

## Calogero-Moser Models. IV

— *Limits to Toda Theory* —

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Calogero-Moser models and Toda models are well-known integrable multi-particle dynamical systems based on root systems associated with Lie algebras. The relation between these two types of integrable models is investigated at the levels of the Hamiltonians and the Lax pairs. The Lax pairs of Calogero-Moser models are specified by the representations of the reflection groups, which are not the same as those of the corresponding Lie algebras. The latter specify the Lax pairs of Toda models. The Hamiltonians of the elliptic Calogero-Moser models tend to those of Toda models as one of the periods of the elliptic function goes to infinity, provided the dynamical variables are properly shifted and the coupling constants are scaled. On the other hand most of Calogero-Moser Lax pairs, for example, the root type Lax pairs, do not have a consistent Toda model limit. The minimal type Lax pairs, which corresponds to the minimal representations of the Lie algebras, tend to the Lax pairs of the corresponding Toda models.

### §1. Introduction

This is the fifth paper in a series devoted to the integrable dynamical systems of Calogero-Moser type. In the first three papers<sup>1)</sup> (hereafter referred to as I, II and III), various Lax pairs were constructed and the symmetries of the systems were elucidated. In the fourth paper<sup>2)</sup> a universal Lax pair operator applicable to the models based on non-crystallographic as well as crystallographic root systems was constructed. When suitable representation spaces are chosen, the universal Lax pair reproduces all the Lax pairs obtained so far and many other representations give new Lax pairs. In this paper we focus on the relations of Calogero-Moser models and Toda models.

Calogero-Moser models<sup>3)</sup> and Toda models<sup>4), 5)</sup> are well-known integrable multi-particle dynamical systems based on the crystallographic root systems, i.e., those associated with Lie algebras. Though they are markedly different at first sight, it has long been known that for some particular root systems Toda models can be obtained as a special limit of the Calogero-Moser models.<sup>6)</sup> The purpose of this paper is to clarify this limit for all the Calogero-Moser models, now that we have a universal framework for the integrable structure of these models. The limit problem can be considered at two levels, or maybe even three. The first is the limits of the dynamical variables and the Hamiltonian. The second is those of the Lax pairs. Since there are many Lax pairs for one and the same Calogero-Moser model, the

result is expected to be varied. The third level would be the limits of the solutions of the equations of motion and of the associated linear problem of the Lax equations. We will not address the problems at the third level in this paper.

While for the Calogero-Moser models the set of all roots is necessary, only the simple roots enter the Toda models. The potentials of Toda models are exponential functions of the dynamical variables  $q$  (the coordinates), with the mass scale parameters being the only physically meaningful parameters at the classical level. The potentials of Calogero-Moser models are more varied. Besides the (independent) coupling constant(s) (for the long and short roots in the non-simply laced theories), the generic elliptic potentials, i.e., the Weierstrass  $\wp$  functions have two primitive periods  $\{2\omega_1, 2\omega_3\}$  as adjustable parameters. The other potentials, the trigonometric, hyperbolic and rational potentials are obtained as degenerate cases of the elliptic ones. As we will show in §2, the Hamiltonian of an elliptic Calogero-Moser model based on the root system of Lie algebra  $\mathfrak{g}$  tends to the Hamiltonian of a Toda model based on  $\mathfrak{g}$ , its dual algebra  $\mathfrak{g}^\vee$ , its affine counterpart  $\mathfrak{g}^{(1)}$  or its dual  $(\mathfrak{g}^{(1)})^\vee$ . The detailed conditions for the limits will be stated there. It should be remarked that the significance of the independent coupling constants in the non-simply laced Calogero-Moser models is somehow lost in the limit. The limits of the Lax pairs are more intriguing.

As shown in a previous paper,<sup>2)</sup> the essential ingredient of Calogero-Moser Lax pairs is the representation of the reflection groups, which are not identical with those of the corresponding Lie algebras. On the other hand the known Lax pairs of Toda models are formulated in terms of the representation of the Lie algebras. This poses an interesting question whether some Toda models Lax pairs which do not belong to any representations of the corresponding Lie algebras could be obtained as limits of Calogero-Moser Lax pairs, for example, the root type ones. The answer turns out to be negative as we will show in detail in §3. In fact most of Calogero-Moser Lax pairs do not have a consistent Toda model limit. The minimal type Lax pairs are shown to have consistent Toda model limits.

The rest of this paper is organised as follows. In §4 we give an intuitive method for constructing the Lax pairs of Toda models by explicit examples. This is inspired by the limit of the minimal type Lax pairs. Section 5 is devoted to summary and discussion. In Appendix A some definitions and useful formulas of the elliptic functions are given. Appendix B gives some explicit forms of the functions entering in Calogero-Moser Lax pairs. These functions are also considered in the limits to Toda models. Appendix C gives a new Lax pair for the  $BC_r$  Calogero-Moser model, which is necessary in §3. The asymptotic forms of various functions appearing in Calogero-Moser Lax pairs are given in Appendix D.

## §2. From the elliptic potentials to the exponential potentials

Let us start with the elliptic Calogero-Moser model, which is a Hamiltonian system associated with a root system  $\Delta$  of rank  $r$ . We consider two types of root systems. The first type is those root systems associated with finite Lie algebras. The second is the so-called  $BC_r$  system which is a union of the  $B_r$  roots and  $C_r$  roots.

The dynamical variables are the coordinates  $\{q^j\}$  and their canonically conjugate momenta  $\{p_j\}$ , which will be denoted by vectors in  $\mathbf{R}^r$

$$q = (q^1, \dots, q^r), \quad p = (p_1, \dots, p_r). \quad (2.1)$$

The Hamiltonian for the elliptic Calogero-Moser model is

$$\mathcal{H} = \frac{1}{2}p^2 + \sum_{\alpha \in \Delta} \frac{g_{|\alpha|}^2}{|\alpha|^2} V_{|\alpha|}(\alpha \cdot q), \quad (2.2)$$

in which the potential functions  $V_{|\alpha|}$  and the real coupling constants  $g_{|\alpha|}$  are defined on orbits of the corresponding finite reflection group, i.e., they are identical for roots in the same orbit. That is  $g_{|\alpha|} = g$  for all roots in simply laced models and  $g_{|\alpha|} = g_L$  for the long roots and  $g_{|\alpha|} = g_S$  for the short roots in non-simply laced models. The potential functions  $V_{|\alpha|}$  are:

1. *Untwisted elliptic potential* This applies to all of the root systems associated with Lie algebras and the potential function is

$$V_L(\alpha \cdot q) = V_S(\alpha \cdot q) = \wp(\alpha \cdot q | \{2\omega_1, 2\omega_3\}), \quad (2.3)$$

in which  $\wp$  is the Weierstrass  $\wp$  function with a pair of primitive periods  $\{2\omega_1, 2\omega_3\}$  (A.1). Throughout this paper we adopt the convention that the Weierstrass  $\wp$ ,  $\zeta$  and  $\sigma$  functions have the above standard periods, unless otherwise stated.

2. *Twisted elliptic potential* This applies to all of the non-simply laced root systems. Except for the  $G_2$  model, the potential functions are

$$V_L(\alpha \cdot q) = \wp(\alpha \cdot q | \{2\omega_1, 2\omega_3\}), \quad V_S(\alpha \cdot q) = \wp(\alpha \cdot q | \{\omega_1, 2\omega_3\}). \quad (2.4)$$

That is, the potential for the short roots has one half of the standard period in one direction, which we choose to be  $\omega_1$ . For the  $G_2$  model,

$$V_L(\alpha \cdot q) = \wp(\alpha \cdot q | \{2\omega_1, 2\omega_3\}), \quad V_S(\alpha \cdot q) = \wp\left(\alpha \cdot q \left| \left\{ \frac{2\omega_1}{3}, 2\omega_3 \right\} \right.\right). \quad (2.5)$$

Derivation of the twisted models from the untwisted ones by folding is given in paper II.<sup>1)</sup>

3. *Untwisted and twisted potentials for the  $BC_r$  system* The  $BC_r$  root system consists of the long, middle and short roots,  $\Delta = \Delta_L \cup \Delta_M \cup \Delta_S$ . The untwisted model has the same potential for all the roots

$$V_L(\alpha \cdot q) = V_M(\alpha \cdot q) = V_S(\alpha \cdot q) = \wp(\alpha \cdot q | \{2\omega_1, 2\omega_3\}). \quad (2.6)$$

The twisted model has potentials with the full, a half and a fourth periods:

$$\begin{aligned} V_L(\alpha \cdot q) &= \wp(\alpha \cdot q | \{2\omega_1, 2\omega_3\}), & V_M(\alpha \cdot q) &= \wp(\alpha \cdot q | \{\omega_1, 2\omega_3\}), \\ V_S(\alpha \cdot q) &= \wp(\alpha \cdot q | \{\omega_1/2, 2\omega_3\}). \end{aligned} \quad (2.7)$$

There are three independent coupling constants  $g_L$ ,  $g_M$  and  $g_S$  in both cases. In the discussion below the root systems associated with the Lie algebras are assumed. Modification for the  $BC_r$  case is straightforward.

On taking the limits to Toda models it is convenient to adopt the following parametrisation of the periods:

$$\omega_1 = -i\pi, \quad \omega_3 \in \mathbf{R}_+, \quad \tau \equiv \frac{\omega_3}{\omega_1} = i\omega_3/\pi. \quad (2.8)$$

Then the above Hamiltonian (2.2) is real for real dynamical variables  $p, q$  and coupling constants  $g_{|\alpha|}$ . If we let  $\omega_3 \rightarrow +\infty$  for fixed  $u$  the elliptic potentials tend to the hyperbolic ones

$$V_L(u) = \frac{1}{12} + \frac{1}{4} \frac{1}{\sinh^2 u/2}, \quad V_S(u) = \frac{1}{3} + \frac{1}{\sinh^2 u}. \quad (2.9)$$

In order to obtain the exponential potential from the elliptic potentials we follow the general prescription as explained in Appendix (A.6)–(A.12).<sup>6), 7)</sup> First we shift the dynamical variable  $q$

$$q = Q - 2\omega_3\delta v, \quad v \in \mathbf{R}^r, \quad (2.10)$$

in which  $\delta$  is a positive parameter and  $v$  is an as yet unspecified vector in  $\mathbf{R}^r$ . Let us require that  $v$  has a non-vanishing scalar product with all the roots in  $\Delta$

$$\alpha \cdot v \neq 0, \quad \forall \alpha \in \Delta. \quad (2.11)$$

Suppose there are some roots which are orthogonal to  $v$ . They form a sub-root system of  $\Delta$ . The potential functions for such roots will tend to the hyperbolic potentials (2.9) as  $\omega_3 \rightarrow +\infty$ , since their arguments  $\alpha \cdot q = \alpha \cdot Q$  are fixed. This justifies the above requirement (2.11). It should be remarked that the above shift (2.10) breaks the Weyl invariance of the Calogero-Moser models, by the introduction of the special vector  $v$ . On the other hand the set of positive roots  $\Delta_+$  and consequently the set of simple roots  $\Pi$  can be defined in terms of  $v$ :

$$\Delta_+ = \{\alpha \in \Delta, \alpha \cdot v > 0\}, \quad \Pi : \text{set of simple roots.} \quad (2.12)$$

Because of the  $2\omega_3$  periodicity and the even parity of the potentials,  $V_{|\alpha|}(u) = V_{|\alpha|}(-u)$ , we can assume, without loss of generality, that

$$\max_{\alpha \in \Delta_+} \delta \alpha \cdot v < 1. \quad (2.13)$$

In fact we require that  $\delta$  should satisfy a stronger condition

$$\delta((\alpha \cdot v)_{\min} + (\alpha \cdot v)_{\max}) \leq 1, \quad (2.14)$$

which implies that

$$\delta(\alpha \cdot v)_{\max} \leq 1 - \delta(\alpha \cdot v)_{\min} \quad (2.15)$$

and

$$2\delta(\alpha \cdot v)_{\min} \leq 1. \quad (2.16)$$

By comparing the shift formula

$$\alpha \cdot q = \alpha \cdot Q - 2\omega_3\delta \alpha \cdot v,$$

with the limit formula of the potentials (A.6)–(A.12), we find that for the following scalings of the coupling constants

$$g_L = m_L e^{\omega_3 \delta |\alpha \cdot v|_{\min}}, \quad g_S = \begin{cases} m_S e^{\omega_3 \delta |\alpha \cdot v|_{\min}} / \sqrt{2}, & \text{untwisted potential,} \\ m_S e^{2\omega_3 \delta |\alpha \cdot v|_{\min}} / 2\sqrt{2}, & \text{twisted potential,} \end{cases} \quad (2.17)$$

the elliptic potentials *vanish* for all  $\alpha \cdot q$  except for those roots having the minimum (and the maximum) value of the scalar product with the fixed vector  $v$ , for which the exponential potentials are obtained. That is we have

$$g_L^2 V_L(\alpha \cdot q) \xrightarrow{\omega_3 \rightarrow +\infty} \begin{cases} m_L^2 e^{\alpha \cdot Q} & \text{for such } \alpha \in \Delta_+ \text{ that } \alpha \cdot v \text{ is minimum,} \\ m_L^2 e^{-\alpha \cdot Q} & \text{for such } \alpha \in \Delta_+ \text{ that } \alpha \cdot v \text{ is maximum,} \\ 0 & \text{otherwise,} \end{cases} \quad (2.18)$$

and the corresponding formula for  $g_S^2 V_S(\alpha \cdot q)$ . It should be noted that for the twisted models the minimum (maximum) of  $(\alpha \cdot v)$  can be different for  $V_L$  and  $V_S$ , since only the long (short) roots contribute to  $V_L$  ( $V_S$ ). However, for the second possibility ( $\alpha \cdot v$  is maximum) to occur, the parameter  $\delta$  must be so chosen as to saturate the inequality in (2.14). As we will see shortly, (2.22) and (2.23), the saturation occurs only for the long root potentials.

In the formula (2.18) we considered only the positive roots  $\alpha$ . The number of non-vanishing potential terms of the resulting theory is determined by those positive roots which give the minimum (and the maximum) of  $\alpha \cdot v$ . Since all the positive roots are linear combination of simple roots with non-negative integer coefficients, the minimum can be attained by the simple roots only and the maximum by the highest root. There are a maximal number of potential terms when the minimum is attained by all the simple roots. All the other cases can be considered as arising from Calogero-Moser models based on some sub-root system of  $\Delta$ .

### 2.1. Models based on the root systems of Lie algebras

First let us discuss the models based on root systems associated with finite Lie algebras. We consider the case that the minimum of  $\alpha \cdot v$  is attained by all the simple roots. We adopt the convention that the long roots have squared length 2,  $\alpha_L^2 = 2$ . It is well-known that the Weyl vector  $\rho$  and its dual  $\rho^\vee$  defined by

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha, \quad \rho^\vee = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha^\vee, \quad \alpha^\vee = 2\alpha/\alpha^2 \quad (2.19)$$

satisfy the above criterion. In fact we have

$$\rho \cdot \alpha_i = \frac{\alpha_i^2}{2}, \quad \rho^\vee \cdot \alpha_i = 1, \quad \forall \alpha_i \in \Pi, \quad (2.20)$$

and

$$\rho \cdot \alpha_h = h^\vee - 1, \quad \rho^\vee \cdot \alpha_h = h - 1, \quad \alpha_h : \text{highest root,} \quad (2.21)$$

in which  $h$  and  $h^\vee$  are the Coxeter number and the dual Coxeter number, respectively. For the choice  $v = \rho^\vee$ , the minimum is always 1 for the long and short simple roots. For the non-simply laced root system, there are two different values of minimum for  $v = \rho$ , the Weyl vector, 1 for the long roots and  $1/2$  for the short roots. This corresponds to the existence of two different coupling constants  $g_L$  and  $g_S$ . For all root systems  $\Delta$  except  $A_1$ , we have  $|\rho \cdot \alpha|_{\min} < |\rho \cdot \alpha|_{\max}$ , with the long roots satisfying

$$|\rho \cdot \alpha|_{\min} + |\rho \cdot \alpha|_{\max} = h^\vee, \quad |\rho^\vee \cdot \alpha|_{\min} + |\rho^\vee \cdot \alpha|_{\max} = h \tag{2.22}$$

and the short roots

$$|\rho \cdot \alpha|_{\min} + |\rho \cdot \alpha|_{\max} < h^\vee, \quad |\rho^\vee \cdot \alpha|_{\min} + |\rho^\vee \cdot \alpha|_{\max} < h. \tag{2.23}$$

That is, the saturation of of the inequality (2.14) occurs only for the long roots for the choices of  $\delta = 1/h$  for  $v = \rho^\vee$  or  $\delta = 1/h^\vee$  for  $v = \rho$ . For  $A_1$  we have

$$\rho = \rho^\vee, \quad |\rho \cdot \alpha|_{\min} = |\rho \cdot \alpha|_{\max} = 1, \quad h = 2. \tag{2.24}$$

In the Hamiltonian of an elliptic Calogero-Moser model based on a root system  $\Delta$  which is associated with a Lie algebra  $\mathfrak{g}$ , we redefine the dynamical variables from  $\{p, q\}$  to  $\{P, Q\}$

$$q = Q - 2\omega_3\delta v, \quad p = P, \quad v = \rho \quad \text{or} \quad \rho^\vee, \tag{2.25}$$

and take the limit  $\omega_3 \rightarrow +\infty$  with  $\omega_1 = -i\pi$ . The coupling constants are also scaled:

$$g_L = m_L e^{\omega_3\delta}, \quad g_S = \begin{cases} m_S e^{\omega_3\delta}/\sqrt{2}, & \text{untwisted potential,} \\ m_S e^{2\omega_3\delta}/2\sqrt{2}, & \text{twisted potential,} \end{cases} \quad \text{for } v = \rho^\vee, \\ g_S = \begin{cases} m_S e^{\omega_3\delta/2}/\sqrt{2}, & \text{untwisted potential,} \\ m_S e^{\omega_3\delta}/2\sqrt{2}, & \text{twisted potential,} \end{cases} \quad \text{for } v = \rho. \tag{2.26}$$

In this limit we arrive at the Hamiltonian of the Toda models associated with the Lie algebra  $\mathfrak{g}$ , its dual algebra  $\mathfrak{g}^\vee$ , the untwisted affine Lie algebra  $\mathfrak{g}^{(1)}$  or its dual  $(\mathfrak{g}^{(1)})^\vee$  depending on the types of the potential, untwisted or twisted, and the values of the parameter  $\delta$ .

For the simply laced root system  $\Delta$ , we have the Toda system associated with the Lie algebra  $\mathfrak{g}$  for  $\delta < 1/h$ :

$$\mathcal{H} = \frac{1}{2}P^2 + m^2 \sum_{\alpha_i \in \Pi} e^{\alpha_i \cdot Q}, \tag{2.27}$$

and the Toda system associated with the untwisted affine Lie algebra  $\mathfrak{g}^{(1)}$  for  $\delta = 1/h$ :

$$\mathcal{H} = \frac{1}{2}P^2 + m^2 \left( \sum_{\alpha_i \in \Pi} e^{\alpha_i \cdot Q} + e^{\alpha_0 \cdot Q} \right), \tag{2.28}$$

in which  $\alpha_0$  is the affine root  $\alpha_0 = -\alpha_h$ . For the  $A_1$  case with  $\delta = 1/2$  the extreme situations (2.24) and (A.12) apply and give the above result (2.28).

For a non-simply laced root system  $\Delta$  and an untwisted potential, we have the Toda system associated with the Lie algebra  $\mathfrak{g}$ :

$$\mathcal{H} = \frac{1}{2}P^2 + m_L^2 \sum_{\alpha_i \in \Pi \cap \Delta_L} e^{\alpha_i \cdot Q} + m_S^2 \sum_{\alpha_i \in \Pi \cap \Delta_S} e^{\alpha_i \cdot Q}, \tag{2.29}$$

for  $v = \rho^\vee$  and  $\delta < 1/h$  or for  $v = \rho$  and  $\delta < 1/h^\vee$ , and the Toda system associated with the untwisted affine Lie algebra  $\mathfrak{g}^{(1)}$ :

$$\mathcal{H} = \frac{1}{2}P^2 + m_L^2 \left( \sum_{\alpha_i \in \Pi \cap \Delta_L} e^{\alpha_i \cdot Q} + e^{\alpha_0 \cdot Q} \right) + m_S^2 \sum_{\alpha_i \in \Pi \cap \Delta_S} e^{\alpha_i \cdot Q}, \tag{2.30}$$

for  $v = \rho^\vee$  and  $\delta = 1/h$  or for  $v = \rho$  and  $\delta = 1/h^\vee$ . In these formulas  $\Delta_L$  ( $\Delta_S$ ) is the set of long (short) roots.

For a non-simply laced root system  $\Delta$  and a twisted potential, we have the Toda system associated with the dual Lie algebra  $\mathfrak{g}^\vee$ :

$$\mathcal{H} = \frac{1}{2}P^2 + m_L^2 \sum_{\alpha_i \in \Pi \cap \Delta_L} e^{\alpha_i \cdot Q} + m_S^2 \sum_{\alpha_i \in \Pi \cap \Delta_S} e^{2\alpha_i \cdot Q}, \tag{2.31}$$

for  $\delta < 1/h$  and  $v = \rho^\vee$  or  $\delta < 1/h^\vee$  and  $v = \rho$ , and the Toda system associated with the twisted affine Lie algebra  $(\mathfrak{g}^{(1)})^\vee$ :

$$\mathcal{H} = \frac{1}{2}P^2 + m_L^2 \left( \sum_{\alpha_i \in \Pi \cap \Delta_L} e^{\alpha_i \cdot Q} + e^{\alpha_0 \cdot Q} \right) + m_S^2 \sum_{\alpha_i \in \Pi \cap \Delta_S} e^{2\alpha_i \cdot Q}, \tag{2.32}$$

for  $v = \rho^\vee$  and  $\delta = 1/h$  or for  $v = \rho$  and  $\delta = 1/h^\vee$ . The formulas for the twisted potential cases (2.31) and (2.32) are valid for all the non-simply laced models except for the one based on  $G_2$ . In this case the last term in (2.31) and (2.32) should be changed to  $e^{3\alpha_i \cdot Q}$  with appropriate scaling of  $g_S$ .

A few remarks are in order. In the Calogero-Moser models the meaning of the coupling constant is clear. They specify the strength of the repulsive potentials near the boundary of the Weyl chambers. Thus the independence of the coupling constants is quite crucial in the Calogero-Moser model. These properties are lost in the transition to the Toda models by the shift of the dynamical variables (2.25) and the scalings of the coupling constants (2.26). The remaining parameters  $m_L$  and  $m_S$ , (2.26), are generally considered as giving mass scales of the Toda theories. For the Toda theories based on finite Lie algebras, this interpretation is not adequate, since these theories are conformally invariant. In fact, in the Toda theories based on finite Lie algebras  $\mathfrak{g}$ , the mass parameters  $m_i^2$  can be changed arbitrarily. Suppose we start from the Hamiltonian

$$\mathcal{H} = \frac{1}{2}P^2 + \sum_{\alpha_i \in \Pi} m_i^2 e^{\alpha_i \cdot Q}$$

and make a shift

$$Q = Q' + \sum_{\alpha_i \in \Pi} \frac{2}{\alpha_i^2} \lambda_i \log \left( \frac{m_i'^2}{m_i^2} \right), \tag{2.33}$$

in which  $\{\lambda_i\}$ ,  $i = 1, \dots, r$  are the fundamental weights, satisfying  $\lambda_i \cdot \alpha_j^\vee = \delta_{ij}$  for  $\alpha_j \in \Pi$ . We arrive at

$$\mathcal{H} = \frac{1}{2} P^2 + \sum_{\alpha_i \in \Pi} m_i'^2 e^{\alpha_i \cdot Q'}. \tag{2.34}$$

2.2. *BC<sub>r</sub> model*

The *BC<sub>r</sub>* root system consists of three parts, long, middle and short roots:

$$\Delta_{BC_r} = \Delta_L \cup \Delta_M \cup \Delta_S,$$

in which the roots are conveniently expressed in terms of an orthonormal basis of  $\mathbf{R}^r$ :

$$\Delta_L = \{\pm 2e_j\}, \quad \Delta_M = \{\pm e_j \pm e_k\}, \quad \Delta_S = \{\pm e_j\} : \quad j = 1, \dots, r. \tag{2.35}$$

The set of simple roots is the same as that of *B<sub>r</sub>*:

$$\Pi = \{e_r\} \cup \{e_j - e_{j+1}, j = 1, \dots, r - 1\}. \tag{2.36}$$

If we define

$$\rho^\vee = \sum_{j=1}^r (r + 1 - j)e_j, \quad h = 2r + 1, \tag{2.37}$$

and the following scalings,

$$g_L = \sqrt{2}m_L e^{\omega_{3\delta}}, \quad g_M = m_M e^{\omega_{3\delta}}, \quad g_S = m_S e^{\omega_{3\delta}/\sqrt{2}}, \tag{2.38}$$

for the untwisted potential we obtain the non-affine *B<sub>r</sub>* Toda model for  $\delta < 1/h$  and *A<sub>2n</sub><sup>(2)</sup>* Toda model for  $\delta = 1/h$ :

$$\mathcal{H} = \frac{1}{2} P^2 + m_L^2 e^{-2Q_1} + m_M^2 \sum_{j=1}^{r-1} e^{Q_j - Q_{j+1}} + m_S^2 e^{Q_r}. \tag{2.39}$$

For the following scalings,

$$g_L = \sqrt{2}m_L e^{\omega_{3\delta}}, \quad g_M = m_M e^{2\omega_{3\delta}/2}, \quad g_S = m_S e^{4\omega_{3\delta}/4\sqrt{2}}, \tag{2.40}$$

and the twisted potentials, we obtain the non-affine *C<sub>r</sub>* Toda model for  $\delta < 1/h$  and another form of the *A<sub>2n</sub><sup>(2)</sup>* Toda model for  $\delta = 1/h$ :

$$\mathcal{H} = \frac{1}{2} P^2 + m_L^2 e^{-2Q_1} + m_M^2 \sum_{j=1}^{r-1} e^{2(Q_j - Q_{j+1})} + m_S^2 e^{4Q_r}. \tag{2.41}$$

This is due to the fact that the *A<sub>2n</sub><sup>(2)</sup>* root system is self-dual.



### §3. Limits of the Lax pairs

In this section we discuss the corresponding limits of the Calogero-Moser Lax pairs. Contrary to the Hamiltonian case which has well-defined limits to the Toda model Hamiltonians, the limits of the Lax pairs are diverse, some having well-defined limits to the Toda model Lax pairs and some not. We consider two different types of Calogero-Moser Lax pairs, the root type and the minimal type,<sup>1)</sup> both expressing the canonical equations of motion in terms of a pair of matrices  $L$  and  $M$

$$\dot{L} = \frac{d}{dt}L = [L, M]. \tag{3.1}$$

Roughly speaking the elements of the  $L$  matrix are square roots of the Hamiltonian, since  $\text{Tr}(L^2) \propto \mathcal{H}$ . These Lax pairs depend on an additional parameter  $\xi$ , the spectral parameter, which requires a shift proportional to  $\omega_3$  as we will see presently. Therefore the existence of a limit of the Hamiltonian does not imply that of a Lax pair, not to mention those of both types, since they correspond to different types of square roots of the Hamiltonian. In the following we examine the limits of various Lax pairs in turn.

#### 3.1. Minimal type Lax pair for simply laced root systems

This type of Lax pairs has the simplest structure and has a well-defined limit to the Lax pair of the corresponding Toda model, as expected. This applies to the models based on the root systems of  $A_r$ ,  $D_r$ ,  $E_6$  and  $E_7$ .

The minimal type Lax pairs have the following form,

$$\begin{aligned} L(q, p, \xi) &= p \cdot H + X, \\ M(q, \xi) &= D + Y. \end{aligned} \tag{3.2}$$

The matrix elements of  $L$  and  $M$  are labelled by the weights of a minimal representation, (I.4.1) in Ref. 1). The matrices  $H$  and  $D$  are diagonal

$$H_{\mu\nu} = \mu\delta_{\mu\nu} \quad \text{and} \quad D_{\mu\nu} = \delta_{\mu\nu}D_\mu, \quad D_\mu = ig \sum_{\Delta \ni \beta = \mu - \nu} \wp(\beta \cdot q). \tag{3.3}$$

The matrices  $X$  and  $Y$  have the form

$$X = ig \sum_{\alpha \in \Delta} x(\alpha \cdot q, \xi)E(\alpha), \quad Y = ig \sum_{\alpha \in \Delta} y(\alpha \cdot q, \xi)E(\alpha), \tag{3.4}$$

in which  $\xi$  is the spectral parameter and  $E(\alpha)_{\mu\nu} = \delta_{\mu-\nu, \alpha}$ . It should be stressed that the  $H$  and  $E(\alpha)$  here are *not* the Lie algebra generators for the associated Lie algebra  $\mathfrak{g}$ , though they satisfy relations

$$\begin{aligned} [H, E(\alpha)] &= \alpha E(\alpha), & [H, [E(\alpha), E(\beta)]] &= (\alpha + \beta)[E(\alpha), E(\beta)], \\ E(-\alpha) &= E(\alpha)^T, & [E(\alpha), E(-\alpha)] &= \alpha \cdot H. \end{aligned} \tag{3.5}$$

The function  $x(u, \xi)$  ( $y(u, \xi) = \partial_u x(u, \xi)$ ) is a solution of a certain functional equation involving the potential function,<sup>8)</sup> (I.2.14),<sup>1)</sup> and it factorises the potential

as

$$x(u, \xi)x(-u, \xi) = -\wp(u) + \wp(\xi). \tag{3.6}$$

It is not unique in the sense that if  $x(u, \xi)$  is a solution then

$$\tilde{x}(u, \xi) = x(u, \xi) e^{b(\xi)u}, \quad b(\xi) : \text{an arbitrary function} \tag{3.7}$$

is also a solution providing another factorisation, (II.2.27).<sup>1)</sup> As shown in the previous section, for

$$q = Q - 2\omega_3\delta\rho, \quad g = m e^{\omega_3\delta}, \tag{3.8}$$

we have the following limits of the potential,

$$g^2\wp(\alpha \cdot q) \xrightarrow{\omega_3 \rightarrow +\infty} \begin{cases} m^2 e^{\alpha \cdot Q}, & \alpha \in \Pi, & \delta < 1/h, \\ m^2 e^{-\alpha_h \cdot Q}, & \alpha_h : \text{highest root}, & \delta = 1/h, \\ 0, & \text{otherwise.} \end{cases} \tag{3.9}$$

(In a simply laced root system  $\rho = \rho^\vee$ .) In order to have a finite limit for  $g^2\wp(\xi)$ ,  $\xi$  needs to be shifted, too:

$$\xi = \log Z - 2\omega_3\epsilon, \quad Z \in \mathbf{R}_+, \quad \delta < \epsilon \leq 1/2, \tag{3.10}$$

in which the  $\omega_3$ -independent part of  $\xi$  is parametrised by a positive number  $Z$  for later convenience. From (A.8) we then find that  $g^2\wp(\xi) \propto e^{2\omega_3(\delta-|\epsilon|)}$  has a finite limit for  $\omega_3 \rightarrow +\infty$  up to a diverging constant. This constant is canceled by the one coming from  $g^2\wp(\alpha \cdot q)$  in the factorisation formula (3.6). For the consistency of the limit of the Lax pair with that of the Hamiltonian, the following limit of  $gx(\alpha \cdot q, \xi)$ ,

$$gx(\alpha \cdot q, \xi) \rightarrow \begin{cases} \text{finite} & \text{for } \pm \alpha_i \in \Pi \quad (\delta \leq 1/h) \quad \text{and } \pm \alpha_h \quad (\delta = 1/h), \\ 0 & \text{otherwise,} \end{cases} \tag{3.11}$$

is necessary. It is obvious that this condition selects, if any, a unique solution among the equivalent ones related by the symmetry transformation (3.7).

We will show that the following solution,<sup>9)</sup>

$$x(u, \xi) = \frac{\sigma(\xi - u)}{\sigma(\xi)\sigma(u)} \exp(\zeta(\xi)u) \tag{3.12}$$

satisfies the above condition (3.11) and gives a minimal type Lax pair which tends to a Toda model Lax pair in the limit  $\omega_3 \rightarrow +\infty$ . The asymptotic form of this solution can be evaluated using formulas (A.13) and (A.15).<sup>\*)</sup> For  $\sigma(\xi)$  only (A.13) is needed and for  $\sigma(\alpha \cdot q)$  and  $\sigma(\xi - \alpha \cdot q)$  both (A.13) and (A.15) are necessary according to the range of the arguments.

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<sup>\*)</sup> The formulas (3.13), (3.14), (3.17) and (3.18) are valid for all the root systems except for  $A_1$ . The  $A_1$  case needs to be considered separately because of (2.24). The Toda Lax pair (3.19) and (3.20) is valid for all the cases including the  $A_1$ .

For positive roots  $\alpha$ :

$$\begin{aligned} gx(\alpha \cdot q, \xi) &\rightarrow -m \exp\left(\frac{\alpha \cdot Q}{2}\right) \exp[\omega_3 \delta(1 - \rho \cdot \alpha)], & 0 < \epsilon - \delta \rho \cdot \alpha < 1, \\ &\rightarrow mZ \exp\left(-\frac{\alpha \cdot Q}{2}\right) \exp[\omega_3(\delta + \delta \rho \cdot \alpha - 2\epsilon)], & -1 < \epsilon - \delta \rho \cdot \alpha \leq 0, \end{aligned} \tag{3.13}$$

whilst for negative roots  $\alpha$ :

$$\begin{aligned} gx(\alpha \cdot q, \xi) &\rightarrow m \exp\left(-\frac{\alpha \cdot Q}{2}\right) \exp[\omega_3 \delta(1 + \rho \cdot \alpha)], & 0 < \epsilon - \delta \rho \cdot \alpha < 1, \\ &\rightarrow -\frac{m}{Z} \exp\left(\frac{\alpha \cdot Q}{2}\right) \exp[\omega_3(2\epsilon + \delta - \delta \rho \cdot \alpha - 2)], & 1 \leq \epsilon - \delta \rho \cdot \alpha < 2. \end{aligned} \tag{3.14}$$

For example, for the parameter ranges

$$\delta(h - 1) < \epsilon \leq 1/2, \quad \text{or} \quad \delta < 1/h, \quad \epsilon = 1/2, \tag{3.15}$$

the function  $gx(\alpha \cdot q, \xi)$  is non-vanishing only for the positive and negative simple roots. This corresponds to the Toda models based on simply laced finite Lie algebras. For

$$\epsilon = 1/2, \quad \delta = 1/h \tag{3.16}$$

the function  $gx(\alpha \cdot q, \xi)$  is non-vanishing only for the positive and negative simple roots and the highest roots. This corresponds to the Toda models based on simply laced affine Lie algebras.

In the other Lax matrix  $M$ , (3.2), there are terms  $D$  and  $Y$ . The limit of  $D$  simply vanishes under the scaling because it contains the term  $g\wp(\alpha \cdot q)$ , which has a power of  $g$  less than that appearing in the potential, which has a finite limit. Calculating the limit for  $Y$  is also not very difficult as we have already calculated the limit of  $gx(\alpha \cdot q)$  and  $gy(\alpha \cdot q) = gx'(\alpha \cdot q)$ . The result is summarised as follows ( $\alpha$  is a positive root):

$$\begin{aligned} gx(\alpha \cdot q, \xi) &\rightarrow -m \exp\left(\frac{\alpha \cdot Q}{2}\right), & \text{simple roots,} \\ &\rightarrow mZ \exp\left(-\frac{\alpha \cdot Q}{2}\right), & \text{highest root,} \\ &\rightarrow 0, & \text{otherwise,} \\ gy(\alpha \cdot q, \xi) &\rightarrow -\frac{m}{2} \exp\left(\frac{\alpha \cdot Q}{2}\right), & \text{simple roots,} \\ &\rightarrow -\frac{mZ}{2} \exp\left(-\frac{\alpha \cdot Q}{2}\right), & \text{highest root,} \\ &\rightarrow 0, & \text{otherwise,} \end{aligned} \tag{3.17}$$

and

$$\begin{aligned}
 gx(-\alpha \cdot q, \xi) &\rightarrow m \exp\left(\frac{\alpha \cdot Q}{2}\right), && \text{simple roots,} \\
 &\rightarrow -\frac{m}{Z} \exp\left(-\frac{\alpha \cdot Q}{2}\right), && \text{highest root,} \\
 &\rightarrow 0, && \text{otherwise,} \\
 gy(-\alpha \cdot q, \xi) &\rightarrow -\frac{m}{2} \exp\left(\frac{\alpha \cdot Q}{2}\right), && \text{simple roots,} \\
 &\rightarrow -\frac{m}{2Z} \exp\left(-\frac{\alpha \cdot Q}{2}\right), && \text{highest root,} \\
 &\rightarrow 0, && \text{otherwise.} \tag{3.18}
 \end{aligned}$$

The Lax pair now reads

$$\begin{aligned}
 L &= P \cdot H - im \sum_{\alpha \in \Pi} \exp\left(\frac{\alpha \cdot Q}{2}\right) [E(\alpha) - E(-\alpha)] \\
 &\quad + im \exp\left(\frac{\alpha_0 \cdot Q}{2}\right) [ZE(-\alpha_0) - Z^{-1}E(\alpha_0)], \tag{3.19}
 \end{aligned}$$

$$\begin{aligned}
 M &= -\frac{i}{2}m \sum_{\alpha \in \Pi} \exp\left(\frac{\alpha \cdot Q}{2}\right) [E(\alpha) + E(-\alpha)] \\
 &\quad -\frac{i}{2}m \exp\left(\frac{\alpha_0 \cdot Q}{2}\right) [ZE(-\alpha_0) + Z^{-1}E(\alpha_0)]. \tag{3.20}
 \end{aligned}$$

For the Toda models based on a finite Lie algebra  $\mathfrak{g}$ , one should drop the terms containing the affine root  $\alpha_0$ . The parameter  $Z$  which is a scaled version of the original spectral parameter  $\xi$  now plays the role of a spectral parameter for the affine Toda model based on  $\mathfrak{g}^{(1)}$ . It should be stressed again that although the matrices  $E(\alpha)$  are not Lie algebra generators as a whole, they satisfy the necessary relations for the Toda model Lax pairs

$$[E(\alpha), E(-\beta)] = 0, \quad \alpha, \beta \in \Pi \cup \{\alpha_0\}, \tag{3.21}$$

on top of those listed in (3.5). The non-vanishing intermediate state  $\kappa$  in the above commutation relations either does not exist (for the case  $\alpha \cdot \beta = -1$ ), or if it exists it forms a pair which cancels with each other (for the case  $\alpha \cdot \beta = 0$ ). This is due to the fact that the weights of a minimal representation form a single Weyl orbit. Thus we find that

$$\begin{aligned}
 \dot{L} &= [L, M] \iff \dot{Q} = P, \\
 \dot{P} &= -m^2 \left( \sum_{\alpha \in \Pi} \exp(\alpha \cdot Q)\alpha + \exp(\alpha_0 \cdot Q)\alpha_0 \right). \tag{3.22}
 \end{aligned}$$

3.2. Root type Lax pair for simply laced root systems

This type of Lax pairs applies universally to all the Calogero-Moser models based on simply laced root systems.<sup>1),2)</sup> Its representation space is the set of roots  $\Delta$  itself. Thus it is not related to any representation of the associated algebra  $\mathfrak{g}$  except for the simplest case of  $A_1$ . This type of Lax pairs does *not* have a well-defined limit to the Lax pair of the corresponding Toda model. Thus the Lax pair of a Toda model which is not Lie algebra valued cannot be constructed in this way.

The root type Lax pair for simply laced root systems reads

$$\begin{aligned} L(q, p, \xi) &= p \cdot H + X + X_d, \\ M(q, \xi) &= D + Y + Y_d. \end{aligned} \tag{3.23}$$

All of the matrices are labelled by the roots,  $\alpha, \beta, \gamma \in \Delta$ . The matrices  $H, D$  and  $X$  have a similar structure to those in the minimal type Lax pair, except that

$$E(\alpha)_{\beta\gamma} = \delta_{\beta-\gamma, \alpha}.$$

The matrices  $X_d$  and  $Y_d$  are special for the root type Lax pair:

$$X_d = 2ig \sum_{\alpha \in \Delta} x_d(\alpha \cdot q, \xi) E_d(\alpha), \quad Y_d = ig \sum_{\alpha \in \Delta} y_d(\alpha \cdot q, \xi) E_d(\alpha), \quad E_d(\alpha)_{\beta\gamma} = \delta_{\beta-\gamma, 2\alpha}. \tag{3.24}$$

The functions  $x(u, \xi)$  and  $x_d(u, \xi)$  are solutions of coupled functional equations (II.2.24) and (II.2.25).<sup>1)</sup> They share similar properties. For example, similar to the factorisation of the potential in terms of  $x(u, \xi)$  as in (3.6), we have another factorisation

$$x_d(u, \xi)x_d(-u, \xi) = -\wp(u) + \wp(2\xi). \tag{3.25}$$

Thus, for the consistent limit of the Lax pair,  $x_d(u, \xi)$  should have the same type of asymptotic behaviour as that of  $x(u, \xi)$  (3.11):

$$gx_d(\alpha \cdot q, \xi) \rightarrow \begin{cases} \text{finite} & \text{for } \pm \alpha_i \in \Pi \text{ and } \pm \alpha_h, \\ 0 & \text{otherwise.} \end{cases} \tag{3.26}$$

However, this is not the case, since it is not compatible with the following functional identity (III.3.21),<sup>1)</sup>

$$x(2u, \xi)x_d(-u, \xi) + x(-2u, \xi)x_d(u, \xi) = -\wp(u) + \wp(\xi), \tag{3.27}$$

which is a simple consequence of the general functional equation (II.2.25).<sup>1)</sup> We multiply  $g^2$  to (3.27) and choose  $\alpha$  to be a simple root

$$u = \alpha \cdot q = \alpha \cdot Q - 2\omega_3 \delta \rho \cdot \alpha, \quad \rho \cdot \alpha = 1.$$

Then the right-hand side is finite as in (3.9). On the left-hand side

$$gx(\pm 2\alpha \cdot q, \xi) \rightarrow 0$$

since both have twice the damping factor. This means that either  $gx_d(\alpha \cdot q, \xi)$  or  $gx_d(-\alpha \cdot q, \xi)$  or both must be divergent for simple roots  $\alpha$ . Thus the desired asymptotic behaviour of  $gx_d(u, \xi)$  (3.26) is not achieved and the Lax pair has no consistent limit. In Appendix D we list the asymptotic forms of  $x_d$  and other functions.

### 3.3. Other root type Lax pairs without Toda limits

The above argument for the non-existence of Toda limits for Calogero-Moser Lax pairs, based as it is on the non-existence of finite  $x$  and  $x_d$  functions for the simple roots, can be applied to other models whose Lax pairs contain the root type Lax pair ( $x$  and  $x_d$  functions). Thus the following Lax pairs do not have a consistent Toda limit:

1. Root type Lax pair based on long roots for untwisted and twisted  $B_r$  model.
2. Root type Lax pair based on short roots for untwisted and twisted  $C_r$  model.
3. Root type Lax pair based on long and short roots for untwisted and twisted  $F_4$  model.
4. Root type Lax pair based on long and short roots for untwisted and twisted  $G_2$  model.

The set of the long roots of  $B_r$  is the same as the set of the roots of  $D_r$ . Thus the  $B_r$  Lax pairs based on the long roots for the untwisted and the twisted models (see §4 of paper III<sup>1)</sup>) contain the functions  $x$  and  $x_d$ .

The set of short roots of  $C_r$  is the same as the set of roots of  $D_r$ . Thus the  $C_r$  Lax pairs based on the short roots for the untwisted model (see §4 of paper III<sup>1)</sup>) contain the functions  $x$  and  $x_d$ . The  $C_r$  Lax pair based on the short roots for the twisted model contains the twisted functions  $x^{(1/2)}$  and  $x_d^{(1/2)}$ . For these functions the twisted analogue of the identity (3·27) reads

$$x^{(1/2)}(2u, \xi)x_d^{(1/2)}(-u, \xi) + x^{(1/2)}(-2u, \xi)x_d^{(1/2)}(u, \xi) = -\wp(u|\{\omega_1, 2\omega_3\}) + f(\xi). \quad (3\cdot28)$$

This can be obtained from (III.5·10),<sup>1)</sup> and  $f(\xi)$  is a  $\xi$  dependent constant of integration. Thus we know that  $x^{(1/2)}(\alpha \cdot q, \xi)$  and  $x_d^{(1/2)}(\alpha \cdot q, \xi)$  for  $\pm$  simple roots cannot be finite at the same time by the argument given in the previous section. It should be stressed that this conclusion does not depend on a particular choice of solutions  $x^{(1/2)}$  and  $x_d^{(1/2)}$ , e.g. (III.5·15),<sup>1)</sup> since (3·28) is a consequence of the functional equations. In fact, it also applies to a different set of solutions  $x^{(1/2)}$  and  $x_d^{(1/2)}$  given in Appendix B which are equivalent to those used in Ref. 7) for the Lax pair of the  $F_4$  model based on short roots.

Since the algebra  $F_4$  is self-dual,  $F_4^\vee = F_4$ , the sets of long roots and of short roots have the same structure as the set of roots for  $D_4$ . Thus the  $F_4$  Lax pair based on the long roots for the untwisted and twisted models (see §4 of paper III<sup>1)</sup>) contains the functions  $x$  and  $x_d$ . The  $F_4$  Lax pair based on the short roots for the untwisted model contains the functions  $x$  and  $x_d$  and the twisted models the twisted functions  $x^{(1/2)}$  and  $x_d^{(1/2)}$ .

The algebra  $G_2$  is also self-dual,  $G_2^\vee = G_2$ . The sets of long roots and of short roots have the same structure as the set of  $A_2$  roots. The same argument as that for  $F_4$  applies to this case, although the twisting is threefold.

### 3.4. Minimal type Lax pair for non-simply laced root systems

This applies to the untwisted and twisted  $B_r$  models based on the spinor representation and the untwisted and twisted  $C_r$  models based on the vector representa-

tion. These can be handled in a unified way. The Toda limits of the Lax pairs exist only for the untwisted models and for both choices  $v = \rho, \rho^\vee$ .

First let us consider the untwisted potential case. The Lax pair contains one function  $x$  only. The choice  $v = \rho$  is suitable for the Toda limit because in this case the minimum of  $\rho \cdot \alpha (= 1/2)$  is achieved by the short simple roots and the long simple roots have  $\rho \cdot \alpha = 1$ . Therefore the different scalings, as explained in (2.26), for  $g_L$  and  $g_S$  would produce a consistent Toda Lax pair. That is  $g_L = m_L e^{\omega_3 \delta}$  and  $g_S = m_S e^{\omega_3 \delta / 2} / \sqrt{2}$  with  $\delta < 1/h^\vee (= 1/h)$  for non-affine (affine) theories.

For the choice  $v = \rho^\vee$ ,  $\delta < 1/h (= 1/h)$ ,  $g_L = m_L e^{\omega_3 \delta}$ ,  $g_S = m_S e^{\omega_3 \delta} / \sqrt{2}$ , the minimum of  $\rho^\vee \cdot \alpha$  is achieved by all the simple roots. Thus the minimal type Lax pair has a finite limit and gives non-affine  $B_r$  ( $C_r$ ) or affine  $B_r^{(1)}$  ( $C_r^{(1)}$ ) Toda models for the same parameter range as in the minimal type Lax pairs for the simply laced root systems.

The Lax pair for the twisted potentials contains two functions,  $x$  and  $x^{(1/2)}$ . These functions satisfy the relations (III. 5.23),<sup>1)</sup>

$$x(u, \xi) x^{(1/2)}(-u, \xi) + x(-u, \xi) x^{(1/2)}(u, \xi) = -2\wp(u) + g(\xi), \quad (3.29)$$

in which  $g(\xi)$  is a  $\xi$ -dependent constant of integration. Based on this formula one can show that the Toda model limit does not exist for either choice of  $v = \rho$  or  $\rho^\vee$  as in the root type Lax pair cases.

### 3.5. Root type Lax pair for non-simply laced root systems

In this subsection the root type Lax pair for non-simply laced root systems which are not treated in §3.3 will be discussed. These are the untwisted and twisted  $B_r$  models based on the short roots and the untwisted and twisted  $C_r$  models based on the long roots. As has been pointed out in paper II,<sup>1)</sup> the root type Lax pair for the  $C_r$  model based on the long roots is equivalent to the minimal type Lax pair. We will therefore discuss Toda model limits of the  $B_r$  model Lax pair based on the short roots.

First let us consider the untwisted  $B_r$  model. In this case the Lax pair contains the function  $x$  for the long roots and  $x_d$  for the short roots. For the choice  $v = \rho^\vee$ , the long roots and short roots are treated equally. Thus the same argument based on (3.27) applies and the Lax pair has no consistent Toda model limit. For the choice  $v = \rho$ , the simple short roots have half the decreasing factor of the simple long roots. This requires  $g_L = m_L e^{\omega_3 \delta}$ ,  $g_S = m_S e^{\omega_3 \delta / 2} / 2\sqrt{2}$  for the finite potentials. By multiplying  $g_L g_S$  to (3.27), we obtain

$$g_L x(2u, \xi) g_S x_d(-u, \xi) + g_L x(-2u, \xi) g_S x_d(u, \xi) = -g_L g_S (\wp(u) - \wp(\xi)).$$

Suppose that  $g_L x(\alpha \cdot q)$  and  $g_S x_d(\alpha \cdot q)$  are both finite for the simple roots. Let us choose  $u = \alpha \cdot q$  with  $\alpha$  being a simple short root. The left-hand side is then finite, but the right-hand side is divergent since  $g_S^2 \wp(\alpha \cdot q)$  is finite by assumption and the right-hand side has an extra divergent factor  $g_L / g_S \propto e^{\omega_3 \delta / 2}$ . Thus the Lax pair for the  $B_r$  untwisted model based on the short roots has no consistent Toda model limit.

Next we consider the twisted model. In this case the Lax pair contains the function  $x$  for the long roots and  $x_d^{(1/2)}$  for the short roots. For  $v = \rho$ ,  $g_L x(\alpha \cdot q, \xi)$  and  $g_S x_d^{(1/2)}(\alpha \cdot q, \xi)$  are non-vanishing for the simple roots with the scalings  $g_L = m_L e^{\omega_{3\delta}}$ ,  $g_S = m_S e^{\omega_{3\delta}}/2\sqrt{2}$ . We have a relation (III. 5.22),<sup>1)</sup>

$$x(2u, \xi)x_d^{(1/2)}(-u, \xi) + x(-2u, \xi)x_d^{(1/2)}(u, \xi) = -\wp(u|\{\omega_1, 2\omega_3\}) + \wp(\xi) - \wp(\omega_1), \tag{3.30}$$

which is consistent with the limit of the Hamiltonian. The result is summarised as follows ( $\alpha$  is a positive root):

$$\begin{aligned} g_L x(\alpha \cdot q, \xi) &\rightarrow -m_L \exp\left(\frac{\alpha \cdot Q}{2}\right), && \text{long simple roots,} \\ &\rightarrow m_L Z \exp\left(-\frac{\alpha \cdot Q}{2}\right), && \text{highest root,} \\ &\rightarrow 0, && \text{otherwise,} \\ g_S x_d^{(1/2)}(\alpha \cdot q, \xi) &\rightarrow -m_S \exp(\alpha \cdot Q), && \text{short simple roots,} \\ &\rightarrow 0, && \text{otherwise.} \end{aligned} \tag{3.31}$$

The resulting Lax pair is that of the  $C_r = (B_r)^\vee$  Toda model for the parameter range

$$\delta(h - 1) < \epsilon \leq 1/2, \quad \text{or} \quad \delta < 1/h, \quad \epsilon = 1/2,$$

and the Lax pair of the Toda model based on the twisted affine algebra  $A_{2r-1}^{(2)} = (B_r^{(1)})^\vee$  for

$$\epsilon = 1/2, \quad \delta = 1/h.$$

The finiteness of the potential for long and short simple roots when  $v = \rho^\vee$  requires different scalings of  $g_L$  and  $g_S$ , namely  $g_L = m_L e^{\omega_{3\delta}}$ ,  $g_S = m_S e^{2\omega_{3\delta}}/2\sqrt{2}$ , (2.26). For  $\delta < 1/h$  ( $= 1/h$ ) we obtain a finite non-affine (affine) Toda Lax pair.

### 3.6. Root type Lax pairs for the $BC_r$ model based on the long and/or short roots

The set of the middle roots of the  $BC_r$  root system is the same as the set of roots of  $D_r$ . Consequently the root type Lax pair for the  $BC_r$  model based on its middle roots does not have a consistent Toda model limit. This leads us to consider the root type Lax pair based on the long or the short roots. Since the  $BC_r$  root system is self-dual, the Lax pair based on the long roots and that based on the short roots are related by the duality transformation. Thus we consider only the root type Lax pair based on the short roots. Our interest lies in the case in which the potentials for the long, middle and short roots survive in the Toda limit. The other situations are the same as in the  $B_r$  or  $C_r$  cases.

First let us consider the untwisted potentials. In this case the Lax pair contains two functions,  $x$  for the long and the middle roots and  $x_d$  for the short roots. For the choice of  $v = \rho^\vee$  given in (2.37),  $\rho^\vee \cdot \alpha = 1$  for the simple middle and the simple short roots. As shown in the root type Lax pair for the non-simply laced case,



$g_M x(\alpha \cdot q, \xi)$  and  $g_S x_d(\alpha \cdot q, \xi)$  cannot be simultaneously finite for the simple roots. Here the scalings of the couplings are determined by the Hamiltonian (2.38). Thus the Lax pair does not have a Toda model limit for untwisted potentials.

Next we consider the most general twisted model, the extended  $BC_r$  model with five independent coupling constants, (III.4.87).<sup>1),10)</sup> We give the Lax pair for this model in Appendix C. It is easy to see that the extended function  $x_d^{(1/2)}$  for the short roots cannot be finite at the same time as  $x^{(1/2)}$  for the middle roots. Similarly, another extended function  $x^{(1/2)}$  for the long roots cannot be finite at the same time as  $x^{(1/2)}$  for the middle roots. Thus for the consistent Toda model limit we have to put  $g_{S_1} = g_{L_2} = 0$  and we are led to the ordinary twisted model with  $g_{L_1} \equiv g_L$ ,  $g_M$  and  $g_{S_2} \equiv g_S$  should be scaled as in (2.40), which is determined by the limit of the Hamiltonian. Now we have three functions  $x$  for the long roots,  $x^{(1/2)}$  for the middle roots and  $x_d^{(1/4)}$  for the short roots in the Lax pair. The coexistence of the function  $x$  and  $x^{(1/2)}$  cannot have a finite limit as discussed in §3.4 (see (3.29)). We thus come to the conclusion that the  $A_{2r}^{(2)}$  Toda Lax pair cannot be obtained as a limit from the Lax pair of the  $BC_r$  Calogero-Moser Lax pair.

Let us close this section with a remark on the Toda model limits of the Lax pairs of Calogero-Moser model in the *generic* representations of the reflection groups (see §5 of Ref. 2)). It is rather straightforward to see that the consistent Toda model limit does not exist in general. The Lax pair contains the function  $x(\alpha \cdot q, \alpha^\vee \cdot \mu \xi)$ , in which  $\mu$  is a generic basis vector of the representation. This means that there are as many different functions of  $\alpha \cdot q$  as there are different values of  $\alpha^\vee \cdot \mu$  corresponding to one potential function  $\wp(\alpha \cdot q)$ . It is in general not possible that all of them have the same Toda model limit.

#### §4. Lax pairs of Toda models

In this section we give a general intuitive method of constructing Lax pairs for Toda models. This method works for systems associated with  $A$ ,  $B$ ,  $C$  and  $D$  series. The method works for those root systems for which the simple roots and the affine root can be written in the form  $\pm 2e_i$ ,  $\pm e_i \pm e_j$  or  $\pm e_i$ , where  $\{e_i\}$  forms an orthonormal basis. A hint for this method actually comes from the limits of the minimal type Calogero-Moser Lax pairs described in the preceding section. We explain the method with simple examples.

##### 4.1. $C_2^{(1)}$

First we discuss the case where the roots are of the form  $\pm 2e_i$  and  $\pm e_i \pm e_j$  (the case of the form  $\pm e_i$  will be discussed later in the section). Let us consider the affine  $C_2^{(1)}$  Toda model. The Hamiltonian is given by

$$\mathcal{H}_{C_2^{(1)}} = \frac{1}{2}(p_1^2 + p_2^2) + m^2 e^{q_1 - q_2} + \frac{m^2}{2}(e^{2q_2} + e^{-2q_1}). \quad (4.1)$$

The canonical equations of motion are

$$\begin{aligned} p_1 &= -\frac{\partial \mathcal{H}}{\partial q_1} = -m^2 e^{q_1 - q_2} + m^2 e^{-2q_1}, \\ p_2 &= -\frac{\partial \mathcal{H}}{\partial q_2} = m^2 e^{q_1 - q_2} - m^2 e^{2q_2}, \end{aligned} \tag{4.2}$$

with  $\dot{q}_1 = p_1$  and  $\dot{q}_2 = p_2$ . In this case the two simple roots are  $2e_2$  and  $e_1 - e_2$ . The affine root is given by  $-2e_1$ . The  $L$  matrix is constructed in the following way. The dimension  $dim$  of the Lax pair would be twice the rank of the algebra (this is not always the case, as we remark afterwards). The diagonal part of  $L$  is given by

$$L_{i,i} = -L_{2r+1-i,2r+1-i} = p_i, \quad i = 1, 2, \dots, r; \quad r : \text{rank.} \tag{4.3}$$

Now corresponding to every simple or affine root we have off-diagonal elements in  $L$ . Suppose we have a simple root of the form  $e_i - e_j$ . Corresponding to this we have

$$L_{i,j} = -L_{j,i} = L_{2r+1-j,2r+1-i} = -L_{2r+1-i,2r+1-j} = im \exp \frac{1}{2}(q_i - q_j), \quad i \neq j. \tag{4.4}$$

For the present example  $C_2$ ,  $2r = 4$  and there is a simple root  $e_1 - e_2$  and correspondingly we have

$$L_{1,2} = -L_{2,1} = L_{3,4} = -L_{4,3} = im \exp \frac{1}{2}(q_1 - q_2). \tag{4.5}$$

Another type of root is of the form  $\pm 2e_i$ . Corresponding to these we have counter diagonal elements in  $L$  as follows:

$$L_{i,2r+1-i} = -L_{2r+1-i,i} = \pm im \exp(\pm q_i). \tag{4.6}$$

In the  $C_2$  case corresponding to the affine root  $-2e_1$  and a simple root  $2e_2$  then one has

$$L_{1,4} = -L_{4,1} = -im \exp(-q_1) \quad \text{and} \quad L_{2,3} = -L_{3,2} = im \exp(q_2), \tag{4.7}$$

respectively. The rest of the matrix elements of  $L$  vanishes. This completes the construction of  $L$  for affine  $C_2^{(1)}$  Toda model. Once  $L$  is constructed  $M$  can be written down very easily in the following way. In  $M$  the diagonal elements are vanishing,

$$M_{i,i} = M_{2r+1-i,2r+1-i} = 0, \tag{4.8}$$

and the off-diagonal elements are determined by those of  $L$ :

$$M_{i,j} = \frac{i}{2} |L_{i,j}|, \quad i \neq j. \tag{4.9}$$

The matrices  $L$  and  $M$  for the affine  $C_2^{(1)}$  model read

$$L = \begin{pmatrix} p_1 & ime^{(q_1 - q_2)/2} & 0 & -ime^{-q_1} \\ -ime^{(q_1 - q_2)/2} & p_2 & ime^{q_2} & 0 \\ 0 & -ime^{q_2} & -p_2 & ime^{(q_1 - q_2)/2} \\ ime^{-q_1} & 0 & -ime^{(q_1 - q_2)/2} & -p_1 \end{pmatrix}, \tag{4.10}$$

$$M = \frac{1}{2} \begin{pmatrix} 0 & ime^{(q_1-q_2)/2} & 0 & ime^{-q_1} \\ ime^{(q_1-q_2)/2} & 0 & ime^{q_2} & 0 \\ 0 & ime^{q_2} & 0 & ime^{(q_1-q_2)/2} \\ ime^{-q_1} & 0 & ime^{(q_1-q_2)/2} & 0 \end{pmatrix}. \tag{4.11}$$

It is easy to check

$$\text{Tr}(L^2) = 2(p_1^2 + p_2^2) + 4m^2e^{q_1-q_2} + 2m^2(e^{2q_2} + e^{-2q_1}) = 4\mathcal{H}_{C_2}^{(1)}, \tag{4.12}$$

and the Lax equation  $\dot{L} = [L, M]$  is identical to the canonical equation given in (4.2).

#### 4.2. $A_3^{(2)}$

The second example is the Toda model Lax pair for the twisted affine algebra  $A_3^{(2)}$ . The simple roots of the affine  $A_3^{(2)}$  are identical to those of  $C_2$  already considered. The affine root is  $-e_1 - e_2$ . We adopt the following Hamiltonian:

$$\mathcal{H}_{A_3}^{(2)} = \frac{1}{2}(p_1^2 + p_2^2) + m^2(e^{q_1-q_2} + e^{-q_1-q_2}) + \frac{m^2}{2}e^{2q_2}. \tag{4.13}$$

The canonical equations of motion are

$$\begin{aligned} \dot{p}_1 &= -\frac{\partial \mathcal{H}}{\partial q_1} = -m^2e^{q_1-q_2} + m^2e^{-q_1-q_2}, \\ \dot{p}_2 &= -\frac{\partial \mathcal{H}}{\partial q_2} = m^2e^{q_1-q_2} + m^2e^{-q_1-q_2} - m^2e^{2q_2}. \end{aligned} \tag{4.14}$$

For simple roots of the form  $\pm(q_i + q_j)$  one writes elements in  $L$  as

$$L_{i,2r+1-j} = -L_{2r+1-j,i} = L_{j,2r+1-i} = -L_{2r+1-i,j} = \pm im \exp \left[ \pm \frac{1}{2}(q_i + q_j) \right]. \tag{4.15}$$

So in this case corresponding to the affine root we have (the other terms are the same as those in the  $C_2$  case)

$$L_{1,3} = -L_{3,1} = L_{2,4} = -L_{4,2} = -im \exp \frac{1}{2}(-q_1 - q_2), \tag{4.16}$$

and the Lax pair is a set of  $4 \times 4$  matrices:

$$L = \begin{pmatrix} p_1 & ime^{(q_1-q_2)/2} & -ime^{(-q_1-q_2)/2} & 0 \\ -ime^{(q_1-q_2)/2} & p_2 & ime^{q_2} & -ime^{(-q_1-q_2)/2} \\ ime^{(-q_1-q_2)/2} & -ime^{q_2} & -p_2 & ime^{(q_1-q_2)/2} \\ 0 & ime^{(-q_1-q_2)/2} & -ime^{(q_1-q_2)/2} & -p_1 \end{pmatrix}, \tag{4.17}$$

$$M = \frac{1}{2} \begin{pmatrix} 0 & ime^{(q_1-q_2)/2} & ime^{(-q_1-q_2)/2} & 0 \\ ime^{(q_1-q_2)/2} & 0 & ime^{q_2} & ime^{(-q_1-q_2)/2} \\ ime^{(-q_1-q_2)/2} & ime^{q_2} & 0 & ime^{(q_1-q_2)/2} \\ 0 & ime^{(-q_1-q_2)/2} & ime^{(q_1-q_2)/2} & 0 \end{pmatrix}. \tag{4.18}$$

It is easy to check

$$\text{Tr}(L^2) = 2(p_1^2 + p_2^2) + 4m^2 e^{q_1 - q_2} + 4m^2 e^{-q_1 - q_2} + 2m^2 e^{2q_2} = 4\mathcal{H}_{A_3}^{(2)}. \tag{4.19}$$

In these two examples  $L$  is hermitian and  $M$  anti-hermitian. Of course both of the Lax pairs constructed here are well-known.

### 4.3. $D_3^{(2)}$ and $B_2^{(1)}$

The third and fourth examples are the duals of those given in the previous two subsections. The dimensions of the Lax pairs for  $B_2^{(1)}$  and  $D_3^{(2)}$  are 5 and 6, respectively. These cases involve some of the simple roots of the form  $e_i$ . First we construct the Lax pair for the  $B_2^{(1)}$  Toda model. We could have written down the simple roots and the affine root of affine  $B_2^{(1)}$  as  $e_1 - e_2$ ,  $e_2$  and  $-e_1 - e_2$ , respectively, by taking the co-roots of  $D_3^{(2)}$ . In this case normalisation for the long roots is 2 whereas in the earlier parametrisation it is 4. For each simple root of the form  $e_i$  one has to add an additional row and column to the Lax pair given above which is initially  $2r$  dimensional i.e., twice the rank. Since in the  $B_2$  case there is only one simple root  $e_2$  of this form the dimension of the Lax pair is 5. We mark the new row and column by 0, and we put the corresponding row and column in the middle of the Lax pair matrices. The diagonal element  $L_{0,0} = 0$ . Corresponding to  $\pm e_i$  we insert elements

$$L_{i,0} = -L_{0,i} = L_{0,2r+1-i} = -L_{2r+1-i,0} = \pm im \exp\left(\pm \frac{1}{2} q_i\right). \tag{4.20}$$

For the  $B_2$  Toda model we then have

$$L_{2,0} = -L_{0,2} = L_{0,3} = -L_{3,0} = im \exp\left(\frac{1}{2} q_2\right). \tag{4.21}$$

The rest of the elements of  $L$  and  $M$  are constructed in the usual manner as explained above and we have the following Lax pair:

$$L = \begin{pmatrix} p_1 & im_1 e^{(q_1 - q_2)/2} & 0 & -im_2 e^{(-q_1 - q_2)/2} & 0 \\ -im_1 e^{(q_1 - q_2)/2} & p_2 & im e^{q_2/2} & 0 & -im_2 e^{(-q_1 - q_2)/2} \\ 0 & -im e^{q_2/2} & 0 & im e^{q_2/2} & 0 \\ im_2 e^{(-q_1 - q_2)/2} & 0 & -im e^{q_2/2} & -p_2 & im_1 e^{(q_1 - q_2)/2} \\ 0 & im_2 e^{(-q_1 - q_2)/2} & 0 & -im_1 e^{(q_1 - q_2)/2} & -p_1 \end{pmatrix}, \tag{4.22}$$

$$M = \frac{1}{2} \begin{pmatrix} 0 & im_1 e^{(q_1 - q_2)/2} & 0 & im_2 e^{(-q_1 - q_2)/2} & 0 \\ im_1 e^{(q_1 - q_2)/2} & 0 & im e^{q_2/2} & 0 & im_2 e^{(-q_1 - q_2)/2} \\ 0 & im e^{q_2/2} & 0 & im e^{q_2/2} & 0 \\ im_2 e^{(-q_1 - q_2)/2} & 0 & im e^{q_2/2} & 0 & im_1 e^{(q_1 - q_2)/2} \\ 0 & im_2 e^{(-q_1 - q_2)/2} & 0 & im_1 e^{(q_1 - q_2)/2} & 0 \end{pmatrix}. \tag{4.23}$$

Once again  $\frac{1}{4} \text{Tr}(L^2)$  gives us the Hamiltonian for the affine  $B_2^{(1)}$  ( $= C_2^{(1)}$ ) Toda model. Notice that in the above we have introduced three coupling constants viz.  $m_1, m_2$

and  $m$ , but as shown in (2.33)–(2.34) these are irrelevant. In a similar fashion we construct a 6 dimensional Lax pair for  $D_3^{(2)}$  ( $= A_3^{(2)}$ ) model. We write the simple and affine roots of  $D_3^{(2)}$  as  $e_2, e_1 - e_2$  and  $-e_1$ , respectively. The Lax pair reads

$$L = \begin{pmatrix} p_1 & i m e^{(q_1 - q_2)/2} & -i m_1 e^{-q_1/2} & 0 & 0 & 0 \\ -i m e^{(q_1 - q_2)/2} & p_2 & 0 & i m_2 e^{q_2/2} & 0 & 0 \\ i m_1 e^{-q_1/2} & 0 & 0 & 0 & 0 & -i m_1 e^{-q_1/2} \\ 0 & -i m_2 e^{q_2/2} & 0 & 0 & i m_2 e^{q_2/2} & 0 \\ 0 & 0 & 0 & -i m_2 e^{q_2/2} & -p_2 & i m e^{(q_1 - q_2)/2} \\ 0 & 0 & i m_1 e^{-q_1/2} & 0 & -i m e^{(q_1 - q_2)/2} & -p_1 \end{pmatrix}, \tag{4.24}$$

$$M = \frac{1}{2} \begin{pmatrix} 0 & i m e^{(q_1 - q_2)/2} & i m_1 e^{-q_1/2} & 0 & 0 & 0 \\ i m e^{(q_1 - q_2)/2} & 0 & 0 & i m_2 e^{q_2/2} & 0 & 0 \\ i m_1 e^{-q_1/2} & 0 & 0 & 0 & 0 & i m_1 e^{-q_1/2} \\ 0 & i m_2 e^{q_2/2} & 0 & 0 & i m_2 e^{q_2/2} & 0 \\ 0 & 0 & 0 & i m_2 e^{q_2/2} & 0 & i m e^{(q_1 - q_2)/2} \\ 0 & 0 & i m_1 e^{-q_1/2} & 0 & i m e^{(q_1 - q_2)/2} & 0 \end{pmatrix}. \tag{4.25}$$

Two of the coupling constants could be redefined and the Hamiltonian could be written in terms of a single coupling constant (mass parameter), for example  $m$ .

Before closing this section let us summarise that the intuitive method gives  $2r$  dimensional Lax pairs for  $C_r^{(1)}$  (vector representation) and  $A_{2r-1}^{(2)}$  and a  $2r + 1$  dimensional (vector representation) Lax pair for  $B_r^{(1)}$  and a  $2r + 2$  dimensional Lax pair for  $D_{r+1}^{(2)}$  Toda models. Of course one can also construct the  $A_r^{(1)}$  and the  $D_r^{(1)}$  Lax pairs in the vector representations in a similar fashion.

### §5. Summary and discussion

As shown in §2, the Hamiltonians of the elliptic Calogero-Moser models tend to those of Toda models as one of the periods of the elliptic function goes to infinity, provided the dynamical variables are properly shifted and the coupling constants are scaled. Although both Calogero-Moser and Toda models are integrable, the corresponding limits of the Lax pairs are subtle. This is partly because of the abundance of Lax pairs, i.e., there are many Lax pairs for a given Calogero-Moser model. Here we list those Lax pairs of Calogero-Moser models which have Toda model limits.

algebras	potential	Lax pair	Toda models
$A_r, D_r, E_6, E_7$	untwisted	minimal	$\mathfrak{g}, \mathfrak{g}^{(1)}$
$B_r$	untwisted	minimal	$B_r, B_r^{(1)}$
$C_r$	untwisted	minimal	$C_r, C_r^{(1)}$
$B_r$	twisted	short roots	$C_r, A_{2r-1}^{(2)}$
$C_r$	untwisted	long roots	$C_r, C_r^{(1)}$

Here we give an intuitive explanation why some Calogero-Moser Lax pairs, for example, the root type Lax pairs for simply laced root systems, do not have Toda

limits. In some Lax pairs, there are two *different* functions, e.g.,  $x$  and  $x_d$ , corresponding to the *same* potential in the Hamiltonian. These functions are related to the potentials by the factorisation formulas, for example (3.6) and (3.25), and the limits of the potentials are unique. That is, these two *different* functions must have the *same* limit for all the possible arguments  $\alpha \cdot q$  belonging to the potential. This is not compatible with the functional equations they must satisfy, thus the postulated limits to Toda models do not exist in such cases.

In Ref. 7) it is claimed that the Lax pair for the twisted  $F_4$  model based on the short roots has a Toda model limit. In §3.3 we have shown that such a limit does not exist. Let us follow their logic here. They use three functions  $x$ ,  $x^{(1/2)}$  and  $x_d^{(1/2)}$  in our notation, and their solution for  $x^{(1/2)}$  and  $x_d^{(1/2)}$  corresponds to (B.1) and (B.2) given in Appendix B. They show that  $g_S x_d^{(1/2)}(\alpha \cdot q, \xi)$  has a non-vanishing finite limit for  $\pm$  simple short roots and at the same time it is claimed that  $g_S x^{(1/2)}(\alpha \cdot q, \xi)$  vanishes for positive short roots. This would clearly violate the factorisation relation (the half period analogue of (3.6)) unless  $g_S x^{(1/2)}(-\alpha \cdot q, \xi)$  diverge for positive short roots. As a result, the Lax pair does not have a Toda model limit.

Let us assume, according to their claim, that  $g_S x^{(1/2)}(\pm \alpha \cdot q, \xi)$  vanishes for short roots. This would lead to another contradiction with the formula  $\text{Tr}(L^2) \propto \mathcal{H}$ , without invoking the factorisation formulas. On the right-hand side the limit of the potential is well defined and unique, whereas on the left-hand side the short root potential has two different sources, one from the function  $x_d^{(1/2)}$  and the other from  $x^{(1/2)}$ . The former has a finite limit while the latter contribution vanishes. This means that the relation  $\text{Tr}(L^2) \propto \mathcal{H}$  is broken in the limiting process.

In Ref. 7) a Toda model Lax pair for the twisted affine Lie algebra  $D_{r+1}^{(2)}$  is obtained as a limit of a Lax pair for the twisted  $C_r$  Calogero-Moser model in a  $2r+2$  dimension representation,<sup>11)</sup> which does not belong to the minimal type or root type Lax pairs developed in our series of papers.<sup>1)</sup>

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### Appendix A

#### — Elliptic Functions —

Here we collect some mathematical formulas which will be used in the main text. For the elliptic functions we follow the notation and conventions of Ref. 12) throughout this paper. The Weierstrass function  $\wp$  is a doubly periodic meromorphic

function with a pair of primitive periods  $\{2\omega_1, 2\omega_3\}$ ,  $\Im(\omega_3/\omega_1) > 0$ :

$$\wp(u) \equiv \wp(u|\{2\omega_1, 2\omega_3\}) = \frac{1}{u^2} + \sum'_{m,n} \left[ \frac{1}{(u - \Omega_{m,n})^2} - \frac{1}{\Omega_{m,n}^2} \right], \tag{A.1}$$

in which  $\Omega_{m,n}$  is a period

$$\Omega_{m,n} = 2m\omega_1 + 2n\omega_3$$

and  $\sum'$  denotes the summation over all integers, positive, negative and zero, excluding  $m = n = 0$ . The Weierstrass sigma function  $\sigma(u)$  is defined from  $\wp(u)$  via the Weierstrass zeta function  $\zeta(u)$  as

$$\begin{aligned} \wp(u) &= -\zeta'(u), & \zeta(u) &= d \log \sigma(u)/du = \sigma'(u)/\sigma(u), \\ \zeta(u) &\equiv \zeta(u|\{2\omega_1, 2\omega_3\}) = \frac{1}{u} + \sum'_{m,n} \left[ \frac{1}{u - \Omega_{m,n}} + \frac{1}{\Omega_{m,n}} + \frac{u}{\Omega_{m,n}^2} \right], \\ \sigma(u) &\equiv \sigma(u|\{2\omega_1, 2\omega_3\}) = u \prod'_{m,n} \left( 1 - \frac{u}{\Omega_{m,n}} \right) \exp \left[ \frac{u}{\Omega_{m,n}} + \frac{u^2}{2\Omega_{m,n}^2} \right], \end{aligned} \tag{A.2}$$

in which  $\prod'$  denotes the product over all integers, positive, negative and zero, excluding  $m = n = 0$ . For the parametrisation of the periods

$$\omega_1 = -i\pi, \quad \omega_3 \in \mathbf{R}_+, \quad \tau \equiv \frac{\omega_3}{\omega_1} = i\omega_3/\pi, \quad q = e^{i\tau\pi} = e^{-\omega_3}$$

the following expansion formulas for the elliptic functions are useful:\*)

$$\zeta(u) = \frac{i\eta_1 u}{\pi} + \frac{1}{2} \coth \frac{u}{2} - 2 \sum_{n=1}^{\infty} \frac{q^{2n}}{1 - q^{2n}} \sinh nu, \tag{A.3}$$

$$\wp(u) = \left( \frac{1}{2\pi} \right)^2 \left[ -4i\eta_1\pi + \frac{\pi^2}{\sinh^2(u/2)} + 8\pi^2 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1 - q^{2n}} \cosh nu \right], \tag{A.4}$$

in which

$$\eta_1 = \zeta(\omega_1) = i\pi \left( \frac{1}{12} - 2 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1 - q^{2n}} \right) \xrightarrow{\omega_3 \rightarrow +\infty} i \frac{\pi}{12}. \tag{A.5}$$

The exponential potentials can be obtained from the elliptic potentials simply by the following way. For a shifted argument

$$u = U - \omega_3 \ell, \quad |\ell| < 1, \tag{A.6}$$

the summations in (A.3) and (A.4) can be neglected in the limit  $\omega_3 \rightarrow +\infty$  to obtain

$$\zeta(u) = \frac{i\eta_1 u}{\pi} + \frac{1}{2} \coth \frac{u}{2}, \tag{A.7}$$

$$V_L(u) \approx \frac{1}{12} + \frac{1}{4} \frac{1}{\sinh^2 u/2}, \quad V_S(u) \approx \frac{1}{3} + \frac{1}{\sinh^2 u}. \tag{A.8}$$

\*) See, for example, Ref. 13), p. 221.

In other words we obtain the following asymptotic formulas:

$$V_L(u) \approx e^{\pm U \mp \omega_3 \ell} + \text{const}, \quad V_S(u) \approx 4e^{\pm 2U \mp 2\omega_3 \ell} + \text{const}, \quad \ell \gtrless 0. \quad (\text{A}\cdot 9)$$

Thus by appropriate scalings of the coupling constants

$$g_L = m_L e^{\omega_3 |\ell|/2}, \quad g_S = m_S e^{\omega_3 |\ell|/2}, \quad |\ell| \leq 1, \quad (\text{A}\cdot 10)$$

the exponential potentials can be obtained

$$g_L^2 V_L(u) \approx m_L^2 e^{\pm U}, \quad g_S^2 V_S(u) \approx m_S^2 e^{\pm 2U}, \quad \ell \gtrless 0, \quad |\ell| < 1, \quad (\text{A}\cdot 11)$$

in which we understand the constant parts are properly subtracted. The extreme case  $|\ell| = 1$  deserves special attention

$$\begin{aligned} g_L^2 V_L(U \pm \omega_3) &\approx m_L^2 (e^U + e^{-U}), \\ g_S^2 V_S(U \pm \omega_3) &\approx 4m_S^2 (e^{2U} + e^{-2U}). \end{aligned} \quad (\text{A}\cdot 12)$$

It should be remarked that any shift of the argument  $u$  of the potentials which is proportional to  $\omega_3$  can always be written in the form (A.6) with  $|\ell| \leq 1$ , due to the  $2\omega_3$  periodicity of the  $\wp$  function. Thus the shifted potential functions always decrease exponentially  $e^{-\omega_3 |\ell|}$ , up to an additive constant term. Only those potential terms which have the minimal decrease can be made finite by appropriate scalings of the coupling constants.

The corresponding approximation formula for the  $\sigma$  function reads

$$\sigma(u) \approx 2 \sinh \frac{u}{2} \exp\left(-\frac{u^2}{24}\right), \quad (\text{A}\cdot 13)$$

in which  $u$  is shifted as in (A.6). Since  $\sigma(u)$  is quasi-periodic in  $u$ , we need two additional asymptotic formulas corresponding to plus (minus) one period shift. For

$$u = u_0 \pm 2\omega_3, \quad u_0 = U - \omega_3 \ell, \quad |\ell| < 1, \quad (\text{A}\cdot 14)$$

we have

$$\sigma(u) \approx -2 \sinh \frac{u_0}{2} e^{\pm u_0 + \omega_3} \exp\left(-\frac{u^2}{24}\right). \quad (\text{A}\cdot 15)$$

## Appendix B

### — Other Solutions of the Functional Equations —

Here we present without proof some new sets of solutions to the functional equations for the twisted functions  $\{x^{(1/2)}, x_d^{(1/2)}\}$  and similar functions for the  $G_2$  model,  $\{x^{(1/3)}, x_d^{(1/3)}, x_t^{(1/3)}\}$ . These solutions are closely related to the ones given in the Appendix of Ref. 2) and the derivation is similar. For  $\{x^{(1/2)}, x_d^{(1/2)}\}$  we have

$$x^{(1/2)}(u, \xi) = \left[ \frac{\sigma(\xi - u)}{\sigma(\xi)\sigma(u)} - \frac{\sigma(\xi - u - \omega_1)}{\sigma(\xi)\sigma(u + \omega_1)} \exp[\eta_1 \xi] \right] \quad (\text{B}\cdot 1)$$



and

$$x_d^{(1/2)}(u, \xi) = \left[ \frac{\sigma(2\xi - u)}{\sigma(2\xi)\sigma(u)} + \frac{\sigma(2\xi - u - \omega_1)}{\sigma(2\xi)\sigma(u + \omega_1)} \exp[2\eta_1 \xi] \right]. \quad (\text{B}\cdot 2)$$

Only the sign of the second term of (B·1) is different from (A·27) of Ref. 2). For the  $G_2$  functions we have

$$x^{(1/3)}(u, \xi) = \left[ \frac{\sigma(\xi - u)}{\sigma(\xi)\sigma(u)} + \lambda \frac{\sigma(\xi - u - \frac{2\omega_1}{3})}{\sigma(\xi)\sigma(u + \frac{2\omega_1}{3})} \exp[(2/3)\eta_1 \xi] \right. \\ \left. + \lambda^2 \frac{\sigma(\xi - u - \frac{4\omega_1}{3})}{\sigma(\xi)\sigma(u + \frac{4\omega_1}{3})} \exp[(4/3)\eta_1 \xi] \right], \quad (\text{B}\cdot 3)$$

in which  $\lambda$  is a cubic root of unity,  $\lambda = e^{\pm 2i\pi/3}$  and

$$x_d^{(1/3)}(u, \xi) = \left[ \frac{\sigma(2\xi - u)}{\sigma(2\xi)\sigma(u)} + \lambda^2 \frac{\sigma(2\xi - u - \frac{2\omega_1}{3})}{\sigma(2\xi)\sigma(u + \frac{2\omega_1}{3})} \exp[(4/3)\eta_1 \xi] \right. \\ \left. + \lambda \frac{\sigma(2\xi - u - \frac{4\omega_1}{3})}{\sigma(2\xi)\sigma(u + \frac{4\omega_1}{3})} \exp[(8/3)\eta_1 \xi] \right], \quad (\text{B}\cdot 4)$$

$$x_t^{(1/3)}(u, \xi) = \left[ \frac{\sigma(3\xi - u)}{\sigma(3\xi)\sigma(u)} + \frac{\sigma(3\xi - u - \frac{2\omega_1}{3})}{\sigma(3\xi)\sigma(u + \frac{2\omega_1}{3})} \exp[2\eta_1 \xi] \right. \\ \left. + \frac{\sigma(3\xi - u - \frac{4\omega_1}{3})}{\sigma(3\xi)\sigma(u + \frac{4\omega_1}{3})} \exp[4\eta_1 \xi] \right]. \quad (\text{B}\cdot 5)$$

They satisfy (III. 5·10) for  $n = 3$  and (III. 5·12), (III. 5·13).<sup>1)</sup>

### Appendix C

#### — Extended Twisted $BC_r$ Root System Lax Pair with Five Independent Couplings Based on Short Roots —

Here we write the root type Lax pair for the twisted  $BC_r$  system based on the short roots.

The pattern of short root – short root is

$$BC_r : \quad \text{short root} - \text{short root} = \begin{cases} \text{long root,} \\ \text{middle root,} \\ 2 \times \text{short root,} \\ \text{non-root.} \end{cases} \quad (\text{C}\cdot 1)$$

Based on this one can construct the Lax pair as

$$L(q, p, \xi) = p \cdot H + X_m + X_d + X_l, \\ M(q, \xi) = D_m + D_L + D_s + Y_m + Y_d + Y_l, \quad (\text{C}\cdot 2)$$

in which  $X_m$  ( $Y_m$ ) corresponds to short root – short root = middle root,  $X_L$  ( $Y_L$ ) corresponds to short root – short root = long root and  $X_d$  ( $Y_d$ ) corresponds to short

root – short root = 2 × short root:

$$\begin{aligned}
 X_d &= 2i \sum_{\lambda \in \Delta_S} \left[ g_{S_1} x_d^{(1/2)}(\lambda \cdot q, \xi) + g_{S_2} x_d^{(1/4)}(\lambda \cdot q, \xi) \right] E_d(\lambda), \\
 X_m &= ig_M \sum_{\alpha \in \Delta_M} x^{(1/2)}(\alpha \cdot q, \xi) E(\alpha), \\
 X_l &= i \sum_{\Xi \in \Delta_L} \left[ g_{L_1} x(\Xi \cdot q, \xi) + g_{L_2} x^{(1/2)}(\Xi \cdot q, \xi) \right] E(\Xi), \\
 Y_d &= i \sum_{\lambda \in \Delta_S} \left[ g_{S_1} y_d^{(1/2)}(\lambda \cdot q, \xi) + g_{S_2} y_d^{(1/4)}(\lambda \cdot q, \xi) \right] E_d(\lambda), \\
 Y_m &= ig_M \sum_{\alpha \in \Delta_M} y^{(1/2)}(\alpha \cdot q, \xi) E(\alpha), \\
 Y_l &= i \sum_{\Xi \in \Delta_L} \left[ g_{L_1} y(\Xi \cdot q, \xi) + g_{L_2} y^{(1/2)}(\Xi \cdot q, \xi) \right] E(\Xi),
 \end{aligned}$$

$$E_d(\lambda)_{\mu\nu} = \delta_{\mu-\nu, 2\lambda}, \quad E(\alpha)_{\mu\nu} = \delta_{\mu-\nu, \alpha}, \quad E(\Xi)_{\mu\nu} = \delta_{\mu-\nu, \Xi}, \tag{C.3}$$

$$(Ds)_{\mu\nu} = \delta_{\mu, \nu} (Ds)_{\mu},$$

$$(Ds)_{\mu} = i [g_{S_1} \wp(\mu \cdot q | \{\omega_1, 2\omega_3\}) + g_{S_2} \wp(\mu \cdot q | \{\omega_1/2, 2\omega_3\})], \tag{C.4}$$

$$(Dm)_{\mu\nu} = \delta_{\mu, \nu} (Dm)_{\mu},$$

$$(Dm)_{\mu} = ig_M \sum_{\alpha \in \Delta_M, \alpha \cdot \mu = 1} \wp(\alpha \cdot q | \{\omega_1, 2\omega_3\}), \tag{C.5}$$

$$(Dl)_{\mu\nu} = \delta_{\mu, \nu} (Dl)_{\mu},$$

$$(Dl)_{\mu} = i [g_{L_1} \wp(2\mu \cdot q) + g_{L_2} \wp(2\mu \cdot q | \{\omega_1, 2\omega_3\})]. \tag{C.6}$$

### Appendix D

— Asymptotic Forms of Various Functions Appearing in the Lax Pair —

In this section we give asymptotic forms of various functions,  $x_d$ ,  $x^{(1/2)}$ ,  $x_d^{(1/2)}$ , etc. The dynamical variables and the spectral parameter are scaled as in (3.8) and (3.10) with  $\omega_1 = -i\pi$  and  $\omega_3 \rightarrow +\infty$ .

i)  $x_d$  for positive roots  $\alpha$ :

$$\begin{aligned}
 x_d(\alpha \cdot q, \xi) &\rightarrow -1, & 0 < 2\epsilon - \delta\rho \cdot \alpha < 1, \\
 &\rightarrow Z^2 \exp(-\alpha \cdot Q) \exp[\omega_3(2\delta\rho \cdot \alpha - 4\epsilon)], & -1 < 2\epsilon - \delta\rho \cdot \alpha \leq 0,
 \end{aligned} \tag{D.1}$$

and for negative roots  $\alpha$ :

$$\begin{aligned} x_d(\alpha \cdot q, \xi) &\rightarrow \exp(-\alpha \cdot Q) \exp[2\omega_3 \delta \rho \cdot \alpha], & 0 < 2\epsilon - \delta \rho \cdot \alpha < 1, \\ &\rightarrow -\frac{1}{Z^2} \exp[\omega_3(4\epsilon - 2)], & 1 \leq 2\epsilon - \delta \rho \cdot \alpha < 2, \end{aligned} \quad (D.2)$$

in which

$$x_d(u, \xi) = \frac{\sigma(2\xi - u)}{\sigma(2\xi)\sigma(u)} \exp(2\zeta(\xi)u).$$

ii)  $x^{(1/2)}$  for positive roots  $\alpha$ :

$$\begin{aligned} x^{(1/2)}(\alpha \cdot q, \xi) &\rightarrow -2 \exp\left(\frac{3\alpha \cdot Q}{2}\right) \exp[-3\omega_3 \delta \rho \cdot \alpha], & 0 < \epsilon/2 - \delta \rho \cdot \alpha < 1, \\ &\rightarrow 2Z \exp\left(-\frac{\alpha \cdot Q}{2}\right) \exp[\omega_3(\delta \rho \cdot \alpha - 2\epsilon)], & -1 < \epsilon/2 - \delta \rho \cdot \alpha \leq 0 \end{aligned} \quad (D.3)$$

and for negative roots  $\alpha$ :

$$\begin{aligned} x^{(1/2)}(\alpha \cdot q, \xi) &\rightarrow 2 \exp\left(-\frac{\alpha \cdot Q}{2}\right) \exp[\omega_3 \delta \rho \cdot \alpha], & 0 < \epsilon/2 - \delta \rho \cdot \alpha < 1, \\ &\rightarrow -\frac{2}{Z} \exp\left(\frac{3\alpha \cdot Q}{2}\right) \exp[\omega_3(2\epsilon - 3\delta \rho \cdot \alpha - 4)], & 1 \leq \epsilon/2 - \delta \rho \cdot \alpha < 2, \end{aligned} \quad (D.4)$$

in which

$$x^{(1/2)}(u, \xi) = \frac{x(u, \xi/2) x(u + \omega_1, \xi/2)}{x(\omega_1, \xi/2)} \exp[u(\zeta(\xi) - 2\zeta(\xi/2))],$$

and  $x(u, \xi)$  is defined in (3.12).

iii)  $x_d^{(1/2)}$  for positive roots  $\alpha$ :

$$\begin{aligned} x_d^{(1/2)}(\alpha \cdot q, \xi) &\rightarrow -2 \exp(\alpha \cdot Q) \exp[-2\omega_3 \delta \rho \cdot \alpha], & 0 < \epsilon - \delta \rho \cdot \alpha < 1, \\ &\rightarrow 2Z^2 \exp(-\alpha \cdot Q) \exp[\omega_3(2\delta \rho \cdot \alpha - 4\epsilon)], & -1 < \epsilon - \delta \rho \cdot \alpha \leq 0 \end{aligned} \quad (D.5)$$

and for negative roots  $\alpha$ :

$$\begin{aligned} x_d^{(1/2)}(\alpha \cdot q, \xi) &\rightarrow 2 \exp(-\alpha \cdot Q) \exp[2\omega_3 \delta \rho \cdot \alpha], & 0 < \epsilon - \delta \rho \cdot \alpha < 1, \\ &\rightarrow -\frac{2}{Z^2} \exp(\alpha \cdot Q) \exp[\omega_3(4\epsilon - 2\delta \rho \cdot \alpha - 4)], & 1 \leq \epsilon - \delta \rho \cdot \alpha < 2, \end{aligned} \quad (D.6)$$

in which

$$x_d^{(1/2)}(u, \xi) = \frac{x(u, \xi) x(u + \omega_1, \xi)}{x(\omega_1, \xi)}.$$

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