Can Commodities Dominate Stock and Bond Portfolios?

July 19, 2018

Abstract

In this article we discuss whether commodities should be included as an asset class when establishing portfolios. By investigating second order stochastic dominance relations, we find that the stock and bond indices used tend to dominate the individual commodities. We further study if we can find a combination of stocks, bonds and commodities that dominate others. Compared to a 60 percent stock and 40 percent bond portfolio mix, portfolios consisting of long positions in gold futures and two different actively managed indices are the only commodity investments to be included as long positions in a stock/ bond portfolio. The results should be of interest for fund managers and traders that seek to improve their risk-return trade off compared to the traditional 60/ 40 portfolio.

Keywords: Commodities, Stochastic Dominance, Asset Allocation
MSC 2010 Subject Classification: 90C15, 62P05

An acknowledgement in the following form should be included, together with a link to the published version on the publisher's website: "This is a post-peer-review, pre-copyedit version of an article published in [insert journal title]. The final authenticated version is available online at: http://dx.doi.org/10.1007/s10479-018-2996-7

1 Introduction and literature review

From 2003 to 2013 there was a strong growth in commodity-linked derivative investments. Institutional and private investors increased their positions in commodity futures from \$ 15 bn in 2003 to \$ 410 bn in 2013 (cf. Bessler and Wolff (2015) and Commodity Futures Trading Commission (2008)). However, from around 2013 falling commodity prices have reversed much of the net flow into commodity markets and increased scepticism to commodities as an attractive asset class.

Our paper is a contribution to the debate on wheter commodities should be included in the investor's portfolio. The academic literature does not agree on this issue. Most previous studies have applied traditional mean-variance, risk parity, Black-Litterman or simple naive strategies (see, e.g., Bessler and Wolff (2015); Pflug et al. (2012) or Pichler (2017)). This study takes a different methodological approach by applying stochastic dominance to verify whether commodities should be included in stock-bond portfolios. In our sample of individual commodities, our findings indicate that gold can be included as long position in a benchmark portfolio of stocks and bonds. We also find evidence that the four commodity indices used in the study are dominated by the stock and bond indices, but they perform better than individual commodities. The most active managed commodity index considered, SummerHaven Dynamic Commodity index, is the only commodity index that increases the risk-adjusted performance compared to the 60/ 40 stock/ bond portfolio.

The growing interest in commodities might be due to the perception that commodities have low correlation with traditional assets. This can be related to the factors driving commodity prices, i.e., the interaction of supply and demand, weather, politics and event risk rather than discounted future cash flows (Gorton and Rouwenhorst (2006) and Kat and Oomen (2007a,b)). Each commodity has also very distinct dynamics and treating each commodity as a single asset class is inappropriate Brooks and Prokopczuk (2013). Further, commodities are seen as an inflation hedge (see, e.g., Geman (2009); Fabozzi et al. (2008), and Roncoroni et al. (2015) for a broad discussion of the co-movement and inflation hedge properties of commodities).

Some academics, market participants and policy makers have been quick to associate the strong inflows into commodity investments with the commodity price spikes between 2007 and 2011 (see, e.g., Masters (2009a,b) for a detailed discussion). However, it is open for debate whether capital inflow to commodity linked products influences the dynamics of the commodity market. Some argue that the increased dependence between stocks and commodities is due to capital inflows from "speculators", the so-called *financialization* of commodities (see, e.g., Tang and Xiong (2012); Cheng and Xiong (2014); Silvennoinen and Thorp (2013) and Henderson et al. (2014). On the other hand, several authors do not support, or even oppose the commodity financialization hypothesis (see, e.g., Stoll and Whaley (2011); Dwyer et al. (2011); Sanders and Irwin (2011); Irwin and Sanders (2012); Steen and Gjolberg (2013); Demirer et al. (2015) and Hamilton and Wu (2015)).

Whether commodity index investing is useful is open for debate. Meyer (2015) discusses this in a recent Financial Times article by reviewing the work by Gorton and Rouwenhorst (2006) and Erb and Harvey (2006). The first paper views commodity indices as a potential attractive asset class, where investors will get paid a risk premium. These risk premiums are paid by producers for providing insurance against future falls in commodity prices. In the long run, the authors claim commodity index investing will generate returns similar to US stocks. Bhardwaj et al. (2015) still support this view even after the large fall of most commodity prices. Other support for commodities as an asset class is also found in You and Daigler (2013).

On the other hand, Sanders and Irwin (2012) find no evidence that commodity futures markets produce positive earnings. Further, Erb and Harvey (2006) argue that the main source of commodity investment returns come from the term structure of futures, which is hard to predict. From a passive, long only commodity index investment strategy there is no reason to expect equity like returns (cf. Miffre (2016) for a review on long-short commodity investing). Additional support for this is found in, e.g., Daskalaki and Skiadopoulos (2011). The argument is that commodities are intended for consumption and can only be stored to a limited extend. Their distinct price dynamics is driven both by demand and supply conditions. Commodity returns are determined by interest paid on collateral held against futures positions and roll-over returns (when one contract expires and gets replaced by another one). This roll-over return can be positive or negative and, as already stated, be very hard to predict.

When including commodities in a portfolio framework, several researchers observe a positive shift in the in-sample efficient frontier (see, e.g., Jensen et al. (2000); Laws and Thompson (2007); Idzorek et al. (2007) and Belousova and Dorfleitner (2012)). However, they report that benefits are time-varying and heterogeneous across different commodities. Cao et al. (2010) report that there was no significant shift in the efficient frontier when adding commodities in the period 2003 to 2010, which they argue might be a consequence of the financialization addressed above.

Results from out-of-sample analysis also vary. You and Daigler (2013) and Daigler et al. (2017) suggest that commodities improve performance, while Daskalaki and Skiadopoulos (2011) find evidence of the opposite. Lombardi and Ravazzolo (2016) find that portfolios become substantially more volatile when commodities are included and thereby experience a decrease in the Sharpe ratio. Bessler and Wolff (2015) identify that benefits of including commodity futures are time-varying and differ considerably across commodities. They further find no evidence of differences in diversification effects for periods with different states of the economy. Using a stochastic dominance (SD) framework, Daskalaki et al. (2016) find that commodities provide diversification benefits and that commodity markets are segmented from the stock and bond market. They employ a stochastic dominance efficiency test and optimize portfolios based on the whole distribution following Scaillet and Topaloglou (2010); in contrast, we employ SD as constraint to cover the entire tail-risk of the portfolios.

A shortcoming of some previous research is that the studies examining the diversification benefits from adding commodities to a stock-bond portfolio are limited to portfolio optimization techniques using rigid assumptions. In contrast, stochastic dominance constitutes a technique that involves the whole distribution rather than selected statistics. The concept of stochastic dominance arises where one *gamble* (here, a probability distribution over possible returns, also known as prospects) can be ranked as superior to another gamble. It is a robust technique based on shared preferences regarding sets of possible outcomes and their associated probabilities. Stochastic dominance has the advantage that it requires only the assumption that investors prefer more return over less and that they are risk-averse for second order stochastic dominance to hold. These are not unrealistic assumptions. Unlike other portfolio choice models, only limited knowledge of preferences is required for determining dominance. In our paper we use stochastic dominance relations as constraints to the objective function of maximizing the portfolio return.

We test a range of commodities and commodity indices against a stock index and a bond index. We find that the individual commodities are dominated by both the stock index and the bond index. Further, when including individual commodities in our optimized portfolios we need to increase the allocation in the stock index due to the poor performance of the individual commodities. When analysing commodity indices we find evidence that the stock index does not dominate the actively managed commodity index. We also observe that only the actively managed commodity indices and gold are interesting investments to be included in order to dominate the 60/40 portfolio. Further, none of the portfolios constructed have significantly higher risk adjusted return than the 60/40 portfolio, except the increased Omega ratio when SummerHaven Dynamic commodity index is included.

Outline of the paper. In Section 2 we describe the data we use and provide descriptive statistics. Section 3 describes the method of portfolio optimization with stochastic dominance constraints. In Section 4 we perform the empirical analysis and discuss the results. Section 5 concludes and discusses the implications of our results for portfolio management and commodity as an asset class.

2 Data and descriptive statistics

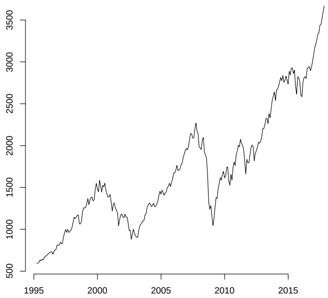
Our dataset consists of monthly end prices from March 1995 to November 2017 and covers 273 observations for price levels.[1] All series are total return series. In this study we use nine different commodity futures series, four different commodity indices, one stock index and one bond index.[2] These commodity futures and indices used are listed in Table 1.

As benchmark for the stock market we use S&P1200 (stocks). This index provides exposure to the global equity market by capturing approximately 70 % of the global market capitalization. For the bond market benchmark we use Bloomberg Barclays Global Aggregate Bond Index (bonds). This index measures the global investment grade debt from 24 local currency markets. This benchmark includes treasury, government-related, corporate and securitized fixed-rate bonds from both developed and emerging markets.

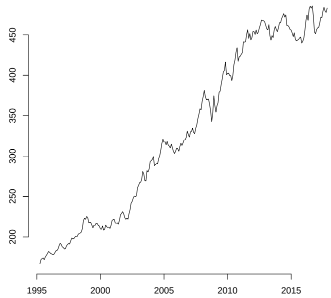
In contrast to the stock and bond market, which are driven by market capitalization, there is no generally accepted way of defining the composition of an aggregate commodity futures market (see e.g., Erb and Harvey (2006)). For this we use two of the most popular commodity indices, the S&P GSCI (GSCI) and the Bloomberg Commodity Index (BCOM). GSCI is a world-production weighted index and, by that, heavily tilted towards energy. GSCI includes 24 commodity nearby

[1]The returns are calculated as the natural logarithm of relative price differences.

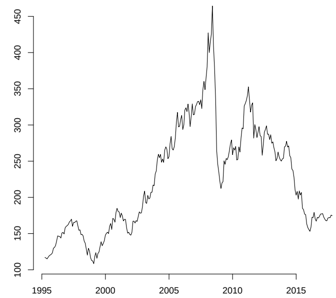
[2]For robustness check, we have compared our data from datastream with data from quandl.com and find similar developments of the futures series. The data set should therefore provide reliable results.



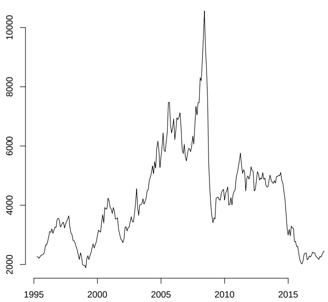
(a) STOCKS



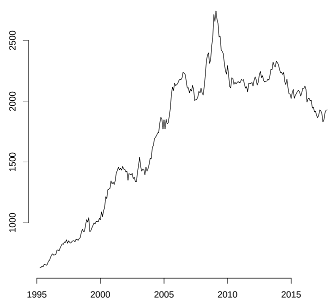
(b) BONDS



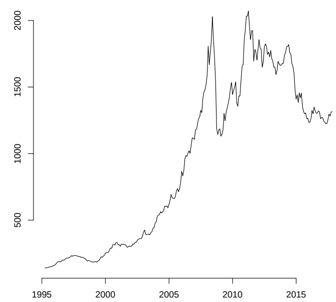
(c) GSCI



(d) BCOM



(e) CYD



(f) SDCI

Figure 1: Indices. The plots show monthly indices from March 1995 to November 2017. The sample size is based on 273 observations. The indices are demoninated in United States dollar and are total return indices. Cf. also Figure 2.

futures contracts with liquidity constraints. BCOM consist of 22 commodities with weighting methodology based on 1/3 world production value and 2/3 market liquidity. It also has constraints with a maximum weight of 15 % and minimum of 2 % in a single commodity, further a sector constraint with maximum weight of 33 % in commodity groups.

Due to the term structure of commodities there are developed indices (investment strategies) that are constructed to avoid the roll-over losses (see Miffre (2014)). There are several such indices with different strategies. We use the CYD Long Only (CYD) and the SummerHaven Dynamic Commodity Index (SDCI) as our benchmarks for strategies that try to mitigate roll-losses. The CYD index holds long positions solely in backwardated commodities which satisfy a given liquidity criteria, also precious metals such as gold and silver are excluded. The SDCI selects 14 commodity contracts based on backwardation and momentum and the commodities are equally weighted in the

Table 1: Description of Indices and Commodity Futures Used

Asset	Abbreviation
S&P 1200	stocks
Bloomberg Barclays Global Aggregate Bond Index	bonds
S&P GSCI	GSCI
Bloomberg Commodity Index	BCOM
CYD Long Only	CYD
SummerHaven Dynamic Commodity Index	SDCI
Corn No.2 Yellow	corn
Soybeans	soybeans
Sugar No.11	sugar
High Grade Copper	copper
Gold 100 oz	gold
Silver 5000 oz	silver
Light Sweet Crude Oil - WTI	crude oil
Chicago SRW Wheat	wheat
Henry Hub Natural Gas	natural gas

The table displays the different assets used in the study and their corresponding abbreviation. The data is downloaded from Thomson Reuters Datastream except bonds and SDCI which is downloaded from Bloomberg and the SummerHaven homepage: https://summerhavenindex.com/sdci/, respectively. For contract specification of the different commodities see http://www.cmegroup.com.

portfolio.[3]

We use several different commodities due to the argument of Brooks and Prokopczuk (2013), discussed in Section 1, that treating each commodity as a single asset class is inappropriate. We use nine different commodity futures to cover different commodity sectors: precious metals, industrial metals, energy and agricultural products. The futures contracts used are continuous front month contracts which switch from the 1st to the 2nd position when the 2nd month future volume exceeds the 1st future month volume (type 3 in Datastream).[4] For the contract specification for the different commodity futures contracts used in this study see the Chicago Mercantile Excange (CME) groups homepage.[5]

The chosen period covers many events that had major impacts on supply and demand in both

[3] For more information on CYD and SDCI see https://www.vescore.com/en-int/what-we-offer/ and https://summerhavenindex.com/sdci/

[4] There are several different rolling techniques one can apply on futures contracts. Thomson Reuters Datastream provides six different (from Type 0 to Type 5).

[5] http://www.cmegroup.com.

financial markets, as well as in the commodity markets. Examples of these are the Asian crisis in 1997, the dot-com bubble around 2000, the terror attacks on 9/ 11 2001, Hurricane Katrina in the Gulf of Mexico in 2005 and the boom and bust of the world economy from 2003 to the financial crises in 2008. Other events having an influence on commodity prices in the period are the wars on Balkan, Iraq and Afghanistan as well as extreme weather conditions (e.g., the very dry weather and wild fires in Russia in 2010 and the extreme heat in the US corn belt in 2012). These and several other events influence both the demand and supply side of commodities and the price formation across different commodities.

Figures 1 and 2 display the evolution of prices for the different indices and commodities used in this study. Further, Figures 3 and 4 display the Value-at-Risk (V@R) and the lower Average Value-at-Risk (AV@R) which depict the volatility and tails for the return series. One observation that is made from these figures and the descriptive statistics in Table 2 is the difference in return dynamics of the various assets.

During the full sample period, the SDCI had the highest annualized mean return of 9.9 %. However, this is not significantly higher than the return for stocks. Looking at stand-alone commodities we see that gold and silver stand out as the two with highest annualized mean return followed by crude oil.

We also see that the standard deviation of 14.4 % for SDCI is lower than for the stocks which has a standard deviation of 15.1 %. When testing for differences in variances for SDCI and the stock we see that SDCI does not significantly differ from stocks. There is a clear pattern that the volatility of the stand-alone commodities, except gold, exhibit greater volatility than the indices. Compared to stocks and bonds, the minimum monthly returns deceed the return of all the commodities and the commodity indices, except CYD and gold. This pattern is also visible in the maximum monthly values where commodities tend to have a greater value. The SDCI exhibits the most skewed distribution and also the fattest tails given the skew and kurtosis values of −1.1 and 6.2. On the other hand, wheat is the only asset with positive skewness.

For the risk-adjusted performance measures (RAPM), we include both Sharpe (Sharpe (1994)) and Omega (Keating and Shadwick (2002)) ratios. The reason for including these two are: (1) A mean variance analysis (M-V) employing Sharpe ratios yields the same result as stochastic dominance with normally distributed variables (cf. Ogryczak and Ruszczyński (1999)) and will help discuss- and compare our results to a M-V setting; (2) As Sharpe assume normally distributed returns, Omega does not require any assumption concerning the distributional properties of the returns. With respect to the ratios we find that the commodity indices CYD and especially SDCI stand out as good investments in the set of commodities, better than both stocks and bonds according to Omega ratio. Gold stands out among the individual commodities, however, it has lower RAPM than both stocks, bonds, CYD and SDCI. Agricultural commodities distinguish themselves in negative direction. Bonds and SDCI are the highest ranked assets when looking at the RAPM.

Table 3 displays Pearson correlations between all assets. The correlation between stocks and the commodity futures and indices range from 0.4 (copper and SDCI) to −0.1 (CYD). The correlation between the bond index and the commodities are in the range of 0.5 (gold) to −0.1 (CYD). The CYD index display close to zero correlation with the other assets, ranging between 0 and −0.2.

In sum, there is substantial difference in the risk characteristics of the different commodities and the indices in our sample data.

Table 2: Descriptive statistics for indices and commodities during full sample period

	mean	std dev	min	max	skewness	kurtosis	Sharpe	Omega
stocks	8,2 %	15,1 %	-21,0 %	10,8 %	-0,9	2,6	0.4	1.5
bonds	4,8 %	5,4 %	-4,1 %	6,0 %	-0,1	0,7	0.4	1.9
GSCI	1,9 %	15,8 %	-23,9 %	12,2 %	-0,7	2,9	-0.0	1.1
BCOM	0,5 %	22,3 %	-33,1 %	18,0 %	-0,7	2,2	-0.1	1.0
CYD	5,0 %	8,6 %	-11,8 %	7,6 %	-0,2	1,9	0.3	1.6
SDCI	9,9 %	14,4 %	-25,6 %	13,1 %	-1,1	6,2	0.5	1.7
corn	1,6 %	27,7 %	-25,9 %	24,8 %	-0,2	0,6	-0.0	1.0
soybeans	2,3 %	26,3 %	-26,6 %	18,9 %	-0,6	1,0	-0.0	1.1
sugar	0,9 %	31,1 %	-35,8 %	27,8 %	-0,1	1,4	-0.0	1.0
copper	3,7 %	27,7 %	-45,7 %	30,3 %	-0,6	4,8	0.0	1.1
gold	5,2 %	16,3 %	-19,7 %	14,9 %	-0,1	1,2	0.2	1.3
silver	5,0 %	29,5 %	-32,7 %	24,9 %	-0,3	1,1	0.1	1.1
crude oil	4,9 %	30,4 %	-38,1 %	30,0 %	-0,4	1,4	0.1	1.1
wheat	1,1 %	29,5 %	-26,2 %	32,0 %	0,1	1,0	-0.0	1.0
natural gas	2,3 %	45,6 %	-53,3 %	38,4 %	-0,2	1,2	0.0	1.0

The table displays descriptive statistics based on monthly returns (the natural logarithm of relative price differences). The calculations are based on return series from March 1995 to November 2017, i.e., 272 observations. Mean, standard deviation (std dev) and Sharpe ratios are annualized numbers. We have used the three-month Treasury Bill: Secondary Market Rate from FRED as the risk-free rate for calculating Sharpe ratios. The threshold in the Omega ratio is set to zero. Abbreviations for the different indices are given in Table 1.

3 Stochastic dominance relations

Order relations constitute a strong tool to model and describe the preference of individual decision makers. A total order is a relation which allows a comparison of all possible occurrences with respect to preference. Order relations have been considered for stochastic outcomes as well. Stochastic order relations are not always total and a direct comparison of stochastic outcomes (random variables) is not always available (cf. the discussion in Section 3.1 below).

Prevalent order relations in stochastic optimization are the first order and the second order stochastic dominance relations. An $\mathbb{R}$-valued random variable Y is said to dominate another random variable X in *first order stochastic dominance* (FSD), abbreviated $X \preccurlyeq_{(1)} Y$, if outcomes exceeding

Table 3: Correlation matrix of monthly returns

	stocks	bonds	GSCI	BCOM	CYD	SDCI	corn	soybeans	sugar	copper	gold	silver	crude oil	wheat
bonds	0,2													
GSCI	0,4	0,3												
BCOM	0,3	0,2	0,9											
CYD	-0,1	-0,1	-0,1	0,0										
SDCI	0,4	0,3	0,9	0,8	0,0									
corn	0,2	0,2	0,5	0,3	-0,1	0,5								
soybeans	0,3	0,2	0,5	0,3	-0,1	0,5	0,7							
sugar	0,1	0,0	0,1	0,1	0,0	0,2	0,0	0,0						
copper	0,4	0,2	0,6	0,5	-0,2	0,5	0,2	0,2	0,1					
gold	0,1	0,5	0,4	0,3	0,0	0,5	0,2	0,2	0,1	0,3				
silver	0,2	0,3	0,5	0,3	-0,1	0,5	0,3	0,2	0,1	0,3	0,7			
crude oil	0,3	0,1	0,8	0,9	0,0	0,6	0,2	0,2	0,1	0,4	0,2	0,3		
wheat	0,2	0,3	0,5	0,3	-0,2	0,4	0,6	0,5	0,1	0,2	0,2	0,2	0,2	
natural gas	0,0	0,2	0,5	0,5	0,1	0,3	0,1	0,1	0,0	0,1	0,1	0,0	0,3	0,1

The table displays the Pearson correlation matrix of monthly returns based on series from March 1995 to November 2017, i.e., 272 observations. See Table 2 for further specifications.

any threshold are more likely for Y than for X, i.e.,

$$X \preccurlyeq_{(1)} Y \quad :\Longleftrightarrow \quad P(X \le y) \ge P(Y \le y) \text{ for all } y \in \mathbb{R}. \tag{1}$$

Equivalent formulations include

$$\mathbb{E}\,\mathbb{1}_{(x,\infty)}(X) \le \mathbb{E}\,\mathbb{1}_{(x,\infty)}(Y) \text{ for all } x \in \mathbb{R}$$

and

$$\mathsf{V@R}_\alpha(X) \le \mathsf{V@R}_\alpha(Y) \text{ for all } \alpha \in [0,1]$$

(cf. Müller and Stoyan (2002)), where $\mathsf{V@R}_\alpha(X) := \inf\{q\colon P(X \le y) \ge \alpha\}$ is the *Value-at-Risk* at level α and $\mathbb{1}_A$ the indicator function of the event A.

A weaker relation than first order stochastic dominance is second order stochastic dominance. The random variable Y is said to dominate X in *second order stochastic dominance* (SSD), $X \preccurlyeq_{(2)} Y$, if

$$\mathbb{E}(y - X)_+ \ge \mathbb{E}(y - Y)_+ \quad \text{for all} \quad y \in \mathbb{R}, \tag{2}$$

where $x_+ := \max(0, x)$. Second order stochastic dominance is weaker than first order stochastic dominance, $\preccurlyeq_{(1)} \subseteq \preccurlyeq_{(2)}$, as we have that

$$\mathbb{E}(y - X)_+ = \int_{-\infty}^{y} P(X \le y')\mathrm{d}y'$$

by involving the relation (1) for first order stochastic dominance.

Müller and Stoyan (2002) (cf. also Pflug and Römisch (2007, Section 1.10) or Gutjahr and Pichler (2013)) characterize these stochastic order relations by employing test functions and expectations. Then

$$X \preccurlyeq_{(1)} Y \text{ if and only if } \mathbb{E}\, u(X) \le \mathbb{E}\, u(Y)$$

for all nondecreasing functions $u\colon \mathbb{R} \to \mathbb{R}$ and

$$X \preccurlyeq_{(2)} Y \text{ if and only if } \mathbb{E}\, u(X) \le \mathbb{E}\, u(Y) \tag{3}$$

for all nondecreasing and concave functions $u\colon \mathbb{R} \to \mathbb{R}$, for which the expectation exists.

It is common practice in economic sciences to maximize a utility function instead of an expectation. It follows from the previous statements and particularly from (3) that second order stochastic dominance is closed under nondecreasing and concave utility functions. A portfolio, dominating in second stochastic order, thus will always be given preference, for every particular utility function with the properties addressed.

It follows from (2) that $X \preccurlyeq_{(2)} Y$ implies

$$\mathsf{AV@R}_\alpha(X) \le \mathsf{AV@R}_\alpha(Y) \text{ for all } 0 < \alpha \le 1, \tag{4}$$

where

$$\mathsf{AV@R}_\alpha(X) := \inf\left\{ y - \frac{1}{\alpha}\, \mathbb{E}\,(y - X)_+ : y \in \mathbb{R} \right\} \qquad (0 < \alpha \le 1)$$

is the (lower) *Average Value-at-Risk* at level α (cf. Pflug and Römisch (2007)). Ogryczak and Ruszczyński (2002) elaborate that the latter relation (4) is actually equivalent to the second order stochastic dominance relation $X \preccurlyeq_{(2)} Y$. Eq. (4) gives also rise to the interpretation that average losses of X are more severe than those of Y given $X \preccurlyeq_{(2)} Y$, independently of the risk level $\alpha \le 1$.

It is well-known that the $\mathsf{V@R}$ is not convex (concave), while the lower $\mathsf{AV@R}$ is concave and the largest coherent risk measure dominated by $\mathsf{V@R}$, i.e., $\mathsf{AV@R}_\alpha(Y) \le \mathsf{V@R}_\alpha(Y)$ for all $\alpha > 0$ (Föllmer and Schied (2004, Theorem 4.61)).

3.1 Adjustment, or the degree of violating stochastic dominance relations

The stochastic order relation $\preccurlyeq_{(1)}$ and $\preccurlyeq_{(2)}$ are not total, i.e., it is possible that two random variables X and Y are *not comparable* in stochastic order and

$$X \not\preccurlyeq_{(1)} Y \text{ and } Y \not\preccurlyeq_{(1)} X, \text{ or} \tag{5}$$

$$X \not\preccurlyeq_{(2)} Y \text{ and } Y \not\preccurlyeq_{(2)} X. \tag{6}$$

In order to ensure a relation in stochastic dominance it has been proposed in the literature (cf. Dentcheva and Ruszczyński (2011)) to employ translation equivariance to quantify the degree to which random variables satisfy, or violate the stochastic order relations (5) or (6).

Indeed, the Value-at-Risk, as well as the Average Value-at-Risk are *translation equivariant*, that is,

$$\mathsf{V@R}_\alpha(Y + c\cdot \mathbb{1}) = \mathsf{V@R}_\alpha(Y) + c$$

and

$$\mathsf{AV@R}_\alpha(Y + c\cdot \mathbb{1}) = \mathsf{AV@R}_\alpha(Y) + c \tag{7}$$

for any $c \in \mathbb{R}$. For this we may measure the gap to an active stochastic dominance relation by considering the quantities

$$c_{(1)} := \inf_{\alpha\in(0,1)} \mathsf{V@R}_\alpha(Y) - \mathsf{V@R}_\alpha(X) \text{ and } c_{(2)} := \inf_{\alpha\in(0,1)} \mathsf{AV@R}_\alpha(Y) - \mathsf{AV@R}_\alpha(X), \tag{8}$$

as it is then immediate that

$$\mathsf{V@R}(X) + c_{(1)} \preccurlyeq_{(1)} \mathsf{V@R}_\alpha(Y) \text{ and } \mathsf{AV@R}_\alpha(X) + c_{(2)} \preccurlyeq_{(2)} \mathsf{AV@R}_\alpha(Y).$$

Note that the relation $X \preccurlyeq_{(1)} Y$ ($X \preccurlyeq_{(2)} Y$, resp.) holds, if $c_{(1)} \geq 0$ ($c_{(2)} \geq 0$, resp.) and conversely, stochastic order relations do *not* hold, if $c_{(1)} < 0$ ($c_{(2)} < 0$, resp.). We thus interpret the quantities $c_{(1)}$ and $c_{(2)}$ as a degree, to which stochastic dominance relations are satisfied. A negative value of $c_{(1)}$ and $c_{(2)}$ indicates that stochastic dominance is violated, while stochastic dominance is given provided $c_{(1)} \geq 0$ or $c_{(2)} \geq 0$. Note that we also have the general bounds

$$\operatorname{ess\,inf}(X - Y) \leq \mathsf{AV@R}_\alpha(X - Y) \leq \mathsf{AV@R}_\alpha(X) - \mathsf{AV@R}_\alpha(Y) \leq -c_{(2)}$$

for all $\alpha \in (0, 1)$ by concavity of the Average Value-at-Risk (cf. Pflug and Römisch (2007)).

3.2 Optimization with stochastic dominance constraints

In what follows we consider an asset allocation problem by investing in J different asset classes with the objective to maximize the return. We denote the weight of the asset class j, $j = 1, \dots J$, within the optimal portfolio by x_j. Every feasible portfolio is supposed to dominate each benchmark variable of returns $Y^{(b)}$, $b = 1 \dots, B$ in second order stochastic dominance. The random return of the portfolio with investments $x := (x_1, \dots, x_J)$ is $\Xi(x)$. The optimization problem thus reads

$$\begin{aligned} \text{maximize } & \mathbb{E}\,\Xi(x) \\ \text{subject to } & \Xi(x) \succcurlyeq_{(2)} Y^{(1)}, \\ & \dots \\ & \Xi(x) \succcurlyeq_{(2)} Y^{(b)}, \\ & \sum_{j=1}^{J} x_j = 1 \quad j = 1, \dots J, \end{aligned} \tag{9}$$

a slight generalization of Dentcheva and Ruszczyński (2011, problem (9.34)–(9.36)), cf. also Dentcheva and Ruszczyński (2003, 2006). The constraint $\sum_{j=1}^{J} x_j = 1$ represents the *budget* *constraint*, while $x_j \geq 0$ is the *short-selling constraint* and each $Y^{(b)}$ is a benchmark, which is often an index. In our study we extend this by using a set of commodities, the commodity index and a 60/40 stock/bond portfolio as the benchmark $Y^{(b)}$. The formulation (9) chooses the investment strategy $\Xi(x)$ to outperform the benchmarks, i.e., $\Xi(x) \succcurlyeq_{(2)} Y^{(b)}$ for each $b = 1, \dots B$.

Problem (9) is not necessarily feasible. To obtain a feasible program we involve the parameter c introduced in (7), which accounts for the degree the stochastic dominance constraints are satisfied. Dentcheva and Ruszczyński (2011) propose a penalty $\kappa > 0$ for violating the stochastic dominance constraints and consider the problem

$$\begin{aligned} &\text{maximize } \mathbb{E}\,\Xi(x) - \kappa \cdot c \\ &\text{subject to } \Xi(x) + c \succcurlyeq_{(2)} Y^{(b)}, \qquad b = 1, \dots B, \text{ and} \\ &\sum_{j=1}^{J} x_j = 1. \end{aligned} \tag{10}$$

As above, (10) becomes feasible provided that the correction or adjustment c is large enough. The constant c is finite for every discrete distribution.

Importantly, by involving the reformulation of second order stochastic dominance constraints as outlined in (2) the problem is

$$\begin{aligned} &\text{maximize } \mathbb{E}\,\Xi(x) - \kappa \cdot c \\ &\text{subject to } \mathbb{E}(y - \Xi(x) - c)_+ \leq \mathbb{E}(y - Y^{(b)})_+ \qquad y \in \mathbb{R},\ b = 1, \dots B \text{ and} \\ &\sum_{j=1}^{J} x_j = 1, \qquad j = 1, \dots J. \end{aligned} \tag{11}$$

For x to be feasible, the constraints in (11) have to hold for every $y \in \mathbb{R}$, i.e., the problem (10) has a continuum of constraints. However, if the benchmark variable $Y^{(b)}$ has only finitely many outcomes, then the constraints can be reduced to a finite number, each possible outcome representing a constraint. This is indeed the case, as $y \mapsto \mathbb{E}\,(y - \Xi(x) - c)_+$ is convex and $y \mapsto \mathbb{E}\left(y - Y^{(b)}\right)_+$ is piecewise linear. In this way problem (10) simplifies to

$$\begin{aligned} &\text{maximize } \mathbb{E}\,\Xi(x) - \kappa \cdot c \\ &\text{subject to } \mathbb{E}\,(y_i - \Xi(x) - c)_+ \leq \mathbb{E}\left(y_i - Y^{(b)}\right)_+, \qquad i = 1, \dots I,\ b = 1, \dots B \text{ and} \\ &\sum_{j=1}^{J} x_j = 1, \qquad j = 1, \dots J, \end{aligned} \tag{12}$$

where y_i is in the range of $Y^{(b)}$, i.e., $Y^{(b)} \in \{y_i : i = 1, \dots, I\}$ for all $b = 1, \dots B$.

To reformulate the convex problem (12) as a linear problem we introduce the random variables $S_i := (y_i - \Xi(x) - c)_+$ (cf. also Fábián et al. (2009) for linear reformulations). Note that $S_i \geq 0$ and

$\Xi(x) + c + S_i \geq y_i$ so that the investment problem (11) rewrites as

$$\begin{aligned}
&\text{maximize } \mathbb{E}\,\Xi(x) - \kappa \cdot c && && (13)\\
&\text{subject to } \mathbb{E}\,S_i \leq \min_{b=1,\dots B} \mathbb{E}\left(y_i - Y^{(b)}\right)_+, && i = 1,\dots I,\\
&\qquad S_i \geq 0,\ \Xi(x) + c + S_i \geq y_i, && i = 1,\dots I,\\
&\qquad \sum_{j=1}^{J} x_j = 1, && j = 1,\dots J.
\end{aligned}$$

The expectation (13) is discretized in the usual way by introducing the probabilities $p_k := \frac{t_k - t_{k-1}}{t_K - t_0}$ and by specifying the annualized returns $\Xi_{k,j} := \frac{t_k - t_{k-1}}{t_K - t_0} \log \frac{S^j_{t_k}}{S^j_{t_{k-1}}}$ ($Y^{(b)}_k$, resp.), where $S^j_{t_k}$ is the price of the asset j observed at time t_k, $k = 0,\dots K$ (the prices $S^j_{t_k}$ for the assets observed and considered in the computations below are depicted in Figures 1 and 2).

The asset allocation based on annualized returns $\Xi_{k,j}$ then is linear, $\Xi(x) = \Xi \cdot x$. We finally deduce from (13) the following formulation as a linear program,

$$\begin{aligned}
&\text{maximize } \sum_{k=1}^{K}\sum_{j=1}^{J} p_k\, \Xi_{k,j}\, x_j - \kappa \cdot c && && (14)\\
&\text{subject to } \sum_{k=1}^{K} p_k\, s_{k,i} \leq \min_{b=1,\dots B} \sum_{k=1}^{K} p_k \left(y_i - Y^{(b)}_k\right)_+ && i = 1,\dots I,\\
&\qquad s_{k,i} \geq 0\ \sum_{j=1}^{J} \Xi_{k,j}\, x_j + c + s_{k,i} \geq y_i, && k = 1,\dots K,\ i = 1,\dots I,\\
&\qquad \sum_{j=1}^{J} x_j = 1, && j = 1,\dots J,
\end{aligned}$$

which is linear in the decision variables x_j, $s_{k,i}$ and c. Involving more than one second order stochastic dominance constraint in the original problem (10) notably increases the problem's dimension, although not its complexity.

3.3 Stochastic dominance within commodities

Stochastic dominance relations are based on the monthly random returns presented in Section 2. Figure 3 displays the Value-at-Risk for the commodities during the period considered in this study.

Stochastic dominance relations are strong relations and strict stochastic dominance relations are not often present when comparing different asset classes. In what follows we discuss first order stochastic dominance relations for the indices used in this study, then we discuss the second order stochastic dominance relations for both all indices and the individual commodities.

First order stochastic dominance

The gray areas in Figures 3 and 4 indicate first order stochastic dominance, as they display the V@R for the risk levels $0 \le \alpha \le 100\,\%$ for the indices and individual commodities. We find that none of the indices or commodities dominates any other in first stochastic order. Table 4 reports the additive adjustment $c_{(1)}$ necessary to achieve stochastic dominance in first order for the indices. We deduce from the table, for example, that GSCI $-2\% \preccurlyeq_{(1)}$ S&P 1200, that is, a uniform adjustment or correction of $c_{(1)} = -2\,\%$ in monthly return has to be accepted to obtain first order stochastic dominance. The average return of the stock index must be increased with 2 % to dominate the GSCI index in first stochastic order.

Table 4: Adjustments $c_{(1)}$ to compare indices in first order stochastic dominance, cf (8)

Adjustment $c_{(1)}$/ %	stocks	bonds	GSCI	BCOM	CYD	SDCI
stocks		-6	-3	-12	-4	-5
bonds	-17		-20	-30	-8	-22
GSCI	-2	-7		-9	-5	-2
BCOM	-7	-12	-6		-10	-5
CYD	-9	-3	-12	-21		-14
SDCI	-3	-7	-3	-8	-5	

Second order stochastic dominance

Second order stochastic dominance is a convex relation and weaker than first order stochastic dominance. The results for commodities are slightly different for second order stochastic dominance.

The lines in Figure 3 and Figure 4 represent the Average Value-at-Risk for the risk levels $0 \le \alpha \le 100\,\%$ for the assets used in this study. In relation, Table 5 displays the adjustments necessary to obtain second order stochastic dominance for the indices and the individual commodities. The adjustment $c_{(2)}$ represents the largest difference in AV@R between the two assets we compare, cf. (8). In other words, the AV@R curves for two assets cannot cross (see (4)) if stochastic dominance of second order holds true. Second order stochastic dominance (SSD) is obtained for those pairs of asset classes, which do not require an adjustment or where the adjustment is nonnegative.

From Table 5 we se that true SSD can be found between several assets and indices. We can observe that bonds and CYD dominate all commodity linked products except the SDCI, which is close to being equal preferable as the bond and CYD index. We also observe that stocks and SDCI are dominating all of the individual commodities, except gold. Further, only small adjustments are needed for stocks to dominate the bond and vice versa. We also note that gold dominates all other commodities in the table, while natural gas is dominated by all other commodities. Conversely, large adjustments have to be accepted in order to dominate gold, while huge adjustments are necessary for natural gas in order to dominate any other commodity. Gold is notably the only commodity which dominates the index GSCI and BCOM.

Table 5: Adjustments $c_{(2)}$ for second order stochastic dominance relations, cf. (8)

Adjustment $c_{(2)}$/ %	stocks	bonds	GSCI	BCOM	CYD	SDCI	corn	soybeans	sugar	copper	gold	silver	crude oil	wheat	natural gas
stocks		−0	−3	−12	−0	−5	−8	−10	−16	−25	−1	−13	−17	−9	−32
bonds	−17		−20	−29	−8	−22	−22	−23	−32	−42	−16	−29	−34	−22	−49
GSCI	−0	0		−9	0	−2	−8	−9	−14	−22	0	−11	−14	−8	−29
BCOM	1	0	0		0	1	−3	−5	−10	−13	0	−8	−7	−4	−21
CYD	−9	0	−12	−21		−14	−15	−17	−25	−34	−8	−21	−26	−17	−41
SDCI	−1	−0	−2	−8	−0		−9	−10	−15	−20	−1	−13	−13	−9	−28
corn	1	0	0	−7	0	0		−2	−10	−20	0	−7	−12	−2	−27
soybeans	1	0	0	−7	0	1	−1		−9	−19	0	−6	−12	−1	−27
sugar	1	0	0	0	0	1	0	0		−10	0	−0	−2	0	−18
copper	0	0	−0	−0	0	1	−2	−1	−2		0	−1	−2	−1	−11
gold	−1	0	−4	−14	0	−6	−9	−11	−17	−26		−14	−19	−10	−34
silver	0	0	−0	−0	0	0	−0	−0	−4	−13	0		−5	−0	−21
crude oil	0	0	−0	−0	0	0	−0	−0	−3	−8	0	−1		−1	−15
wheat	1	0	0	−7	0	1	−1	−2	−10	−20	0	−7	−12		−27
natural gas	1	0	0	−0	0	1	−0	0	−0	0	0	0	0	−0	

4 Asset allocation including commodities

This section addresses two strategies of asset allocation involving commodities. First, we consider an investor investing only in stocks and bonds (see the following Subsection 4.1 below). This investor plans to benefit from commodities by adjusting his positions so that his portfolio dominates a given commodity index, a selected commodity or all commodities in second order stochastic dominance.

In our second approach (Subsection 4.2) the objective is to create portfolios of stocks, bonds and commodities that dominate a 60 % stocks and 40 % bonds portfolio in second order stochastic dominance. The 60/ 40 is an ad-hoc portfolio choice with benefits. Many investors do not have the risk appetite to bear the risk of being in an all equity portfolio and a stable element like bonds may give the opportunity to buy when stock market is low, i.e., it uses the bonds exposure to hedge part of the equity position (for a discussion on the 60/ 40 allocation see, e.g., Ambachtsheer (1987), Chaves et al. (2011) , Qian (2011), Roncalli (2013) and Faber (2015)). Further, a close to 60/ 40 portfolio is argued to replicate the world capital markets (see Erb and Harvey (2013) and Doeswijk et al. (2014)).

4.1 Investments in stocks and bonds

Table 6 displays the optimal split of stocks and bonds which provides the highest return on average by dominating the given commodity index or individual commodity, i.e.,

$$x_S \cdot \text{Stocks} + x_B \cdot \text{Bonds} \succcurlyeq_{(2)} \text{Commodity}. \tag{15}$$

From Table 6 the optimal portfolio mix dominating the "naive" commodity indices GSCI and BCOM have allocation of stocks of 69% and 61% with an annual return of about 7%. Looking at the more active commodity indices, the CYD needs a relatively small position in the stock index and a larger proportion of bonds, mainly due to the low risk of the CYD index. From the descriptive statistics in Table 2 we observe that SDCI and the stock index are the two assets with the most similar risk/ return characteristics. This is also visible in Table 6, where one needs 82% in stocks and 18% in bonds to dominate the SDCI.

The results in Table 6 also indicate that individual commodities do not impose strong constraints on the portfolio composition. For most of the portfolios a short position in bonds is needed to better participate in the return of stocks, exceptions are stock/ bond portfolios dominating copper and gold.

We further compose an optimal portfolio which dominates *all* commodities collectively. This portfolio consist of about 54 % stocks and 46 % bonds with a return of 6.6% , relatively close to our 60/ 40 benchmark portfolio in section 4.2 and we find no significant difference in the returns and the variance between the portfolios.[6]

Table 6 also indicate that a for a given stock/bond portfolio to dominate the individual commodity a leveraged position in stocks are required. This is consistent with borrowing to meet a required return or risk profile, and this will entail an additional dimension of risk. However, a short position is not required for the portfolio that includes all commodities, the commodity indices, nor the individual commodities gold and copper. The reason that the stock/bond portfolio not impose a leveraged position to dominate the commodity portfolios may be that these portfolios are diversified and have less tail-risk than the individual commodities.

4.2 Additional investments in commodities

This section addresses an investor investing in stocks, bonds and additionally in commodities. We start by asking the question of how to add commodities to a portfolio mix of 60 % stocks and 40 % bonds. The resulting portfolio thus satisfies

$$x_S \cdot \text{Stocks} + x_B \cdot \text{Bonds} + x_C \cdot \text{Commodity} \succcurlyeq_{(2)} 60\,\%\ \text{Stocks} + 40\,\%\ \text{Bonds}, \tag{16}$$

where x_S, x_B and x_C are the shares invested in stocks, bonds and commodity, respectively.

Table 7 presents results from asset allocation which satisfy (16). We find a similar pattern compared to Table 6. The table indicates that the two actively managed commodity indices (CYD

[6]Test for difference in returns yield a t-statistic of 1.1, and the F-test for differences in variance exhibit a F-value of 1.17 with a F-critical of 1.22.

Table 6: Allocation of stocks and bonds dominating an individual commodity, the commodity index and all commodities with relaxed short-selling constraints

commodity	stocks	bonds	return	std dev	Sharpe	Omega
GSCI	69 %	31 %	7.1 %	11.0 %	0.4	1.6
BCOM	61 %	39 %	6.9 %	9.9 %	0.5	1.7
CYD	15 %	85 %	5.3 %	5.6 %	0.5	2.0
SDCI	82 %	18 %	7.6 %	12.7 %	0.4	1.6
corn	128 %	-28 %	9.2 %	19.0 %	0.4	1.4
soybeans	131 %	-31 %	9.4 %	19.5 %	0.4	1.4
sugar	166 %	-66 %	10.5 %	24.5 %	0.3	1.4
copper	64 %	36 %	6.9 %	10.2 %	0.4	1.7
gold	32 %	68 %	5.9 %	6.8 %	0.5	1.9
silver	161 %	-61 %	10.4 %	23.8 %	0.3	1.4
crude oil	141 %	-41 %	9.6 %	20.9 %	0.3	1.4
wheat	130 %	-30 %	9.3 %	19.3 %	0.4	1.4
natural gas	156 %	-56 %	10.1 %	23.1 %	0.3	1.4
all commodities	54 %	46 %	6.6 %	9.1 %	0.5	1.7

The table show allocation of stocks and bonds that dominate the given commodity in second order, see Equation (15) . The calculations are based on monthly return series from March 1995 to November 2017, i.e., 272 observations. Mean, standard deviation (std dev) and Sharpe ratios are annualized numbers. We have used the 3-Month Treasury Bill: Secondary Market Rate from FRED as the risk-free rate for calculating Sharpe ratios. The threshold in the Omega ratio is set to zero. Abbreviations for the different indices are shown in Table 1.

and SDCI) can be included as long positions to dominate the benchmark portfolio. For the individual commodities, only gold futures are included with a long position, but only with 4% capital allocated and the portfolio is not outperforming the 60/ 40 portfolio when comparing the risk adjusted performance measures (RAPM). However, the portfolio including long position in SDCI have both slightly higher Sharpe- and Omega ratio. On the other hand, when long position in CYD is included, the RAPM are lower compared with the 60/ 40 portfolio. From both Tables 7 and 6 we see that when commodities are included we have to increase the position in the stock index to compensate for the low commodity performance.

Out-of-sample test: For robustness we have also tested the stochastic dominance relation in (16) out-of-sample. As the in-sample window we use observations from April 1995 to November 2011 (200 observations) and test our model with a recursive window with annual re-balancing until November 2017 (72 out-of-sample observations). The optimization procedure was relative stable according to weights. This is also visible in Table 8, as the 60/ 40 portfolio does not need any adjustment to dominate the given portfolio of stocks, bonds and the given commodity except for very minor adjustments when comparing to the portfolios consisting of GSCI and BCOM. On the other hand we see that the stock/ bond/ commodity portfolios only need minor changes to dominate

Table 7: Allocation of stocks, bonds and the given commodity dominating the 60/40 portfolio.

commodity	stocks	bonds	commodity	return	std dev	Sharpe	Omega
GSCI	56 %	87 %	−43 %	8.0 %	9.5 %	0.6	1.9
BCOM	55 %	74 %	−29 %	7.9 %	9.7 %	0.6	1.8
CYD	87 %	−21 %	34 %	7.9 %	12.8 %	0.4	1.6
SDCI	12 %	43 %	45 %	7.5 %	8.5 %	0.6	2.0
corn	68 %	52 %	−20 %	7.7%	11.2 %	0.5	1.7
soybeans	73 %	46 %	−19 %	7.8 %	11.5 %	0.5	1.7
sugar	71 %	34 %	−14 %	7.9 %	11.8 %	0.4	1.6
copper	87 %	30 %	−17 %	8.0 %	12.4 %	0.5	1.6
gold	81 %	15 %	4 %	7.6 %	14.8 %	0.5	1.6
silver	84 %	22 %	−5 %	7.6 %	12.7 %	0.4	1.6
crude oil	81 %	23 %	−4 %	7.5 %	12.3 %	0.4	1.6
wheat	70 %	49 %	−19 %	7.9 %	11.5 %	0.5	1.7
natural gas	77 %	29 %	−6 %	7.6 %	12.2 %	0.4	1.6
60/40 portfolio	60 %	40 %	–	6.8 %	9.8 %	0.5	1.7

The table show allocation of stocks, bonds and commodities that dominate the 60/40 portfolio in second order, see Equation (16) . The calculations are based on monthly return series from March 1995 to November 2017, i.e., 272 observations. Mean, standard deviation (std dev) and Sharpe ratios are annualized numbers. We have used the 3-Month Treasury Bill: Secondary Market Rate from FRED as the risk-free rate for calculating Sharpe ratios. The threshold in the Omega ratio is set to zero. Abbreviations for the different indices are shown in Table 1.

the the 60/40 portfolio, except GSCI and BCOM. This indicates that our results are stable over time and that our in-sample results are valid.

5 Conclusion and summary

This paper analyzes possible benefits from adding commodities to stock-bond portfolios. We analyze nine different individual commodity futures contracts and four different commodity indices to a traditional stock/bond portfolio using data from April 1995 to November 2017. We investigate the inclusion of commodities to stock bond portfolios by employing stochastic dominance relations. This method is employed as it involves the whole distribution rather than selected statistics. The advantage of stochastic dominance is that we do not need to know the investors utility functions, their preference or aversion for skewness and fat tails.

We do not find any cases of first order stochastic dominance relations. The criteria for first order stochastic are strict and it is not surprising that this criterion is too strong to degrade assets to the inefficient set.

For the second order stochastic dominance analysis performed on the individual commodities we find that gold dominates all other commodities and also the "naive" commodity indices BCOM

Table 8: Out-of-sample second order stochastic dominance relations

Adjustments, $c_{(2)}$/ % , necessary for 60/ 40 in order to dominate stock, bond and commodity portfolio												
GSCI	BCOM	CYD	SDCI	corn	soybeans	sugar	copper	gold	silver	crude oil	wheat	natural gas
−0	−0	0	1	1	0	0	0	1	1	0	0	0

Adjustments, $c_{(2)}$/ % , necessary for stock, bond commodity portfolio to dominate the 60/ 40 portfolio												
GSCI	BCOM	CYD	SDCI	corn	soybeans	sugar	copper	gold	silver	crude oil	wheat	natural gas
0	0	−0	−1	−0	−0	0	−0	−1	−1	−0	−0	−0

The table shows the out-of-sample monthly additive adjustment $c_{(2)}$ necessary to achieve second order stochastic dominance ($\preccurlyeq_{(2)}$) between the 60/ 40 portfolio and the optimized portfolios consisting of stocks, bonds and the given commodity. Cf. the defining equation (8) and Table 7 for full period in-sample results.

and GSCI. On the other hand we find that natural gas is dominated by all of the other assets used in this study. We also find that the more "actively" managed commodity indices, stocks and bonds are dominating most of the individual commodities. The bond index also dominates three out of four commodity indices and only a small adjustment is needed to dominate the fourth.

We further address two approaches to asset allocation involving commodities. One, where the investor intends to benefit from commodities by adjusting the portfolio in such way that the portfolio dominates the commodity indices or individual commodity futures, and the second where the objective is to outperform a 60/ 40-portfolio by including commodities. In the first approach we find that individual commodities do not impose strong constraints on the portfolio composition, except for gold and copper. When all commodities are included, the optimal mix that dominates all commodities collectively is a portfolio consisting of 54 % stocks and 46 % bonds, this portfolio is not statistically different from our 60/ 40 benchmark portfolio in terms of return and volatility. The results for the commodity indices also impose no short position in bonds to leverage up the position in stocks to dominate the given index. For the second approach, gold is the only individual commodity that is to be included as long position. For the commodity indices, the CYD and SDCI are included as long positions in the stock/bond portfolio dominating the 60/ 40 benchmark. However, the SDCI is the only that increase the RAPM of the portfolio.

Our main conclusion is that most of the individual commodities and the GSCI and BCOM are dominated by stocks, bonds and more actively managed commodity indices, and hence need a positive shift in the AV@R curve. Further, stock/ bond portfolios including long buy-and-hold positions in commodities or baskets of commodities do not outperform the 60/ 40 benchmark portfolio. However, the SDCI, which uses several selection criteria (as backwardation and momentum) for inclusion of commodities, is included as long position in the stock/ bond portfolio. The portfolio including the SDCI also has higher RAPM than the 60/ 40 portfolio.

6 Acknowledgment

We would like to thank the editor of the journal and the referees for their commitment to assess and improve the paper. We would also thank the discussants at ECOMFIN2016 in Paris.

References

K. P. Ambachtsheer. Pension fund asset allocation: In defense of a 60/40 equity/debt asset mix. *Financial Analysts Journal*, 43(5):14–24, 1987. 15

J. Belousova and G. Dorfleitner. On the diversification benefits of commodities from the perspective of euro investors. *Journal of Banking & Finance*, 36(9):2455–2472, 2012. 3

W. Bessler and D. Wolff. Do commodities add value in multi-asset portfolios? An out-of-sample analysis for different investment strategies. *Journal of Banking & Finance*, 60:1–20, 2015. ISSN 0378-4266. doi:10.1016/j.jbankfin.2015.06.021. 2, 3

G. Bhardwaj, G. Gorton, and G. Rouwenhorst. Facts and fantasies about commodity futures ten years later. Technical report, The National Bureau of Economic Research, June 2015. URL `http://dx.doi.org/10.3386/w21243`. 3

C. Brooks and M. Prokopczuk. The dynamics of commodity prices. *Quantitative Finance*, 13(4): 527–542, 2013. doi:10.1080/14697688.2013.769689. 2, 6

B. Cao, S. Jayasuriya, W. Shambora, et al. Holding a commodity futures index fund in a globally diversified portfolio: A placebo effect. *Economics Bulletin*, 30(3):1842–1851, 2010. 3

D. Chaves, J. Hsu, F. Li, and O. Shakernia. Risk parity portfolio vs. other asset allocation heuristic portfolios. *The Journal of Investing*, 20(1):108–118, 2011. 15

I.-H. Cheng and W. Xiong. Financialization of commodity markets. *Annu. Rev. Financ. Econ.*, 6(1): 419–441, 2014. 2

Commodity Futures Trading Commission. Staff Report on Commodity Swap Dealers & Index Traders with Commission Recommendations. Technical Report, CFTC, September 2008. 2

R. T. Daigler, B. Dupoyet, and L. You. Spicing up a portfolio with commodity futures: Still a good recipe? *The Journal of Alternative Investments*, 19(4):8–23, 2017. 3

C. Daskalaki and G. Skiadopoulos. Should investors include commodities in their portfolios after all? new evidence. *Journal of Banking & Finance*, 35(10):2606–2626, 2011. doi:10.1016/j.jbankfin.2011.02.022. 3

C. Daskalaki, G. S. Skiadopoulos, and N. Topaloglou. Diversification benefits of commodities: A stochastic dominance efficiency approach. 2016. 3

R. Demirer, H.-T. Lee, and D. Lien. Does the stock market drive herd behavior in commodity futures markets? *International Review of Financial Analysis*, 39:32–44, 2015. 2

D. Dentcheva and A. Ruszczyński. Optimization with stochastic dominance constraints. *SIAM Journal on Optimization*, 14(2):548–566, 2003. doi:10.1137/S1052623402420528. 11

D. Dentcheva and A. Ruszczyński. Portfolio optimization with stochastic dominance constraints. *Journal of Banking & Finance*, 30:433–451, 2006. doi:10.1016/j.jbankfin.2005.04.024. 11

D. Dentcheva and A. Ruszczyński. Portfolio optimization with risk control by stochastic dominance constraints. In G. Infanger, editor, *Stochastic Programming*, volume 150 of *International series in Operations Research & Management Science*, chapter 9, pages 189–211. Springer Science+Business Media, LLC, 2011. doi:10.1007/978-1-4419-1642-6. 10, 11, 12

R. Doeswijk, T. Lam, and L. Swinkels. The global multi-asset market portfolio, 1959–2012. *Financial Analysts Journal*, 70(2):26–41, 2014. 15

A. Dwyer, G. Gardner, T. Williams, et al. Global commodity markets–price volatility and financialisation. *Reserve Bank of Australia Bulletin, June*, pages 49–57, 2011. 2

C. B. Erb and C. R. Harvey. The strategic and tactical value of commodity futures. *Financial Analysts Journal*, 62(2):69–97, 2006. doi:10.2469/faj.v62.n2.4084. 2, 3, 4

C. B. Erb and C. R. Harvey. The golden dilemma. *Financial Analysts Journal*, 69(4):10–42, 2013. 15

M. Faber. *Global Asset Allocation: A Survey of the World's Top Asset Allocation Strategies*. The Idea Farm, 2015. 15

C. I. Fábián, G. Mitra, and D. Roman. Processing second order stochastic dominance models using cutting plane representations. *Mathematical Programming*, 130(1):33–37, 2009. doi:10.1007/s10107-009-0326-1. 12

F. J. Fabozzi, R. Fuss, and D. G. Kaiser. *The handbook of commodity investing*, volume 156. John Wiley & Sons, 2008. 2

H. Föllmer and A. Schied. *Stochastic Finance: An Introduction in Discrete Time*. de Gruyter Studies in Mathematics 27. Berlin, Boston: De Gruyter, 2004. URL `http://books.google.com/books?id=cL-bZSOrqWoC`. 10

H. Geman. *Commodities and commodity derivatives: modeling and pricing for agriculturals, metals and energy*. John Wiley & Sons, 2009. ISBN 978-0-470-01218-5. 2

G. Gorton and K. G. Rouwenhorst. Facts and fantasies about commodity futures. *Financial Analysts Journal*, 62(2):47–68, 2006. doi:10.2469/faj.v62.n2.4083. 2

W. J. Gutjahr and A. Pichler. Stochastic multi-objective optimization: a survey on non-scalarizing methods. *Annals of Operations Research*, 236(2):1–25, 2013. doi:10.1007/s10479-013-1369-5. 10

J. D. Hamilton and J. C. Wu. Effects of index-fund investing on commodity futures prices. *International Economic Review*, 56(1):187–205, 2015. 2

B. J. Henderson, N. D. Pearson, and L. Wang. New evidence on the financialization of commodity markets. *Review of Financial Studies*, pages 1285–1311, 2014. 2

T. M. Idzorek, H. Till, and J. Eagleeye. Commodities and strategic asset allocation. *Intelligent Commodity Investing*, pages 113–177, 2007. 3

S. H. Irwin and D. R. Sanders. Financialization and structural change in commodity futures markets. *Journal of agricultural and applied economics*, 44(03):371–396, 2012. 2

G. R. Jensen, R. R. Johnson, and J. M. Mercer. Efficient use of commodity futures in diversified portfolios. *Journal of Futures Markets*, 20(5):489–506, 2000. 3

H. M. Kat and R. C. Oomen. What every investor should know about commodities part i: univariate return analysis. *CFA Digest*, 2007a. 2

H. M. Kat and R. C. Oomen. What every investor should know about commodities, part ii: multivariate return analysis. *Journal of Investment Management*, 5(3), 2007b. 2

C. Keating and W. F. Shadwick. A universal performance measure. *Journal of performance measurement*, 6(3):59–84, 2002. 7

J. Laws and J. Thompson. Portfolio diversification and commodity futures. *Unpublished paper, CIBEF, Liverpool John Moores University, Liverpool*, 2007. 3

M. J. Lombardi and F. Ravazzolo. On the correlation between commodity and equity returns: implications for portfolio allocation. *Journal of Commodity Markets*, 2(1):45–57, 2016. 3

M. W. Masters. Testimony of Michael W. Masters. June 4:1–29, 2009a. 2

M. W. Masters. Testimony of Michael W. Masters. Aug 5:1–21, 2009b. 2

G. Meyer. Commodity index investing debate reignites. *Financial Times*, September 2015. 2

J. Miffre. Comparing first, second and third generation commodity indices. *Alternative Investment Analyst Review*, 3(2):22–33, 2014. 5

J. Miffre. Long-short commodity investing: A review of the literature. *Journal of Commodity Markets*, 1(1):3–13, 2016. 3

A. Müller and D. Stoyan. *Comparison methods for stochastic models and risks*. Wiley series in probability and statistics. Wiley, Chichester, 2002. URL https://books.google.com/books?id=a8uPRWteCeUC. 9, 10

W. Ogryczak and A. Ruszczyński. From stochastic dominance to mean-risk models: Semideviations as risk measures. *European Journal of Operational Research*, 116:33–50, 1999. doi:10.1016/S0377-2217(98)00167-2. 7

W. Ogryczak and A. Ruszczyński. Dual stochastic dominance and related mean-risk models. *SIAM Journal on Optimization*, 13(1):60–78, 2002. doi:10.1137/S1052623400375075. 10

G. Ch. Pflug and W. Römisch. *Modeling, Measuring and Managing Risk*. World Scientific, River Edge, NJ, 2007. doi:10.1142/9789812708724. 10, 11

G. Ch. Pflug, A. Pichler, and D. Wozabal. The 1/ N investment strategy is optimal under high model ambiguity. *Journal of Banking & Finance*, 36(2):410–417, 2012. doi:10.1016/j.jbankfin.2011.07.018. URL http://dx.doi.org/10.1016/j.jbankfin.2011.07.018. 2

A. Pichler. A quantitative comparison of risk measures. *Annals of Operations Research*, 254(1): 251–275, 2017. doi:10.1007/s10479-017-2397-3. 2

E. Qian. Risk parity and diversification. *Journal of Investing*, 20(1):119, 2011. 15

T. Roncalli. *Introduction to risk parity and budgeting*. CRC Press, 2013. 15

A. Roncoroni, G. Fusai, and M. Cummins. *Handbook of Multi-Commodity Markets and Products: Structuring, Trading and Risk Management*. John Wiley & Sons Ltd UK, 2015. doi:10.1002/9781119011590. 2

D. R. Sanders and S. H. Irwin. The impact of index funds in commodity futures markets: A systems approach. *The Journal of Alternative Investments*, 14(1):40–49, 2011. doi:10.3905/jai.2011.14.1.040. 2

D. R. Sanders and S. H. Irwin. A reappraisal of investing in commodity futures markets. *Applied Economic Perspectives and Policy*, 34(3):515–530, 2012. doi:10.1093/aepp/pps026. 3

O. Scaillet and N. Topaloglou. Testing for stochastic dominance efficiency. *Journal of Business & Economic Statistics*, 28(1):169–180, 2010. 3

W. F. Sharpe. The Sharpe ratio. *The journal of portfolio management*, 21(1):49–58, 1994. doi:10.3905/jpm.1994.40950. 7

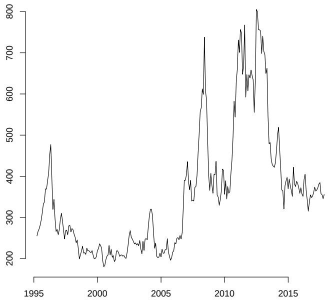
A. Silvennoinen and S. Thorp. Financialization, crisis and commodity correlation dynamics. *Journal of International Financial Markets, Institutions and Money*, 24:42–65, 2013. doi:10.1016/j.intfin.2012.11.007. 2

M. Steen and O. Gjolberg. Are commodity markets characterized by herd behaviour? *Applied Financial Economics*, 23(1):79–90, 2013. doi:10.1080/09603107.2012.707770. 2

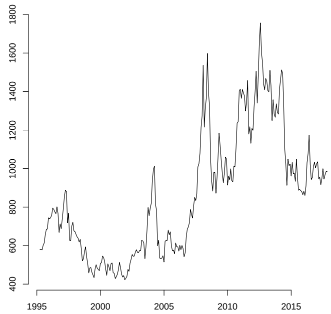
H. R. Stoll and R. E. Whaley. Commodity index investing: speculation or diversification? *The Journal of Alternative Investments*, 14(1):50–60, 2011. doi:10.2139/ssrn.1633908. 2

K. Tang and W. Xiong. Index investment and the financialization of commodities. *Financial Analysts Journal*, 68(5):54–74, 2012. 2

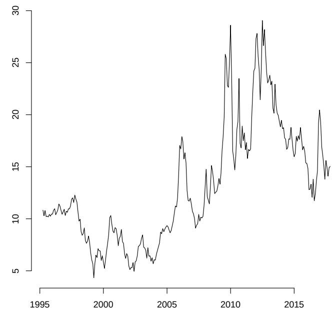
L. You and R. T. Daigler. A Markowitz optimization of commodity futures portfolios. *Journal of Futures Markets*, 33(4):343–368, 2013. doi:10.1002/fut.21553. 3



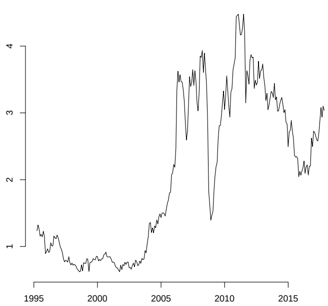
(a) corn



(b) soybeans



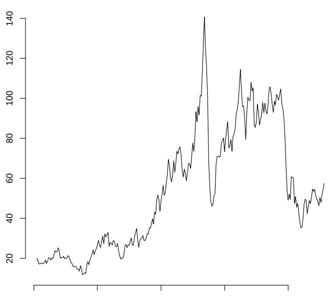
(c) sugar



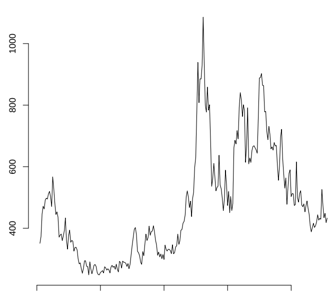
(d) copper

(e) gold

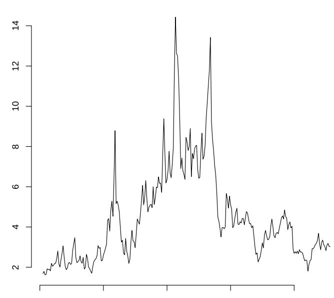
(f) silver



(g) crude oil

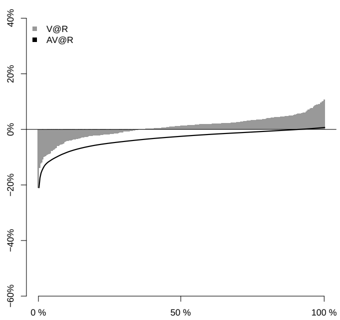


(h) wheat

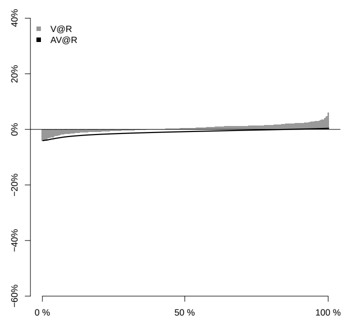


(i) natural gas

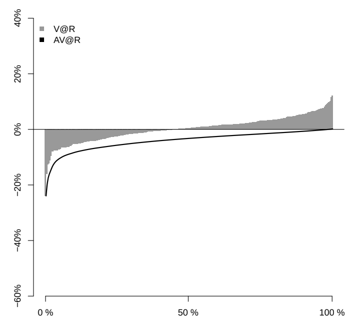
Figure 2: Prices. The graphs show monthly prices from March 1995 to November 2017. The sample size is based on 273 observations. Corn and soybeans are measured in cent per bushel, sugar in dollars per pound, copper in cents per pound, gold in dollar per troy ounce, silver in cent per troy ounce, crude oil in dollar per barrel, natural gas in dollars per million British thermal units. See Table 1 for more information. Cf. also Figure 1.



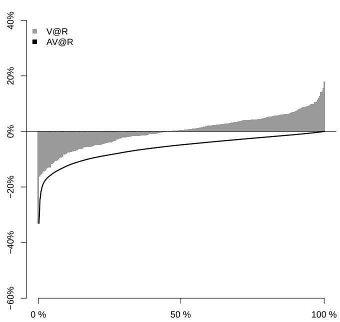
(a) stocks



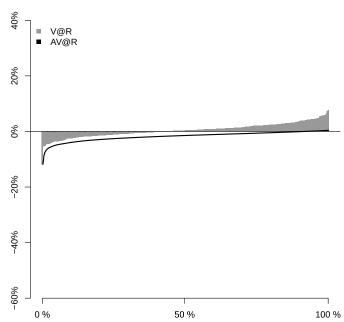
(b) bonds



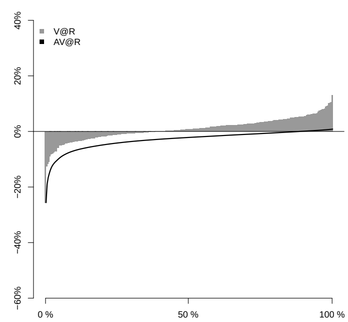
(c) GSCI



(d) BCOM

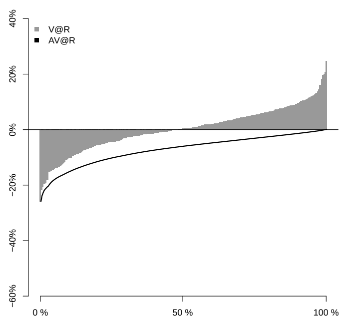


(e) CYD

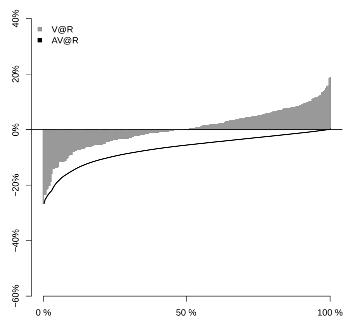


(f) SDCI

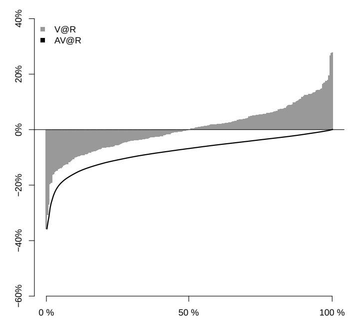
Figure 3: The Value-at-Risk (area) and the (lower) Average Value-at-Risk (line) of all indexes considered for the levels $\alpha \in (0, 1)$. Compare with Figure 4.



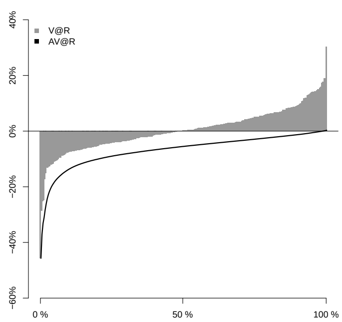
(a) corn



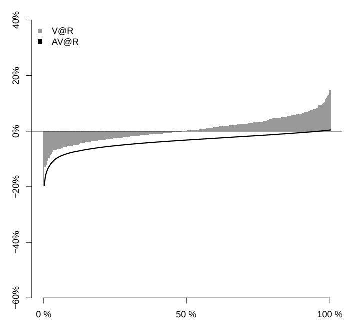
(b) soybeans



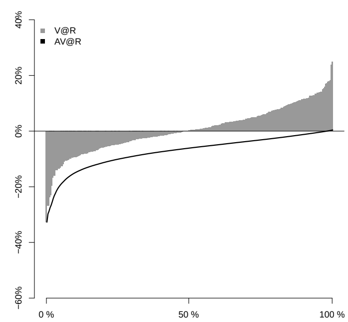
(c) sugar



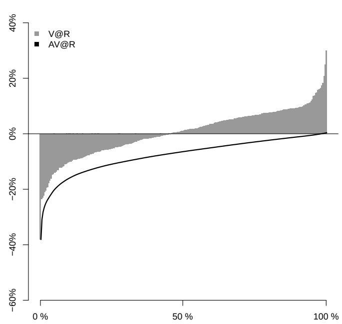
(d) copper



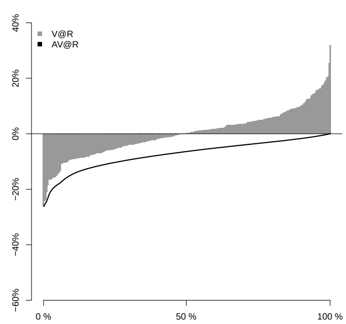
(e) gold



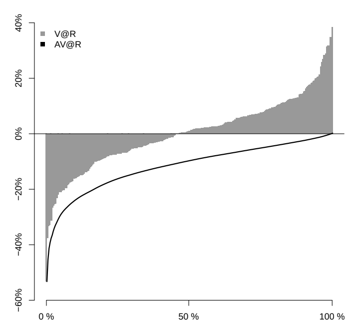
(f) silver



(g) crude oil



(h) wheat



(i) natural gas

Figure 4: The Value-at-Risk (area) and the (lower) Average Value-at-Risk (line) of all commodities considered for the levels $\alpha \in (0, 1)$.