldots\cdot e_p=e_1\wedge\ldots\wedge e_p$ denote the unit volume element for $\mathbf{R}^p$.

Also, let $(\ ,\ )$ denote the $\mathbf{C}$-symmetric bilinear form on $\bigwedge\mathbf{C}^p$ extending $(\ ,\ )$ and identify:

$$\mathrm{Cl}_{\mathbf{C}}(p)\cong\bigwedge\mathbf{C}^p \tag{2.2}$$

(the complexified version of (2.1)). Define

$$\mathbf{P}_{\mathbf{C}}\equiv\bigwedge\mathbf{C}^p\cong\mathrm{Cl}_{\mathbf{C}}\cong(\bigwedge\mathbf{R}^p)\otimes_{\mathbf{R}}\mathbf{C}\cong\mathrm{Cl}(p)\otimes_{\mathbf{R}}\mathbf{C}.$$

Define a set of $2p$ *γ-matrices* $\gamma_1,\ldots,\gamma_{2p}\in\mathrm{End}_{\mathbf{C}}(\mathbf{P}_{\mathbf{C}})$ by:

$$\gamma_j(w)\equiv e_j\cdot w\quad\text{and}\quad\gamma_{p+j}(w)\equiv i\tilde{w}\cdot e_j,\quad \text{for } w\in\mathbf{P}_{\mathbf{C}} \text{ and } j=1,\ldots,p.$$

Equivalently,

$$\gamma_j\equiv E_{e_j}-I_{e_j}\quad\text{and}\quad\gamma_{p+j}=i(E_{e_j}+I_{e_j}),\quad j=1,\ldots,p.$$

Identify $V(2p)$ with the $\mathrm{span}_{\mathbf{R}}$ of $\gamma_1,\ldots,\gamma_{2p}$ by choosing an orthonormal basis for $V(2p)$.

PROPOSITION 2.1. *The real subalgebra of* $\mathrm{End}_{\mathbf{C}}(\mathbf{P}_{\mathbf{C}})$ *generated by* $\gamma_1, \dots, \gamma_{2p}$ *is isomorphic to the real Clifford algebra* $\mathrm{Cl}(2p)$.

Proof. First verify that amstex$\gamma_i\gamma_j = -\delta_{ij}$. The fundamental lemma of Clifford algebras provides an extension of the embedding of V into $\mathrm{End}_{\mathbf{C}}$ to an algebra homomorphism of $\mathrm{Cl}(V)$ into $\mathrm{End}_{\mathbf{C}}(\mathbf{P}_{\mathbf{C}})$. Since $\mathrm{Cl}(2p)$ is simple this is an injection. □

Let c denote conjugation on $\mathbf{P}_{\mathbf{C}} = (\bigwedge \mathbf{R}^p) \otimes_{\mathbf{R}} \mathbf{C}$, i.e. $c(w) \equiv \bar{w}$. Let $r_\sigma \in \mathrm{End}_{\mathbf{C}}(\mathbf{P}_{\mathbf{C}})$ denote right multiplication by σ, i.e. $r_\sigma(w) \equiv w \cdot \sigma$. Define $R \in \mathrm{End}_{\mathbf{C}}(\mathbf{P}_{\mathbf{C}})$ by $R \equiv r_\sigma \circ c = c \circ r_\sigma$. Since $p \equiv 0 \mod 4$, $\sigma^2 = 1$ and hence $R^2 = 1$. Also, note that $R \circ i = -i \circ R$, so that if $\mathbf{P}$ denotes the $+1$ eigenspace of R then $i\mathbf{P}$ is the -1 eigenspace. Thus $\mathbf{P}_{\mathbf{C}} = \mathbf{P} \oplus i\mathbf{P}$, with R a second *conjugation* on $\mathbf{P}_{\mathbf{C}}$. Define a *conjugation* or *reality operator* $\mathcal{R}$ on $\mathrm{End}_{\mathbf{C}}(\mathbf{P}_{\mathbf{C}})$ by $\mathcal{R}a \equiv RaR$ for all $a \in \mathrm{End}_{\mathbf{C}}(\mathbf{P}_{\mathbf{C}})$. Note that $\mathcal{R}i = -i$ so that $\mathcal{R}$ is a conjugation on $\mathrm{End}_{\mathbf{C}}(\mathbf{P}_{\mathbf{C}})$.

PROPOSITION 2.2. *The Clifford algebra* $\mathrm{Cl}(2p) \subset \mathrm{End}_{\mathbf{C}}(\mathbf{P}_{\mathbf{C}})$ *generated by* $\gamma_1, \dots, \gamma_{2p}$ *is the fixed set of* $\mathcal{R}$, *i.e.*

$$\mathrm{Cl}(2p) = \{a \in \mathrm{Cl}_{\mathbf{C}}(2p) \cong \mathrm{End}_{\mathbf{C}}(\mathbf{P}_{\mathbf{C}}) : \mathcal{R}a = RaR = a\}.$$

Proof. By a dimension count it suffices to prove that $\mathcal{R}$ fixes $\gamma_1, \dots, \gamma_{2p}$. First, note that for $j = 1, \dots, p$ both c and r_σ commute with γ_j and hence $\mathcal{R}\gamma_j \equiv R\gamma_j R = \gamma_j R^2 = \gamma_j$. Second note that for $p < j \leqslant 2p$, c and r_σ both anticommute with γ_j so that $\mathcal{R}\gamma_j = \gamma_j$ as before. □

Define

$$\textstyle\bigwedge_+ \mathbf{R}^p \equiv \{x \in \bigwedge \mathbf{R}^p : x \cdot \sigma = x\}, \quad \text{the \textit{self dual elements},}$$

and

$$\textstyle\bigwedge_- \mathbf{R}^p \equiv \{x \in \bigwedge \mathbf{R}^p : x \cdot \sigma = -\sigma\}, \quad \text{the \textit{anti-self dual elements}.}$$

PROPOSITION 2.3. *The* $+1$ *eigenspace of* R *is given by*

$$\mathbf{P} = \textstyle\bigwedge_+ \mathbf{R}^p \oplus i \bigwedge_- \mathbf{R}^p.$$

Proof. If $x \in \bigwedge_+ \mathbf{R}^p$ then $Rx = \bar{x} \cdot \sigma = x$. If $x \in \bigwedge_- \mathbf{R}^p$ then $Rix = -iRx = -i\bar{x} \cdot \sigma = ix$.□

COROLLARY 2.4 (The pinor representation). *The subalgebra* $\mathrm{Cl}(2p)$ *of* $\mathrm{End}_{\mathbf{C}}(\mathbf{P}_{\mathbf{C}})$ *is equal to the subalgebra* $\mathrm{End}_{\mathbf{R}}(\mathbf{P})$ *of* $\mathrm{End}_{\mathbf{R}}(\mathbf{P}_{\mathbf{C}})$ *with* $\mathbf{P} \equiv \bigwedge_+ \mathbf{R}^p \oplus i \bigwedge_- \mathbf{R}^p$.

The vector space $\mathbf{P}$ *is called the space of pinors for* V, $\langle\ ,\ \rangle$.

The unit volume element $\lambda \equiv \gamma_1 \dots \gamma_{2p}$ is given by $\lambda(w) \equiv \sigma \cdot w\sigma = \tilde{w}$. Therefore, the *positive spinors* $\mathbf{S}^+$ are given by

$$\mathbf{S}^+ \equiv \textstyle\bigwedge_+^{\text{even}} \mathbf{R}^p \oplus i \bigwedge_-^{\text{even}} \mathbf{R}^p$$

and the *negative spinors* $\mathbf{S}^-$ are given by

$$\mathbf{S}^- \equiv \textstyle\bigwedge_+^{\text{odd}} \mathbf{R}^p \oplus i \bigwedge_-^{\text{odd}} \mathbf{R}^p.$$

Since λ is also the unit volume element for $\mathrm{Cl}_{\mathbf{C}}(2p) \cong \mathrm{End}(\mathbf{P}_{\mathbf{C}})$, the positive and negative spinors for the complex Clifford algebra are given by

$$\mathbf{S}_{\mathbf{C}}^+ = \textstyle\bigwedge^{\text{even}} \mathbf{C}^p \quad \text{and} \quad \mathbf{S}_{\mathbf{C}}^- = \bigwedge^{\text{odd}} \mathbf{C}^p.$$

PROPOSITION 2.5. *The form $\varepsilon(z,w) \equiv (z, w\cdot\sigma)$, for all $z, w \in \mathbf{P}_{\mathbf{C}}$, is a complex symmetric bilinear form on $\mathbf{P}_{\mathbf{C}}$, called the pinor inner product, which satisfies:*

$$\varepsilon(az, w) = \varepsilon(z, \hat{a}w), \tag{2.3}$$

for all $a \in \mathrm{End}_{\mathbf{C}}(\mathbf{P}_{\mathbf{C}}) \cong \mathrm{Cl}_{\mathbf{C}}(2p)$. That is, the hat anti-isomorphism on $\mathrm{Cl}_{\mathbf{C}}(2p)$ is the adjoint with respect to ε.

Note that $\varepsilon(z,w) = \pm(z,w)$ on $\bigwedge_\pm \mathbf{C}^p$, and that $\bigwedge_+ \mathbf{C}^p$ and $\bigwedge_- \mathbf{C}^p$ are orthogonal.

Proof. It suffices to verify (2.3) for $a = \gamma_j$, $j = 1, ..., 2p$. First, for $j = 1, ..., p$,

$$\varepsilon(\gamma_j z, w) = (e_j \cdot z, w\cdot\sigma) = -(z, e_j \cdot w \cdot \sigma) = -\varepsilon(z, \gamma_j w).$$

Second, for $j = 1, ..., p$,

$$\begin{aligned} \varepsilon(\gamma_{p+j} z, w) &= (i\tilde{z}\cdot e_j, w\cdot\sigma) = -(\tilde{z}, iw\cdot\sigma\cdot e_j) = (\tilde{z}, iw\cdot e_j\cdot\sigma) = -(z, i\tilde{w}\cdot e_j\cdot\sigma) \\ &= -\varepsilon(z, \gamma_{p+j} w). \end{aligned}$$

□

COROLLARY 2.6. *On $\mathbf{P} \equiv \mathbf{S}^+ \oplus \mathbf{S}^-$, ε is a real valued positive definite inner product, with $\mathbf{S}_+$ and $\mathbf{S}_-$ orthogonal, and $\varepsilon(az, w) = \varepsilon(z, \hat{a}w)$, for all $z, w \in \mathbf{P}$ and $a \in \mathrm{Cl}(2p)$.*

Pure spinors and complex structures. Let $V_{\mathbf{C}}$ denote $V \otimes_{\mathbf{R}} \mathbf{C}$ with the complex symmetric bilinear form $(\ ,\ )$ extending $(\ ,\ )$ on V. Let $\mathrm{Cpx}(p)$ denote the set of orthogonal complex structures on V, i.e.

$$\mathrm{Cpx}(p) = \{J \in \mathrm{O}(V) : J^2 = -1\} \cong \mathrm{O}(2p)/\mathrm{U}(p).$$

Given $J \in \mathrm{Cpx}(p)$, let $V_{1,0}$ denote the $+i$ eigenspace of J in $V_{\mathbf{C}}$ and $V_{0,1} = \overline{V}_{1,0}$ the $-i$ eigenspace.

Note that $V_{0,1}$ is a complex p-dimensional totally null subspace of $V_{\mathbf{C}}$. Let $\mathbf{N}$ denote the space of all complex p-dimensional totally null subspaces of $V_{\mathbf{C}}$. Given $W \in \mathbf{N}$, the subspace W must be a graph over V in $V_{\mathbf{C}} = V \oplus iV$, since iV does not contain any null vectors. Let J denote the linear map $J: V \to V$ with $W \equiv \operatorname{graph} J$. Since W is totally null, $J \in \mathrm{Cpx}(p)$. Thus

$$\mathbf{N} \cong \mathrm{Cpx}(p) \cong \mathrm{O}(2p)/\mathrm{U}(p).$$

Let $\mathbf{N}^{\pm} \cong \mathrm{Cpx}(p)^{\pm}$ denote the two connected components ($W \in \mathbf{N}^+$ are called α-planes and $W \in \mathbf{N}^-$ are called β-planes).

A pinor $S \in \mathbf{P_C}$ is said to be *pure* (i.e. $S \in \mathrm{PURE}_\mathbf{C}$) if:

$$N_S \equiv \{Z \in V_\mathbf{C} : Z(S) = 0\} \in \mathbf{N}.$$

Since N_S is automatically totally null, S is pure if and only if N_S has dimension p. Note that $\mathrm{Spin}(2p)$ acts on $\mathrm{PURE}_\mathbf{C}$.

The map $S \mapsto N_S$ from

$$\mathrm{PURE}_\mathbf{C}/\mathbf{C}^* \text{ to } \mathbf{N}$$

is equivariant with

$$\mathrm{PURE}_\mathbf{C}/\mathbf{C}^* \cong \mathrm{Pin}(2p)/\tilde{\mathrm{U}}(p).$$

This isotropy subgroup $\tilde{\mathrm{U}}(p)$ is obtained from $\mathrm{U}(p)$ by lifting $\mathrm{U}(p) \subset \mathrm{SO}(2p)$ to $\mathrm{Spin}(2p)$ under the vector representation. In the following we will exhibit a pure spinor in $\mathbf{S}^+_\mathbf{C}$. It follows that, of the two connected components $\mathrm{PURE}^{\pm}_\mathbf{C}$ of $\mathrm{PURE}_\mathbf{C}$, one of these $\mathrm{PURE}^+_\mathbf{C} \subset \mathbf{S}^+_\mathbf{C}$, while the other $\mathrm{PURE}^-_\mathbf{C} \subset \mathbf{S}^-_\mathbf{C}$.

THEOREM 2.7 ($p \equiv 0 \mod 4$). *Suppose $S \in \mathrm{PURE}^+_\mathbf{C}$ is a pure spinor with $S = s_1 + is_2$ and $s_1, s_2 \in \mathbf{S}^+$ unit positive spinors. Let $J \in \mathrm{Cpx}(2p)$ denote the associated complex structure with Kähler form ω. There exists a unitary basis $\gamma_1, \ldots, \gamma_p$ for V (with the complex structure J) such that (with $Z_j \equiv \gamma_j + i\gamma_{p+j}$ and $\gamma_{p+j} \equiv J\gamma_j$, $j = 1, \ldots, p$),*

$$2^p s_1 \odot s_1 = \mathrm{Re}\, Z_1 \wedge \ldots \wedge Z_p + 1 - \frac{1}{2!}\omega^2 + \frac{1}{4!}\omega^4 - \ldots \tag{2.4}$$

$$2^p s_2 \odot s_2 = -\mathrm{Re}\, Z_1 \wedge \ldots \wedge Z_p + 1 - \frac{1}{2!}\omega^2 + \frac{1}{4!}\omega^4 - \ldots \tag{2.5}$$

$$2^p s_1 \odot s_2 = \mathrm{Im}\, Z_1 \wedge \ldots \wedge Z_p + \omega - \frac{1}{3!}\omega^3 + \ldots \tag{2.6}$$

$$2^p s_2 \odot s_1 = \mathrm{Im}\, Z_1 \wedge \ldots \wedge Z_p - \omega + \frac{1}{3!}\omega^3 - \ldots \tag{2.7}$$

In particular,

$$\tfrac{1}{2} \cdot 2^p S \odot S = Z_1 \wedge \ldots \wedge Z_p \quad \textit{and} \quad \tfrac{1}{2} \cdot 2^p \bar{S} \odot \bar{S} = \bar{Z}_1 \wedge \ldots \wedge \bar{Z}_p. \tag{2.8}$$

Proof. Because of the equivariance it suffices to prove the theorem for a particular $S \in \mathrm{PURE}^+_\mathbf{C}$. Utilizing the model for $\mathrm{Cl}(2p) \cong \mathrm{End}_\mathbf{R}(\mathbf{P})$ introduced earlier, let

$$Z_j \equiv \gamma_j + i\gamma_{p+j} = -2I_{e_j} \quad \text{and} \quad \bar{Z}_j \equiv \gamma_j - i\gamma_{p+j} = 2E_{e_j}, \quad j = 1, \ldots, p,$$

where $j \in \mathrm{Cpx}(2p)$ is defined by $J\gamma_j = \gamma_{p+j}$, $j=1,\dots,p$. Therefore, $1 \in \mathbf{S}^+_{\mathbf{C}} = \bigwedge^{\text{even}} \mathbf{C}^p$ is a pure spinor. Since, in our model $\mathbf{P} \equiv \bigwedge_+ \mathbf{R}^p \oplus i \bigwedge_- \mathbf{R}^p$, the pure spinor $S \equiv \sqrt{2} \in \mathbf{S}^+_{\mathbf{C}}$ has real part s_1 and imaginary part s_2 given by:

$$s_1 = \frac{1}{\sqrt{2}}(1+\sigma) \quad \text{and} \quad s_2 = \frac{1}{\sqrt{2}\, i}(1-\sigma)$$

which are both unit positive spinors for $\mathrm{Cl}(2p)$. Let

$$\omega_j \equiv \gamma_j \gamma_{p+j}, \quad j=1,\dots,p,$$

so that

$$\omega \equiv \omega_1 + \dots + \omega_p$$

is the Kähler form on V with respect to the complex structure J. The theorem is immediate from the next lemma. □

LEMMA 2.8. (i) $2^p 1 \odot 1 = Z_1 \dots Z_p$,
(ii) $2^p \sigma \odot \sigma = \bar{Z}_1 \dots \bar{Z}_p$,
(iii) $2^p \sigma \odot 1 = \prod_{j=1}^p (1 + i\omega_j)$,
(iv) $2^p 1 \odot \sigma = \prod_{j=1}^p (1 - i\omega_j)$, *and*
(v) $\prod_{j=1}^p (1+i\omega_j) = \big(1 - (1/2!)\omega^2 + (1/4!)\omega^4 - \dots\big) + i\big(\omega - (1/3!)\omega^3 + \dots\big)$.

Proof. Since $Z_j = -2I_{e_j}$ and $p \equiv 0 \mod 4$,

$$Z_1 \dots Z_p(w) = \begin{cases} 2^p, & \text{if } w = \sigma \\ 0, & \text{if } w \perp \sigma. \end{cases}$$

Also, for $1 \in \bigwedge \mathbf{C}^p = \mathbf{P}_{\mathbf{C}}$,

$$16^m (1 \odot 1)(w) = 2^p \varepsilon(1, w) \cdot 1 = (1, w \cdot \sigma) \cdot 1 = (\sigma, w) \cdot 1.$$

This proves part (i).

Part (ii) can be proven in the same manner as part (i). Alternatively,

$$2^p \sigma \odot \sigma = 2^p \gamma_{p+1} \dots \gamma_{2p} (1 \odot 1) \gamma_{p+1} \dots \gamma_{2p} = \gamma_{p+1} \dots \gamma_{2p} Z_1 \dots Z_p \gamma_{p+1} \dots \gamma_{2p} = \bar{Z}_1 \dots \bar{Z}_p.$$

Similarly,

$$\begin{aligned} 2^p \sigma \odot 1 &= 2^p \gamma_1 \dots \gamma_p (1 \odot 1) = \gamma_1 \dots \gamma_p Z_1 \dots Z_p = \gamma_p \dots \gamma_1 Z_1 \dots Z_p \\ &= \prod_{j=1}^p (-1 + i\omega_j) = \prod_{j=1}^p (1 - i\omega_j), \end{aligned}$$

and

$$\begin{aligned} 2^p (1 \odot \sigma) &= 2^p (1 \odot 1) \gamma_1 \dots \gamma_p = Z_1 \dots Z_p \gamma_1 \dots \gamma_p Z_1 \dots Z_p \gamma_p \dots \gamma_1 \\ &= \prod_{j=1}^p (-1 - i\omega_j) = \prod_{j=1}^p (1 + i\omega_j). \end{aligned}$$

□

COROLLARY 2.9. $\mathrm{PURE}_{\mathbf{C}}^{+}\cong\mathrm{Spin}(2p)/\tilde{\mathrm{SU}}(p)$ *with* $\tilde{\mathrm{SU}}(p)\cong\mathrm{SU}(p)$ *under the vector representation.*

Proof. The lift of $\mathrm{SU}(p)$ to a subgroup of $\mathrm{Spin}(2p)$ has two components since $\mathrm{SU}(p)$ is simply connected. Since $-I\in\mathrm{Spin}(2p)$ does not fix $1\in\mathbf{S}_{\mathbf{C}}^{+}$ only the component of the identity can be in the isotropy $\tilde{\mathrm{SU}}(p)$. □

Remark. Note that, under the vector representation, $(1/\sqrt{2})(1+\omega_j)\in\mathrm{Spin}(2p)$ maps to the orthogonal transformation equal to J acting on the (γ_j,γ_{p+j})-plane and equal to the identity on the orthogonal complement of this plane. Thus, under the vector representation,

$$\varrho=2^{-p/2}\prod_{j=1}^{p}(1+\omega_j)=2^{-p/2}\left(1+\omega+\frac{1}{2!}\omega^2+\ldots+\frac{1}{p!}\omega^p\right)$$

maps to $J\in\mathrm{SO}(2p)$. Also, $\varrho^2=\lambda$, the unit volume element.

COROLLARY 2.10. *The isotropy subgroup of* $\mathrm{Spin}(2p)$ *at* $s_1\in\mathbf{S}^+$ *has two components* $\tilde{\mathrm{SU}}(p)$ *and* $\tilde{\mathrm{SU}}(p)\cdot r_\sigma$ *(where* r_σ *denotes the right multiplication by* σ*). Under the vector representation this isotropy group maps to* $\mathrm{SU}(p)\cup\mathrm{SU}(p)\cdot c$ *(where* c *is the conjugation on* $V(2p)$ *given by* $c(\gamma_j)=\gamma_j$ *and* $c(\gamma_{p+j})=-\gamma_{p+j}$, $j=1,\ldots,p$*).*

Remark (A differential system). Theorem 2.7 can be used to provide a concrete example of a differential system guaranteed by Theorem 1.1. Consider the form $\Phi=2^p s_1\odot s_1=\mathrm{Re}\,Z_1\ldots Z_p+1-(1/2!)\omega^2+(1/4!)\omega^4+\ldots$. The main Theorem 2.7 is applicable. The calibrations

$$\frac{1}{k!}\omega^k \quad\text{and}\quad \mathrm{Re}\,Z_1\ldots Z_p+(-1)^k\frac{1}{k!}\omega^k,\quad k=p/2,$$

are well known (cf. [4]). Using our model with $\mathbf{S}^+\equiv\bigwedge_+^{\mathrm{even}}\mathbf{R}^p\oplus i\bigwedge_-^{\mathrm{even}}\mathbf{R}^p$, the unit spinors $s_1\equiv(1/\sqrt{2})(1+\sigma)$, $s_2\equiv(1/i\sqrt{2})(1-\sigma)$ can be completed to an orthonormal basis $s_1,\ldots,s_N$, for $\mathbf{S}^+$ $(N=\frac{1}{2}\cdot2^p)$, consisting of elements of the form $(1/\sqrt{2})(e_I+e_I\cdot\sigma)$ and $(1/i\sqrt{2})(e_I-e_I\cdot\sigma)$. The products $\Psi_j\equiv2^p s_1\odot s_j$ can be easily computed using Lemma 2.8 yielding an example of a differential system described by Theorem 1.1.

Remark (An alternate differential system). The conjugation $Rx\equiv\bar{x}$ on $\bigwedge\mathbf{C}^p$ converts the $\mathbf{C}$-symmetric bilinear form $\varepsilon_{\mathbf{C}}(x,y)$ on $\bigwedge\mathbf{C}$ to a positive definite ($\mathbf{C}$-Hermitian) inner product $\varepsilon_{\mathbf{C}}(\bar{x},y)$. Suppose x is of unit length with respect to this inner product, i.e. $\varepsilon_{\mathbf{C}}(\bar{x},x)=1$, and $x_1\equiv x,x_2,\ldots,x_N$ is an orthonormal basis for $\mathbf{S}_{\mathbf{C}}^{+}$ with respect to this positive definite inner product. Then, for each $\xi\in G(k,V(2p))$,

$$\sum_{j=1}^{N}2^p\left|(\bar{x}_j\odot x)(\xi)\right|^2=|\xi|^2. \tag{2.9}$$

This equality can be proved as follows. Note that $(\bar{x}_j \odot x)(\xi) = 2^{-p} \varepsilon_{\mathbf{C}}(\bar{x}_j, \xi(x))$, by Proposition I.2.1 (6). Also, one can show that Spin$(2p)$ preserves this positive definite inner product on $\mathbf{P}_{\mathbf{C}} \equiv \bigwedge \mathbf{C}^p$. Therefore,

$$|\xi|^2 = 1 = \varepsilon_{\mathbf{C}}(\overline{\xi(x)}, \xi(x)) = \sum_j |\varepsilon_{\mathbf{C}}(\bar{x}_j, \xi(x))|^2$$
$$= \sum_j 2^p |(\bar{x}_j \odot x)(\xi)|^2$$

as desired.

Now consider the basis for $\mathbf{S}^+_{\mathbf{C}} = \bigwedge^{\text{even}} \mathbf{C}^p$ given by $x_\varnothing \equiv 1$ and $x_I = e_I$ for $|I|$ even. Note that $\bar{x}_I = x_I \cdot \sigma$. Then:

$$2^p \bar{x}_I \odot x_\varnothing = Z_I \cdot \prod_{j=1}^{p} (1 + i\omega_j).$$

In particular,

$$2^p \bar{x}_{1,\ldots,p} \odot x_\varnothing = Z_1 \ldots Z_p.$$

Consequently, with this choice of spinor basis the equality (2.9) is the same as the equality discovered in [4], Theorem II.6.11.

Remark. Given a pure spinor $S \in \mathbf{S}^+_{\mathbf{C}}$ the real oriented two plane $s_1 \wedge s_2 \in G(2, \mathbf{S}^+)$, with $S \equiv s_1 + i s_2$, will be referred to as a *pure spinor two plane* and the collection of all such 2-planes denoted by PURE$(2, \mathbf{S}^+)$. One can show that

$$\mathrm{PURE}(2, \mathbf{S}^+) \cong \mathrm{Spin}(2p)/\tilde{\mathrm{U}}(p) \cong \mathrm{Cpx}(p),$$

since the degree 2 part of $2^p s_1 \odot s_2 - 2^p s_2 \odot s_2$ is the Kähler form ω.

III.3. Spinors in dimensions $\not\equiv 0 \mod 8$

As explained in the introduction one may construct from a calibration ϕ on $\mathbf{R}^p$ calibrations on lower dimensional Euclidean spaces. In this section we shall show how to completely analyze calibrations on $\mathbf{R}^{15}$ that come from squares of 15-dimensional spinors.

Let $e_0 \in V(16) \subset \mathrm{Cl}(16)$ be a unit vector. Let $V(15, e_0) = e_0^\perp$ be the orthocomplement of e_0 in $V(16)$. Recall [3] that sending a vector $u \in V(15, e_0)$ to $e_0 \cdot u \in \mathrm{Cl}(16)$ induces an isomorphism Ψ_{e_0} of $\mathrm{Cl}(V(15, e_0))$ with $\mathrm{Cl}(16)^{\text{even}}$. In particular Ψ_{e_0} gives a vectorspace isomorphism between $\bigwedge(e_0^\perp)$ and $\mathrm{Cl}(16)^{\text{even}}$. Furthermore,

$$\Psi_{e_0} : \mathrm{Pin}(15, e_0) \to \mathrm{Spin}(15, e_0) \cup \mathrm{Spin}(15, e_0)\lambda \subset \mathrm{Spin}(16)$$

where λ is the unit volume element in Cl(16). The Grassmannians $G(4k, e_0^\perp)$ form the set of elements of order two in $\mathrm{Spin}(15, e_0)$ while $G(4k-1, e_0^\perp)$ exhaust all elements of order two in $\mathrm{Spin}(15, e_0)\lambda$. The unit volume element λ is central and of order two. It is easily seen that multiplication by λ is up to a sign the Hodge $*$ operator on $\bigwedge^p(e_0^\perp)$.

For $s \in \mathbf{S}^+(16)$ let $\varphi = 16^2 s \circ s \in \bigwedge(V(16))$. We may view $s \in \mathbf{S}^+(16)$ also as a pinor for $\mathrm{Cl}(15, e_0)$ and let $\psi = 16^2 s \circ s \in \bigwedge(e_0^\perp)$. The relationship between φ and ψ is given by:

THEOREM 3.1 ([3], Theorem 14.129). *Let $\alpha = e_0 \llcorner \varphi$ and write $\varphi = e_0^* \wedge \alpha + \beta$. Then $\psi = \alpha + \beta$. Furthermore, the Grassmannian elements calibrated by β are the elements of order two in* $\mathrm{Spin}(15, e_0)_s = \{g \in \mathrm{Spin}(16)_s : \chi_g(e_0) = e_0\}$. *The Hodge $*$ (in $e_0^\perp$) of these calibrated planes are the planes calibrated by α.*

The above theorem combined with our analysis of squares of spinors $S(\lambda) \in D$ enables us to compute the planes in $e_0^\perp$ calibrated by $\psi = 16^2 S(\lambda) \circ S(\lambda)$ explicitly. However, not every $\mathrm{Spin}(15, e_0)$ orbit in $\mathbf{S}^+(16)$ meets D. To deal with the general ψ we shall use:

PROPOSITION 3.2. *Let $\psi \in \bigwedge(e_0^\perp) \cong \mathrm{Cl}(15)$ be the square of a 15-pinor $s \in \mathbf{S}^+(16)$, $\psi = 16^2 s \circ s$ as described above. Let $S(\lambda) \in D$ and $e \in V(16)$ determine forms $\alpha, \beta \in \bigwedge(e^\perp)$ by the decomposition of $S(\lambda) \circ S(\lambda) \in \mathrm{Cl}(16)$, $16^2 S(\lambda) \circ S(\lambda) = e^* \wedge \alpha + \beta$. For each ψ there is a choice of $S(\lambda) \in D$ and $e \in V(16)$ such that ψ is the pullback of $\alpha + \beta$ under a linear isometry of $V(16)$ sending e_0 to e. In particular, ψ and $\alpha + \beta$ determine the same geometry.*

Proof. Choose $g \in \mathrm{Spin}(16)$ so that $g \cdot s \in D$. Set $S(\lambda) = g \cdot s$. In Cl(16) we have by Proposition I.2.1

$$s \circ s = g^{-1} S(\lambda) \circ g^{-1} S(\lambda) = \hat{g}(S(\lambda) \circ S(\lambda)) g.$$

As an element of $\bigwedge V(16)^*$, $\hat{g} S(\lambda) \circ S(\lambda) g$ is the pullback of $S(\lambda) \circ S(\lambda)$ by χ_g^*. Next set $e = \chi_g(e_0)$ and let α and β be defined by the decomposition

$$16^2 S(\lambda) \circ S(\lambda) = e^* \wedge \alpha + \beta.$$

Applying χ_g^* to both sides now yields the Cl(16) identity

$$16^2 s \circ s = 16^2 \chi_g^*(S(\lambda) \circ S(\lambda)) = e_0 \wedge \chi_g^* \alpha + \chi_g^* \beta. \qquad \square$$

According to Proposition 3.2 to study the calibrations arising from squares of 15-pinors $s \in \mathbf{S}^+(16)$ it suffices to decompose the squares of 16 spinors $S(\lambda) \in D$ according to $e \in V(16)$. Is is also clear from the proof of the proposition that for a fixed $S(\lambda) \in D$ the decompositions according to $e_1, e_2 \in V(16)$ are isometrically equivalent if e_1 and e_2 are on the same $K(\lambda)$ orbit, where $K(\lambda) \subset \mathrm{Spin}(16)$ is the isotropy at $S(\lambda)$. For constructing calibrations whose faces are maximal among calibrations obtained from squares of

15-pinors we need to choose $S(\lambda)$ with $K(\lambda)$ maximal, and $e\in V(16)$ such that the isotropy $K(\lambda)_e$ is maximal in $K(\lambda)$ (cf. Theorem 3.1). In every maximal geometry discussed in Section II.4 with the exception of the $\mathrm{Sp}(2)\times\mathrm{U}(2)$ and $\mathrm{Sp}(2)\times\mathrm{Sp}(1)\times\mathrm{Sp}(1)$ geometries the action of $K(\lambda)$ on $V(16)$ is polar and so finding maximal isotropy subgroups of $K(\lambda)$ is straightforward.

Example: *The* Spin(8) *geometry.* Let $\lambda=(1/\sqrt{8})(1,-1,-1,-1,-1,-1,-1,-1)$. For the action of $K(\lambda)=\mathrm{Spin}(8)\cup\mathrm{Spin}(8)\xi$ on $V(16)=V_8\oplus V_8'$ there is only one orbit for which the isotropy subgroups are maximal. This is the orbit of $e=\varepsilon_1\in V_8$. The isotropy $K(\lambda)_e\cong$ Spin(7): Recall that $(g_0,g_+,g_-)\in\mathrm{Spin}(8)$ acts on V_8 by g_+ and V_8' by g_-. Hence

$$K(\lambda)_e=\{(g_0,g_+,g_-)\in\mathrm{Spin}(8):g_+\varepsilon_1=\varepsilon_1\}.$$

As we have seen in Section I.3 there is only one nontrivial conjugacy class of elements of order two in Spin(7). For an element (g_0,g_+,g_-) in this class g_+ is a reflection along a 4-plane in $\varepsilon_1^\perp\subset V_8$ while g_- is a reflection along a Cayley plane in V_8'. This defines an isometry f between $G(4,\varepsilon_1^\perp)$ and $\mathrm{CAY}(V_8')$. Summarizing:

$$16S(\lambda)\circ S(\lambda)=\varepsilon_1^*\wedge\alpha+\beta,$$

$$\beta=1+\beta_8,$$

$$\alpha=\mathrm{vol}(\varepsilon_1^\perp)+\alpha_7,\qquad \alpha_7=*\beta_8.$$

The 8-form β_8 calibrates planes of the form $\pi\wedge f(\pi)$, $\pi\in G(4,\varepsilon_1^\perp)$. The planes calibrated by α_7 are obtained by Hodge $*$ from the planes calibrated by β_8. They are of the form $\pi\wedge h(\pi)$ where $\pi\in G(3,\varepsilon_1^\perp)$ and $h: G(3,\varepsilon_1^\perp)\to\mathrm{CAY}(V_8')$ is the isometry $h=*f*$.

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Received July 3, 1990
Received in revised form October 1, 1991