

Calogero-Moser Models. II

— *Symmetries and Foldings* —

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(Received October 26, 1998)

Universal Lax pairs (the root type and the minimal type) are presented for Calogero-Moser models based on simply laced root systems, including E_8 . They exist with and without spectral parameters and they work for all of the four choices of potentials: the rational, trigonometric, hyperbolic and elliptic. For the elliptic potential, the discrete symmetries of the simply laced models, originating from the automorphism of the extended Dynkin diagrams, are combined with the periodicity of the potential to derive a class of Calogero-Moser models known as the ‘twisted non-simply laced models’. For untwisted non-simply laced models, two kinds of root type Lax pairs (based on long roots and short roots) are derived which contain independent coupling constants for the long and short roots. The BC_n model contains three independent couplings, for the long, middle and short roots. The G_2 model based on long roots exhibits a new feature which deserves further study.

§1. Introduction

In a previous paper¹⁾ a new and universal formulation of Lax pairs of Calogero-Moser models based on simply laced root systems was presented. This paper is devoted to further developments and refinements of the Lax pairs²⁾⁻⁵⁾ and the Calogero-Moser models themselves, with an emphasis on the symmetries of the simply laced as well as the twisted and untwisted non-simply laced models. The Calogero-Moser models⁶⁾ are a collection of completely integrable one-dimensional dynamical systems characterised by root systems and a choice of four long-range interaction potentials: (i) $1/L^2$, (ii) $1/\sin^2 L$, (iii) $1/\sinh^2 L$ and (iv) $\wp(L)$, where L is the inter-particle “distance”.

Besides various direct applications of the models to lower dimensional physics ranging from solid state to particle physics,⁷⁾ elliptic Calogero-Moser models are attracting attention owing to their connection with (supersymmetric) gauge theory, classical soliton dynamics,³⁾ Toda theories and infinite dimensional algebras. The Seiberg-Witten curve and differential and $\mathcal{N} = 2$ supersymmetric gauge theory are analysed in terms of elliptic Calogero-Moser models with the same Lie algebra.^{8), 11)} The untwisted and twisted Calogero-Moser models are known to reduce to Toda models in a certain limit.^{4), 5)} The affine algebras acting on Toda models are relatively well understood. This fosters the expectation that the elliptic Calogero-Moser models (with Lie algebraic properties from the root system and toroidal properties from the potential) open a way to a greater symmetry algebra than the affine algebras.¹²⁾

In this paper we address the problem of the symmetries of the Calogero-Moser

models and the associated Lax pairs, in particular, the amalgamation of the Lie algebraic properties originating from the root structure and the toroidal properties from the elliptic potential. As a first step we present the universal Lax pairs with and without spectral parameters for all four choices of potential for Calogero-Moser models based on simply laced root systems. There are two types of universal Lax pairs, the root type and the minimal type.¹⁾ The root type Lax pair is represented on the set of roots itself. It is intrinsic to the root system and it applies to all of the models based on root systems, including E_8 , for which construction of a Lax pair had been a mystery for more than twenty years. The minimal type Lax pair is represented on the set of weights belonging to a minimal representation.^{1), 13)} Every Lie algebra, except for E_8 , has at least one minimal representation. The minimal type Lax pair provides a unified description of all known examples of Calogero-Moser Lax pairs and adds some new ones.^{1)–5)}

As a second step, we uncover a discrete symmetry of elliptic Calogero-Moser models based on simply laced root systems. All simply laced root systems, except for E_8 , have a symmetry under the automorphism(s) of the Dynkin diagram or its extended version. By combining the symmetry under the automorphism with the periodicity of the elliptic potential, a non-trivial discrete symmetry of the models is obtained. New integrable dynamical systems can be derived from the elliptic Calogero-Moser models by restricting the dynamical variables to the invariant subspace of the discrete symmetry. This process is known as ‘reduction’ or ‘folding’. It is an important and useful tool in Toda lattices and field theories,^{14), 15)} another class of integrable models based on root systems. In the present case we obtain so-called twisted non-simply laced Calogero-Moser models.

The untwisted non-simply laced models can also be obtained by folding the simply laced models.¹⁾ In the reduced models, however, the coupling constants for the long and short roots have a fixed ratio, since the simply laced models have only one coupling. In order to exhibit the fuller symmetry of the untwisted non-simply laced models, root type Lax pairs with independent coupling constants are constructed as a third step. There are two kinds of root type Lax pairs for non-simply laced models, one based on long roots and another on short roots. Both are straightforward generalisations of the root type Lax pair for simply laced systems, except for the G_2 case based on long roots. This case requires a new set of functions in the Lax pair. A simple example of the new set of functions is given. The BC_n model contains three independent couplings, for the long, middle and short roots.

This paper is organised as follows. In §2 we present the universal Lax pairs with and without spectral parameters for all four choices of potential for Calogero-Moser models based on simply laced root systems. In §3 certain discrete symmetries of Calogero-Moser models based on simply laced root systems are introduced. Twisted non-simply laced Calogero-Moser models are derived by folding with respect to this symmetry. In §4 two kinds of root type Lax pairs with independent coupling constants, one based on long roots and the other on short roots, are constructed for all of the untwisted non-simply laced models. The BC_n model has three independent couplings. Section 5 is devoted to summary and comments.

§2. Universal Lax pairs for Calogero-Moser models based on simply laced algebras

In order to set the stage and introduce notation, let us recapitulate our previous results on the universal Lax pairs for the Calogero-Moser models based on a *simply laced* root system Δ . For the elliptic potential, the universal Lax pairs without a spectral parameter were reported in a previous paper.¹⁾ Here we include those with a spectral parameter.

The basic ingredient of the model is a root system Δ associated with semi-simple and *simply laced* Lie algebra \mathfrak{g} with rank r . The roots $\alpha, \beta, \gamma, \dots$ are real r dimensional vectors and are normalised, without loss of generality, to 2:

$$\Delta = \{\alpha, \beta, \gamma, \dots\}, \quad \alpha \in \mathbb{R}^r, \quad \alpha^2 = \alpha \cdot \alpha = 2, \quad \forall \alpha \in \Delta. \quad (2.1)$$

We denote by Dim the total number of roots of Δ .

The dynamical variables are the canonical coordinates $\{q^j\}$ and their canonical conjugate momenta $\{p_j\}$ with the Poisson brackets:

$$q^1, \dots, q^r, \quad p_1, \dots, p_r, \quad \{q^j, p_k\} = \delta_{j,k}, \quad \{q^j, q^k\} = \{p_j, p_k\} = 0. \quad (2.2)$$

In most cases we denote them by r dimensional vectors q and p ,^{*)}

$$q = (q^1, \dots, q^r) \in \mathbb{R}^r, \quad p = (p_1, \dots, p_r) \in \mathbb{R}^r,$$

so that the scalar products of q and p with the roots $\alpha \cdot q$, $p \cdot \beta$, etc., can be defined. The Hamiltonian is given by (g is a real coupling constant)

$$\mathcal{H} = \frac{1}{2}p^2 - \frac{g^2}{2} \sum_{\alpha \in \Delta} x(\alpha \cdot q)x(-\alpha \cdot q), \quad (2.3)$$

in which $x(t)$ is given (2.11)–(2.19) for various choices of potentials.

As is well known, with the help of a Lax pair, L and M , which expresses the canonical equation of motion derived from the Hamiltonian (2.3) in an equivalent matrix form,

$$\dot{L} = \frac{d}{dt}L = [L, M], \quad (2.4)$$

a sufficient number of conserved quantities can be obtained from the trace:

$$\frac{d}{dt} \text{Tr}(L^k) = 0, \quad k = 1, \dots. \quad (2.5)$$

Two types of universal Lax pairs, the root type and the minimal type, were constructed. The matrices used in the root type Lax pair bear a resemblance to the adjoint representation of the associated Lie algebra, and they exist for all models. Thus the root type Lax pair provides a universal tool for proving the integrability

^{*)} For A_r models, it is customary to introduce one more degree of freedom, q^{r+1} and p_{r+1} and embed all of the roots in \mathbb{R}^{r+1} .

of Calogero-Moser models. The ‘minimal’ types provide a unified description of all known examples of Calogero-Moser Lax pairs. They are based on the set of weights of minimal representations of the associated Lie algebras. The important guiding principle for deriving these Lax pairs is the Weyl invariance of the set of roots Δ and of the Hamiltonian. For details, see Ref. 1).

2.1. *Root type Lax pair*

The detailed structure of the simply laced root system Δ is very different from one type of algebra to another, so it is hardly universal. One universal feature is the root difference pattern, i.e., which multiples of roots appear in the difference of two roots:

$$\text{Simply laced root system : } \quad \text{root} - \text{root} = \begin{cases} \text{root,} \\ 2 \times \text{root,} \\ \text{non-root.} \end{cases} \quad (2.6)$$

To be more specific, there can be no terms like $3 \times \text{root}$, etc. on the right-hand side. This then determines the root type Lax pair for simply laced root systems (we choose L to be hermitian and M anti-hermitian):

$$\begin{aligned} L(q, p, \xi) &= p \cdot H + X + X_d, \\ M(q, \xi) &= D + Y + Y_d. \end{aligned} \quad (2.7)$$

Here ξ is a spectral parameter, which is relevant only for the elliptic potential. In Eq. (2.7), L, H, X, X_d, D, Y and Y_d are $Dim \times Dim$ matrices whose indices are labelled by the roots themselves, denoted here by $\alpha, \beta, \gamma, \eta$ and κ . H and D are diagonal:

$$H_{\beta\gamma} = \beta\delta_{\beta,\gamma}, \quad D_{\beta\gamma} = \delta_{\beta,\gamma}D_\beta, \quad D_\beta = -ig \left(z(\beta \cdot q) + \sum_{\kappa \in \Delta, \kappa \cdot \beta=1} z(\kappa \cdot q) \right). \quad (2.8)$$

X and Y correspond to the first line of (2.6):

$$X = ig \sum_{\alpha \in \Delta} x(\alpha \cdot q, \xi)E(\alpha), \quad Y = ig \sum_{\alpha \in \Delta} y(\alpha \cdot q, \xi)E(\alpha), \quad E(\alpha)_{\beta\gamma} = \delta_{\beta-\gamma,\alpha}. \quad (2.9)$$

X_d and Y_d are associated with the ‘double root’ in the second line of (2.6):

$$X_d = 2ig \sum_{\alpha \in \Delta} x_d(\alpha \cdot q, \xi)E_d(\alpha), \quad Y_d = ig \sum_{\alpha \in \Delta} y_d(\alpha \cdot q, \xi)E_d(\alpha), \quad E_d(\alpha)_{\beta\gamma} = \delta_{\beta-\gamma,2\alpha}. \quad (2.10)$$

The matrix $E(\alpha)$ ($E_d(\alpha)$) might be called a (double) root discriminator. It takes the value 1 only when the difference of the two indices is equal to (twice) the root α . These matrices correspond to the first and the second line of (2.6), respectively. In §4 we encounter a triple root discriminator corresponding to the ‘ $3 \times$ short root’ part of (4.57). The functions x, y, z (x_d, y_d, z_d) depend on the choice of the inter-particle

potential. For the rational potential, $1/L^2$, they are

$$x(t) = x_d(t) = \frac{1}{t}, \quad y(t) = y_d(t) = -\frac{1}{t^2}, \quad z(t) = z_d(t) = -\frac{1}{t^2}, \quad (2.11)$$

for the trigonometric potential, $1/\sin^2 L$, they are

$$\begin{aligned} x(t) = x_d(t) &= a \cot at, & y(t) = y_d(t) &= -\frac{a^2}{\sin^2 at}, \\ z(t) = z_d(t) &= -\frac{a^2}{\sin^2 at}, & a &: \text{const}, \end{aligned} \quad (2.12)$$

for the hyperbolic potential, $1/\sinh^2 L$, they are

$$\begin{aligned} x(t) = x_d(t) &= a \coth at, & y(t) = y_d(t) &= -\frac{a^2}{\sinh^2 at}, \\ z(t) = z_d(t) &= -\frac{a^2}{\sinh^2 at}, & a &: \text{const}. \end{aligned} \quad (2.13)$$

For the elliptic potential, $\wp(L)$, the functions x and x_d generally differ. There are several choices of the functions. They are related to each other by a modular transformation. A first choice is ^{*})

$$\begin{aligned} x(t) &= \frac{c}{2} \left[\frac{1 + k \operatorname{sn}^2(ct/2, k)}{\operatorname{sn}(ct/2, k)} - i \frac{(1+k)(1 - k \operatorname{sn}^2(ct/2, k))}{\operatorname{cn}(ct/2, k) \operatorname{dn}(ct/2, k)} \right], \\ y(t) = x'(t), \quad z(t) &= -\wp(t), \quad c = \sqrt{e_1 - e_3}, \end{aligned} \quad (2.14)$$

and

$$x_d(t) = \frac{c}{\operatorname{sn}(ct, k)}, \quad y_d(t) = -c^2 \frac{\operatorname{cn}(ct, k) \operatorname{dn}(ct, k)}{\operatorname{sn}^2(ct, k)}, \quad z_d(t) = -\wp(t), \quad (2.15)$$

in which k is the modulus of the elliptic function. This set of functions ^{**)} is obtained by setting $\xi = \omega_3$ in the spectral parameter dependent functions (2.22) for $j = 3$.

A second choice is

$$\begin{aligned} x(t) &= \frac{c}{2} \left[\frac{\operatorname{cn}^2(ct/2, k) - k' \operatorname{sn}^2(ct/2, k)}{\operatorname{sn}(ct/2, k) \operatorname{cn}(ct/2, k)} + (1+k') \frac{\operatorname{cn}^2(ct/2, k) + k' \operatorname{sn}^2(ct/2, k)}{\operatorname{dn}(ct/2, k)} \right], \\ y(t) = x'(t), \quad z(t) &= -\wp(t), \end{aligned} \quad (2.16)$$

and

$$x_d(t) = c \frac{\operatorname{cn}(ct, k)}{\operatorname{sn}(ct, k)}, \quad y_d(t) = -c^2 \frac{\operatorname{dn}(ct, k)}{\operatorname{sn}^2(ct, k)}, \quad z_d(t) = -\wp(t), \quad (2.17)$$

in which $k' = \sqrt{1 - k^2}$.

^{*}) We denote the fundamental periods of the Weierstrass' functions by $\{2\omega_1, 2\omega_3\}$ and $e_j = \wp(\omega_j)$, $j = 1, \dots, 3$.

^{**)} The detailed properties of the functions in the elliptic potential cases will be discussed elsewhere.

A third choice is

$$\begin{aligned} x(t) &= \frac{c}{2} \left[\frac{\operatorname{dn}^2(ct/2, k) + ikk' \operatorname{sn}^2(ct/2, k)}{\operatorname{sn}(ct/2, k) \operatorname{dn}(ct/2, k)} + \frac{k \operatorname{cn}^2(ct/2, k) - ik'}{\operatorname{cn}(ct/2, k)} \right], \\ y(t) &= x'(t), \quad z(t) = -\wp(t), \end{aligned} \tag{2.18}$$

and

$$x_d(t) = c \frac{\operatorname{dn}(ct, k)}{\operatorname{sn}(ct, k)}, \quad y_d(t) = -c^2 \frac{\operatorname{cn}(ct, k)}{\operatorname{sn}^2(ct, k)}, \quad z_d(t) = -\wp(t). \tag{2.19}$$

For the elliptic Lax pair with spectral parameter we find several sets of functions which are closely related to each other. The first set is

$$\begin{aligned} x(t, \xi) &= \frac{\sigma(\xi/2 - t)}{\sigma(\xi/2)\sigma(t)}, \quad y(t, \xi) = x(t, \xi) [\zeta(t - \xi/2) - \zeta(t)], \\ z(t, \xi) &= -[\wp(t) - \wp(\xi/2)], \end{aligned} \tag{2.20}$$

$$\begin{aligned} x_d(t, \xi) &= \frac{\sigma(\xi - t)}{\sigma(\xi)\sigma(t)}, \quad y_d(t, \xi) = x_d(t, \xi) [\zeta(t - \xi) - \zeta(t)], \\ z_d(t, \xi) &= -[\wp(t) - \wp(\xi)]. \end{aligned} \tag{2.21}$$

Here σ and ζ are Weierstrass' sigma and zeta functions. The other sets of functions are related to the above one by simple shifts of the parameter ξ and the 'gauge transformation' (2.27) explained below:

$$\begin{aligned} x(t, \xi) &= \frac{\sigma(\xi/2 + \omega_j - t)}{\sigma(\xi/2 + \omega_j)\sigma(t)} \exp[t(\eta_j + \zeta(\xi)/2)], & \eta_j &= \zeta(\omega_j), \\ y(t, \xi) &= x'(t, \xi), \quad z(t, \xi) = -[\wp(t) - \wp(\xi/2 + \omega_j)], & j &= 1, 2, 3, \\ x_d(t, \xi) &= \frac{\sigma(\xi - t)}{\sigma(\xi)\sigma(t)} \exp[t\zeta(\xi)], \\ y_d(t, \xi) &= x'_d(t, \xi), \quad z_d(t, \xi) = -[\wp(t) - \wp(\xi)]. \end{aligned} \tag{2.22}$$

$$\tag{2.23}$$

In the trigonometric and hyperbolic functions the constant a is a free parameter setting the scale of the theory. One obtains the rational potential in the limit $a \rightarrow 0$. The trigonometric ($k \rightarrow 0$) and hyperbolic ($k \rightarrow 1$) limits of the elliptic cases give other sets of functions for these cases. One important property is that they all satisfy the *sum rule*

$$y(u)x(v) - y(v)x(u) = x(u + v)[z(u) - z(v)], \quad u, v \in \mathbb{C}. \tag{2.24}$$

The functions x_d, y_d and z_d satisfy the same relations, including those containing the spectral parameter. These functions also satisfy a *second sum rule*

$$\begin{aligned} x(-v)y(u) - x(u)y(-v) + 2[x_d(u)y(-u - v) - y(u + v)x_d(-v)] \\ + x(u + v)y_d(-v) - y_d(u)x(-u - v) = 0, \end{aligned} \tag{2.25}$$

which is essentially the same as the condition (3.29) in Ref. 1). In all of these cases the inter-particle potential V is proportional to $-z + \text{const}$ (see the Hamiltonian (2.3)), and y (y_d) is the derivative of x (x_d), and z is always an even function:

$$y(t) = x'(t), \quad z(t) = x(t)x(-t) + \text{const}, \quad z(-t) = z(t). \tag{2.26}$$

It should be remarked that the set of functions $\{x(t), x_d(t)\}$ has a kind of ‘gauge freedom’. If $\{x(t), x_d(t)\}$ satisfies the first and the second sum rules, then

$$\{\tilde{x}(t) = x(t)e^{tb}, \quad \tilde{x}_d(t) = x_d(t)e^{2tb}\} \tag{2.27}$$

also satisfies the same sum rules. Here b is an arbitrary t -independent constant, which can depend on ξ . The function $z(t)$ is gauge invariant.

For the rational (2.11), trigonometric (2.12) and hyperbolic (2.13) cases, x is an odd function and y is an even function, but they do not have definite parity for the elliptic potentials (2.14)–(2.22). The Hamiltonian (2.3) is proportional to the lowest conserved quantity up to a constant:

$$\text{Tr}(L^2) = 2I_{\text{Adj}}\mathcal{H} = 4h\mathcal{H}, \tag{2.28}$$

where I_{Adj} is the *second Dynkin index* for the adjoint representation and h is the Coxeter number.

2.2. Minimal type Lax pair

The minimal type Lax pair is represented in the set of weights of a minimal representation

$$\Lambda = \{\mu, \nu, \rho, \dots\}, \tag{2.29}$$

of a semi-simple *simply laced* algebra \mathfrak{g} with root system Δ of rank r . It is characterised^{1), 13)} by the condition that any weight $\mu \in \Lambda$ has scalar products with the roots restricted as follows:

$$\frac{2\alpha \cdot \mu}{\alpha^2} = 0, \pm 1, \quad \forall \mu \in \Lambda \quad \text{and} \quad \forall \alpha \in \Delta. \tag{2.30}$$

Corresponding to (2.6), we have the following universal pattern for the minimal representations:

$$\text{Minimal Representation :} \quad \text{weight} - \text{weight} = \begin{cases} \text{root,} \\ \text{non-root.} \end{cases} \tag{2.31}$$

Due to the definition of the minimal weights (2.30), there can be no terms like $2 \times \text{root}$, etc., on the right-hand side of (2.31). This determines the structure of the minimal type Lax pairs:

$$\begin{aligned} L(q, p, \xi) &= p \cdot H + X, \\ M(q, \xi) &= D + Y. \end{aligned} \tag{2.32}$$

The matrices H , X and Y have the same form as before,

$$X = ig \sum_{\alpha \in \Delta} x(\alpha \cdot q, \xi)E(\alpha), \quad Y = ig \sum_{\alpha \in \Delta} y(\alpha \cdot q, \xi)E(\alpha), \tag{2.33}$$

corresponding to the first line of (2.31). We need only the functions x , y and z (no x_d , etc.) and they need only satisfy (2.24) but not (2.25). Thus, along with

those listed in §2 (2.11)–(2.19), there are additional choices of these functions, for example: ²⁾

$$\begin{aligned}
 x(t) &= \frac{a}{\sin at}, & \frac{a}{\sinh at}, & \quad a : \text{const}, & \quad \frac{c}{\text{sn}(ct, k)}, & \quad c \frac{\text{cn}(ct, k)}{\text{sn}(ct, k)}, & \quad c \frac{\text{dn}(ct, k)}{\text{sn}(ct, k)}, \\
 c &= \sqrt{e_1 - e_3}
 \end{aligned}
 \tag{2.34}$$

for the trigonometric, hyperbolic and elliptic potentials. As in the case of the root type Lax pair, the three choices of functions for the elliptic potentials are related to each other by modular transformations. For the elliptic Lax pair with spectral parameter: ^{3), 5)}

$$\begin{aligned}
 x(t, \xi) &= \frac{\sigma(\xi - t)}{\sigma(\xi)\sigma(t)}, & y(t, \xi) &= x(t, \xi) [\zeta(t - \xi) - \zeta(t)], \\
 z(t, \xi) &= -(\wp(t) - \wp(\xi)).
 \end{aligned}
 \tag{2.35}$$

The difference with the root type Lax pair is that their matrix elements are labeled by the weights instead of the roots:

$$H_{\mu\nu} = \mu\delta_{\mu,\nu}, \quad E(\alpha)_{\mu\nu} = \delta_{\mu-\nu,\alpha}.$$

In the diagonal matrix D , the terms related to the double roots are dropped:

$$D_{\mu\nu} = \delta_{\mu,\nu}D_\mu, \quad D_\mu = -ig \sum_{\Delta \ni \beta = \mu - \nu} z(\beta \cdot q).
 \tag{2.36}$$

Here the summation is over roots β such that for $\exists \nu \in \Lambda$

$$\mu - \nu = \beta \in \Delta.$$

The Hamiltonian (2.3) is proportional to the lowest conserved quantity for the minimal type Lax pair, too:

$$\text{Tr}(L^2) = 2I_\Lambda \mathcal{H}.
 \tag{2.37}$$

Here I_Λ is the second Dynkin index (2.28) of the representation Λ . For the proof of the equivalence of the Lax equation

$$\dot{L} = \frac{d}{dt}L = [L, M]$$

and the canonical equation for the Hamiltonian (2.3) see Ref. 1).

§3. Symmetries and reductions of elliptic Calogero-Moser models

In this section we discuss the symmetries of the Calogero-Moser models with the elliptic potential

$$\mathcal{H} = \frac{1}{2}p^2 + \frac{g^2}{2} \sum_{\alpha \in \Delta} \wp(\alpha \cdot q),
 \tag{3.1}$$

based on the root system Δ of a semi-simple simply laced algebra \mathfrak{g} . As is well known, the root system Δ is characterised by its Dynkin diagram. The Dynkin diagrams (and the extended ones with the affine root attached) of simply laced algebras have various *automorphisms* A that map one root to another:

$$A\alpha \in \Delta, \quad \forall \alpha \in \Delta. \quad (3.2)$$

Thus by combining transformation by an automorphism with the periodicity of the elliptic potential, we find that the above Hamiltonian (3.1) is invariant under the following discrete transformation of the dynamical variables:

$$\begin{aligned} q &\rightarrow q' = Aq + 2\omega\lambda, \\ p &\rightarrow p' = Ap, \end{aligned} \quad (3.3)$$

where 2ω is any one of the periods ($2\omega_1, 2\omega_2, 2\omega_3$) of the Weierstrass elliptic function \wp and λ is an arbitrary element of the *weight lattice*, that is, it satisfies

$$\alpha \cdot \lambda \in \mathbb{Z}, \quad \forall \alpha \in \Delta.$$

By restricting the dynamical variables to the *invariant subspace* of the transformation

$$\begin{aligned} q &= Aq + 2\omega\lambda, \\ p &= Ap, \end{aligned} \quad (3.4)$$

we obtain a *reduced model* of the elliptic Calogero-Moser model. In terms of the roots, this corresponds to folding the simply laced root system Δ by the automorphism A (see Ref. 14) for the corresponding examples in Toda theories). If we choose the automorphism of the ordinary (un-extended) Dynkin diagram A_u , an untwisted non-simply laced root system is obtained. For the automorphism of the extended Dynkin diagram A_e one obtains a twisted non-simply laced root system. According to the nature of the automorphism A and the choice of λ we have the following two different cases:

(i) **Untwisted non-simply laced model.** We choose the automorphism of the unextended Dynkin diagram A_u and $\lambda \equiv 0$. Since $\lambda \equiv 0$, the periodicity is irrelevant and the model is defined for all four choices of the potential. This gives the well known Calogero-Moser models based on untwisted non-simply laced root systems. Various examples of this reduction for minimal type Lax pairs were presented in a previous paper.¹⁾ The Lax pair for these reduced models with as many independent coupling constants as independent Weyl orbits in the root system will be fully discussed in §4.

(ii) **Twisted non-simply laced model.** Let us choose the automorphism of the extended Dynkin diagram A_e and some special weight λ (in most cases it is a minimal weight λ_{\min} or a linear combination of them). In this case some of the roots vanish in the invariant subspace.^{14), 15)} In order to avoid the singularity of the elliptic function, a non-vanishing weight vector λ is necessary and it should have non-vanishing scalar products with the roots that are mapped to zero. Some of these models have been introduced in 5) in a different context. A twisted BC_n model (3.61) is obtained by

the folding $D_{2n+2}^{(1)} \rightarrow A_{2n}^{(2)}$. (In this paper we use notation like $D_{2n}^{(1)}$ only to indicate the extended Dynkin diagram but not the affine Lie algebra.) This will be derived in §3.5. One might be tempted to combine the automorphism of the unextended Dynkin diagram A_u with a non-vanishing weight λ . So far as we have tried this does not lead to a new integrable model.

It should be noted that all of the models derived in this section are a subsystem of the Calogero-Moser models based on simply laced root systems. Thus the integrability of these models is inherited from the original models.

In the remainder of this section we consider the reduction by automorphisms A_e of the extended Dynkin diagrams. The corresponding reductions of the Dynkin diagrams are

$$\begin{aligned} D_{2n}^{(1)} &\rightarrow A_{2n-1}^{(2)}, & D_{n+2}^{(1)} &\rightarrow D_{n+1}^{(2)}, & E_7^{(1)} &\rightarrow E_6^{(2)}, \\ E_6^{(1)} &\rightarrow D_4^{(3)} & \text{and} & & D_{2n+2}^{(1)} &\rightarrow A_{2n}^{(2)}. \end{aligned} \tag{3.5}$$

These automorphisms, which we denote by A for brevity, satisfy

$$A^2 = 1 \quad \text{for} \quad D_{2n}^{(1)} \rightarrow A_{2n-1}^{(2)}, \quad D_{n+2}^{(1)} \rightarrow D_{n+1}^{(2)}, \quad E_7^{(1)} \rightarrow E_6^{(2)}, \tag{3.6}$$

$$A^3 = 1 \quad \text{for} \quad E_6^{(1)} \rightarrow D_4^{(3)}, \quad A^4 = 1 \quad \text{for} \quad D_{2n+2}^{(1)} \rightarrow A_{2n}^{(2)}. \tag{3.7}$$

For these automorphisms we consider Eq. (3.4) determining the invariant subspace of the discrete transformation (3.3). The projector to the invariant subspace is given by

$$Pr = \frac{1}{2}(1 + A), \quad Pr = \frac{1}{3}(1 + A + A^2) \quad \text{and} \quad Pr = \frac{1}{4}(1 + A + A^2 + A^3), \tag{3.8}$$

respectively. By multiplying the first equation determining the invariant subspace (3.4) by A (and A^2), we obtain

$$Aq = A^2q + 2\omega A\lambda, \quad A^2q = A^3q + 2\omega A^2\lambda.$$

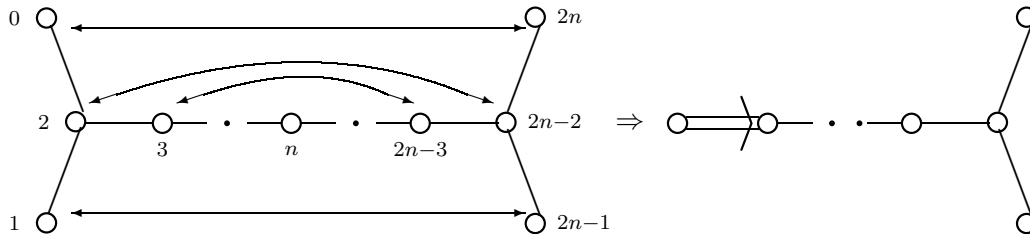
This puts a restriction on the possible choice of the weight vector λ :

$$Pr\lambda = 0. \tag{3.9}$$

Let us consider the reductions listed in (3.5) in turn.

3.1. Twisted C_n model

The Dynkin diagram of $A_{2n-1}^{(2)}$ is obtained from that of $D_{2n}^{(1)}$ by the following folding:



The C_n Dynkin diagram is contained in the $A_{2n-1}^{(2)}$ Dynkin diagram. The automorphism is given by

$$A\alpha_j = \alpha_{2n-j}, \quad j = 0, 1, \dots, 2n, \tag{3.10}$$

in which $\{\alpha_j\}$, $j = 1, \dots, 2n$ are D_{2n} simple roots in an orthonormal basis of \mathbb{R}^{2n} :

$$\begin{aligned} \alpha_1 &= e_1 - e_2, & \dots, & & \alpha_{2n-2} &= e_{2n-2} - e_{2n-1}, \\ \alpha_{2n-1} &= e_{2n-1} - e_{2n}, & \alpha_{2n} &= e_{2n-1} + e_{2n} \end{aligned} \tag{3.11}$$

and $\alpha_0 = -(e_1 + e_2)$. In terms of the orthonormal basis, the automorphism A has a simple expression:

$$Ae_j = -e_{2n+1-j}, \quad j = 1, \dots, 2n. \tag{3.12}$$

Among the $2n(4n - 2)$ roots of D_{2n} , $2n$ roots,

$$\pm(e_j - e_{2n+1-j}), \quad j = 1, \dots, n, \tag{3.13}$$

belong to the invariant subspace of A ; that is, these $2n$ roots remain long roots after folding. There are $2n$ roots which are eigenvectors of A with eigenvalue -1 :

$$\pm(e_j + e_{2n+1-j}), \quad j = 1, \dots, n, \tag{3.14}$$

$$A(e_j + e_{2n+1-j}) = -(e_j + e_{2n+1-j}). \tag{3.15}$$

These $2n$ roots are mapped to the null vector in the invariant subspace. The remaining $8n(n - 1)$ roots are mapped to the $2n(n - 1)$ short roots of C_n four to one. In fact ($j, k = 1, \dots, n$)

$$\begin{aligned} \text{(a)} \quad e_j + e_k, & & \text{(b)} \quad -e_{2n+1-j} + e_k, \\ \text{(c)} \quad -e_{2n+1-k} + e_j, & & \text{(d)} \quad -e_{2n+1-j} - e_{2n+1-k} \end{aligned} \tag{3.16}$$

are all mapped to the short root

$$Pr(e_j + e_k) = \frac{1}{2}(e_j - e_{2n+1-j} + e_k - e_{2n+1-k}), \tag{3.17}$$

which has $(\text{length})^2 = 1$. There is a unique minimal weight (the spinor weight λ_{2n}) which is annihilated by Pr :

$$Pr\lambda_{2n} = \frac{1}{2}Pr(e_1 + e_2 + \dots + e_{2n}) = 0. \tag{3.18}$$

As expected, λ_{2n} has a scalar product 1 (mod 2) with all the roots (3.14) which are mapped to the origin of the invariant subspace. Thus the singularity of the elliptic potential is avoided after folding. It is easy to see that λ_{2n} has a scalar product 1 (mod 2) with two of the short roots in (3.16) and scalar product 0 (mod 2) with the other two. It is easy to see that the solution of (3.4) is given by

$$q = \sum_{j=1}^n Q^j v_j + \omega \lambda_{2n}, \quad v_j = \frac{1}{\sqrt{2}}(e_j - e_{2n+1-j}), \tag{3.19}$$

$$p = \sum_{j=1}^n P_j v_j, \quad \lambda_{2n} = \sum_{j=1}^n \frac{1}{2}(e_1 + \dots + e_{2n-1} + e_{2n}), \tag{3.20}$$

where $\{Q^j, P_j\}$ are the canonical variables for the reduced system. By substituting the above solution into the original Hamiltonian we arrive at the twisted C_n Calogero-Moser model

$$\mathcal{H} = \frac{1}{2} \sum_{j=1}^n P_j^2 + \frac{g^2}{2} \sum_{\alpha \in \Delta_l} \wp(\alpha \cdot Q) + g^2 \sum_{\mu \in \Delta_s} [\wp(\mu \cdot Q) + \wp(\mu \cdot Q + \omega)], \quad (3.21)$$

in which the sets of long and short roots are

$$\Delta_l = \{\pm\sqrt{2}v_j : j = 1, \dots, n\}, \quad \Delta_s = \left\{ \frac{1}{\sqrt{2}}(\pm v_j \pm v_k) : j, k = 1, \dots, n \right\}. \quad (3.22)$$

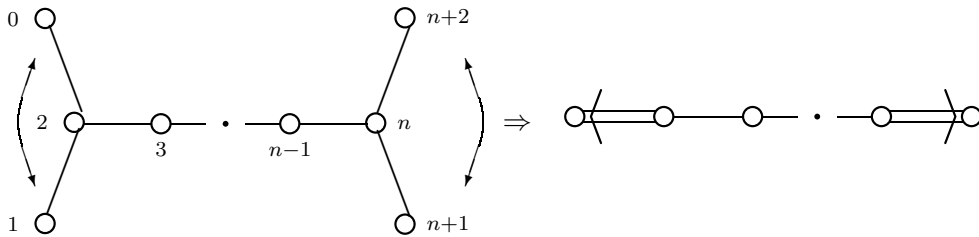
It is well known that the combination of elliptic functions appearing in the short root potential can be expressed in terms of an elliptic function with a half period. For example, the $\wp^{(1/2)}$ function,

$$\wp^{(1/2)}(x) \equiv \wp(x) + \wp(x + \omega_1) - \wp(\omega_1),$$

has the set of fundamental periods $\{\omega_1, 2\omega_3\}$, instead of the original $\{2\omega_1, 2\omega_3\}$.

3.2. Twisted B_n model

The Dynkin diagram of $D_{n+1}^{(2)}$ is obtained from that of $D_{n+2}^{(1)}$ by the following folding:



The B_n Dynkin diagram is contained in the $D_{n+1}^{(2)}$ Dynkin diagram. The automorphism is given by

$$\begin{aligned} A\alpha_0 &= \alpha_1, & A\alpha_{n+1} &= \alpha_{n+2}, & A\alpha_j &= \alpha_j, & j &= 2, \dots, n, \\ A\alpha_1 &= \alpha_0, & A\alpha_{n+2} &= \alpha_{n+1}, \end{aligned} \quad (3.23)$$

where $\alpha_j, j = 1, \dots, n + 2$, are D_{n+2} simple roots and α_0 is the affine root. By using the expression of the simple roots in terms of an orthonormal basis of \mathbb{R}^{n+2} (see (3.11)) the automorphism A can be expressed as

$$Ae_1 = -e_1, \quad Ae_{n+2} = -e_{n+2}, \quad Ae_j = e_j, \quad j = 2, \dots, n + 1. \quad (3.24)$$

This means that the invariant subspace of the automorphism A is spanned by e_j ($j = 2, \dots, n + 1$), and the two dimensional subspace spanned by $\{e_1, e_{n+2}\}$ is annihilated by the projector Pr :

$$Pr e_j = e_j, \quad j = 2, \dots, n + 1, \quad (3.25)$$

$$Pr e_1 = Pr e_{n+2} = 0. \quad (3.26)$$

Among the $2n(n+2)(n+1)$ roots of D_{n+2} , the following $2n(n-1)$ roots

$$\pm e_j \pm e_k, \quad j, k = 2, \dots, n+1, \tag{3.27}$$

remain long and become the long roots of B_n . There are four roots that are mapped to the origin of the invariant subspace:

$$\pm e_1 \pm e_{n+2}. \tag{3.28}$$

The remaining $8n$ roots are mapped to short roots four to one:

$$\left. \begin{array}{l} \pm e_1 \pm e_j \\ \pm e_{n+2} \pm e_j \end{array} \right\} \rightarrow \pm e_j, \quad j = 2, \dots, n+1. \tag{3.29}$$

It is easy to see that

$$q = Q + \omega\lambda, \quad Q = \sum_{j=2}^{n+1} q^j e_j, \tag{3.30}$$

$$p = P, \quad P = \sum_{j=2}^{n+1} p_j e_j,$$

$$\lambda = \lambda_1 = e_1, \quad \text{or} \quad \lambda = e_{n+2} = \lambda_{n+2} - \lambda_{n+1}, \tag{3.31}$$

are solutions of (3.4). In other words, this means that

$$\{q^1 = 0, \quad q^{n+2} = \omega, \quad \text{or} \quad q^1 = \omega, \quad q^{n+2} = 0\} \quad \text{and} \quad p_1 = 0, \quad p_{n+2} = 0 \tag{3.32}$$

are valid restrictions of the elliptic D_{n+2} Calogero-Moser model. The above λ (3.31) has non-vanishing scalar products with the roots that are mapped to zero:

$$\lambda \cdot (\pm e_1 \pm e_{n+2}) = 1 \pmod{2},$$

it has a scalar product 1 (mod 2) with one half of the roots that are mapped to short roots:

$$\lambda_1 \cdot (\pm e_1 \pm e_j) = 1 \pmod{2},$$

and zero with the rest:

$$\lambda_1 \cdot (\pm e_{n+2} \pm e_j) = 0.$$

By substituting the solution (3.30) into the Hamiltonian, we obtain

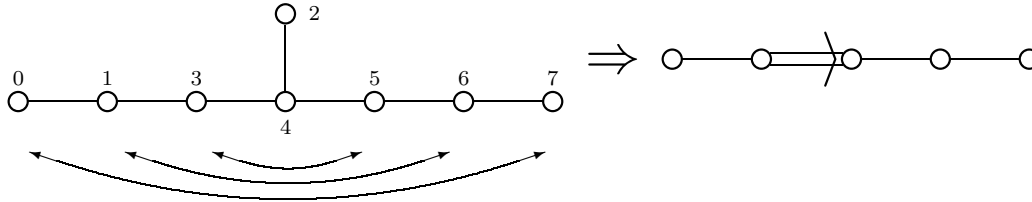
$$\mathcal{H} = \frac{1}{2} \sum_{j=2}^{n+1} p_j^2 + \frac{g^2}{2} \sum_{\alpha \in \Delta_l} \wp(\alpha \cdot q) + g^2 \sum_{\mu \in \Delta_s} [\wp(\mu \cdot q) + \wp(\mu \cdot q + \omega)] + \text{const}, \tag{3.33}$$

in which the sets of long and short roots are

$$\Delta_l = \{\pm e_j \pm e_k : j, k = 2, \dots, n+1\}, \quad \Delta_s = \{\pm e_j : j = 2, \dots, n+1\}. \tag{3.34}$$

3.3. Twisted F_4 model

The Dynkin diagram of $E_6^{(2)}$ is obtained from that of $E_7^{(1)}$ by the following folding:



The F_4 Dynkin diagram is contained in the $E_6^{(2)}$ Dynkin diagram. As indicated in the diagram, the automorphism A is given by

$$\begin{aligned}
 A\alpha_1 &= \alpha_6, & A\alpha_2 &= \alpha_2, & A\alpha_3 &= \alpha_5, \\
 A\alpha_4 &= \alpha_4, & A\alpha_5 &= \alpha_3, & A\alpha_6 &= \alpha_1, \\
 A\alpha_7 &= \alpha_0, & A\alpha_0 &= \alpha_7.
 \end{aligned}
 \tag{3.35}$$

Let us adopt the following representation of the simple roots of E_7 in terms of an orthonormal basis of \mathbb{R}^7 :

$$\begin{aligned}
 \alpha_1 &= \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 + \sqrt{2}e_7), & \alpha_2 &= e_1 + e_2, \\
 \alpha_3 &= -e_1 + e_2, & \alpha_4 &= -e_2 + e_3, \\
 \alpha_5 &= -e_3 + e_4, & \alpha_6 &= -e_4 + e_5, \\
 \alpha_7 &= -e_5 + e_6, & \alpha_0 &= -\sqrt{2}e_7.
 \end{aligned}
 \tag{3.36}$$

By an analysis similar to that applied above, we find that among the 126 roots of E_7 , the following 24 roots remain long:

$$\begin{aligned}
 &\pm(e_1 + e_2), & \pm(e_2 - e_3), & \pm(e_3 + e_4), \\
 &\frac{1}{2}(\pm(e_1 + e_2) \pm (e_3 + e_4) \pm (e_5 - e_6 + \sqrt{2}e_7)), \\
 &\pm(e_1 + e_3), & \pm(e_2 + e_4), & \pm(e_1 - e_4), \\
 &\frac{1}{2}(\pm(e_1 - e_4) \mp (e_2 - e_3) \pm (e_5 - e_6 + \sqrt{2}e_7)).
 \end{aligned}
 \tag{3.37}$$

The following 6 roots are mapped to 0:

$$\pm(e_5 + e_6), \quad \frac{1}{2}(\pm(e_1 - e_2 - e_3 + e_4) \pm (e_5 - e_6 + \sqrt{2}e_7)).
 \tag{3.38}$$

The remaining 96 roots are mapped to F_4 short roots four to one. It is easy to see that the solution of (3.4) is given by

$$q = \sum_{j=1}^4 Q^j v_j + \omega \lambda, \quad \lambda = \lambda_7 = e_6 + \frac{1}{\sqrt{2}}e_7,
 \tag{3.39}$$

$$p = \sum_{j=1}^4 P_j v_j, \quad \text{or} \quad \lambda = \lambda_3 - \lambda_5 = \frac{1}{2}(-e_1 + e_2 + e_3 - e_4 - e_5 - e_6), \quad (3.40)$$

where $\{v_j\}$, $j = 1, \dots, 4$ is a new orthonormal basis of the four-dimensional invariant subspace:

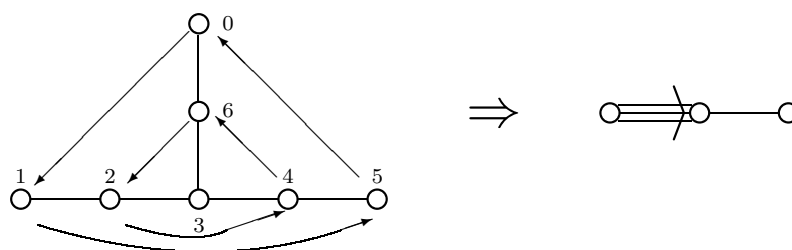
$$\begin{aligned} v_1 &= \frac{1}{\sqrt{2}}(e_1 + e_2), & v_2 &= \frac{1}{\sqrt{6}}(e_1 - e_2 + 2e_3), \\ v_3 &= \frac{1}{\sqrt{12}}(-e_1 + e_2 + e_3 + 3e_4), & v_4 &= \frac{1}{2}(e_5 - e_6 + \sqrt{2}e_7). \end{aligned}$$

Both choices of λ have a scalar product 1 (mod 2) with all the roots (3.38) which are mapped to zero. It is straightforward to check that both choices of λ have a scalar product 1 (mod 2) with one half of the short roots and 0 with the rest. By substituting the above solution into the original Hamiltonian, we arrive at the twisted F_4 Calogero-Moser model,

$$\mathcal{H} = \frac{1}{2} \sum_{j=1}^4 P_j^2 + \frac{g^2}{2} \sum_{\alpha \in \Delta_l} \wp(\alpha \cdot Q) + g^2 \sum_{\mu \in \Delta_s} [\wp(\mu \cdot Q) + \wp(\mu \cdot Q + \omega)] + \text{const.} \quad (3.41)$$

3.4. Twisted G_2 model

The Dynkin diagram of $D_4^{(3)}$ is obtained from that of $E_6^{(1)}$ by the triple folding:



The G_2 Dynkin diagram is contained in the $D_4^{(3)}$ Dynkin diagram. As indicated in the diagram, the automorphism A is given by

$$\begin{aligned} A\alpha_1 &= \alpha_5, & A\alpha_2 &= \alpha_4, & A\alpha_3 &= \alpha_3, \\ A\alpha_4 &= \alpha_6, & A\alpha_5 &= \alpha_0, & A\alpha_6 &= \alpha_2, \\ A\alpha_0 &= \alpha_1. \end{aligned} \quad (3.42)$$

Let us adopt the following representation of the simple roots of E_6 in terms of an orthonormal basis of \mathbb{R}^6 :

$$\begin{aligned}
 \alpha_1 &= \frac{1}{2}(e_1 - e_2 - e_3 - e_4 + e_5 - \sqrt{3}e_6), & \alpha_2 &= e_4 - e_5, \\
 \alpha_3 &= e_3 - e_4, & \alpha_4 &= e_4 + e_5, \\
 \alpha_5 &= \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 + \sqrt{3}e_6), & \alpha_6 &= e_2 - e_3, \\
 \alpha_0 &= -(e_1 + e_2).
 \end{aligned}
 \tag{3.43}$$

By a similar analysis as before, we find that among the 72 roots of E_6 , the following 6 roots remain long:

$$\pm(e_2 + e_3), \quad \pm(e_2 + e_4), \quad \pm(e_3 - e_4).
 \tag{3.44}$$

The following 12 roots are mapped to 0:

$$\pm e_1 \pm e_5, \quad \frac{1}{2}(\pm e_1 \pm (e_2 - e_3 - e_4) \pm e_5 \pm \sqrt{3}e_6).
 \tag{3.45}$$

The remaining 54 roots are mapped to the 6 short roots of G_2 nine to one. The short roots have $(\text{length})^2=2/3$ because of the third order folding. The invariant subspace of the automorphism A is spanned by two vectors,

$$v_1 = \frac{1}{\sqrt{2}}(e_3 - e_4), \quad v_2 = \frac{1}{\sqrt{6}}(2e_2 + e_3 + e_4).
 \tag{3.46}$$

Let us consider Eq. (3.4) determining the invariant subspace, with λ being one of the minimal weights λ_1 (or λ_5):

$$\begin{aligned}
 q &= Aq + 2\omega\lambda_1, & \lambda_1 &= e_1 - \frac{1}{\sqrt{3}}e_6, & \lambda_5 &= e_1 + \frac{1}{\sqrt{3}}e_6, \\
 p &= Ap.
 \end{aligned}$$

It is elementary to check that the following satisfies the above equation:

$$\begin{aligned}
 q &= \sum_{j=1}^2 Q^j v_j + \frac{2\omega}{3}(\lambda_1 + \lambda_5), & \lambda_1 + \lambda_5 &= 2e_1, \\
 p &= \sum_{j=1}^2 P_j v_j.
 \end{aligned}
 \tag{3.47}$$

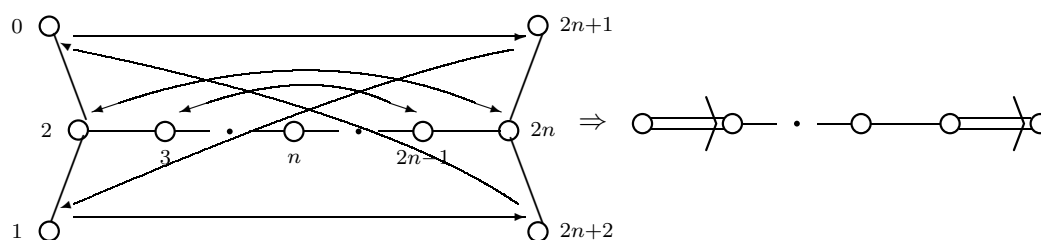
Here $\{Q^j, P_j\}$ are the canonical variables of the reduced system and $\{v_1, v_2\}$ are given in (3.46). It should be noted that $\lambda_1 + \lambda_5 = 2e_1$ has a scalar product 1 and 2 (mod 3) with all the roots (3.45) which are mapped to 0. Moreover, it has a scalar product 0, 1 and 2 (mod 3) with each one third of the roots that are mapped to G_2 short roots. By substituting the solution (3.47) into the elliptic E_6 Calogero-Moser Hamiltonian, we obtain the twisted G_2 Hamiltonian:

$$\begin{aligned}
 \mathcal{H} &= \frac{1}{2} \sum_{j=1}^2 P_j^2 + \frac{g^2}{2} \sum_{\alpha \in \Delta_l} \wp(\alpha \cdot Q) \\
 &+ \frac{3g^2}{2} \sum_{\mu \in \Delta_s} \left[\wp(\mu \cdot Q) + \wp\left(\mu \cdot Q + \frac{2\omega}{3}\right) + \wp\left(\mu \cdot Q + \frac{4\omega}{3}\right) \right] + \text{const.}
 \end{aligned}
 \tag{3.48}$$

3.5. $A_{2n}^{(2)}$ model or twisted BC_n model

This model is associated with the twisted affine algebra $A_{2n}^{(2)}$. It is obtained by folding the $D_{2n+2}^{(1)}$ diagram using the fourth order automorphism. After rescaling the $A_{2n}^{(2)}$ algebra has $2n$ long and short roots of the form $\{\pm 2e_j\}$, $\{\pm e_j\}$, $j = 1, \dots, n$ and $2n(n - 1)$ middle roots of the form $\{\pm e_j \pm e_k\}$, $j, k = 1, \dots, n$. Thus it could be understood as a twisted BC_n model. This will provide a Lie algebraic interpretation of the BC_n model, as we now show.

The Dynkin diagram of $A_{2n}^{(2)}$ is obtained from that of $D_{2n+2}^{(1)}$ by the fourth order folding:



As shown above, the D_{2n+2} root system is invariant under the following automorphism:

$$\begin{aligned} A\alpha_0 &= \alpha_{2n+1}, & A\alpha_1 &= \alpha_{2n+2}, \\ A\alpha_{2n+1} &= \alpha_1, & A\alpha_{2n+2} &= \alpha_0, \\ A\alpha_j &= \alpha_{2n+2-j}, & j &= 2, \dots, 2n. \end{aligned} \tag{3.49}$$

In terms of the standard orthonormal basis of \mathbb{R}^{2n+2} it is simply expressed as

$$\begin{aligned} Ae_1 &= e_{2n+2}, & Ae_{2n+2} &= -e_1, \\ Ae_j &= -e_{2n+3-j}, & j &= 2, \dots, 2n+1. \end{aligned} \tag{3.50}$$

That is, the automorphism A satisfies

$$A^4 = 1, \tag{3.51}$$

in the two-dimensional subspace spanned by $\{e_1, e_{2n+2}\}$ and in the rest of the space it satisfies

$$A^2 = 1. \tag{3.52}$$

Among the $4(n + 1)(2n + 1)$ roots of D_{2n+2} , the following $2n$ roots remain long:

$$\pm(e_j - e_{2n+3-j}), \quad j = 2, \dots, n + 1. \tag{3.53}$$

The following $8n(n - 1)$ roots are mapped to middle roots with $(\text{length})^2=1$:

$$\pm e_j \pm e_k, \quad j + k \neq 2n + 3, \quad j, k = 2, \dots, 2n + 1. \tag{3.54}$$

In this case four different roots are mapped to one middle root. There are $16n$ roots that are mapped to short roots with $(\text{length})^2=1/2$:

$$\pm e_1 \pm e_j, \quad \pm e_{2n+2} \pm e_j \quad j, k = 2, \dots, 2n + 1. \tag{3.55}$$

In this case eight different roots are mapped to one short root. Finally, there are $2n + 4$ roots which are mapped to zero:

$$\pm(e_j + e_{2n+3-j}), \quad \pm e_1 \pm e_{2n+2} \quad j, k = 2, \dots, n + 1. \tag{3.56}$$

We look for a solution of Eq. (3.4) with $\lambda = \lambda_{2n+1}$, the anti-spinor weight which is a minimal weight:

$$\begin{aligned} q &= Aq + 2\omega\lambda_{2n+1}, & \lambda_{2n+1} &= \frac{1}{2}(e_1 + \dots + e_{2n+1} - e_{2n+2}), \\ p &= Ap. \end{aligned}$$

It is elementary to verify that

$$q = \sum_{j=2}^{n+1} Q^j v_j + \omega \tilde{\lambda}, \quad \tilde{\lambda} = e_1 + \frac{1}{2}(e_2 + \dots + e_{2n+1}), \tag{3.57}$$

$$p = \sum_{j=2}^{n+1} P_j v_j \tag{3.58}$$

is a solution. In the above expression, $\{Q^j, P_j\}$, $j = 2, \dots, n + 1$ are the canonical variables of the reduced system and

$$v_j = \frac{1}{\sqrt{2}}(e_j - e_{2n+3-j}), \quad j = 2, \dots, n + 1 \tag{3.59}$$

is an orthonormal basis of the invariant subspace of A . It is easy to see that $\tilde{\lambda}$ has a vanishing scalar product with all the long roots (3.53). As with the middle roots, $\tilde{\lambda}$ has a scalar product 1 (mod 2) with one half of them and 0 with the rest. An interesting situation arises when we consider the scalar products of $\tilde{\lambda}$ with the short roots (3.55):

$$\alpha \cdot \tilde{\lambda} = \begin{cases} \frac{1}{2} \pmod{2} & \text{for } \alpha = \pm e_1 - e_j \quad \text{and} \quad \pm e_{2n+2} + e_j, \\ \frac{3}{2} \pmod{2} & \text{for } \alpha = \pm e_1 + e_j \quad \text{and} \quad \pm e_{2n+2} - e_j. \end{cases} \tag{3.60}$$

It should be noted that $\alpha \cdot \tilde{\lambda} = 1, 0 \pmod{2}$ do not occur for short roots α . Finally $\tilde{\lambda}$ has a scalar product 1 (mod 2) with all the roots (3.56) that are mapped to zero.

By substituting the above solution (3.57) into the Hamiltonian of elliptic D_{2n+2} Calogero-Moser model, we obtain

$$\mathcal{H} = \frac{1}{2} \sum_{j=2}^{n+1} P_j^2 + \frac{g^2}{2} \sum_{\Xi \in \Delta_l} \wp(\Xi \cdot Q)$$

$$\begin{aligned}
& + g^2 \sum_{\alpha \in \Delta_m} [\wp(\alpha \cdot Q) + \wp(\alpha \cdot Q + \omega)] \\
& + 2g^2 \sum_{\mu \in \Delta_s} \left[\wp\left(\mu \cdot Q + \frac{\omega}{2}\right) + \wp\left(\mu \cdot Q + \frac{3\omega}{2}\right) \right] + \text{const.} \quad (3.61)
\end{aligned}$$

Here Δ_l , Δ_m and Δ_s are the sets of long, middle and short roots of the BC_n system, respectively. This model was previously described in 4).

Before closing this subsection, some remarks are in order. First, the twisted models derived in this subsection inherit the integrability of the original simply laced models. The conserved quantities of the twisted models are obtained from those of the simply laced theory by substitution of variables. Second, the Hamiltonian of the ordinary BC_n system could be obtained from the Hamiltonian of D_{2n+2} theory by the same folding as above with $\lambda = 0$, if we ignore the singularities of the potential caused by the vanishing roots. Third, as we have remarked at the beginning of this section, we have utilised only the automorphism of the extended Dynkin diagram as relevant to the ordinary root vectors. The actual connection with the underlying affine algebras is rather subtle, established only in the limit of the affine Toda theories.^{4),5)} However, the very fact that the affine root system (without the null roots) plays a fundamental role here seems to suggest the existence of an infinite dimensional algebra (perhaps a kind of toroidal algebra.¹²⁾) in the elliptic Calogero-Moser systems. This algebra is thought to play the same or similar role as that played by affine algebras in the affine Toda theories.

§4. Independent coupling constants

In the previous section we showed that all of the Calogero-Moser models based on non-simply laced root systems, the untwisted as well as the twisted models, are obtained by folding (reduction) of the models based on simply laced root systems. These non-simply laced models inherit integrability as well as restrictions from the original simply laced theories. In these cases the ratio of the coupling constants for the long and short root potentials is fixed by the order of the automorphism used for the folding. In fact, these models are integrable even when these coupling constants are independent.

In this section we give the root type Lax pairs of the untwisted non-simply laced models with as many independent coupling constants as independent Weyl orbits in the set of roots. The independence of the coupling constants stems from the independence of the Weyl orbits of the roots with different lengths. Thus the root type Lax pair based on the set of roots itself is conceptually most suitable for the purpose of verifying the independence of the coupling constants. For most theories this implies two independent coupling constants, one for the long and the other for the short root potentials. However, for the model based on the BC_n root system, there are three independent coupling constants.

We give two different root type Lax pairs for most of the untwisted non-simply laced models, one based on the set of long roots and the other on the set of short roots. Both give the identical Hamiltonian and equation of motion. The list of Lax

pairs is complete in the sense that it contains all the models with all four choices of potential and with and without the spectral parameter, except for the G_2 model based on the long roots. In this case, new functions satisfying constraints related to the third order folding are necessary. For the rational, trigonometric and hyperbolic potentials the functions given in §2 also satisfy the new constraints. A new set of functions with and without spectral parameter is obtained for the elliptic potential case. The actual verification that these Lax pairs are equivalent to the canonical equation of motion is almost parallel to that of the root type Lax pairs based on simply laced root systems.¹⁾ The functions appearing in the root type Lax pairs are the same for the simply laced and the untwisted non-simply laced cases, except for the G_2 case mentioned above. Thus we only give the explicit forms of the Lax pairs for each of the Calogero-Moser models based on untwisted non-simply laced root systems.

To this time the Lax pairs for untwisted non-simply laced models have been given in some of the minimal representations only.^{2),5),1)} The situation was a bit confusing: the allowed number of independent coupling constants can be different for two different representations of the minimal type Lax pair for one and the same theory. Now we have the universal root type Lax pairs for the untwisted non-simply laced models with independent coupling constants.

In most cases we normalise the $(\text{length})^2 = 2$ for the long roots, except for the C_n and BC_n system, in which $(\text{length})^2 = 4$ is used. They are denoted by the subscript L . For the G_2 case we choose to normalise $(\text{length})^2 = 3$ for the long roots and $(\text{length})^2 = 1$ for the short roots only for convenience. The coupling constant g without a suffix is reserved for the long roots, except for the C_n and BC_n systems, in which it is used for the short- and middle-root coupling and the long-root coupling constant is denoted by g_L . The short-root coupling is denoted by g_s .

4.1. B_n model

The set of B_n roots consists of two parts, long roots and short roots:

$$\Delta_{B_n} = \Delta \cup \Delta_s. \tag{4.1}$$

Here the roots are conveniently expressed in terms of an orthonormal basis of \mathbb{R}^n :

$$\begin{aligned} \Delta &= \{\alpha, \beta, \gamma, \dots\} = \{\pm e_j \pm e_k : j, k = 1, \dots, n\}, & 2n(n-1) \text{ roots,} \\ \Delta_s &= \{\lambda, \mu, \nu, \dots\} = \{\pm e_j : j = 1, \dots, n\}, & 2n \text{ roots.} \end{aligned} \tag{4.2}$$

From this we know the root difference pattern:

$$B_n : \quad \text{short root} - \text{short root} = \begin{cases} \text{long root,} \\ 2 \times \text{short root,} \\ \text{non-root,} \end{cases} \tag{4.3}$$

and

$$B_n : \quad \text{long root} - \text{long root} = \begin{cases} \text{long root,} \\ 2 \times \text{long root,} \\ 2 \times \text{short root,} \\ \text{non-root.} \end{cases} \tag{4.4}$$

From this knowledge only we can construct the root type Lax pair for the B_n model by following the recipe of the root type Lax pair for simply laced models.

4.1.1. Root type Lax pair for untwisted B_n model based on short roots Δ_s

The Lax pair is given in terms of the short roots. The matrix elements of L_s and M_s are labeled by indices μ, ν , etc.:

$$\begin{aligned} L_s(q, p, \xi) &= p \cdot H + X + X_d, \\ M_s(q, \xi) &= D + Y + Y_d. \end{aligned} \quad (4.5)$$

Here X and Y correspond to the part of the “short root – short root = long root” of (4.3),

$$X = ig \sum_{\alpha \in \Delta} x(\alpha \cdot q, \xi) E(\alpha), \quad Y = ig \sum_{\alpha \in \Delta} y(\alpha \cdot q, \xi) E(\alpha), \quad E(\alpha)_{\mu\nu} = \delta_{\mu-\nu, \alpha}, \quad (4.6)$$

and X_d and Y_d correspond to the “short root – short root = $2 \times$ short root” of (4.3),

$$X_d = 2ig_s \sum_{\lambda \in \Delta_s} x_d(\lambda \cdot q, \xi) E_d(\lambda), \quad Y_d = ig_s \sum_{\lambda \in \Delta_s} y_d(\lambda \cdot q, \xi) E_d(\lambda), \quad E_d(\lambda)_{\mu\nu} = \delta_{\mu-\nu, 2\lambda}. \quad (4.7)$$

The diagonal parts of L_s and M_s are given by

$$H_{\mu\nu} = \mu \delta_{\mu, \nu}, \quad D_{\mu\nu} = \delta_{\mu, \nu} D_\mu, \quad D_\mu = -i \left(g_s z(\mu \cdot q) + g \sum_{\gamma \in \Delta, \gamma \cdot \mu = 1} z(\gamma \cdot q) \right). \quad (4.8)$$

The functions x, y, z and x_d, y_d, z_d are the same as those given in §2. It is easy to verify

$$\text{Tr}(L_s^2) = 4\mathcal{H}_{B_n}, \quad (4.9)$$

in which the B_n Hamiltonian is given by

$$\mathcal{H}_{B_n} = \frac{1}{2} p^2 - \frac{g^2}{2} \sum_{\alpha \in \Delta} x(\alpha \cdot q) x(-\alpha \cdot q) - g_s^2 \sum_{\lambda \in \Delta_s} x(\lambda \cdot q) x(-\lambda \cdot q). \quad (4.10)$$

4.1.2. Root type Lax pair for untwisted B_n model based on long roots Δ

The Lax pair is given in terms of the long roots. The matrix elements of L_l and M_l are labeled by indices α, β , etc.:

$$\begin{aligned} L_l(q, p, \xi) &= p \cdot H + X + X_d + X_s, \\ M_l(q, \xi) &= D + D_s + Y + Y_d + Y_s. \end{aligned} \quad (4.11)$$

Here X and Y correspond to the part of the “long root – long root = long root” of (4.4),

$$X = ig \sum_{\alpha \in \Delta} x(\alpha \cdot q, \xi) E(\alpha), \quad Y = ig \sum_{\alpha \in \Delta} y(\alpha \cdot q, \xi) E(\alpha), \quad E(\alpha)_{\beta\gamma} = \delta_{\beta-\gamma, \alpha}, \quad (4.12)$$

and X_d and Y_d correspond to the “long root – long root = $2 \times$ long root” of (4.4),

$$X_d = 2ig \sum_{\alpha \in \Delta} x_d(\alpha \cdot q, \xi) E_d(\alpha), \quad Y_d = ig \sum_{\alpha \in \Delta} y_d(\alpha \cdot q, \xi) E_d(\alpha), \quad E_d(\alpha)_{\beta\gamma} = \delta_{\beta-\gamma, 2\alpha}. \tag{4.13}$$

The additional term in L_l (M_l), X_s (Y_s), corresponds to “long root – long root = $2 \times$ short root” of (4.4),

$$X_s = 2ig_s \sum_{\lambda \in \Delta_s} x_d(\lambda \cdot q, \xi) E_d(\lambda), \quad Y_s = ig_s \sum_{\lambda \in \Delta_s} y_d(\lambda \cdot q, \xi) E_d(\lambda), \quad E_d(\lambda)_{\beta\gamma} = \delta_{\beta-\gamma, 2\lambda}. \tag{4.14}$$

The diagonal parts of L_l and M_l are given by

$$H_{\beta\gamma} = \beta\delta_{\beta,\gamma}, \quad D_{\beta\gamma} = \delta_{\beta,\gamma} D_\beta, \quad D_\beta = -ig \left(z(\beta \cdot q) + \sum_{\kappa \in \Delta, \kappa \cdot \beta = 1} z(\kappa \cdot q) \right), \tag{4.15}$$

and

$$(Ds)_{\beta\gamma} = \delta_{\beta,\gamma} (Ds)_\beta, \quad (Ds)_\beta = -ig_s \sum_{\lambda \in \Delta_s, \beta \cdot \lambda = 1} z(\lambda \cdot q). \tag{4.16}$$

The functions x, y, z and x_d, y_d, z_d are also the same as those given in §2. It is easy to verify that

$$\text{Tr}(L_l^2) = 8(n - 1)\mathcal{H}_{B_n}, \tag{4.17}$$

in which the B_n Hamiltonian is the same as that given above (4.10). In both cases the reduction from D_{n+1} fixes $g_s = g$. Clearly, the consistency of the root type Lax pairs (4.5) and (4.11) does not depend on the explicit representation of the roots in terms of the orthonormal basis (4.2). This remark applies to the other models as well.

4.2. C_n model

The set of C_n roots consists of two parts, long roots and short roots:

$$\Delta_{C_n} = \Delta_L \cup \Delta. \tag{4.18}$$

Here the roots are conveniently expressed in terms of an orthonormal basis of \mathbb{R}^n :

$$\Delta_L = \{\Xi, \Upsilon, \Omega, \dots\} = \{\pm 2e_j : j = 1, \dots, n\}, \quad 2n \text{ roots}, \tag{4.19}$$

$$\Delta = \{\alpha, \beta, \gamma, \dots\} = \{\pm e_j \pm e_k : j, k = 1, \dots, n\}, \quad 2n(n - 1) \text{ roots}. \tag{4.20}$$

The root difference pattern is

$$C_n : \quad \text{short root} - \text{short root} = \begin{cases} \text{short root,} \\ 2 \times \text{short root,} \\ \text{long root,} \\ \text{non-root.} \end{cases} \tag{4.21}$$

and

$$C_n : \quad \text{long root} - \text{long root} = \begin{cases} 2 \times \text{long root,} \\ 2 \times \text{short root,} \\ \text{non-root.} \end{cases} \quad (4.22)$$

From this knowledge only we can construct the root type Lax pair for the C_n model by following the recipe of the root type Lax pair for simply laced models.

4.2.1. Root type Lax pair for untwisted C_n model based on short roots Δ

The Lax pair is given in terms of short roots. The matrix elements of L_s and M_s are labeled by indices β, γ , etc.:

$$\begin{aligned} L_s(q, p, \xi) &= p \cdot H + X + X_d + X_L, \\ M_s(q, \xi) &= D + D_L + Y + Y_d + X_L. \end{aligned} \quad (4.23)$$

Here X and Y correspond to the part of the “short root – short root = short root” of (4.21),

$$X = ig \sum_{\alpha \in \Delta} x(\alpha \cdot q, \xi) E(\alpha), \quad Y = ig \sum_{\alpha \in \Delta} y(\alpha \cdot q, \xi) E(\alpha), \quad E(\alpha)_{\beta\gamma} = \delta_{\beta-\gamma, \alpha}, \quad (4.24)$$

and X_d and Y_d correspond to the “short root – short root = 2 × short root” of (4.21),

$$X_d = 2ig \sum_{\alpha \in \Delta} x_d(\alpha \cdot q, \xi) E_d(\alpha), \quad Y_d = ig \sum_{\alpha \in \Delta} y_d(\alpha \cdot q, \xi) E_d(\alpha), \quad E_d(\alpha)_{\beta\gamma} = \delta_{\beta-\gamma, 2\alpha}. \quad (4.25)$$

The additional term in L_s (M_s), X_L (Y_L), corresponds to the “short root – short root = long root” of (4.22),

$$X_L = ig_L \sum_{\Xi \in \Delta_L} x(\Xi \cdot q, \xi) E(\Xi), \quad Y_L = ig_L \sum_{\Xi \in \Delta_L} y(\Xi \cdot q, \xi) E(\Xi), \quad E(\Xi)_{\beta\gamma} = \delta_{\beta-\gamma, \Xi}. \quad (4.26)$$

The diagonal parts of L_s and M_s are given by

$$H_{\beta\gamma} = \beta \delta_{\beta, \gamma}, \quad D_{\beta\gamma} = \delta_{\beta, \gamma} D_\beta, \quad D_\beta = -ig \left(z(\beta \cdot q) + \sum_{\kappa \in \Delta, \kappa \cdot \beta = 1} z(\kappa \cdot q) \right), \quad (4.27)$$

and

$$(D_L)_{\beta\gamma} = \delta_{\beta, \gamma} (D_L)_\beta, \quad (D_L)_\beta = -ig_L \sum_{\Upsilon \in \Delta_L, \beta \cdot \Upsilon = 2} z(\Upsilon \cdot q). \quad (4.28)$$

The functions x, y, z and x_d, y_d, z_d are the same as those given in §2. It is easy to verify

$$\text{Tr}(L_s^2) = 8(n - 1)\mathcal{H}_{C_n}, \quad (4.29)$$

in which C_n Hamiltonian is given by

$$\mathcal{H}_{C_n} = \frac{1}{2}p^2 - \frac{g^2}{2} \sum_{\alpha \in \Delta} x(\alpha \cdot q)x(-\alpha \cdot q) - \frac{g_L^2}{4} \sum_{\Xi \in \Delta_L} x(\Xi \cdot q)x(-\Xi \cdot q). \quad (4.30)$$

4.2.2. Root type Lax pair for untwisted C_n model based on long roots Δ_L

The Lax pair is given in terms of long roots. The matrix elements of L_L and M_L are labeled by indices Υ, Ω , etc.:

$$\begin{aligned} L_L(q, p, \xi) &= p \cdot H + X_d + X_s, \\ M_L(q, \xi) &= D + Y_d + Y_s. \end{aligned} \tag{4.31}$$

Here X_d and Y_d correspond to the part of the “long root – long root = $2 \times$ long root” of (4.22),

$$\begin{aligned} X_d &= 2ig_L \sum_{\Xi \in \Delta_L} x_d(\Xi \cdot q, \xi) E_d(\Xi), \quad Y_d = ig_L \sum_{\Xi \in \Delta_L} y_d(\Xi \cdot q, \xi) E_d(\Xi), \\ E_d(\Xi)_{\Upsilon\Omega} &= \delta_{\Upsilon-\Omega, 2\Xi}, \end{aligned} \tag{4.32}$$

and X_s and Y_s correspond to the “long root – long root = $2 \times$ short root” of (4.22),

$$X_s = 2ig \sum_{\alpha \in \Delta} x_d(\alpha \cdot q, \xi) E_d(\alpha), \quad Y_s = ig \sum_{\alpha \in \Delta} y_d(\alpha \cdot q, \xi) E_d(\alpha), \quad E_d(\alpha)_{\Upsilon\Omega} = \delta_{\Upsilon-\Omega, 2\alpha}. \tag{4.33}$$

The diagonal parts of L_L and M_L are given by

$$H_{\Upsilon\Omega} = \Upsilon \delta_{\Upsilon, \Omega}, \quad D_{\Upsilon\Omega} = \delta_{\Upsilon, \Omega} D_{\Upsilon}, \quad D_{\Upsilon} = -i \left(g_L z(\Upsilon \cdot q) + g \sum_{\kappa \in \Delta, \kappa \cdot \Upsilon = 2} z(\kappa \cdot q) \right). \tag{4.34}$$

Since only the functions x_d and y_d appear and neither function x nor y appears in the Lax pair, we can safely use x, y and z , which are used in the minimal-type Lax pairs, in place of x_d, y_d and z_d . It should be noted that the set of long roots (4.19) is 2 times the set of vector weights of C_n :

$$\Lambda = \{\pm e_j : j = 1, \dots, n\}.$$

In fact, the above L_L matrix is twice L for the vector representation, with two independent coupling constants^{2), 1)} (with proper identification):

$$L_L = 2L_{\text{vec}}, \quad M_L = M_{\text{vec}}.$$

In other words the root type Lax pair based on C_n long roots is equivalent to the vector representation Lax pair. This explains why two independent coupling constants are allowed in the vector representation Lax pair of the C_n model.²⁾

4.3. F_4 model

The set of F_4 roots consists of two parts, long and short roots:

$$\Delta_{F_4} = \Delta \cup \Delta_s. \tag{4.35}$$

Here the roots are conveniently expressed in terms of an orthonormal basis of \mathbb{R}^4 :

$$\begin{aligned} \Delta &= \{\alpha, \beta, \gamma, \dots\} = \{\pm e_j \pm e_k : j, k = 1, \dots, 4\}, \quad 24 \text{ roots}, \\ \Delta_s &= \{\lambda, \mu, \nu, \dots\} = \{\pm e_j, \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4) : j = 1, \dots, 4\}, \quad 24 \text{ roots}. \end{aligned} \tag{4.36}$$

The set of long roots has the same structure as the D_4 roots and the set of short roots has the same structure as the union of D_4 vector, spinor and anti-spinor weights. From this we know the root difference pattern:

$$F_4 : \quad \text{short root} - \text{short root} = \begin{cases} \text{long root,} \\ \text{short root,} \\ 2 \times \text{short root,} \\ \text{non-root,} \end{cases} \quad (4.37)$$

and

$$F_4 : \quad \text{long root} - \text{long root} = \begin{cases} \text{long root,} \\ 2 \times \text{long root,} \\ 2 \times \text{short root,} \\ \text{non-root.} \end{cases} \quad (4.38)$$

From this knowledge only we can construct the root type Lax pair for the F_4 model by following the same recipe as above.

4.3.1. Root type Lax pair for untwisted F_4 model based on short roots Δ_s

The Lax pair is given in terms of short roots. The matrix elements of L_s and M_s are labeled by indices μ, ν etc.:

$$\begin{aligned} L_s(q, p, \xi) &= p \cdot H + X + X_d + X_l, \\ M_s(q, \xi) &= D + D_l + Y + Y_d + Y_l. \end{aligned} \quad (4.39)$$

Here X and Y correspond to the part of the “short root – short root = short root” of (4.37),

$$X = ig_s \sum_{\lambda \in \Delta_s} x(\lambda \cdot q, \xi) E(\lambda), \quad Y = ig_s \sum_{\lambda \in \Delta_s} y(\lambda \cdot q, \xi) E(\lambda), \quad E(\lambda)_{\mu\nu} = \delta_{\mu-\nu, \lambda}, \quad (4.40)$$

and X_d and Y_d correspond to the “short root – short root = 2 × short root” of (4.37),

$$X_d = 2ig_s \sum_{\lambda \in \Delta_s} x_d(\lambda \cdot q, \xi) E_d(\lambda), \quad Y_d = ig_s \sum_{\lambda \in \Delta_s} y_d(\lambda \cdot q, \xi) E_d(\lambda), \quad E_d(\lambda)_{\mu\nu} = \delta_{\mu-\nu, 2\lambda}. \quad (4.41)$$

The additional terms X_l and Y_l correspond to the “short root – short root = long root” of (4.37),

$$X_l = ig \sum_{\alpha \in \Delta} x(\alpha \cdot q, \xi) E(\alpha), \quad Y_l = ig \sum_{\alpha \in \Delta} y(\alpha \cdot q, \xi) E(\alpha), \quad E(\alpha)_{\mu\nu} = \delta_{\mu-\nu, \alpha}. \quad (4.42)$$

The diagonal parts of L_s and M_s are given by

$$H_{\mu\nu} = \mu\delta_{\mu,\nu}, \quad D_{\mu\nu} = \delta_{\mu,\nu} D_\mu, \quad D_\mu = -ig_s \left(z(\mu \cdot q) + \sum_{\lambda \in \Delta_s, \lambda \cdot \mu = 1/2} z(\lambda \cdot q) \right), \quad (4.43)$$

and

$$(D_l)_{\mu\nu} = \delta_{\mu,\nu}(D_l)_\mu, \quad (D_l)_\mu = -ig \sum_{\alpha \in \Delta, \alpha \cdot \mu = 1} z(\alpha \cdot q). \tag{4.44}$$

The functions x, y, z and x_d, y_d, z_d are the same as those given in §2. It is easy to verify that

$$\text{Tr}(L_s^2) = 12\mathcal{H}_{F_4}, \tag{4.45}$$

in which the F_4 Hamiltonian is given by

$$\mathcal{H}_{F_4} = \frac{1}{2}p^2 - \frac{g^2}{2} \sum_{\alpha \in \Delta} x(\alpha \cdot q)x(-\alpha \cdot q) - g_s^2 \sum_{\lambda \in \Delta_s} x(\lambda \cdot q)x(-\lambda \cdot q). \tag{4.46}$$

It should be noted that this has the same general structure as the Hamiltonian of the B_n theory (4.10).

4.3.2. Root type Lax pair for untwisted F_4 model based on long roots Δ

The Lax pair is given in terms of long roots. The general structure of this Lax pair is essentially the same as that of the B_n theory, since the pattern of the long root–long root (4.38) is the same as that of B_n (4.4). This reflects the universal nature of the root type Lax pairs. For this reason we list the general form only without further explanation. This consists of the matrices with indices β, γ , etc.:

$$\begin{aligned} L_l(q, p, \xi) &= p \cdot H + X + X_d + X_s, \\ M_l(q, \xi) &= D + D_s + Y + Y_d + Y_s. \end{aligned} \tag{4.47}$$

$$X = ig \sum_{\alpha \in \Delta} x(\alpha \cdot q, \xi)E(\alpha), \quad Y = ig \sum_{\alpha \in \Delta} y(\alpha \cdot q, \xi)E(\alpha), \quad E(\alpha)_{\beta\gamma} = \delta_{\beta-\gamma, \alpha}. \tag{4.48}$$

$$X_d = 2ig \sum_{\alpha \in \Delta} x_d(\alpha \cdot q, \xi)E_d(\alpha), \quad Y_d = ig \sum_{\alpha \in \Delta} y_d(\alpha \cdot q, \xi)E_d(\alpha), \quad E_d(\alpha)_{\beta\gamma} = \delta_{\beta-\gamma, 2\alpha}. \tag{4.49}$$

$$X_s = 2ig_s \sum_{\lambda \in \Delta_s} x_d(\lambda \cdot q, \xi)E_d(\lambda), \quad Y_s = ig_s \sum_{\lambda \in \Delta_s} y_d(\lambda \cdot q, \xi)E_d(\lambda), \quad E_d(\lambda)_{\beta\gamma} = \delta_{\beta-\gamma, 2\lambda}. \tag{4.50}$$

The diagonal parts of L_l and M_l are given by

$$H_{\beta\gamma} = \beta\delta_{\beta,\gamma}, \quad D_{\beta\gamma} = \delta_{\beta,\gamma}D_\beta, \quad D_\beta = -ig \left(z(\beta \cdot q) + \sum_{\kappa \in \Delta, \kappa \cdot \beta = 1} z(\kappa \cdot q) \right), \tag{4.51}$$

and

$$(D_s)_{\beta\gamma} = \delta_{\beta,\gamma}(D_s)_\beta, \quad (D_s)_\beta = -ig_s \sum_{\lambda \in \Delta_s, \beta \cdot \lambda = 1} z(\lambda \cdot q). \tag{4.52}$$

The functions x, y, z and x_d, y_d, z_d are also the same as those given in §2. It is easy to verify that

$$\text{Tr}(L_l^2) = 24\mathcal{H}_{F_4}, \tag{4.53}$$

in which the F_4 Hamiltonian is the same as given above (4.10). In both cases the reduction from E_6 fixes $g_s = g$.

4.4. G_2 model

The set of G_2 roots consists of two parts, long and short roots:

$$\Delta_{G_2} = \Delta \cup \Delta_s. \tag{4.54}$$

Here the roots are conveniently expressed in terms of an orthonormal basis of \mathbb{R}^2 :

$$\begin{aligned} \Delta &= \{\alpha, \beta, \gamma, \dots\} = \{\pm(-3e_1 + \sqrt{3}e_2)/2, \pm(3e_1 + \sqrt{3}e_2)/2, \pm\sqrt{3}e_2\}, & 6 \text{ roots,} \\ \Delta_s &= \{\lambda, \mu, \nu, \dots\} = \{\pm e_1, \pm(-e_1 + \sqrt{3}e_2)/2, \pm(e_1 + \sqrt{3}e_2)/2\}, & 6 \text{ roots.} \end{aligned} \tag{4.55}$$

The sets of long and short roots have the same structure as the A_2 roots, scaled [(long root)² : (short root)² = 3 : 1] and rotated by $\pi/6$. The root difference pattern is

$$G_2 : \quad \text{short root} - \text{short root} = \begin{cases} \text{long root,} \\ \text{short root,} \\ 2 \times \text{short root,} \\ \text{non-root,} \end{cases} \tag{4.56}$$

$$G_2 : \quad \text{long root} - \text{long root} = \begin{cases} \text{long root,} \\ 2 \times \text{long root,} \\ 3 \times \text{short root,} \\ \text{non-root.} \end{cases} \tag{4.57}$$

The appearance of the $3 \times$ short root in (4.57) is a new feature.

4.4.1. Root type Lax pair for untwisted G_2 model based on short roots Δ_s

The Lax pair is given in terms of short roots. The general structure of this Lax pair is essentially the same as that of the F_4 theory, since the pattern of the short root – short root (4.56) is the same as that of the F_4 (4.37). Hence we list the general form only without further explanation. The matrix elements of L_s and M_s are labeled by indices μ, ν , etc.:

$$\begin{aligned} L_s(q, p, \xi) &= p \cdot H + X + X_d + X_l, \\ M_s(q, \xi) &= D + D_l + Y + Y_d + Y_l, \end{aligned} \tag{4.58}$$

$$X = ig_s \sum_{\lambda \in \Delta_s} x(\lambda \cdot q, \xi) E(\lambda), \quad Y = ig_s \sum_{\lambda \in \Delta_s} y(\lambda \cdot q, \xi) E(\lambda), \quad E(\lambda)_{\mu\nu} = \delta_{\mu-\nu, \lambda}. \tag{4.59}$$

$$\begin{aligned} X_d &= 2ig_s \sum_{\lambda \in \Delta_s} x_d(\lambda \cdot q, \xi) E_d(\lambda), \quad Y_d = ig_s \sum_{\lambda \in \Delta_s} y_d(\lambda \cdot q, \xi) E_d(\lambda), \\ E_d(\lambda)_{\mu\nu} &= \delta_{\mu-\nu, 2\lambda}. \end{aligned} \tag{4.60}$$

$$X_l = ig \sum_{\alpha \in \Delta} x(\alpha \cdot q, \xi) E(\alpha), \quad Y_l = ig \sum_{\alpha \in \Delta} y(\alpha \cdot q, \xi) E(\alpha), \quad E(\alpha)_{\mu\nu} = \delta_{\mu-\nu, \alpha}. \tag{4.61}$$

The diagonal parts of L_s and M_s are given by

$$H_{\mu\nu} = \mu\delta_{\mu,\nu}, \quad D_{\mu\nu} = \delta_{\mu,\nu}D_\mu, \quad D_\mu = -ig_s \left(z(\mu \cdot q) + \sum_{\lambda \in \Delta_s, \lambda \cdot \mu = 1/2} z(\lambda \cdot q) \right), \tag{4.62}$$

and

$$(D_l)_{\mu\nu} = \delta_{\mu,\nu}(D_l)_\mu, \quad (D_l)_\mu = -ig \sum_{\alpha \in \Delta, \alpha \cdot \mu = 3/2} z(\alpha \cdot q). \tag{4.63}$$

The functions x, y, z and x_d, y_d, z_d are the same as those given in §2. It is easy to verify that

$$\text{Tr}(L_s^2) = 6\mathcal{H}_{G_2}, \tag{4.64}$$

in which the G_2 Hamiltonian is given by

$$\mathcal{H}_{G_2} = \frac{1}{2}p^2 - \frac{g^2}{3} \sum_{\alpha \in \Delta} x(\alpha \cdot q)x(-\alpha \cdot q) - g_s^2 \sum_{\lambda \in \Delta_s} x(\lambda \cdot q)x(-\lambda \cdot q). \tag{4.65}$$

4.4.2. Root type Lax pair for untwisted G_2 model based on long roots Δ

This Lax pair is different from the others because of the ‘triple root’ term in (4.57). The matrix elements of L_l and M_l are labeled by indices β, γ , etc.:

$$\begin{aligned} L_l(q, p, \xi) &= p \cdot H + X + X_d + X_t, \\ M_l(q, \xi) &= D + Dt + Y + Y_d + Y_t. \end{aligned} \tag{4.66}$$

The terms X, Y and X_d, Y_d are the same as before:

$$X = ig \sum_{\alpha \in \Delta} x(\alpha \cdot q, \xi)E(\alpha), \quad Y = ig \sum_{\alpha \in \Delta} y(\alpha \cdot q, \xi)E(\alpha), \quad E(\alpha)_{\beta\gamma} = \delta_{\beta-\gamma, \alpha}. \tag{4.67}$$

$$X_d = 2ig \sum_{\alpha \in \Delta} x_d(\alpha \cdot q, \xi)E_d(\alpha), \quad Y_d = ig \sum_{\alpha \in \Delta} y_d(\alpha \cdot q, \xi)E_d(\alpha), \quad E_d(\alpha)_{\beta\gamma} = \delta_{\beta-\gamma, 2\alpha}. \tag{4.68}$$

The terms X_t and Y_t are associated with the ‘triple root’ with new functions x_t and y_t :

$$X_t = 3ig_s \sum_{\lambda \in \Delta_s} x_t(\lambda \cdot q, \xi)E_t(\lambda), \quad Y_t = ig_s \sum_{\lambda \in \Delta_s} y_t(\lambda \cdot q, \xi)E_t(\lambda), \quad E_t(\lambda)_{\beta\gamma} = \delta_{\beta-\gamma, 3\lambda}. \tag{4.69}$$

The diagonal parts of L_l and M_l are given by

$$H_{\beta\gamma} = \beta\delta_{\beta,\gamma}, \quad D_{\beta\gamma} = \delta_{\beta,\gamma}D_\beta, \quad D_\beta = -ig \left(z(\beta \cdot q) + \sum_{\kappa \in \Delta, \kappa \cdot \beta = 3/2} z(\kappa \cdot q) \right), \tag{4.70}$$

and

$$(Dt)_{\beta\gamma} = \delta_{\beta,\gamma}(Dt)_\beta, \quad (Dt)_\beta = -ig_s \sum_{\lambda \in \Delta_s, \beta \cdot \lambda = 3/2} z(\lambda \cdot q). \tag{4.71}$$

The pairs of functions $\{x, y\}$, $\{x_d, y_d\}$ and $\{x_t, y_t\}$ should each satisfy the *sum rule* (2.24). As in the other cases, $\{x, y\}$ and $\{x_d, y_d\}$ should satisfy the *second sum rule* (2.25). There is also a *third sum rule* to be satisfied by all of these functions:

$$\begin{aligned} 0 &= x(2u - v)y(u - 2v) - x(u - 2v)y(2u - v) - x(3v)y_t(u - 2v) \\ &\quad + y_t(2u - v)x(-3u) - 2x_d(3u)y_t(-u - v) + 2y_t(u + v)x_d(-3v) \\ &\quad - 3x_t(2u - v)y(-3u) + 3y(3v)x_t(u - 2v) \\ &\quad - 3x_t(u + v)y_d(-3v) + 3y_d(3u)x_t(-u - v). \end{aligned} \quad (4.72)$$

For the rational, trigonometric and hyperbolic potentials, all three sum rules are satisfied by the same set of functions as before. We have

$$\begin{aligned} x(t) = x_d(t) = x_t(t) &= \frac{1}{t}, & y(t) = y_d(t) = y_t(t) &= -\frac{1}{t^2}, & z(t) &= -\frac{1}{t^2}, \\ x(t) = x_d(t) = x_t(t) &= a \cot at, & y(t) = y_d(t) = y_t(t) &= -\frac{a^2}{\sin^2 at}, \\ z(t) = z_d(t) = z_t(t) &= -\frac{a^2}{\sin^2 at}, & a &: \text{const}, \\ x(t) = x_d(t) = x_t(t) &= a \coth at, & y(t) = y_d(t) = y_t(t) &= -\frac{a^2}{\sinh^2 at}, \\ z(t) = z_d(t) = z_t(t) &= -\frac{a^2}{\sinh^2 at}, & a &: \text{const}, \end{aligned} \quad (4.73)$$

and the Lax pair (4.66) is equivalent to the canonical equation of motion. For the elliptic potential with spectral parameter, a simple set of solutions is obtained in analogy with the solutions (2.21):

$$\begin{aligned} x(t, \xi) &= \frac{\sigma(\xi/3 - t)}{\sigma(\xi/3)\sigma(t)}, & y(t, \xi) &= x(t, \xi) [\zeta(t - \xi/3) - \zeta(t)], \\ & & z(t, \xi) &= -[\wp(t) - \wp(\xi/3)], \\ x_d(t, \xi) &= \frac{\sigma(2\xi/3 - t)}{\sigma(2\xi/3)\sigma(t)}, & y_d(t, \xi) &= x_d(t, \xi) [\zeta(t - 2\xi/3) - \zeta(t)], \\ & & z_d(t, \xi) &= -[\wp(t) - \wp(2\xi/3)], \\ x_t(t, \xi) &= \frac{\sigma(\xi - t)}{\sigma(\xi)\sigma(t)}, & y_t(t, \xi) &= x_t(t, \xi) [\zeta(t - \xi) - \zeta(t)], \\ & & z_t(t, \xi) &= -[\wp(t) - \wp(\xi)]. \end{aligned} \quad (4.74)$$

The spectral parameter independent functions are obtained by setting $\xi = \omega_j$, $j = 1, 2, 3$ with appropriate exponential factors:

$$\begin{aligned} x(t) &= \frac{\sigma(\omega_j/3 - t)}{\sigma(\omega_j/3)\sigma(t)} e^{\eta_j t/3}, & y(t) &= x(t) [\zeta(t - \omega_j/3) - \zeta(t) + \eta_j/3], \\ & & z(t) &= -[\wp(t) - \wp(\omega_j/3)], \\ x_d(t) &= \frac{\sigma(2\omega_j/3 - t)}{\sigma(2\omega_j/3)\sigma(t)} e^{2\eta_j t/3}, & y_d(t) &= x_d(t) [\zeta(t - 2\omega_j/3) - \zeta(t) + 2\eta_j/3], \end{aligned}$$

$$\begin{aligned}
 x_t(t) &= \frac{\sigma(\omega_j - t)}{\sigma(\omega_j)\sigma(t)} e^{\eta_j t}, & z_d(t) &= -[\wp(t) - \wp(2\omega_j/3)], \\
 & & y_t(t) &= x_t(t) [\zeta(t - \omega_j) - \zeta(t) + \eta_j], \\
 & & z_t(t) &= -[\wp(t) - \wp(\omega_j)]. \tag{4.75}
 \end{aligned}$$

They are doubly periodic meromorphic functions and may be viewed as generalisations of co- \wp functions. For example, for $j = 1$, the functions $x(t), x_d(t)$ and $x_t(t)$ have fundamental periods $\{2\omega_1, 12\omega_3\}, \{2\omega_1, 6\omega_3\}$ and $\{2\omega_1, 4\omega_3\}$, respectively. The properties of these functions will be discussed in a future publication.

4.5. BC_n root system Lax pair with three independent couplings

The BC_n root system consists of three parts, long, middle and short roots:

$$\Delta_{BC_n} = \Delta_L \cup \Delta \cup \Delta_s. \tag{4.76}$$

Here the roots are conveniently expressed in terms of an orthonormal basis of \mathbb{R}^n :

$$\Delta_L = \{\Xi, \Upsilon, \Omega, \dots\} = \{\pm 2e_j : j = 1, \dots, n\}, \quad 2n \text{ roots}, \tag{4.77}$$

$$\Delta = \{\alpha, \beta, \gamma, \dots\} = \{\pm e_j \pm e_k : j, k = 1, \dots, n\}, \quad 2n(n - 1) \text{ roots}, \tag{4.78}$$

$$\Delta_s = \{\lambda, \mu, \nu, \dots\} = \{\pm e_j : j = 1, \dots, n\}, \quad 2n \text{ roots}. \tag{4.79}$$

Here we consider the Lax pair based on the middle roots only. The pattern of middle root – middle root is

$$BC_n : \quad \text{middle root} - \text{middle root} = \begin{cases} \text{long root,} \\ \text{middle root,} \\ 2 \times \text{middle root,} \\ 2 \times \text{short root,} \\ \text{non-root.} \end{cases} \tag{4.80}$$

From this knowledge only we can construct the root type Lax pair for the BC_n root system:

$$\begin{aligned}
 L_m(q, p, \xi) &= p \cdot H + X + X_d + X_L + X_s, \\
 M_m(q, \xi) &= D + D_L + Y + Y_d + Y_L + D_s + Y_s. \tag{4.81}
 \end{aligned}$$

The matrix elements of L_m and M_m are labeled by indices β, γ , etc. Here $p \cdot H + X + X_d + X_L$ ($D + D_L + Y + Y_d + Y_L$) is exactly the same as L_s (M_s) matrix of the C_n models with two coupling constants based on short roots. So we give only the terms related to short roots: An additional term in L_m (M_m), X_s (Y_s) corresponds to “middle root – middle root = $2 \times$ short root” of (4.80):

$$X_s = 2ig_s \sum_{\lambda \in \Delta_s} x_d(\lambda \cdot q, \xi) E_d(\lambda), \quad Y_s = ig_s \sum_{\lambda \in \Delta_s} y_d(\lambda \cdot q, \xi) E_d(\lambda), \quad E_d(\lambda)_{\beta\gamma} = \delta_{\beta - \gamma, 2\lambda}. \tag{4.82}$$

$$D_s_{\beta\gamma} = \delta_{\beta, \gamma} D_s_{\beta}, \quad D_s_{\beta} = -ig_s \sum_{\lambda \in \Delta_s, \beta \cdot \lambda = 1} z(\lambda \cdot q). \tag{4.83}$$

The functions x, y, z and x_d, y_d, z_d are the same as those given in §2. It is easy to verify that

$$\mathrm{Tr}(L_m^2) = 8(n-1)\mathcal{H}_{BC_n}, \quad (4.84)$$

in which the BC_n Hamiltonian is the C_n Hamiltonian (4.30) plus the contribution from the short root potential with “renormalisation” of the short root coupling constant:

$$\begin{aligned} \mathcal{H}_{BC_n} = & \frac{1}{2}p^2 - \frac{g^2}{2} \sum_{\alpha \in \Delta} x(\alpha \cdot q)x(-\alpha \cdot q) - \frac{g_L^2}{4} \sum_{\Xi \in \Delta_L} x(\Xi \cdot q)x(-\Xi \cdot q) \\ & - \tilde{g}_s^2 \sum_{\lambda \in \Delta_s} x(\lambda \cdot q)x(-\lambda \cdot q), \quad \tilde{g}_s^2 = g_s(g_s + g_L/2). \end{aligned} \quad (4.85)$$

§5. Summary and comments

Universal Lax pairs for Calogero-Moser models based on simply laced root systems were presented for all of the four choices of potentials: rational, trigonometric, hyperbolic and elliptic, both with and without a spectral parameter (§2). These are the root type Lax pairs and the minimal type Lax pairs. The Calogero-Moser models based on simply laced root systems have discrete symmetries generated by the automorphisms of the Dynkin diagrams and the extended Dynkin diagrams of the root system. By combining the discrete symmetry arising from the automorphism of the extended Dynkin diagram with the periodicity of the elliptic potential, Calogero-Moser models for various twisted non-simply laced root systems are derived from those based on simply laced root systems (§3). The model associated with the affine Dynkin diagram $A_{2n}^{(2)}$ can be interpreted as a twisted version of the BC_n Calogero-Moser model.

The idea of the universal root type Lax pairs has been successfully generalised to all of the untwisted non-simply laced Calogero-Moser models (§4). For non-simply laced root systems, there are two kinds of root type Lax pairs: one based on the set of long roots, and the other on the set of short roots. They both contain as many independent coupling constants as independent Weyl orbits in the set of roots. For the BC_n root system, this implies that there are three independent coupling constants. The consistency of the G_2 root type Lax pair based on long roots requires a new set of functions when the potential is elliptic. A simple set of these functions is given.

We have not discussed the unified Lax pairs, root type as well as minimal type, of the twisted non-simply laced Calogero-Moser models (with independent coupling constants) derived in §3. This is an interesting subject because of its connection with (affine) Toda (lattice or field) theories.

Acknowledgements

This work is partially supported by a Grant-in-aid from the Ministry of Education, Science and Culture, Priority Area “Supersymmetry and unified theory of

elementary particles" (#707). A. J. B. is supported by the Japan Society for the Promotion of Science and the National Science Foundation under grant No. 9703595.

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