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Calogero-Moser Models. I

—— A New Formulation ——

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A new formulation of Calogero-Moser models based on root systems and their Weyl group is presented. The general construction of the Lax pairs applicable to all models based on the simply-laced algebras (ADE) are given for two types which we call 'root' and 'minimal'. The root type Lax pair is new; the matrices used in its construction bear a resemblance to the adjoint representation of the associated Lie algebra, and exist for all models, but they do not contain elements associated with the zero weights corresponding to the Cartan subalgebra. The root type provides a simple method of constructing sufficiently many number of conserved quantities for all models, including the one based on E_8, whose integrability had been an unsolved problem for more than twenty years. The minimal types provide a unified description of all known examples of Calogero-Moser Lax pairs and add some more. In both cases, the root type and the minimal type, the formulation works for all of the four choices of potentials: the rational, trigonometric, hyperbolic and elliptic.

§1. Introduction

The Calogero-Moser models are one dimensional dynamical systems with long-range pair-wise interactions. They are completely integrable when the two-body interaction potential[1)-4)] is proportional to (i) $1/L^2$, (ii) $1/\sin^2 L$, (iii) $1/\sinh^2 L$ and (iv) $\wp(L)$, in which L is the inter-particle "distance". The types of integrable many-particle interactions are governed by Lie algebras or rather their root systems: there are Calogero-Moser models based on the root systems of all of the semi-simple Lie algebras. The total number of the particles in the system is equal to the rank of the algebra and it is an arbitrary integer r for the classical Lie algebras: A_r, B_r, C_r and D_r. But it is quite limited for the exceptional algebras: E_6, E_7, E_8, F_4 and G_2.

Since the models are generic they find various physical applications ranging from solid state physics to particle physics.[5)] Because of their Lie algebraic structure, elliptic Calogero-Moser models are analysed intensively in connection with the Seiberg-Witten curve and differential for $\mathcal{N}=2$ supersymmetric gauge theory with the same Lie algebra.[6)-11)]

In this paper we address the fundamental problems of the Calogero-Moser models rather than the applications. These are the issues of integrability and the universal framework for the construction of the Lax pairs. These have been a mystery from the early days of the Calogero-Moser models. From the very beginning, the structure of the integrable Hamiltonians for all the models based on root systems of semi-simple

Lie algebras was understood but the catalogue of the Lax pairs necessary for the proof of the integrability remained the same for some twenty years. It contained only the vector representations of the classical algebras A_r, B_r, C_r, D_r and the BC_r root system. A general principle governing all the models based on the classical as well as the exceptional algebras was yet to be found. Recently D'Hoker and Phong [12] succeeded in constructing Lax pairs for the exceptional algebras but the method was not complete enough to cover the one based on E_8.

We present in this paper a new formulation of Calogero-Moser Lax pairs based on root systems and their Weyl group. It is applicable to all models based on semi-simple Lie algebras including E_8. Constructions of Lax pairs for models based on simply-laced root systems are given as two types which we will call 'minimal' and 'root'. The root type of Lax pair is new; the matrices used in its construction bear a resemblance to the adjoint representation of the associated Lie algebra, and exist for all models, but they do not contain elements associated with the zero weights corresponding to the Cartan subalgebra. The 'minimal' types provide a unified description of all known examples of Calogero-Moser Lax pairs and reveal some new ones.

This paper is organised as follows. In §2 the basic ingredients of the models are introduced and the principle of Weyl invariance is stated. In §3 the Lax pair of the 'root' type for simply-laced root systems is given and its consistency is proved. The Lax pairs of the models for the non-simply laced root systems can be obtained by reduction or folding, [13] which is a well-known procedure in Toda lattice (field) theory. [14), 15] In §4 the formulation of the Lax pairs of the minimal type is given and consistency is proved in a similar way as in the root type. In both cases, the root type and the minimal type, the Lax pair exists for all the four types of the interaction potentials. Various known examples of Lax pairs are derived as special cases of the minimal type. Some Lax pairs of non-simply laced root systems are derived from those of simply-laced ones by reduction or folding. The spinor and the anti-spinor representations of D_N are discussed in some detail for two purposes; the first to exemplify the relationship between the exponents of the algebra and conserved quantities and the second to derive the B_N Lax pair in the spinor representation by reduction. The Lax pairs to be discussed in this paper are those without spectral parameter. Introduction of the spectral parameter to the elliptic potential case in the present scheme is rather straightforward.[*] We will discuss the Lax pair with spectral parameter in connection with folding in a future publication. Section 5 is for summary and discussion.

§2. Calogero-Moser models

Let us start by defining the Calogero-Moser model based on a semi-simple and *simply-laced* Lie algebra $\mathfrak{g}$ with rank r. In fact we only need the data of its roots. We denote the set of all roots by Δ. They are real r dimensional vectors and are

*) For the Lax-pair with a spectral parameter, see for example, Refs. 12) and 16).

normalised, without loss of generality, to 2:

$$\Delta=\{\alpha,\beta,\gamma,\cdots\},\quad \alpha\in\mathbb{R}^r,\quad \alpha^2=\alpha\cdot\alpha=2,\quad \forall\alpha\in\Delta. \tag{2·1}$$

We denote by Dim the total number of roots of Δ. It is $r(r+1)$ and $2r(r-1)$ for A_r and D_r and 72, 126 and 240 for E_6, E_7 and E_8, respectively.

The dynamical variables are canonical coordinates $\{q^j\}$ and their canonical conjugate momenta $\{p_j\}$ with the Poisson brackets:

$$q^1,\cdots,q^r,\quad p_1,\cdots,p_r,\quad \{q^j,p_k\}=\delta_{j,k},\quad \{q^j,q^k\}=\{p_j,p_k\}=0. \tag{2·2}$$

In most cases we denote them by r dimensional vectors q and p,*)

$$q=(q^1,\cdots,q^r)\in\mathbb{R}^r,\quad p=(p_1,\cdots,p_r)\in\mathbb{R}^r,$$

so that the scalar products of q and p with the roots $\alpha\cdot q$, $p\cdot\beta$, etc. can be defined.

Another ingredient of the theory is the Weyl reflections. Let ξ be an $\mathbb{R}^r$ vector and $\beta\in\Delta$. The Weyl reflection by a root β is defined by

$$W_\beta(\xi)=\xi-\frac{2(\beta\cdot\xi)\beta}{\beta^2}. \tag{2·3}$$

Obviously $W_\beta^2=1$ and $W_\beta=W_{-\beta}=W_\beta^{-1}$ and the totality of the Weyl reflections form a group called the Weyl group. The root systems of the semi-simple Lie algebras are invariant under any Weyl reflection:

$$W_\beta(\alpha)\in\Delta,\quad \forall\alpha,\beta\in\Delta. \tag{2·4}$$

In fact, the set of roots invariant under the Weyl reflection (2·4) is the fundamental ingredient for constructing a Calogero-Moser model. The root system need not belong to a Lie algebra. The Lie algebra structure is important for most cases but not essential. This can be seen most clearly in the BC_r Calogero-Moser model, in which the set of roots Δ is the union of B_r and C_r roots.

Next we introduce the functions appearing in the Lax pair. They depend on the choice of the inter-particle potential. For the rational, $1/L^2$, potential they are:

$$x(t)=x_d(t)=\frac{1}{t},\quad y(t)=y_d(t)=-\frac{1}{t^2},\quad z(t)=z_d(t)=-\frac{1}{t^2}. \tag{2·5}$$

For the trigonometric, $1/\sin^2 L$, potential they are:

$$x(t)=x_d(t)=a\cot at,\quad y(t)=y_d(t)=-\frac{a^2}{\sin^2 at},\quad z(t)=z_d(t)=-\frac{a^2}{\sin^2 at},\quad a:\text{const.} \tag{2·6}$$

For the hyperbolic, $1/\sinh^2 L$, potential they are:

$$x(t)=x_d(t)=a\coth at,\quad y(t)=y_d(t)=-\frac{a^2}{\sinh^2 at},\quad z(t)=z_d(t)=-\frac{a^2}{\sinh^2 at}. \tag{2·7}$$

*) For A_r models, it is customary to introduce one more degree of freedom, q^{r+1} and p_{r+1} and embed all of the roots in $\mathbb{R}^{r+1}$.

For the elliptic, $\wp(L)$, potential there are several choices of the functions. Generally the functions x and x_d differ. A first choice is

$$x(t) = \frac{a}{2}\left[\frac{1+k\,\mathrm{sn}^2(at/2,k)}{\mathrm{sn}(at/2,k)} - i\frac{(1+k)(1-k\,\mathrm{sn}^2(at/2,k))}{\mathrm{cn}(at/2,k)\,\mathrm{dn}(at/2,k)}\right],$$

$$y(t) = x'(t), \quad z(t) = -b^2\wp(bt), \quad b = a/\sqrt{e_1-e_3}, \tag{2·8}$$

and

$$x_d(t) = \frac{a}{\mathrm{sn}(at,k)}, \quad y_d(t) = -a^2\frac{\mathrm{cn}(at,k)\,\mathrm{dn}(at,k)}{\mathrm{sn}^2(at,k)}, \quad z_d(t) = -b^2\wp(bt),$$

$$b = a/\sqrt{e_1-e_3}, \tag{2·9}$$

in which k is the modulus of the elliptic function.*)

A second choice is

$$x(t) = \frac{a}{2}\left[\frac{\mathrm{cn}^2(at/2,k) - k'\mathrm{sn}^2(at/2,k)}{\mathrm{sn}(at/2,k)\,\mathrm{cn}(at/2,k)} + (1+k')\frac{\mathrm{cn}^2(at/2,k)+k'\mathrm{sn}^2(at/2,k)}{\mathrm{dn}(at/2,k)}\right],$$

$$y(t) = x'(t), \quad z(t) = -b^2\wp(bt), \tag{2·10}$$

and

$$x_d(t) = a\frac{\mathrm{cn}(at,k)}{\mathrm{sn}(at,k)}, \quad y_d(t) = -a^2\frac{\mathrm{dn}(at,k)}{\mathrm{sn}^2(at,k)}, \quad z_d(t) = -b^2\wp(bt), \tag{2·11}$$

in which $k' = \sqrt{1-k^2}$.

A third choice is

$$x(t) = \frac{a}{2}\left[\frac{\mathrm{dn}^2(at/2,k) + ikk'\mathrm{sn}^2(at/2,k)}{\mathrm{sn}(at/2,k)\,\mathrm{dn}(at/2,k)} + \frac{k\,\mathrm{cn}^2(at/2,k) - ik'}{\mathrm{cn}(at/2,k)}\right],$$

$$y(t) = x'(t), \quad z(t) = -b^2\wp(bt), \tag{2·12}$$

and

$$x_d(t) = a\frac{\mathrm{dn}(at,k)}{\mathrm{sn}(at,k)}, \quad y_d(t) = -a^2\frac{\mathrm{cn}(at,k)}{\mathrm{sn}^2(at,k)}, \quad z_d(t) = -b^2\wp(bt). \tag{2·13}$$

The trigonometric ($k \to 0$) and hyperbolic ($k \to 1$) limits of the elliptic cases give other sets of functions for these cases. One important property is that they all satisfy the *sum rule*

$$y(u)x(v) - y(v)x(u) = x(u+v)[z(u)-z(v)], \quad u,v \in \mathbb{C}. \tag{2·14}$$

The functions x_d, y_d and z_d satisfy the same relations. In all these cases the inter-particle potential V is proportional to $-z+\mathrm{const}$ and y (y_d) is the derivative of x (x_d) and z is always an even function:

$$y(t) = x'(t), \quad z(t) = x(t)x(-t) + \mathrm{constant}, \quad z(-t) = z(t). \tag{2·15}$$

*) The detailed properties of the elliptic potential cases will be discussed elsewhere.

For the rational (2·5), trigonometric (2·6) and hyperbolic cases (2·7) x is an odd function and y is an even function but they do not have definite parity for the elliptic potentials (2·8) and (2·9).

The Hamiltonian is given by (g is a real coupling constant)

$$\mathcal{H} = \frac{1}{2}p^2 - \frac{g^2}{2}\sum_{\alpha\in\Delta} x(\alpha\cdot q)x(-\alpha\cdot q), \qquad (2\cdot 16)$$

which is invariant under the Weyl reflection of the dynamical variables:

$$q \to q' = W_\beta(q), \quad p \to p' = W_\beta(p), \qquad \forall\beta\in\Delta, \qquad (2\cdot 17)$$

forming a discrete subgroup of $O(r)$. In fact, $p'^2 = p^2$ and $x(\alpha\cdot q') = x(W_\beta(\alpha)\cdot q)$ and the invariance of Δ under Weyl group (2·4) is used. Let us compare the situation with the Toda lattice (field theory),[14),15)] another well-known integrable system based on the root systems of (affine) Lie algebras. In the latter, only the simple roots are used and Weyl invariance does not exist.[*)] It should be stressed that in both cases, Calogero-Moser and Toda, the Hamiltonian is not invariant under the Lie algebra $\mathfrak{g}$ associated with the root system. However, in Toda theories the Lax pairs and the classical R-matrices are constructed from Lie algebra generators and therefore automatically work in any representation. For Calogero-Moser models several attempts[2),3),12)] to generate the Lax pairs based on Lie algebra generators and/or symmetric space ideas have not achieved the desired goal.

In this paper we propose to adopt the root systems and their Weyl invariance rather than the Lie algebraic structure as the basic principle of the Calogero-Moser models. Thus in order to find the Lax pair for the above Hamiltonian we should look for a space in which *the Weyl reflections rather than the Lie algebra generators* are conveniently represented. Obviously the simplest and thus the best choice is the set of roots Δ itself. The Lax pairs thus constructed will be called of the 'root' type. It should be stressed that this is *different* from the adjoint representation. The adjoint representation has $Dim + r$ dimensions. That is, it has rank (r) number of zero weights corresponding to the Cartan subalgebra. These zero weights cause severe problems in representing the Weyl reflection in a $Dim + r$ dimensional linear space when $r > 1$, because the representation matrix can never be uniquely determined in the r dimensional subspace. This is the main obstacle for the proof of the integrability of the E_8 theory, for which the lowest dimensional Lie-algebra representation is the adjoint representation and as we will see in §4 the minimal representation does not exist.

§3. Lax pair of the 'root' type

In this section we present the construction of the Lax pair applicable to all of the Calogero-Moser models based on semi-simple and simply-laced algebras. This

[*)] Though Weyl invariance is absent, the Coxeter element (a product of Weyl transformations corresponding to simple roots) plays an important role.[17)] provides the basic ingredients for a unified proof of integrability of all Calogero-Moser models based on root systems of semi-simple Lie algebras, including those based on non-simply laced root systems. The Lax pairs of non-simply laced theories are obtained from the corresponding simply-laced ones by *reduction* or *folding*, a well-known procedure in Toda lattice (field) theory.[13),15)] The non-simply laced Lax pairs obtained by reduction of the simply laced ones have only one coupling constant. The direct formulation of the root type Lax pairs for non-simply laced theories and the BC_r root system (with two or more independent coupling constants) could be given in a similar way as in this paper.

The goal is to express the canonical equation of motion derived from the Hamiltonian (2·16) in an equivalent matrix form:

$$\dot{L} = \frac{d}{dt}L = [L, M], \qquad (3\cdot1)$$

so that a sufficient number of conserved quantities could be obtained by the trace:

$$\frac{d}{dt}\mathrm{Tr}\,(L^k) = 0, \quad k = 1, \cdots. \qquad (3\cdot2)$$

It should be noted that the Lax pair in all theories and in all representations has the gauge freedom (similarity transformation):

$$\begin{aligned} L \to L^U = U^{-1}LU, &\quad M \to M^U = U^{-1}MU + U^{-1}\dot{U}, \\ \dot{L} = [L, M] &\iff \dot{L^U} = [L^U, M^U]. \end{aligned} \qquad (3\cdot3)$$

The following Lax pair in Δ is believed to be in the simplest gauge (we choose L to be hermitian and M anti-hermitian):

$$\begin{aligned} L(q,p) &= p \cdot H + X + X_d, \\ M(q) &= D + Y + Y_d. \end{aligned} \qquad (3\cdot4)$$

Here L, H, X, X_d, D, Y and Y_d are $Dim \times Dim$ matrices whose indices are labelled by the roots themselves, usually denoted by α, β, γ, η and κ. H and D are diagonal:

$$H_{\beta\gamma} = \beta\delta_{\beta,\gamma}, \quad D_{\beta\gamma} = \delta_{\beta,\gamma}D_\beta, \quad D_\beta = -ig\left(z(\beta\cdot q) + \sum_{\kappa\in\Delta,\ \kappa\cdot\beta=1} z(\kappa\cdot q)\right). \qquad (3\cdot5)$$

X and Y have the same form, differing only by the dependence on the coordinates q:

$$X = ig\sum_{\alpha\in\Delta} x(\alpha\cdot q)E(\alpha), \quad Y = ig\sum_{\alpha\in\Delta} y(\alpha\cdot q)E(\alpha), \quad E(\alpha)_{\beta\gamma} = \delta_{\beta-\gamma,\alpha}. \qquad (3\cdot6)$$

X_d and Y_d are necessary only in the 'root' type Lax pair

$$X_d = 2ig\sum_{\alpha\in\Delta} x_d(\alpha\cdot q)E_d(\alpha), \quad Y_d = ig\sum_{\alpha\in\Delta} y_d(\alpha\cdot q)E_d(\alpha), \quad E_d(\alpha)_{\beta\gamma} = \delta_{\beta-\gamma,2\alpha}. \qquad (3\cdot7)$$

The functions x, y, z (x_d, y_d, z_d) are listed in (2·5)–(2·13). The matrix $E(\alpha)$ ($E_d(\alpha)$) might be called a (double) root discriminator. It takes the value one only when the difference of the two indices is equal to (twice) the root α. Though the matrices H and $E(\alpha)$ satisfy relations

$$[H, E(\alpha)] = \alpha E(\alpha), \quad [H, [E(\alpha), E(\beta)]] = (\alpha+\beta)[E(\alpha), E(\beta)],$$
$$E(-\alpha) = E(\alpha)^T, \quad [E(\alpha), E(-\alpha)] + 2[E_d(\alpha), E_d(-\alpha)] = \alpha \cdot H, \quad (3·8)$$

they are *not Lie algebra generators.* The matrix elements $X_{\beta\gamma}$ and $Y_{\beta\gamma}$ are non-vanishing only when $\beta-\gamma$ is a root. For simply-laced root systems with $(\text{length})^2 = 2$, this can be rephrased as

$$X_{\beta\gamma} = 0 \quad \text{and} \quad Y_{\beta\gamma} = 0 \quad \text{if} \quad \beta \cdot \gamma \neq 1. \quad (3·9)$$

It is easy to rewrite D in a form similar to X and Y,

$$D = -ig \sum_{\alpha\in\Delta} z(\alpha \cdot q) K(\alpha), \quad K(\alpha)_{\beta\gamma} = \delta_{\beta,\gamma}\left(\delta_{\alpha,\beta} + \theta(\alpha\cdot\beta)\right), \quad (3·10)$$

in which $\theta(t)$ has a support only on 1

$$\theta(t) = \begin{cases} 1, & t = 1, \\ 0, & \text{otherwise.} \end{cases} \quad (3·11)$$

It is straightforward to represent the Weyl reflections in Δ. By $\boldsymbol{\alpha}$ we denote a *Dim* dimensional vector whose elements are the roots themselves. The Weyl reflection can be represented by a $Dim\times Dim$ matrix $S(\beta)$ as follows: Under the Weyl reflection in terms of a root β, W_β (2·3), each root α is mapped to $\alpha \to \alpha' = W_\beta(\alpha)$. We express the transformation of the totality of the roots as

$$\boldsymbol{\alpha} \to \boldsymbol{\alpha}' = S(\beta)\,\boldsymbol{\alpha}. \quad (3·12)$$

It is easy to see that the elements of $S(\beta)$ are expressed as

$$S(\beta)_{\gamma\eta} = \delta_{\gamma, W_\beta(\eta)}, \quad \forall \beta, \gamma, \eta \in \Delta. \quad (3·13)$$

The matrices $E(\alpha)$ ($E_d(\alpha)$) and $K(\alpha)$ transform

$$S(\beta)^{-1}E(\alpha)S(\beta) = E(W_\beta(\alpha)), \quad S(\beta)^{-1}K(\alpha)S(\beta) = K(W_\beta(\alpha)), \quad \forall\beta \in \Delta. \quad (3·14)$$

The Weyl covariance of L and M is a simple consequence of (3·14). That is for

$$q \to q' = W_\beta(q), \quad p \to p' = W_\beta(p), \qquad \forall\beta\in\Delta,$$
$$L(q', p') = S(\beta)^{-1}L(q,p)S(\beta), \quad \text{and} \quad M(q') = S(\beta)^{-1}M(q)S(\beta). \quad (3·15)$$

Then the Lax equation

$$\dot{L} = \frac{d}{dt}L = [L, M]$$

is invariant.

It should be remarked that the covariance requirement on the matrix $E(\alpha)$ ($E_d(\alpha)$) and $K(\alpha)$ (3·14) is very strong. Once $E(\alpha_0)$ ($E_d(\alpha_0)$) and $K(\alpha_0)$ are given for one root α_0 all the other $E(\alpha)$ ($E_d(\alpha)$) and $K(\alpha)$ for all the roots α lying on the same Weyl orbit of α_0 are determined uniquely. In the case of simply-laced root systems there is only one Weyl orbit, so everything is determined. Changing the signs of some of the matrices $E(\alpha)$ (except for the overall similarity transformation (3·3)) would destroy the Weyl covariance and thus the Lax pair. This is in contrast with the Lie algebra or symmetric space representations. In these cases the choices of the generators are much less restricted.

In the rest of the section we show that the Lax equation

$$\dot{L} = \frac{d}{dt}L = [L, M] \tag{3·16}$$

is equivalent to the canonical equations of motion for the Hamiltonian (2·16):

$$\dot{q} = p, \quad \dot{p} = -\frac{\partial}{\partial q}\mathcal{H} = -\frac{g^2}{2}\sum_{\alpha\in\Delta}\Big(x(\alpha\cdot q)y(-\alpha\cdot q) - x(-\alpha\cdot q)y(\alpha\cdot q)\Big)\,\alpha. \tag{3·17}$$

The Lax equation (3·16) is decomposed into three parts:

$$\frac{d}{dt}(X + X_d) = [p\cdot H, Y + Y_d], \tag{3·18}$$

$$\frac{dp}{dt}\cdot H = [X + X_d, Y + Y_d]_{\text{diagonal part}}, \tag{3·19}$$

$$0 = [X + X_d, D + Y + Y_d]_{\text{off-diagonal part}}. \tag{3·20}$$

It is easy to see that (3·18) is equivalent to the first set of the canonical equations of motion $\dot{q} = p$. In fact, by taking (β, γ) element of (3·18), we obtain

$$\begin{aligned}
[p\cdot H, Y]_{\beta\gamma} &= ig\sum_{\alpha\in\Delta} y(\alpha\cdot q)E(\alpha)_{\beta\gamma}\, p\cdot(\beta-\gamma) \\
&= ig\sum_{\alpha\in\Delta} y(\alpha\cdot q)E(\alpha)_{\beta\gamma}\, p\cdot\alpha \\
&= \frac{d}{dt}X_{\beta\gamma},
\end{aligned} \tag{3·21}$$

in which the relations $\dot{q} = p$ and $x' = y$ are used. Similar relation holds for X_d.

By using (3·9), we arrive at

$$\begin{aligned}
[X, Y]_{\beta\beta} &= \sum_{\kappa\in\Delta,\ \kappa\cdot\beta=1} (X_{\beta\kappa}Y_{\kappa\beta} - Y_{\beta\kappa}X_{\kappa\beta}) \\
&= -g^2 \sum_{\kappa\in\Delta,\ \kappa\cdot\beta=1} \left[x((\beta-\kappa)\cdot q)y((\kappa-\beta)\cdot q) - y((\beta-\kappa)\cdot q)x((\kappa-\beta)\cdot q)\right].
\end{aligned}$$

Likewise we obtain

$$\begin{aligned}
[X, Y_d]_{\beta\beta} &= [X_d, Y]_{\beta\beta} = 0, \\
[X_d, Y_d]_{\beta\beta} &= -2g^2\left(x_d(\beta\cdot q)y_d(-\beta\cdot q) - x_d(-\beta\cdot q)y_d(\beta\cdot q)\right) \\
&= -2g^2\left(x(\beta\cdot q)y(-\beta\cdot q) - x(-\beta\cdot q)y(\beta\cdot q)\right).
\end{aligned}$$

Thus (3·19) reads

$$\dot{p}\cdot\beta = -g^2\left(\sum_{\kappa\in\Delta,\ \kappa\cdot\beta=1} x((\beta-\kappa)\cdot q)y((\kappa-\beta)\cdot q) - y((\beta-\kappa)\cdot q)x((\kappa-\beta)\cdot q)\right.$$

$$\left. +2x(\beta\cdot q)y(-\beta\cdot q) - 2x(-\beta\cdot q)y(\beta\cdot q)\right)$$

$$= -g^2\left(\sum_{\alpha\in\Delta,\ \alpha\cdot\beta=1} x(\alpha\cdot q)y(-\alpha\cdot q) - x(-\alpha\cdot q)y(\alpha\cdot q)\right.$$

$$\left. +2x(\beta\cdot q)y(-\beta\cdot q) - 2x(-\beta\cdot q)y(\beta\cdot q)\right), \tag{3·22}$$

in which the dummy variable is changed from κ to $\alpha = \beta - \kappa$. This equation is obtained from the second set of canonical equations of motion (3·17) by multiplying β on both sides

$$\dot{p}\cdot\beta = -\frac{g^2}{2}\sum_{\alpha\in\Delta}\Big(x(\alpha\cdot q)y(-\alpha\cdot q) - x(-\alpha\cdot q)y(\alpha\cdot q)\Big)\,\alpha\cdot\beta.$$

Only those terms corresponding to $\alpha\cdot\beta = \pm 2$, i.e., $\alpha = \pm\beta$ and $\alpha\cdot\beta = \pm 1$ contribute and we obtain (3·22). This leaves us to show the vanishing of (3·20), which we decompose into four cases: (A) $\beta\cdot\gamma = 1$ case, (B) $\beta\cdot\gamma = 0$ case, (C) $\beta\cdot\gamma = -1$ case and (D) $\beta\cdot\gamma = -2$ case. Let us evaluate (3·20) in turn.

3.1. *Consistency of the root type Lax pair*

3.1.1. (A) $\beta\cdot\gamma = 1$ case

Let us start with

$$[X, D]_{\beta\gamma} = X_{\beta\gamma}(D_\gamma - D_\beta),$$

in which

$$D_\gamma - D_\beta = -ig\left(z(\gamma\cdot q) - z(\beta\cdot q) + \sum_{\kappa\cdot\gamma=1} z(\kappa\cdot q) - \sum_{\kappa'\cdot\beta=1} z(\kappa'\cdot q)\right).$$

First we simplify the above expression by removing all of the cancelling terms. The first summation ($\kappa\cdot\gamma = 1$) is decomposed into four groups according to the value of $\kappa\cdot\beta = \{2, 1, 0, -1\}$. The term $\kappa\cdot\beta = -2$ does not exist, since it means $\kappa = -\beta$ which is incompatible with $\beta\cdot\gamma = 1$ and $\kappa\cdot\gamma = 1$. The term $-z(\beta\cdot q)$ cancels the $\kappa = \beta$ ($\kappa\cdot\beta = 2$) term in the first summation. Likewise, $z(\gamma\cdot q)$ term cancels the $\kappa' = \gamma$ term in the second summation. The second group $\kappa\cdot\beta = 1$ in the first summation can be dropped since it is canceled by the term in the second sum having $\kappa'\cdot\beta = 1$. The fourth group $\kappa\cdot\beta = -1$ with $\kappa\cdot\gamma = 1$ consists of one term, since this means $\kappa\cdot(\gamma-\beta) = 2$ implying $\kappa = \gamma - \beta$. Then the term $z((\gamma-\beta)\cdot q)$ is cancelled by

$z((\beta-\gamma)\cdot q)$ term in the second summation (z is an even function). Thus only the third group survives:

$$D_\gamma - D_\beta = -ig\left(\sum_{\kappa\cdot\gamma=1,\ \kappa\cdot\beta=0} z(\kappa\cdot q) - \sum_{\kappa'\cdot\beta=1,\ \kappa'\cdot\gamma=0} z(\kappa'\cdot q)\right).$$

It is easy to see that there is a one to one correspondence between the two summations. For each κ appearing in the first sum we define $\kappa' = \kappa+\beta-\gamma$. Then it is a root $\kappa' = W_\beta(\kappa-\gamma)$ and satisfies $\kappa'\cdot\beta = 1$ and $\kappa'\cdot\gamma = 0$. Thus we arrive at

$$D_\gamma - D_\beta = -ig \sum_{\kappa\cdot\gamma=1,\ \kappa\cdot\beta=0} \left[z(-\kappa\cdot q) - z((\kappa+\beta-\gamma)\cdot q)\right],$$

in which the even parity of z is used. Now we have

$$\begin{aligned}[X,D]_{\beta\gamma}, &= X_{\beta\gamma}(D_\gamma - D_\beta)\\ &= g^2 \sum_{\kappa\cdot\gamma=1,\ \kappa\cdot\beta=0} x((\beta-\gamma)\cdot q)\left[z(-\kappa\cdot q) - z((\kappa+\beta-\gamma)\cdot q)\right]\\ &= g^2 \sum_{\kappa\cdot\gamma=1,\ \kappa\cdot\beta=0} \left[y((\kappa+\beta-\gamma)\cdot q)x(-\kappa\cdot q) - y(-\kappa\cdot q)x((\kappa+\beta-\gamma)\cdot q)\right],\end{aligned} \tag{3·23}$$

in which the sum rule of the function x, y and z, (2·14) is used.

Next we evaluate $[X,Y]_{\beta\gamma}$:

$$\begin{aligned}[X,Y]_{\beta\gamma} &= \sum_{\kappa\in\Delta,\ \kappa\cdot\beta=1,\ \kappa\cdot\gamma=1} (X_{\beta\kappa}Y_{\kappa\gamma} - Y_{\beta\kappa}X_{\kappa\gamma})\\ &= -g^2 \sum_{\kappa\cdot\beta=1,\ \kappa\cdot\gamma=1} \left[x((\beta-\kappa)\cdot q)y((\kappa-\gamma)\cdot q) - y((\beta-\kappa)\cdot q)x((\kappa-\gamma)\cdot q)\right].\end{aligned}$$

By changing the dummy variable from κ to $\kappa' = \gamma-\kappa$, we arrive at

$$\begin{aligned}&[X,Y]_{\beta\gamma}\\ &\quad = -g^2 \sum_{\kappa'\cdot\gamma=1,\ \kappa'\cdot\beta=0} \left[x((\beta-\gamma+\kappa')\cdot q)y(-\kappa'\cdot q) - y((\beta-\gamma+\kappa')\cdot q)x(-\kappa'\cdot q)\right],\end{aligned} \tag{3·24}$$

which cancels the previous contribution (3·23). It is trivial to see that the other terms vanish:

$$[X_d, D]_{\beta\gamma} = [X, Y_d]_{\beta\gamma} = [X_d, Y]_{\beta\gamma} = [X_d, Y_d]_{\beta\gamma} = 0.$$

This completes the consistency check for the group $\beta\cdot\gamma = 1$.

3.1.2. (B) $\beta\cdot\gamma = 0$ case

In this case

$$[X,D]_{\beta\gamma} = X_{\beta\gamma}(D_\gamma - D_\beta) = 0,$$

since $X_{\beta\gamma}=0$. The main part also vanishes:

$$[X,Y]_{\beta\gamma}=\sum_{\kappa\in\Delta}(X_{\beta\kappa}Y_{\kappa\gamma}-Y_{\beta\kappa}X_{\kappa\gamma})=0. \tag{3·25}$$

Suppose there exists a root κ_1 such that $\beta-\kappa_1$ and $\kappa_1-\gamma$ are roots (i.e., $\kappa_1\cdot\beta=1$ and $\kappa_1\cdot\gamma=1$), then $\kappa_2=\beta+\gamma-\kappa_1=W_\gamma(\beta-\kappa_1)$ is a root and satisfies $\beta\cdot\kappa_2=1=\kappa_2\cdot\gamma$. They always exist as a pair and their contributions cancel each other ($\beta-\kappa_2=\kappa_1-\gamma$ and $\kappa_2-\gamma=\beta-\kappa_1$):

$$g^2\left[x((\beta-\kappa_1)\cdot q)y((\kappa_1-\gamma)\cdot q)-y((\beta-\kappa_1)\cdot q)x((\kappa_1-\gamma)\cdot q)\right.$$
$$\left.+\,x((\beta-\kappa_2)\cdot q)y((\kappa_2-\gamma)\cdot q)-y((\beta-\kappa_2)\cdot q)x((\kappa_2-\gamma)\cdot q)\right]=0.$$

It is trivial to see that the other terms vanish:

$$[X_d,D]_{\beta\gamma}=[X,Y_d]_{\beta\gamma}=[X_d,Y]_{\beta\gamma}=[X_d,Y_d]_{\beta\gamma}=0.$$

This completes the consistency check for the group $\beta\cdot\gamma=0$.

3.1.3. (C) $\beta\cdot\gamma=-1$ case

This case is a little bit tricky and requires some attention. In this case again we have

$$[X,D]_{\beta\gamma}=X_{\beta\gamma}(D_\gamma-D_\beta)=0=[X_d,D]_{\beta\gamma},$$

since $X_{\beta\gamma}=0$ ($(X_d)_{\beta\gamma}=0$). But three other terms $[X,Y]$, $[X_d,Y]$, $[X,Y_d]$ have non-vanishing contributions. As for the main term

$$[X,Y]_{\beta\gamma}=\sum_{\kappa\in\Delta}(X_{\beta\kappa}Y_{\kappa\gamma}-Y_{\beta\kappa}X_{\kappa\gamma}),$$

there is only one intermediate state κ. For $\kappa\cdot\beta=1$ and $\kappa\cdot\gamma=1$ mean $\kappa\cdot(\beta+\gamma)=2$, implying $\kappa=\beta+\gamma$. The other two terms $[X_d,Y]$, $[X,Y_d]$ have only one intermediate state, too. Thus we obtain

$$[X,Y]_{\beta\gamma}=-g^2\left[x(-\gamma\cdot q)y(\beta\cdot q)-x(\beta\cdot q)y(-\gamma\cdot q)\right], \tag{3·26}$$
$$[X_d,Y]_{\beta\gamma}=-2g^2\left[x_d(\beta\cdot q)y(-(\beta+\gamma)\cdot q)-y((\beta+\gamma)\cdot q)x_d(-\gamma\cdot q)\right], \tag{3·27}$$
$$[X,Y_d]_{\beta\gamma}=-g^2\left[x((\beta+\gamma)\cdot q)y_d(-\gamma\cdot q)-y_d(\beta\cdot q)x(-(\beta+\gamma)\cdot q)\right]. \tag{3·28}$$

For the functions listed in §2 (2·5)–(2·9), the contribution of the above three terms cancel:

$$[X,Y]_{\beta\gamma}+[X_d,Y]_{\beta\gamma}+[X,Y_d]_{\beta\gamma}=0. \tag{3·29}$$

The nature of the above condition and its general solutions for the elliptic potential case will be discussed elsewhere.

3.1.4. (D) $\beta\cdot\gamma=-2$ case

In this case it is very trivial to see that every term vanishes:

$$[X,D]_{\beta\gamma}=[X,Y]_{\beta\gamma}=[X_d,D]_{\beta\gamma}=[X,Y_d]_{\beta\gamma}=[X_d,Y]_{\beta\gamma}=[X_d,Y_d]_{\beta\gamma}=0.$$

Thus the consistency of the Lax pair of the root type is proved for all the four choices of the potentials, the rational, trigonometric, hyperbolic and elliptic.

At the end of this section, let us demonstrate that the lowest conserved quantity is proportional to the Hamiltonian (2·16) up to a constant:

$$\mathrm{Tr}\,(L^2) = 2I_{\mathrm{Adj}}\mathcal{H} = 4h\mathcal{H}, \qquad (3·30)$$

in which I_{Adj} is the *second Dynkin index* for the adjoint representation and h is the Coxeter number, $r+1$, $2(r-1)$ for A_r and D_r and 12, 18 and 30 for E_6, E_7 and E_8. The second Dynkin index I_Λ of *any* representation Λ is related to the *quadratic Casimir invariant* C_Λ of the representation by

$$I_\Lambda = \frac{d_\Lambda}{d} C_\Lambda, \qquad (3·31)$$

in which d_Λ is the dimension of the representation Λ and d is the dimension of the algebra. Let us fix a root α. Let χ be the number of such roots β which have unit scalar product with α:

$$\alpha \cdot \beta = 1.$$

(By the Weyl invariance of the set of the roots, χ is the same for all roots in Δ in the simply-laced theory.) Then it is easy to see

$$\begin{aligned}\mathrm{Tr}\,(E(\alpha)E(\alpha')) &= \sum_{\beta,\kappa\in\Delta} E(\alpha)_{\beta\kappa}E(\alpha')_{\kappa\beta} = \sum_{\beta,\kappa\in\Delta} \delta_{\beta-\kappa,\alpha}\delta_{\kappa-\beta,\alpha'} \\ &= \delta_{\alpha,-\alpha'} \sum_{\beta,\kappa\in\Delta} \delta_{\beta-\kappa,\alpha} \\ &= \chi\delta_{\alpha,-\alpha'}. \end{aligned} \qquad (3·32)$$

Similarly we have

$$\mathrm{Tr}\,(E_d(\alpha)E_d(\alpha')) = \delta_{\alpha,-\alpha'}. \qquad (3·33)$$

The rest of the trace formulas are trivial:

$$\mathrm{Tr}\,(E(\alpha)) = \mathrm{Tr}\,(E_d(\alpha)) = \mathrm{Tr}\,(HE(\alpha)) = \mathrm{Tr}\,(HE_d(\alpha)) = \mathrm{Tr}\,(E(\alpha)E_d(\alpha)) = 0.$$

We evaluate

$$\begin{aligned}\mathrm{Tr}\,(L^2) &= \mathrm{Tr}\,[(p\cdot H + X + X_d)^2] \\ &= \mathrm{Tr}\,[(p\cdot H)^2] + \mathrm{Tr}\,(X^2) + \mathrm{Tr}\,(X_d^2) \\ &\quad +2\mathrm{Tr}\,(p\cdot HX) + 2\mathrm{Tr}\,(p\cdot HX_d) + 2\mathrm{Tr}\,(XX_d).\end{aligned}$$

The last three terms vanish. Next we have

$$\begin{aligned}\mathrm{Tr}\,(X^2) &= -g^2 \sum_{\alpha\in\Delta}\sum_{\alpha'\in\Delta} x(\alpha\cdot q)x(\alpha'\cdot q)\mathrm{Tr}\,(E(\alpha)E(\alpha')) \\ &= -g^2\chi \sum_{\alpha\in\Delta} x(\alpha\cdot q)x(-\alpha\cdot q)\end{aligned}$$

and

$$\mathrm{Tr}\,(X_d^2) = -4g^2 \sum_{\alpha\in\Delta} x_d(\alpha\cdot q)x_d(-\alpha\cdot q) = -4g^2 \sum_{\alpha\in\Delta} x(\alpha\cdot q)x(-\alpha\cdot q) + \text{const.}$$

In order to evaluate

$$\mathrm{Tr}\,[(p\cdot H)^2] = \sum_{\beta\in\Delta}(p\cdot\beta)^2,$$

let us choose p to be proportional to a fixed root α

$$p = \alpha|p|/\sqrt{2}.$$

Then we have

$$\mathrm{Tr}\,(p\cdot\beta)^2 = \frac{p^2}{2}\sum_{\beta\in\Delta}(\alpha\cdot\beta)^2 = \frac{p^2}{2}(2^2+2^2+\chi+\chi).$$

In the last expression the first two terms are from $\beta = \pm\alpha$ and the last two terms are the contributions from $\alpha\cdot\beta = \pm 1$. Thus we arrive at

$$\mathrm{Tr}\,(L^2) = (\chi+4)\left(p^2 - g^2\sum_{\alpha\in\Delta} x(\alpha\cdot q)x(-\alpha\cdot q)\right) + \text{const} = 2(\chi+4)\mathcal{H} + \text{const.} \tag{3·34}$$

By using the known formula

$$\chi + 4 = 2h = I_{\mathrm{Adj}} \tag{3·35}$$

we arrive at the result (3·30).

§4. Lax pair of the minimal type

In this section we present a formulation and a proof of consistency of the minimal type Lax pair. The proof is valid for all four types of potentials. This provides a unified framework for all the Calogero-Moser Lax pairs known to date. However, as we will show in a subsequent paper [18] it is possible to construct Lax pairs other than the root or the minimal types. Another motivation for this section is to show the close relationship between the exponents of the algebra and conserved quantities. This will be demonstrated explicitly in §4.4 for the D_N models. The minimal type Lax pairs have very similar forms to the root type Lax pairs. Their matrix elements are again severely constrained by the requirement of Weyl covariance.

4.1. *Minimal representations*

Let us begin with the definition of the minimal representations in the theory of Lie algebra representations. Let $\mathfrak{g}$ be a semi-simple Lie algebra (simply or non-simply laced) with rank r and the root system Δ. A *minimal* representation Λ of $\mathfrak{g}$ is an irreducible representation such that any weight $\mu\in\Lambda$ has scalar products with the roots restricted as follows:

$$\frac{2\alpha\cdot\mu}{\alpha^2} = 0, \pm 1, \qquad \forall\mu\in\Lambda \quad \text{and} \quad \forall\alpha\in\Delta. \tag{4·1}$$

The minimal representations have played important roles in various branches of physics including conformal field theory.[19] It is known that the minimal representations are characterised by the Coxeter labels and their duals. For any root $\alpha \in \Delta$ we define its dual $\alpha^\vee$ by $\alpha^\vee = 2\alpha/\alpha^2$. Next we introduce the *simple roots* $\{\alpha_1, \cdots, \alpha_r\}$ and the *fundamental weights* $\{\lambda_1, \cdots, \lambda_r\}$ as the dual basis to each other:

$$\frac{2\alpha_j \cdot \lambda_k}{\alpha_j^2} = \delta_{jk}, \quad j, k = 1, \cdots, r. \tag{4·2}$$

The Coxeter labels and their duals are the integers n_j and $n_j^\vee$ appearing in the expansion of the highest root α_0 in terms of the simple roots $\{\alpha_1, \cdots, \alpha_r\}$ and their duals:

$$\alpha_0 = \sum_{j=1}^{r} n_j \alpha_j, \qquad \alpha_0^\vee = \sum_{j=1}^{r} n_j^\vee \alpha_j^\vee. \tag{4·3}$$

A fundamental representation with the highest weight λ_j is *minimal* when the corresponding (dual) Coxeter label is unity,

$$n_j = 1 \quad \text{or} \quad n_j^\vee = 1. \tag{4·4}$$

For the A_r algebra, all the fundamental representations are minimal, $n_j = 1$, $j = 1, \cdots, r$. The Lax pairs for the A_r vector and its conjugate representation are the first known examples.[1),2)] The Lax pairs for the other fundamental representations of A_r were constructed recently by D'Hoker and Phong.[12] There are three minimal representations of D_r. The vector, spinor and anti-spinor representations. The Lax pair for the vector representation has been known for many years,[2] but those for the (anti) spinor representations are new.[12] There are three minimal representations belonging to the simply-laced exceptional algebras. The **27** and $\overline{\mathbf{27}}$ of E_6 and **56** of E_7. The Lax pairs for these representation were also constructed recently.[12] The fact that E_8 has no minimal representations is largely to be blamed for the fact that its integrability has not been understood earlier. Now the integrability of the E_8 model has been demonstrated above using the root-type Lax pair.

Among the non-simply laced algebras, the vector representations of B_r and C_r both have unit dual Coxeter labels $n_V^\vee = 1$. For these, the Lax pairs have been known for many years.[2] D'Hoker and Phong[12] constructed the Lax pair for the spinor representation of B_r which has unit dual Coxeter number $n_{\mathrm{Sp}}^\vee = 1$. The Lax pair of the spinor representation of B_r with two coupling constants can also be obtained easily by folding the minimal representation Lax pair of the spinor and anti-spinor representation of D_{r+1}, as we will show presently, see also Ref. 18). All the fundamental representations of C_r have unit dual Coxeter labels, $n_j^\vee = 1$, $j = 1, \cdots, r$. To the best of our knowledge, the Lax pair is known only for the vector representation mentioned above. It is now clear that the Lax pairs for all these representations can be easily obtained by folding the minimal representation Lax pairs of the corresponding representations of the A_{2r-1} algebra. Lax pairs corresponding to the **7** dimensional representation of G_2 and **26** of F_4 are also given in Ref. 12). These representations have unit dual Coxeter labels. The **7** dimensional representation of G_2 can be obtained by the 3-fold reduction of the vector, spinor and anti-spinor representations of D_4. Thus it can also be obtained by folding the minimal representation D_4 Lax pairs. Likewise the **26** representation of F_4 is obtained by folding the **27** and $\overline{\mathbf{27}}$ of E_6.

4.2. *Lax pair*

Next we construct a Lax pair in the minimal representation Λ

$$\Lambda = \{\mu, \nu, \rho, \cdots\}, \tag{4·5}$$

of a semi-simple *simply-laced* algebra $\mathfrak{g}$ with root system Δ of rank r. It is invariant under the Weyl group: $W_\alpha(\mu) \in \Lambda$, $\forall\mu \in \Lambda$, $\forall\alpha \in \Delta$. It is known that Λ contains no zero weights and that it consists of a single Weyl orbit. The Lax pairs have similar forms to those of the root type:

$$L(q,p) = p \cdot H + X,$$
$$M(q) = D + Y. \tag{4·6}$$

Note that, unlike the root type Lax pairs, X_d and Y_d related with the double roots do not appear. The matrices H, X and Y have the same form as before

$$X = ig \sum_{\alpha\in\Delta} x(\alpha \cdot q) E(\alpha), \quad Y = ig \sum_{\alpha\in\Delta} y(\alpha \cdot q) E(\alpha). \tag{4·7}$$

We need only functions x, y and z (no x_d, etc.) and they need only satisfy (2·14) but not (3·29). Thus, besides those listed in §2 (2·5)–(2·13), there are more choices of these functions, for example:[2)]

$$x(t) = \frac{a}{\sin at}, \quad \frac{a}{\sinh at}, \quad \frac{a}{\mathrm{sn}(at,k)}, \quad a\frac{\mathrm{cn}(at,k)}{\mathrm{sn}(at,k)}, \quad a\frac{\mathrm{dn}(at,k)}{\mathrm{sn}(at,k)} \tag{4·8}$$

for the trigonometric, hyperbolic and elliptic potentials. In this section we assume, without loss of generality, that x is an odd function while y is even:

$$x(-t) = -x(t), \quad y(-t) = y(t), \quad y(t) = x'(t).$$

The difference with the root type Lax pair is that their matrix elements are labeled by the weights instead of the roots:

$$H_{\mu\nu} = \mu\delta_{\mu,\nu}, \quad E(\alpha)_{\mu\nu} = \delta_{\mu-\nu,\alpha}.$$

In the diagonal matrix D the terms related to the double roots are dropped:

$$D_{\mu\nu} = \delta_{\mu,\nu} D_\mu, \quad D_\mu = -ig \sum_{\Delta\ni\beta=\mu-\nu} z(\beta \cdot q). \tag{4·9}$$

Here the summation is over roots β such that for $\exists\nu \in \Lambda$

$$\mu - \nu = \beta \in \Delta.$$

By multiplying β on both sides, we obtain

$$\beta^2 = 2 = \beta \cdot \mu - \beta \cdot \nu.$$

It follows from the assumption of the minimal representation that the conditions

$$\beta \cdot \mu = 1 \quad \text{and} \quad \beta \cdot \nu = -1 \tag{4·10}$$

must be met in all the terms of D_μ.

As in the case of the root type Lax pair we rewrite D as

$$D = -ig \sum_{\alpha \in \Delta} z(\alpha \cdot q) K(\alpha), \quad K(\alpha)_{\mu\nu} = \delta_{\mu,\nu} \theta(\alpha \cdot \mu), \tag{4·11}$$

in which $\theta(t)$ is defined in (3·11). Then the Weyl transformation takes the same form as for the root type Lax pair:

$$S(\beta)^{-1} E(\alpha) S(\beta) = E(W_\beta(\alpha)), \quad S(\beta)^{-1} K(\alpha) S(\beta) = K(W_\beta(\alpha)), \quad \forall \beta \in \Delta. \tag{4·12}$$

Here $S(\beta)$ is the representation matrix of the Weyl reflection W_β in the weight space Λ:

$$S(\beta)_{\mu\nu} = \delta_{\mu, W_\beta(\nu)}. \tag{4·13}$$

Thus the Weyl covariance of the Lax pair (3·15) is guaranteed.

As before we can decompose the Lax equation $\dot{L} = [L, M]$ into three parts:

$$\frac{d}{dt} X = [p \cdot H, Y], \tag{4·14}$$

$$\frac{dp}{dt} \cdot H = [X, Y]_{\text{diagonal part}}, \tag{4·15}$$

$$0 = [X, D + Y]_{\text{off-diagonal part}}. \tag{4·16}$$

The first equation (4·14) is equivalent to the first half of the canonical equations of motion (3·17) $\dot{q} = p$ and it can be shown in the same way as for the root type Lax pairs. Next we show that the second equation (4·15) is equivalent to the second half of the canonical equations of motion (3·17). The proof is valid for all four types of potentials.

$$\begin{aligned} [X, Y]_{\mu\mu} &= \sum_{\nu \in \Lambda} (X_{\mu\nu} Y_{\nu\mu} - Y_{\mu\nu} X_{\nu\mu}) \\ &= -g^2 \sum_{\nu \in \Lambda} \sum_{\alpha \in \Delta} \sum_{\beta \in \Delta} x(\alpha \cdot q) y(\beta \cdot q) \left(\delta_{\mu-\nu,\alpha} \delta_{\nu-\mu,\beta} - \delta_{\mu-\nu,\beta} \delta_{\nu-\mu,\alpha} \right) \\ &= -2g^2 \sum_{\alpha \in \Delta,\ \alpha \cdot \mu = 1} x(\alpha \cdot q) y(\alpha \cdot q), \end{aligned}$$

in which the parity of functions x (odd) and y (even) is used. Thus we obtain from (4·15):

$$\dot{p} \cdot \mu = -2g^2 \sum_{\alpha \in \Delta,\ \alpha \cdot \mu = 1} x(\alpha \cdot q) y(\alpha \cdot q). \tag{4·17}$$

On the other hand, by multiplying μ on both sides of the second Hamiltonian equation (3·17), we obtain:

$$\dot{p} \cdot \mu = -g^2 \sum_{\alpha \in \Delta} x(\alpha \cdot q) y(\alpha \cdot q) \alpha \cdot \mu.$$

By assumption of the minimal representation $\alpha\cdot\mu$ takes value 0 or ±1 and only the latter contribute:

$$\dot{p}\cdot\mu = -g^2\left(\sum_{\alpha\in\Delta,\ \alpha\cdot\mu=1} x(\alpha\cdot q)y(\alpha\cdot q) - \sum_{\alpha\in\Delta,\ \alpha\cdot\mu=-1} x(\alpha\cdot q)y(\alpha\cdot q)\right)$$
$$= -2g^2\sum_{\alpha\in\Delta,\ \alpha\cdot\mu=1} x(\alpha\cdot q)y(\alpha\cdot q), \tag{4·18}$$

in which the parity of the functions is used. Thus the equivalence to the canonical equations of motion is proved.

Next we show (4·16), or the consistency of the Lax pair. First we have ($\mu\neq\nu$)

$$[X,D]_{\mu\nu} = X_{\mu\nu}(D_\nu - D_\mu)$$
$$= g^2\sum_{\alpha\in\Delta} x(\alpha\cdot q)E(\alpha)_{\mu\nu}\left(\sum_{\beta\in\Delta,\ \beta\cdot\nu=1} z(\beta\cdot q) - \sum_{\beta\in\Delta,\ \beta\cdot\mu=1} z(\beta\cdot q)\right), \tag{4·19}$$

which vanishes if $\mu-\nu$ is not a root. Likewise the main part

$$[X,Y]_{\mu\nu}$$
$$= -g^2\sum_{\rho\in\Lambda,\ \mu-\rho\in\Delta,\ \rho-\nu\in\Delta}[x((\mu-\rho)\cdot q)y((\rho-\nu)\cdot q) - y((\mu-\rho)\cdot q)x((\rho-\nu)\cdot q)] \tag{4·20}$$

vanishes if $\mu-\nu$ is not a root. This can be shown in a similar way to (3·25). If $\mu-\nu\in\Delta$ we can use the sum rule of the functions (2·14) to obtain

$$[X,Y]_{\mu\nu} = -g^2\sum_{\rho\in\Lambda} x((\mu-\nu)\cdot q)\,[z((\rho-\nu)\cdot q) - z((\mu-\rho)\cdot q)]$$
$$= -g^2 x((\mu-\nu)\cdot q)\sum_{\rho\in\Lambda}[z((\nu-\rho)\cdot q) - z((\mu-\rho)\cdot q)]$$
$$= -g^2\sum_{\alpha\in\Delta} x(\alpha\cdot q)E(\alpha)_{\mu\nu}\left(\sum_{\beta\in\Delta,\ \beta\cdot\nu=1} z(\beta\cdot q) - \sum_{\beta\in\Delta,\ \beta\cdot\mu=1} z(\beta\cdot q)\right). \tag{4·21}$$

This cancels the above expression (4·19) and the consistency is proved.

At the end of this subsection, let us remark that the relationship between the lowest conserved quantity and the Hamiltonian (2·16) takes the same form as in the root type Lax formulation (3·30):

$$\mathrm{Tr}\,(L^2) = 2I_\Lambda\mathcal{H}, \tag{4·22}$$

in which I_Λ is as before the second Dynkin index (3·31) of the representation Λ. The derivation is similar and rather easier than the case of the root type Lax pairs and therefore it will not be repeated. One only has to note the following relation,

$$\chi_\Lambda = I_\Lambda, \tag{4·23}$$

in which χ_Λ is the number of such weights $\mu \in \Lambda$ that they have a unit scalar product with a fixed root α:

$$\alpha \cdot \mu = 1.$$

In the rest of this section we show that the minimal representations give the known examples of Calogero-Moser Lax pairs by choosing some typical cases. We also remark that the correspondence between the conserved quantities and the exponents of the algebra can be seen most clearly in the Lax pairs of the spinor and anti-spinor representations of the D_N theory.

4.3. A_{N-1} *vector representation*

We introduce an N dimensional orthonormal basis of $\mathbb{R}^N$

$$e_j \cdot e_k = \delta_{j,k}, \quad j,k = 1, \cdots, N. \tag{4·24}$$

Then the sets of roots and vector weights*) are:

$$\Delta = \{e_j - e_k : \quad j,k = 1, \cdots, N\},$$
$$\Lambda = \{e_j : \quad j = 1, \cdots, N\}. \tag{4·25}$$

The Weyl group is represented simply by a permutation of N elements:

$$S(e_j - e_k) = P(j,k), \tag{4·26}$$

in which $P(j,k)$ is the $N \times N$ matrix for permuting j and k. The matrices E and K are:

$$E(e_j - e_k)_{lm} = \delta_{j,l}\delta_{k,m}, \quad K(e_j - e_k)_{lm} = \delta_{l,m}\left(\delta_{j,l} + \delta_{k,m}\right). \tag{4·27}$$

In this basis the Lax pair takes the well-known form: [2]

$$L_{jk} = p_j\delta_{j,k} + ig(1-\delta_{j,k})\, x(q^j - q^k), \quad M_{jk} = D_j\delta_{j,k} + ig(1-\delta_{j,k})\, y(q^j - q^k),$$
$$D_j = -ig\sum_{k\neq j} z(q^j - q^k). \tag{4·28}$$

4.4. D_N

The set of roots in the above orthonormal basis (4·24) is

$$\Delta = \{e_j - e_k, \quad \pm(e_j + e_k) : \quad j,k = 1, \cdots, N\}. \tag{4·29}$$

4.4.1. Vector representation

In the vector representation Λ has $2N$ dimensions

$$\Lambda = \{e_j, -e_j : \quad j = 1, \cdots, N\}.$$

*) To be more precise, the weight is $e_j - \mu_0$, $\mu_0 = (e_1 + \cdots + e_N)/N$. We assume, without loss of generality, that the system is in the center of mass frame: $p_1 + \cdots + p_N = 0$. Then the μ_0 part does not contribute to the Lax equation or to the Hamiltonian (4·22).

The Weyl group consists of permutations of N elements and a sign change:

$$S(e_j - e_k) = 1 \otimes P(j,k), \quad S(\pm(e_j + e_k)) = P(+,-) \otimes P(j,k), \tag{4·30}$$

in the first set the permutation acts on both positive and negative weights. In the second set the positive weights and the negative weights are permuted together with an exchange of j and k. The matrices E are those given in the literature:

$$E(e_j - e_k)_{lm} = \delta_{l,j}\delta_{m,k} + \delta_{l,k+N}\delta_{m,j+N}, \tag{4·31}$$
$$E(e_j + e_k)_{lm} = \delta_{l,j}\delta_{m,k+N} + \delta_{l,k}\delta_{m,j+N}, \quad E(-e_j - e_k) = E(e_j + e_k)^T, \tag{4·32}$$
$$K(\pm e_j \pm e_k)_{lm} = \delta_{l,m}\left(\delta_{j,l} + \delta_{k,m} + \delta_{l,j+N} + \delta_{m,k+N}\right). \tag{4·33}$$

This gives the well-known Lax pair in the block notation:[2)]

$$L = \begin{pmatrix} A_1 & B_1 \\ -B_1 & -A_1 \end{pmatrix}, \quad M = \begin{pmatrix} A_2 & B_2 \\ B_2 & A_2 \end{pmatrix}, \tag{4·34}$$

in which

$$(A_1)_{jk} = p_j\delta_{j,k} + ig(1-\delta_{j,k})\,x(q^j - q^k), \quad (B_1)_{jk} = ig(1-\delta_{j,k})\,x(q^j + q^k),$$
$$(A_2)_{jk} = D_j\delta_{j,k} + ig(1-\delta_{j,k})\,y(q^j - q^k), \quad (B_2)_{jk} = ig(1-\delta_{j,k})\,y(q^j + q^k),$$
$$D_j = -ig\sum_{k\neq j}\left[z(q^j - q^k) + z(q^j + q^k)\right]. \tag{4·35}$$

4.4.2. Spinor plus anti-spinor representations

Each of the spinor and anti-spinor representations has 2^{N-1} dimensions. Instead of writing down the matrix elements of L and M in each of these representations, we choose to express the Lax pair in a more conventional form using the Pauli matrices acting on the tensor product of two component spinors, $\mathbb{C}^2\otimes\cdots\otimes\mathbb{C}^2$ (N times). It is a reducible representation of spinor⊕anti-spinor representations. The set of weights Λ has 2^N dimensions:

$$\Lambda = \left\{\frac{1}{2}\sum_{j=1}^{N}\epsilon_j e_j : \quad \epsilon_j = \pm 1, \quad j = 1,\cdots,N\right\}.$$

The matrices E and K are:

$$E(e_j - e_k) = \sigma_+^{(j)}\sigma_-^{(k)}, \quad E(e_j + e_k) = \sigma_+^{(j)}\sigma_+^{(k)}, \quad E(-e_j - e_k) = \sigma_-^{(j)}\sigma_-^{(k)}, \tag{4·36}$$

$$K(e_j - e_k) = \frac{1}{4}(1 - \sigma_z^{(j)}\sigma_z^{(k)}), \quad K(\pm(e_j + e_k)) = \frac{1}{4}(1 + \sigma_z^{(j)}\sigma_z^{(k)}). \tag{4·37}$$

In the above expressions σ_x, σ_z and $\sigma_\pm$ are Pauli sigma matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The superscripts on the Pauli matrices denote on which space to act. The Weyl group consists of permutations of N elements and an exchange of plus and minus signs:

$$S(e_j - e_k) = P(j,k), \quad S(\pm(e_j + e_k)) = P(+,-) \otimes \sigma_x^{(j)}\sigma_x^{(k)}. \tag{4·38}$$

The Lax pair is expressed simply as

$$\begin{aligned} L_{D_N} &= \frac{1}{2}\sum_{j=1}^{N} p_j \sigma_z^{(j)} + ig\sum_{j<k} x(q^j - q^k)(\sigma_+^{(j)}\sigma_-^{(k)} - \sigma_+^{(k)}\sigma_-^{(j)}) \\ &\qquad + ig\sum_{j<k} x(q^j + q^k)(\sigma_+^{(j)}\sigma_+^{(k)} - \sigma_-^{(k)}\sigma_-^{(j)}), \\ M_{D_N} &= -ig\sum_{j<k} z(q^j - q^k)\frac{1}{2}(1 - \sigma_z^{(j)}\sigma_z^{(k)}) - ig\sum_{j<k} z(q^j + q^k)\frac{1}{2}(1 + \sigma_z^{(j)}\sigma_z^{(k)}) \\ &\quad + ig\sum_{j<k} y(q^j - q^k)(\sigma_+^{(j)}\sigma_-^{(k)} + \sigma_+^{(k)}\sigma_-^{(j)}) \\ &\quad + ig\sum_{j<k} y(q^j + q^k)(\sigma_+^{(j)}\sigma_+^{(k)} + \sigma_-^{(j)}\sigma_-^{(k)}). \end{aligned} \tag{4·39}$$

The spinor and anti-spinor representations are characterised by the eigenvalues (± 1) of the following matrix Γ (γ_{2N+1}, the analogue of γ_5 in four dimensions):

$$\Gamma = \prod_{j=1}^{N} \sigma_z^{(j)}, \quad \Gamma^2 = 1, \quad \Gamma^\dagger = \Gamma. \tag{4·40}$$

It commutes with all the matrices appearing in L and M. Thus by using projectors

$$P_\pm = \frac{1}{2}(1 \pm \Gamma), \quad P_\pm^\dagger = P_\pm^2 = P_\pm, \tag{4·41}$$

the Lax pairs in the spinor and anti-spinor representations are obtained as

$$LP_\pm \quad \text{and} \quad MP_\pm.$$

Thus we have two sets of conserved quantities for the D_N Calogero-Moser model,

$$\mathrm{Tr}\,(L^k P_+) \quad \text{and} \quad \mathrm{Tr}\,(L^k P_-), \quad k = 1, \cdots, \tag{4·42}$$

derived from the spinor and anti-spinor representations, respectively. However, for most values of k, except for N, they give the same conserved quantities since the difference vanishes:

$$\mathrm{Tr}\,(L^k \Gamma) = 0, \quad \text{except for} \quad k = N. \tag{4·43}$$

Among the conserved quantities $\mathrm{Tr}\,(L^k)$ of Toda theories and Calogero-Moser models based an a Lie algebra $\mathfrak{g}$, the independent ones are known to occur at k equal to the *exponent* of $\mathfrak{g}$ plus 1. (At the other values of k, $\mathrm{Tr}\,(L^k)$ either vanishes or is a polynomial of the lower order conserved quantities.) For every Lie algebra 1 is always an exponent. This corresponds to the universal fact that the lowest conserved quantity $\mathrm{Tr}(L^2)$ is (proportional to) the Hamiltonian (3·30), (4·22). For D_N the exponents are $(1, 3, \cdots, 2N-3, N-1)$ and in the present case the exponents $(1, 3, \cdots, 2N-3)$ correspond to the conserved quantities $\mathrm{Tr}(L^{2k})$, $(k = 1, 2, \cdots, N-1)$ obtained in the spinor or anti-spinor representations; the exponent $N-1$ corresponds to the extra conserved quantity derived above

$$\mathrm{Tr}(L^N \Gamma). \tag{4·44}$$

4.5. *E_6 and E_7*

The **27** and $\overline{\mathbf{27}}$ dimensional representations of E_6 are minimal. In both cases Λ is decomposed into **1** + **10** + **16** or the singlet plus the vector plus the spinor representations of D_5.

The minimal **56** dimensional representation of E_7 is decomposed into **12** + **32** + **12**. That is the sum of two vector representations and a spinor representation of D_6.

In both cases the structure of the Lax pair in each sector is described as above. We have not yet found a more succinct way of representing their Lax pair than the general form of the minimal type (4·6).

Let us give some simple examples of the Lax pairs of the non-simply laced algebras obtained by reduction (folding) of the minimal representation ones for the simply-laced algebras.

4.6. *Some B_N Lax pairs by reduction*

It is possible to obtain $2N+2$ dimensional representation of the B_N Lax pair in the vector representation. One only has to impose restrictions on the dynamical variables in the Lax pair of D_{N+1} in the vector representation:

$$q^{N+1} = p_{N+1} = 0. \tag{4·45}$$

This representation can be easily reduced to the well-known one[2] in $2N+1$ dimensions with the coupling constant of the short roots given by $g_1 = \sqrt{2}g$.

It is more interesting to derive the Lax pair of the spinor representation of B_N from the spinor⊕anti-spinor representation of D_{N+1} given above. Together with the restriction of the dynamical variables as above (4·45) one can also impose $\sigma^{(N+1)} \to 1$ to obtain the 2^N dimensional representation:

$$L_{B_N} = L_{D_N} + ig_1 \sum_{j=1}^{N} x(q^j)(\sigma_+^{(j)} - \sigma_-^{(j)}), \quad M_{B_N} = M_{D_N} + ig_1 \sum_{j=1}^{N} y(q^j)(\sigma_+^{(j)} + \sigma_-^{(j)}). \tag{4·46}$$

It is elementary to verify that the coupling constant of the short roots g_1 can be independent of g. (The reduction itself gives the relation $g_1 = g$.)

4.7. *Vector representation of C_N Lax pair by reduction*

The simplest example of a Lax pair derived by reduction is that of the vector representation of C_N. Starting from the A_{2N-1} vector representation Lax pair and imposing restrictions on the dynamical variables:

$$q^{2N+1-j} = -q^j, \quad p_{2N+1-j} = -p_j, \quad j = 1, \cdots, N, \tag{4·47}$$

we obtain the well-known form of the Lax pair:[2)]

$$L = \begin{pmatrix} A_1 & B_1 \\ -B_1 & -A_1 \end{pmatrix}, \quad M = \begin{pmatrix} A_2 & B_2 \\ B_2 & A_2 \end{pmatrix}, \tag{4·48}$$

in which

$$\begin{aligned}
(A_1)_{jk} &= p_j\delta_{j,k} + ig(1-\delta_{j,k})\, x(q^j - q^k), \\
(B_1)_{j,k} &= ig(1-\delta_{j,k})\, x(q^j + q^k) + ig_4\, x(2q^j)\delta_{j,k}, \\
(A_2)_{jk} &= D_j\delta_{j,k} + ig(1-\delta_{j,k})\, y(q^j - q^k), \\
(B_2)_{jk} &= ig(1-\delta_{j,k})\, y(q^j + q^k) + ig_4\, y(2q^j)\delta_{j,k}, \\
D_j &= -ig\sum_{k\neq j}\Big[z(q^j - q^k) + z(q^j + q^k)\Big] - ig_4\, z(2q^j).
\end{aligned} \tag{4·49}$$

In this case the representation space has dimension $2N$ and it is elementary to verify that the coupling constant of the long roots g_4 can be independent of g.

Other types of reductions and the formulation of minimal representation in non-simply laced algebra will be discussed elsewhere.[18)]

§5. Summary and comments

A simple and *universal Lax pair* for the Calogero-Moser models based on any semi-simple Lie algebras, including E_8, is presented. It is based on the root system and Weyl invariance only, suggesting the possibility of generalising Calogero-Moser models to a wider class of root systems beyond those associated with Lie algebras. The key idea is the representation of the Weyl reflections on the set of roots itself for the root type Lax pair. Thus it is applicable, in principle, to all Calogero-Moser models. The proof of the consistency of the Lax pair is elementary and it has been checked for all the four types of interaction potentials. As for the representation of the Weyl reflections, the root type Lax pair is conceptually better than the adjoint representation which consists of the set of roots and the zero weights corresponding to the Cartan subalgebra. If the zero weights were included, the representation matrices of Weyl reflections could not be unique on them. This does not mean, however, that the Lax pairs in the adjoint representations do not exist. We will report some examples of Calogero-Moser Lax pairs in adjoint representations and symmetric tensor representations in a subsequent paper.[18)]

Another type of Calogero-Moser Lax pair, called minimal type, is introduced. The minimal types provide a unified description of all Calogero-Moser Lax pairs known to date and reveals some new ones. Lax pairs belonging to the minimal type of non-simply laced theories are related to those of the simply-laced theories by reduction. The spinor⊕anti-spinor representations of D_N models are discussed in some detail in connection with an alternative representation of the conserved quantities and with the reduction to the B_N spinor representation.

Since the non-Lie algebraic aspects of the Calogero-Moser models are highlighted, it would be interesting to see if these models could be obtained by reduction of self-dual Yang-Mills equations related with some Lie algebras.[20)]

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