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CAN A PULSE EXCITATION SMALLER THAN kT BE DETECTED?*

by

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ABSTRACT. A "one kT-pulse" excitation of a high Q LC-ringing circuit is defined, and a filter configuration is described which allows the detection of such a pulse with good SNR (signal-to-noise ratio) and excellent τ_r (resolving time). SNR's considerably larger than unity were found experimentally, in reasonable agreement with theoretical calculations.

INTRODUCTION. In discussing the sensitivity limit of a high-Q mechanical resonator for the detection of a very short pulse[1], it was pointed out that a suitable filtering technique should allow a measurable output of relatively short duration of an individual pulse, even though energy transfer to the resonator might be much smaller than kT. As the combination of ultra-high sensitivity with a short resolving time may seem paradoxical, some skeptical comments prompted this writer to work out a more rigorous version of Gibbons and Hawkings' assertion. I wish to present the result in the simplest form, namely, in terms of a purely electronic system (a single low-loss LC ringing circuit, no transducers) that can be realized easily and experimentally tested. The LC circuit is connected directly across the input of an electrometer amplifier whose noise contribution will be calculated from two equivalent sources having constant densitities u^2_{amp} and i^2_{amp} per unit bandwidth (Fig. 1). Amplifier gain is assumed flat over the frequency band of interest; the filtering requires a band-pass and a compensated delay-line clipper[2,3] whose desirable characteristics will be defined without going into technical problems.

DEFINITION OF A kT-PULSE. For a given L and C (including the amplifier input capacitance), let $\omega_o = 1/\sqrt{LC}$ and $R_o = \sqrt{L/C} = 1/\omega_o C$. If all losses were concentrated in a single equivalent parallel resistance R_p, its value would be $R_p = QR_o$, where Q is assumed $\gg 1$. The associated Nyquist noise current source, operating into the combined parallel impedance $Z(R_p \,||C||L)$, generates a noise voltage of spectral density[4]

$$S_u = \frac{4kT}{R_p} |Z(\omega)|^2 = \frac{4kT}{C} \cdot \frac{\omega^2 \omega_o / Q}{(\omega^2 - \omega_o^2)^2 + (\omega\omega_o/Q)^2} . \tag{1}$$

Integration over all frequencies yields

$$\overline{U^2} = \frac{1}{2\pi} \int_0^\infty S_u(\omega)\, d\omega = \frac{kT}{C} \tag{2}$$

as expected from the equipartition theorem.[4]

A current pulse $I_p \delta t$ fed into this curcuit (e.g., from a voltage generator in series with R_p) will set up a slowly decaying oscillation with an initial RMS value of

$$U^{rms}_{initial} = \frac{\sqrt{2}\, I_p \delta t}{2C} , \text{ if } \delta t \ll 1/\omega_o . \tag{3}$$

Equating the result of eqn. 3 with the square root of eqn. 2 defines a " 1 kT pulse " as follows:

$$\left(I_p\,\delta t\right)_{1\,kT} = \sqrt{2CkT} \tag{4a}$$

or similarly,

$$\left(U_p\,\delta t\right)_{1\,kT} = \sqrt{2LkT} \tag{4b}$$

in the case of an induced voltage pulse.

If such a pulse could be applied at an instant when the LC system had zero total energy, the pulse would supply it with an amount of energy just equal to 1kT. In thermal equilibrium, a 1kT-pulse occurring at random will increase or decrease the LC energy with equal probability, as illustrated in Fig. 2.

DELAY-LINE FILTER. The input waveform, U(t), may be represented as a sine wave of frequency ω_o with a slowly varying amplitude vector. By forming the difference, $U(t) - U(t + n\,2\pi/\omega_o)$, an output amplitude equal to the absolute value of the input vector change over the last n cycles is obtained.[1] We may even choose multiples of a half-cycle as the delay time and form a difference,

$$U(t) - (-1)^n\, U(t + n\pi/\omega_o), \tag{5}$$

with increased flexibility for optimizing the resolving time

$$\tau_r = n\pi/\omega_o,\ n_{min} = 1 \qquad \text{(in Fig. 1, n = 5)}. \tag{6}$$

It is important to realize that an individual pulse-excited signal is transmitted with shortened duration but without any loss in amplitude. To calculate noise transmission, I shall use the response function in frequency domain,

$$|F(\omega)|^2 = 4\sin^2\left(n\frac{\pi}{2}x\right),\ \text{where } x \equiv \frac{\omega - \omega_o}{\omega_o}\,. \tag{7}$$

BANDPASS FILTER. To reduce the amplifier noise, a bandpass centered around ω_o is needed. The full width should not be made smaller than ω_o/n in order to pass the signals of chosen duration with negligible attenuation (see eqn. 6). Accordingly, I shall integrate the noise spectrum between $\omega_o(1 - \frac{1}{2n})$ and $\omega_o(1 + \frac{1}{2n})$. In this range, expression $7 \approx n^2\pi^2x^2$, and the total spectral density becomes approximately

$$\frac{\pi}{2}\,\omega_o n^2 x^2 \left[\left(\frac{kTR_o}{Q} + \frac{1}{4}\, i^2_{amp}\, R_o^2\right)\frac{1+x}{x^2 + (1/2Q)^2} + u^2_{amp}\right] \qquad (8)$$

per unit dx.

TOTAL OUTPUT NOISE AND SIGNAL-TO-NOISE RATIO (SNR).

Integration of eqn. 8 yields

$$\overline{U^2_{total}} = \frac{\pi}{2C}\left\{ n\left[\frac{kT}{Q} + \frac{1}{4}\, i^2_{amp} R_o\right]\left(1 - \frac{\pi}{2}\frac{n}{Q} \pm \cdots\right) + \frac{u^2_{amp}}{12nR_o}\right\} \qquad (9)$$

If amplifier noise could be neglected, this would lead essentially to the statement made in Reference 1 but with substantial improvement, due to the choice of a half-period as the shortest resolving time:

$$SNR^{ideal}_{(1kT)} = \sqrt{\frac{2Q}{\pi}} \quad ; \quad n = n_{min} = 1.$$

In the general case, eqn. 9 can be minimized by choosing

$$n_{opt} = \frac{1}{2}\sqrt{\left(u^2_{amp}\, Q/3R_o kT\right)\left(1 + i^2_{amp}\, R_o Q/4kT\right)} \qquad (10)$$

which leads to

$$\overline{U^{2(min)}_{total}} = \frac{\pi}{2C}\sqrt{\frac{u^2_{amp}\, kT}{3R_o Q}\left(1 + i^2_{amp}\, R_o Q/4kT\right)}\,. \qquad (11)$$

The optimum SNR value can be obtained as follows:

$$SNR^{opt}_{(1kT)} \approx \sqrt{2/\pi}\quad \sqrt[4]{3R_o QkT/\left(u^2_{amp}\left[1 + \frac{i^2_{amp} R_o Q}{4kT}\right]\right)} \quad ,$$

with a resolving time $\tau_r^{opt} = n_{opt}\, \pi/\omega_o$. (12)

DEMONSTRATION EXPERIMENT. Two ferrite-pot core inductors (size 42/29 mm, material 3B7) having Q-values near 500 were chosen for experimenting with a 66μs delay-line clipper in the n = 1 and n = 3 modes (7.6 kHz and 22.8 kHz ringing frequencies, respectively). A room-temperature 1kT-pulse was induced in a small ferrite toroid through which the inductor ground lead passed. Corresponding RMS voltages, calculated from the respective parallel capacitances according to eqn. 2, are 0.57 μV and 1.0 μV. Typical waveforms at the amplifier output (gain 0.8×10^4 at 22.8 kHz, 20 kHz bandwidth) are reproduced in Fig. 2 (upper 3 traces), where the pulse was always applied at 2 divisions from the left screen end. Note that the pulse may leave the stored energy unchanged depending on relative phase (trace B displays a phase jump rather than an amplitude change). To display the pulse response proper, undisturbed by thermal noise, oscilloscope sensitivity was reduced to 1/100 of its original value while pulse amplitude was increased 100 fold (trace D).

When a 1kT pulse was applied to the 22.8 kHz LC-circuit, the delay-line clipper output (Fig. 3) provided a distinct 10 mV peak-to-peak signal, while total noise was reduced to 0.75 mV RMS. In the notation used to arrive at eqn. 11, this corresponds to $SNR \sim 10/(2 \times \sqrt{2} \times 0.75) = 4.7$. Inserting the noise characteristics of our SFB8558 FET amplifier ($u_{amp} \approx 1.4\,nV/\sqrt{Hz}$, $i_{amp} \approx 8 \times 10^{-15} A/\sqrt{Hz}$ and $R_o = \sqrt{L/C} = 1900\,\Omega$ into eqn. 9, we would expect theoretically $SNR_{(1kT)} = 6.7$ for n = 3.

Similarly, SNR in the 7.6 kHz experiment was ~ 2.6, again somewhat below theoretical prediction (4.9 for $R_o = 1750\,\Omega$, n = 1). The differences are mainly due to the non-ideal bandpass filters used in the experiments. For all practical purposes, the i^2_{amp} term in eqns. 10 and 11 may be neglected when modern FET's are used.

CONCLUSION. A 1kT-pulse can be detected in a high-Q resonator, with SNR substantially greater than unity, if amplifier noise is

$$u^2_{amp}/R_o \ll QkT, \text{ and } i^2_{amp}R_o \ll kT/Q.$$

Under these conditions, SNR = 1 corresponds to a sensitivity limit $\ll$ 1 kT. A more detailed paper is to be published elsewhere.

FOOTNOTES AND REFERENCES

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FIGURE CAPTIONS

Fig. 1 Basic filter configuration for extracting a small pulse-excited signal against high-Q ringing noise. To show the effect of the filter on the pulse-excited signal clearly, waveforms were drawn for $n = 5$, but as if $u_{amp} = 0$. The bandpass would not be required in this hypothetical case, in which one would rather choose $n = n_{min} = 1$.

Fig. 2 Thermal noise at the capacitor (3700 pF) of a 22.8 kHz LC circuit at T = 290 K°. A 1 kT pulse is superimposed on each of the traces, A, B, C, at the time marked t_{pulse}. To obtain trace D, pulse amplitude was increased 100 fold. Sensitivities indicated at the left refer to the FET input.

Fig. 3 Three typical samples of delay-line clipper output, via a 22.8 kHz bandpass of 6.6 kHz width. 1 kT-pulses produce essentially a 3 half-cycle signal of 10 mV peak-to-peak, while noise is reduced to 0.75 mV RMS, so that noise peaks would seldom exceed discriminator levels set at + and -3 mV.

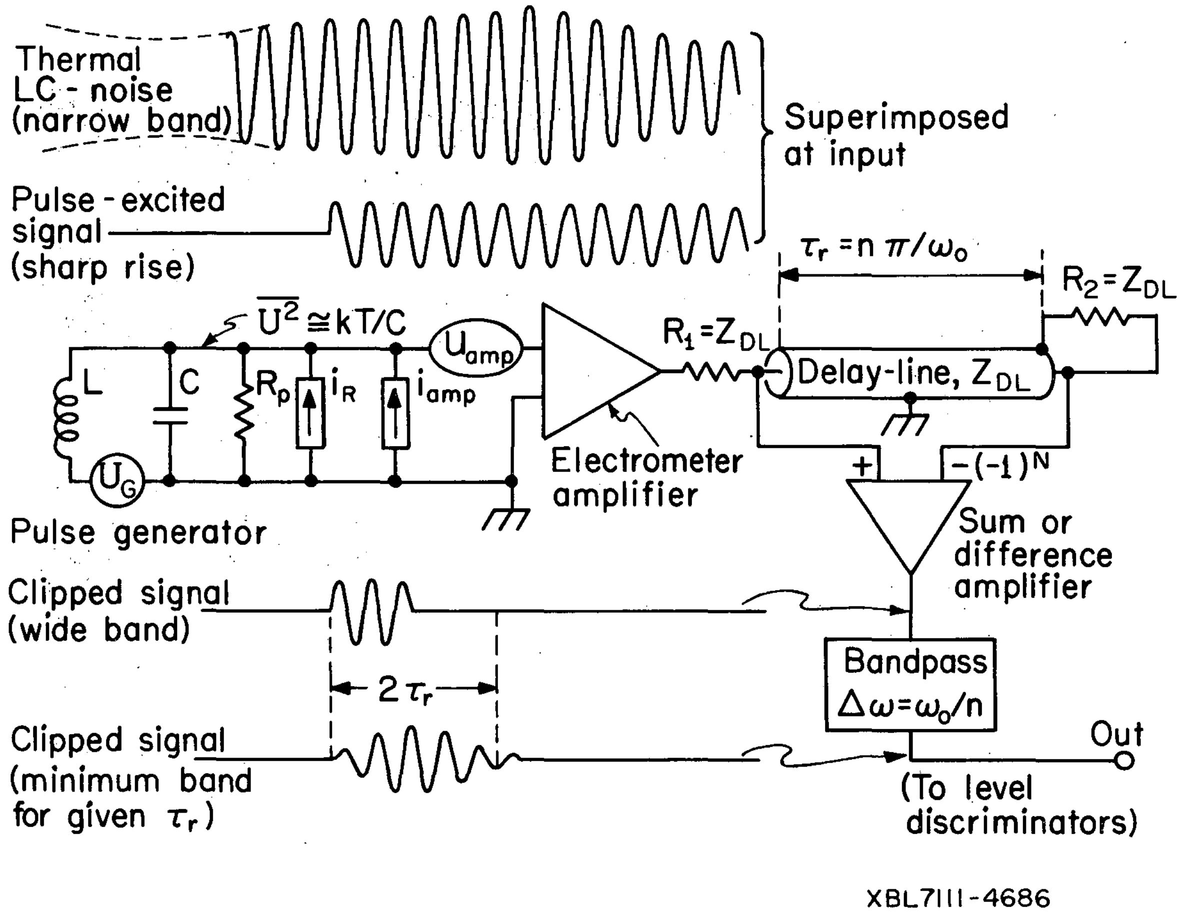
Thermal
LC - noise
(narrow band)
Superimposed
at input
Pulse - excited
signal
(sharp rise)
$\tau_r = n\,\pi/\omega_o$
$R_2 = Z_{DL}$
$\overline{U^2} \cong kT/C$
U_{amp}
$R_1 = Z_{DL}$
Delay-line, Z_{DL}
L
C
R_p
i_R
i_{amp}
U_G
Electrometer
amplifier
+
$-(-1)^N$
Pulse generator
Sum or
difference
amplifier
Clipped signal
(wide band)
Bandpass
$\Delta\omega = \omega_o/n$
$2\tau_r$
Clipped signal
(minimum band
for given τ_r)
Out
(To level
discriminators)
XBL7111-4686

Fig. 1

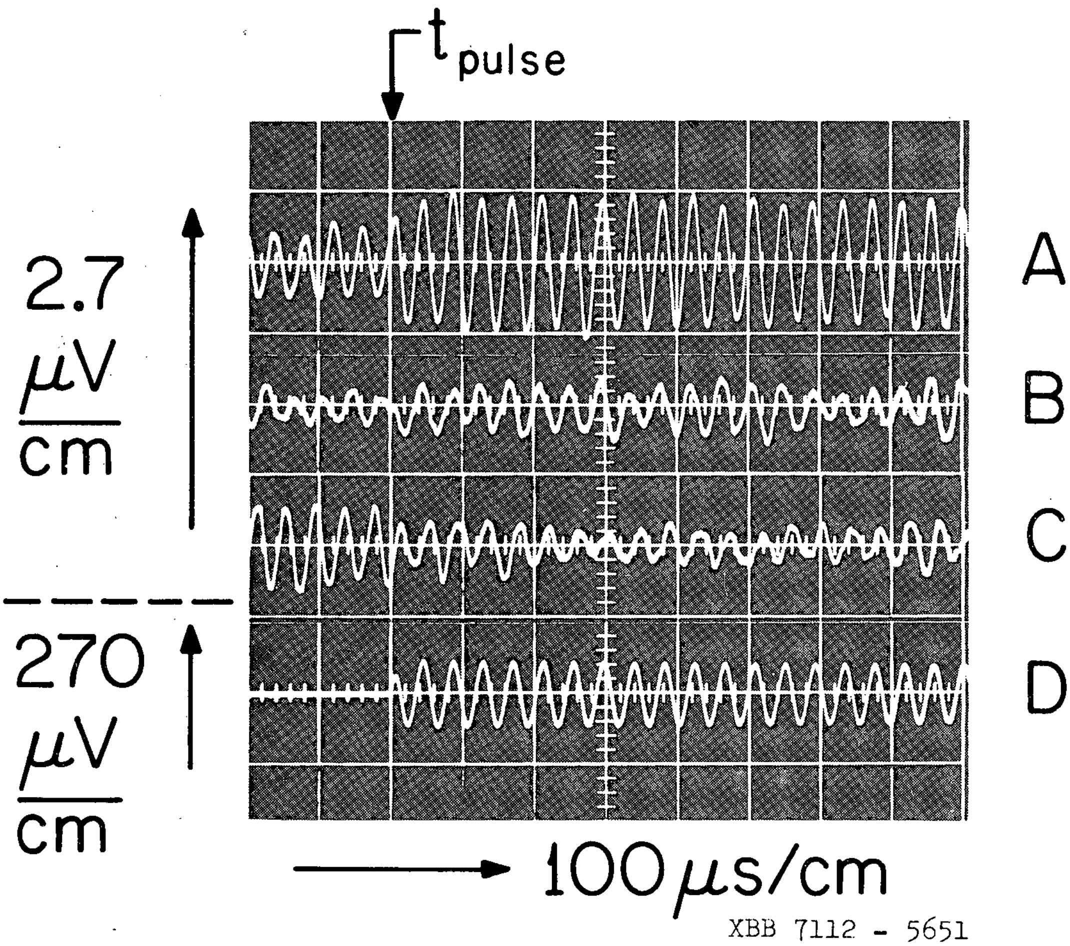

Fig. 2

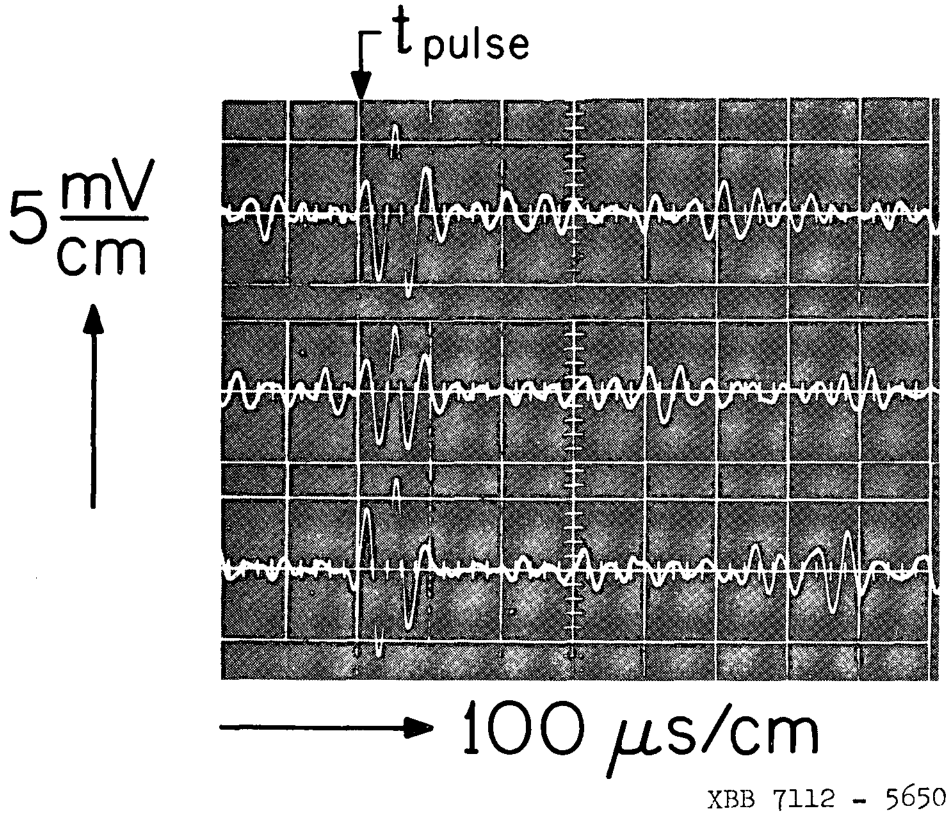

Fig. 3

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