

Can scalars have asymptotic symmetries?

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Recently it has been understood that certain soft factorization theorems for scattering amplitudes can be written as Ward identities of new asymptotic symmetries. This relationship has been established for soft particles with spins $s > 0$, most notably for soft gravitons and photons. Here we study the remaining case of soft scalars. We show that a class of Yukawa-type theories, where a massless scalar couples to massive particles, have an infinite number of conserved charges. This raises the question as to whether one can associate asymptotic symmetries to scalars.

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I. INTRODUCTION

Recently there has been a renewed interest in soft theorems [1–35] for scattering amplitudes and their relation to asymptotic symmetries [36–45], see [46] for a review. First studies that sparked this activity were in the context of gauge and gravity theories [47–50]. It was argued that factorization of an amplitude when one of the momenta becomes soft can be thought of as a consequence of Ward identities for a new set of symmetries acting on asymptotic fields. In the case of gauge theory, this symmetry is generated by large gauge transformations that do not decay at null infinity [47,49]. In the case of gravity, they are the Bondi–van der Burg–Metzner–Sachs supertranslations [36,37].

Soon after, the ideas were extended in multiple directions, including higher dimensions [51,52], subleading soft theorems [53–59], massive particles [60–62], and other theories [63–66]. For gauge and gravity theories, the soft particle in the soft theorems are of spin-1 (photon) and spin-2 (graviton). In [63] and [64,65] the analysis was generalized to the case where the soft particles have spin-1/2 (photino) and spin-3/2 (gravitino) respectively. An analysis for higher spins was given in [66]. Here, we would like to complete this list by studying the case of a spin-0 (scalar) soft particle.

There are two main motivations for this work. First, there is a large number of theories for which soft theorems have been studied [1–35]. It is however not clear to what extent

a relation to new asymptotic conserved charges can be identified in all these cases. It is then natural to try to map the space of theories for which this identification can be made. We believe this will bring us closer to understanding the nature of asymptotic symmetries in general. Our second motivation comes from a Minkowski holography approach [67,68] that has received fresh attention [69,70] after the soft theorem/asymptotic symmetry developments. The potential use of this approach for scattering amplitudes is being explored in the simpler setting of scalar fields [70,71]. We hope that an asymptotic symmetry description of scalar soft theorems could be of help in this program.

In this paper we study a number of field theories where a massless scalar field φ couples to a massive field ψ through a Yukawa-type interaction,

$$\mathcal{L}_{\text{int}} \sim \varphi \psi^2. \quad (1.1)$$

This provides the simplest example where a soft theorem can be associated to an asymptotic charge. Cubic φ^3 interactions in four dimensions are in conflict with the asymptotic expansion at null infinity [72] and are therefore excluded in our analysis. See also [73] for a different—and phenomenologically more relevant—model where soft scalars lead to asymptotic charges.

For concreteness the field ψ will be taken to be either a scalar or a Dirac field, but other fields can be treated similarly. In the theory (1.1), the tree-level amplitude of n hard particles of either type, and one soft φ scalar factorizes as

$$\mathcal{A}_{n+1}(p_1, \dots, p_n; q) \propto \sum_{i \in \psi} \frac{1}{p_i \cdot q} \mathcal{A}_n(p_1, \dots, p_n) + \dots, \quad (1.2)$$

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with a proportionality factor that depends on the spin of ψ . Hard particles have momenta p_i , and the soft one has momentum q . The dots denote subleading terms in q .

Following the by now standard procedure [47], one can recast (1.2) in the form of a Ward identity,

$$Q_{\hat{q}}^- S = S Q_{\hat{q}}^+, \quad (1.3)$$

for appropriately defined charges $Q_{\hat{q}}^\pm$ parametrized by the soft momentum direction \hat{q} . The situation is completely analogous to Ward identities associated to similar soft theorems arising in other theories. There is however a qualitative difference in theories where the spin of the soft particle is greater or equal than one: In those cases the charges are associated to large local symmetries. Specifically, the spin-1 soft theorem is associated to gauge symmetries, the spin-2 theorem to diffeomorphisms, and the spin-3/2 theorem to local supersymmetry. The spin zero and one-half cases fall in a different category and do not appear to be associated to any underlying local symmetry.

Even in the absence of local symmetry, the soft theorem (1.2) still predicts the existence of conserved charges. Why are they conserved? What is the underlying symmetry? In this paper we take the first steps toward answering these questions.

Our analysis will be restricted to tree-level amplitudes. The inclusion of loops would require us to regard (1.1) as a part of a larger theory where the field φ remains massless at quantum level. One possibility could be to realize φ as a “dilaton” (the Nambu-Goldstone boson of spontaneously broken scale symmetry [74,75]), which is known to satisfy the soft theorem (1.2) [12].

The organization of the paper is as follows. In Sec. II we describe the soft scalar theorem of interest and rewrite it as in Eq. (1.3). In Sec. III we express the charges in terms of the asymptotic fields and establish their conservation as a consequence of the field equations (assuming regular fall-off conditions). In Sec. IV we take first steps towards unraveling the symmetries underlying the charges: We compute their action on asymptotic fields and realize the charges in terms of a spacetime current. We also point out several conceptual difficulties which are similar to ones encountered in “magnetic” charges associated to spin 1 and 2 soft theorems. We conclude in Sec. V with a summary of results and open questions. Some calculations and side comments are left to Appendices.

Conventions: We use mostly plus signature spacetime metric. The normalization of Fock operators is such that $[a(\vec{p}), a^\dagger(\vec{p}')] = (2\pi)^3 (2E_p) \delta^3(\vec{p} - \vec{p}')$. We use the following convention for various tensor indices: a, b, \dots for spacetime abstract indices; μ, ν, \dots for cartesian Minkowski indices; α, β, \dots for the (space or time-like) 3-hyperboloid and A, B, \dots for the 2-sphere.

II. SPIN ZERO SOFT THEOREM AS A WARD IDENTITY

Consider first the case where the massive field ψ is a scalar. The coupling with the massless φ is given by the interaction Lagrangian

$$\mathcal{L}_{\text{int}} = \frac{g}{2} \varphi \psi^2. \quad (2.1)$$

A “soft theorem” for this theory may be established along the same lines as the photon or graviton case. One concludes that the tree-level amplitude of n particles of either type and one soft φ -particle factorizes as (see Appendix A for the derivation):

$$\begin{aligned} \mathcal{A}_{n+1}(p_1, \dots, p_n; \omega q) &\stackrel{\omega \rightarrow 0}{=} \frac{g}{2\omega} \sum_{i \in \psi} \frac{1}{p_i \cdot q} \\ &\times \mathcal{A}_n(p_1, \dots, p_n) + O(\omega^0). \end{aligned} \quad (2.2)$$

Following Strominger *et al.*, it is possible to reexpress (2.2) in a Ward identity form (1.3). If $a(\vec{p})$ and $b(\vec{p})$ are the Fock operators associated to φ and ψ respectively, the appropriate charge is

$$Q_{\hat{q}} := \lim_{\omega \rightarrow 0} \frac{\omega}{2} (a(\omega \hat{q}) + a^\dagger(\omega \hat{q})) - \frac{g}{2} \int \widetilde{d^3 p} \frac{b^\dagger(\vec{p}) b(\vec{p})}{p \cdot q} \quad (2.3)$$

where $\widetilde{d^3 p} \equiv \frac{d^3 \vec{p}}{(2\pi)^3 2E_p}$ and

$$q^\mu = (1, \hat{q}) \quad (2.4)$$

is the future-pointing null vector associated to the direction \hat{q} . To simplify notation we are omitting \pm superscripts. The charges $Q_{\hat{q}}^\pm$, acting on the “out” (+) and “in” (−) Fock spaces, have both the form (2.3) with the corresponding in/out Fock operators.

In order to establish the equivalence between the $Q_{\hat{q}}$ Ward identity and the soft theorem one also needs, as in the gauge and gravity cases, a relation between incoming and outgoing soft particles. In the present case the relevant relation is

$$\lim_{\omega \rightarrow 0} \omega \langle \text{out} | S a^\dagger(\omega \hat{q}) | \text{in} \rangle = - \lim_{\omega \rightarrow 0} \omega \langle \text{out} | a(\omega \hat{q}) S | \text{in} \rangle. \quad (2.5)$$

With these ingredients it is straightforward to show the desired equivalence,

$$\begin{aligned} \langle \text{out} | [Q_{\hat{q}}, S] | \text{in} \rangle = 0 &\Leftrightarrow \lim_{\omega \rightarrow 0} \omega \mathcal{A}_{n+1}(p_1, \dots, p_n; \omega q) \\ &= \frac{g}{2} \sum_{i \in \psi} \frac{1}{p_i \cdot q} \mathcal{A}_n(p_1, \dots, p_n). \end{aligned} \quad (2.6)$$

Following standard nomenclature, we write the charge (2.3) as

$$Q_{\hat{q}} = Q_{\hat{q}}^{\text{soft}} + Q_{\hat{q}}^{\text{hard}} \quad (2.7)$$

where $Q_{\hat{q}}^{\text{soft}}$ and $Q_{\hat{q}}^{\text{hard}}$ are respectively the terms linear and quadratic in the Fock operators.

Whereas the interest in this paper is in four dimensions, we note that the above expressions are valid in arbitrary spacetime dimensions.

Similar analysis can be repeated in Yukawa theory in four dimensions, where the massive field is of Dirac type. If we take the interaction Lagrangian as

$$\mathcal{L}_{\text{int}} = g\varphi\bar{\Psi}\Psi, \quad (2.8)$$

the analogous soft theorem takes the form (see Appendix A)

$$\begin{aligned} \mathcal{A}_{n+1}(p_1, \dots, p_n; \omega q) &\stackrel{\omega \rightarrow 0}{\approx} \frac{gm}{\omega} \sum_{i \in \{\Psi, \bar{\Psi}\}} \frac{1}{p_i \cdot q} \\ &\times \mathcal{A}_n(p_1, \dots, p_n) + O(\omega^0) \end{aligned} \quad (2.9)$$

where the sum now is over particles and antiparticles of the Ψ field. The charge whose Ward identity reproduces the soft theorem is of the form (2.7) with $Q_{\hat{q}}^{\text{soft}}$ as before and

$$Q_{\hat{q}}^{\text{hard}} = -gm \sum_s \int \widetilde{d\vec{p}} \frac{1}{p \cdot q} (b_s^\dagger(\vec{p})b_s(\vec{p}) + d_s^\dagger(\vec{p})d_s(\vec{p})) \quad (2.10)$$

where b_s (d_s) are the Fock operators for particles (antiparticles) with spin $s = \pm 1/2$.

III. CONSERVED CHARGES

A. Charges in terms of asymptotic fields

In this section we write the charges $Q_{\hat{q}}^\pm$ in terms of asymptotic future/past fields. In the following discussion and for the remainder of the paper we restrict attention to four spacetime dimensions.

The first step is to express the Fock operators in terms of asymptotic fields. For the massless field φ , this is captured in the null-infinity limit:

$$r \rightarrow \infty \quad \text{with} \quad u = t - r = \text{constant}, \quad \hat{x} = \text{constant}. \quad (3.1)$$

Assuming that in this limit $\varphi(x)$ is given by the free-field expression

$$\varphi(x) \approx \int \widetilde{d\vec{p}} a(\vec{p}) e^{ip \cdot x} + \text{c.c.} \quad (3.2)$$

a standard saddle point argument (see e.g., [76]) tells

$$\varphi(x) = \varphi_{\mathcal{I}^+}(u, \hat{x})/r + \dots \quad (3.3)$$

with

$$\varphi_{\mathcal{I}^+}(u, \hat{x}) = \frac{1}{4\pi i} \int_0^\infty \frac{dE}{2\pi} a(\vec{p} = E\hat{x}) e^{-iEu} + \text{c.c.} \quad (3.4)$$

The field $\varphi_{\mathcal{I}^+}(u, \hat{x})$ is to be regarded as a field living on future null infinity \mathcal{I}^+ . From (3.4) it follows that the soft part of the charge $Q_{\hat{q}}^+$ can be written as

$$Q_{\hat{q}}^{+\text{soft}} = -4\pi \int_{-\infty}^{\infty} du \partial_u \varphi_{\mathcal{I}^+}(u, \hat{q}) \quad (3.5)$$

$$= 4\pi(\varphi_{\mathcal{I}^+}(u = -\infty, \hat{q}) - \varphi_{\mathcal{I}^+}(u = +\infty, \hat{q})). \quad (3.6)$$

For massive fields one needs to consider the time-infinity limit

$$t \rightarrow \infty \quad \text{with} \quad r/t = \text{constant}, \quad \hat{x} = \text{constant}. \quad (3.7)$$

It is convenient to write this limit in terms of hyperbolic coordinates

$$\tau = \sqrt{t^2 - r^2}, \quad \rho = \frac{r}{\sqrt{t^2 - r^2}} \quad (3.8)$$

as

$$\tau \rightarrow \infty \quad \text{with} \quad y^\alpha := (\rho, \hat{x}) = \text{constant}. \quad (3.9)$$

Assuming again an asymptotic free field expression

$$\psi(x) \approx \int \widetilde{d\vec{p}} b(\vec{p}) e^{ip \cdot x} + \text{c.c.} \quad (3.10)$$

$$\Psi(x) \approx \sum_s \int \widetilde{d\vec{p}} (b_s(\vec{p}) u_s(\vec{p}) e^{ip \cdot x} + d_s^\dagger(\vec{p}) v_s(\vec{p}) e^{-ip \cdot x}) \quad (3.11)$$

a standard saddle point argument tells (see e.g., [61]):

$$\psi(x) = \frac{\sqrt{m}}{2(2\pi\tau)^{3/2}} b(\vec{p} = m\rho\hat{x}) e^{-i\tau m} + \text{c.c.} + \dots \quad (3.12)$$

$$\begin{aligned} \Psi(x) &= \frac{\sqrt{m}}{2(2\pi\tau)^{3/2}} \sum_s (b_s(\vec{p}) u_s(\vec{p}) e^{-i\tau m} \\ &+ d_s^\dagger(\vec{p}) v_s(\vec{p}) e^{i\tau m})|_{\vec{p}=m\rho\hat{x}} + \dots \end{aligned} \quad (3.13)$$

where we omitted an unimportant overall phase. Similar to the gauge and gravity cases, the hard part of the charge will be given by the ‘‘source’’ for the field φ . The field equations for φ are

$$\square\varphi = -\frac{g}{2}\psi^2 \quad (3.14)$$

in the scalar ψ theory and

$$\square\varphi = -g\bar{\Psi}\Psi \quad (3.15)$$

in the Yukawa theory. The $t \rightarrow \infty$ behavior of the massive fields implies that in both cases the leading term of the source falls as $1/\tau^3$,¹

$$\square\varphi = \frac{j(y)}{\tau^3} + \dots \quad (3.16)$$

In the scalar ψ theory the leading source is given by

$$j(y) = -g\frac{m}{4(2\pi)^3}b^\dagger(\vec{p})b(\vec{p})|_{\vec{p}=m\rho\hat{x}}, \quad (3.17)$$

whereas in the Yukawa theory

$$j(y) = -2mg\frac{m}{4(2\pi)^3}\sum_s(b_s^\dagger(\vec{p})b_s(\vec{p}) + d_s^\dagger(\vec{p})d_s(\vec{p}))|_{\vec{p}=m\rho\hat{x}}. \quad (3.18)$$

$j(y)$ is to be regarded as a field living on the future time infinity hyperboloid \mathcal{H}^+ (see Appendix B 1). We now show that the charge $Q_{\hat{q}}^{\text{hard}}$ takes a universal form when written in terms of $j(y)$.

Let

$$Y^\mu := \left(\sqrt{1 + \rho^2}, \rho\hat{x} \right) \quad (3.19)$$

be the unit timelike vector defined by y^α . Under the change of variables $\vec{p} \rightarrow y^\alpha$: $\vec{p} = m\rho\hat{x}$ we have

$$p^\mu = mY^\mu, \quad \frac{d^3\vec{p}}{E_p} = m^2 d^3V \quad (3.20)$$

with $d^3V = \sqrt{h}d\rho d^2\hat{x}$ the volume element on \mathcal{H}^+ . Using this and (3.17), (3.18) one finds that the hard charge in either theory is given by

$$Q_{\hat{q}}^{\text{hard}} = \int d^3V \frac{j(y)}{Y \cdot \hat{q}}. \quad (3.21)$$

We emphasize that (3.21) is valid for both the scalar ψ or the Yukawa theory. In fact, one of the conclusions one can draw from this paper is that (3.21) is the form of the hard charge regardless the spin of the massive particles.

¹In the scalar ψ case, the dots in (3.16) include terms of the form $e^{2im\tau}/\tau^3$, $e^{-2im\tau}/\tau^3$. In the Yukawa case such terms are absent and the dots start at $O(\tau^{-4})$.

As another example in Appendix A we discuss the soft theorem for spin 1 massive particles. Most of the remainder of the paper follows from Eq. (3.5), (3.16) and (3.21) and is thus insensitive to the nature of the massive particles.

Similar analysis applies to the asymptotic past. In terms of advanced time $v = t + r$, the asymptotic form of $\varphi(x)$ near past null infinity is

$$\varphi(x) = \varphi_{\mathcal{I}^-}(v, \hat{x})/r + \dots \quad (3.22)$$

with

$$\varphi_{\mathcal{I}^-}(v, \hat{x}) = -\frac{1}{4\pi i} \int_0^\infty \frac{dE}{2\pi} a(\vec{p} = -E\hat{x}) e^{-iEv} + \text{c.c.} \quad (3.23)$$

The soft part of $Q_{\hat{q}}^-$ is then:

$$Q_{\hat{q}}^{\text{soft}} = 4\pi(\varphi_{\mathcal{I}^-}(v = +\infty, -\hat{q}) - \varphi_{\mathcal{I}^-}(v = -\infty, -\hat{q})). \quad (3.24)$$

One can similarly obtain $Q_{\hat{q}}^{\text{hard}}$ in terms of the asymptotic fields at past time infinity.

B. Charge conservation

In the previous section we found expressions for the charges in terms of the asymptotic fields. The Ward identities discussed in Sec. II tell us these charges are conserved. The aim of this section is to understand this conservation from the perspective of the classical field theory.

The strategy is as follows. First, by studying the late-time field equations one can show that

$$Q_{\hat{q}}^+ = 4\pi\varphi_{\mathcal{I}^+}(u = -\infty, \hat{q}), \quad (3.25)$$

$$Q_{\hat{q}}^- = 4\pi\varphi_{\mathcal{I}^-}(v = +\infty, -\hat{q}). \quad (3.26)$$

Next, one studies the asymptotic field equations at spatial infinity to show that

$$\varphi_{\mathcal{I}^+}(u = -\infty, \hat{q}) = \varphi_{\mathcal{I}^-}(v = \infty, -\hat{q}) \quad (3.27)$$

from which the classical conservation $Q_{\hat{q}}^+ = Q_{\hat{q}}^-$ follows. Equation (3.27) is the spin zero version of Strominger's "matching" condition. The outlined strategy was used in [77] to treat the analogue problem in electrodynamics. The way we link future and past null infinity through spatial infinity is inspired from [78–80].

1. Field equations at time-infinity

Let us establish (3.25). From Eqs. (2.7) and (3.6) we see that (3.25) is equivalent to the condition:

$$Q_{\hat{q}}^{\text{hard}} = 4\pi\varphi_{\mathcal{I}^+}(u = +\infty, \hat{q}). \quad (3.28)$$

To show (3.28) we study the field equations in the asymptotic time-infinity limit. The field equation (3.16) implies that in this limit

$$\varphi(x) = \varphi_{\mathcal{H}^+}(y)/\tau + \dots \quad (3.29)$$

with

$$(D^2 + 1)\varphi_{\mathcal{H}^+} = j. \quad (3.30)$$

Here D^2 is the Laplacian operator on the time-infinity hyperboloid \mathcal{H}^+ (see Appendix B 1). Thus, (3.30) is a Poisson-type equation on \mathcal{H}^+ that determines $\varphi_{\mathcal{H}^+}$ for a given source j . The solution can be given in terms of appropriate Green's function,

$$\varphi_{\mathcal{H}^+}(y) = \int d^3V' \mathcal{G}(y; y') j(y'). \quad (3.31)$$

As shown in Appendix B 1, the properties of this Green's function imply:

$$\lim_{\rho \rightarrow \infty} \rho \varphi_{\mathcal{H}^+}(\rho, \hat{q}) = \frac{1}{4\pi} \int d^3V \frac{j(y)}{Y \cdot q} \quad (3.32)$$

$$= \frac{1}{4\pi} Q_{\hat{q}}^{\text{hard}}. \quad (3.33)$$

Finally, consistency between the expansions at null (3.3) and time (3.29) infinities imply the following continuity condition²:

$$\lim_{\rho \rightarrow \infty} \rho \varphi_{\mathcal{H}^+}(\rho, \hat{x}) = \varphi_{\mathcal{I}^+}(u = +\infty, \hat{x}), \quad (3.34)$$

from which (3.28) [and hence (3.25)] follows.

Similar treatment applies to asymptotic past. The analogue of relation (3.28) in this case is:

$$Q_{\hat{q}}^{\text{hard}} = 4\pi\varphi_{\mathcal{I}^-}(v = -\infty, -\hat{q}), \quad (3.35)$$

which together with (3.24) leads to Eq. (3.26).

2. Field equations at spatial infinity

In order to relate the future and past charges we now look at the field φ near spatial infinity. Hyperbolic coordinates adapted to this end are

$$\rho := \sqrt{r^2 - t^2}, \quad \tau := \frac{t}{\sqrt{r^2 - t^2}}. \quad (3.36)$$

The behavior of regular massless scalar at $\rho \rightarrow \infty$ is [81],

$$\varphi(x) = \varphi_{\mathcal{H}^o}(y)/\rho + \dots \quad (3.37)$$

whereas the massive field ψ falls-off as $e^{-m\rho}$. Thus asymptotically φ satisfies the free wave equation, which in turn implies $\varphi_{\mathcal{H}^o}(y)$ satisfies

$$(D^2 - 1)\varphi_{\mathcal{H}^o} = 0 \quad (3.38)$$

where D^2 is the wave operator in the space-infinity hyperboloid \mathcal{H}^o (see Appendix B 2). Equation (3.38) can be seen to imply either $O(1/\tau)$ or $O(\ln \tau/\tau)$ fall-off at $\tau \rightarrow \infty$. The latter however is not consistent with the assumed behavior at infinity. In fact, a continuity argument as in Eq. (3.34) tells

$$\lim_{\tau \rightarrow \infty} \tau \varphi_{\mathcal{H}^o}(\tau, \hat{x}) = \varphi_{\mathcal{I}^+}(u = -\infty, \hat{x}), \quad (3.39)$$

$$\lim_{\tau \rightarrow -\infty} -\tau \varphi_{\mathcal{H}^o}(\tau, \hat{x}) = \varphi_{\mathcal{I}^-}(v = +\infty, \hat{x}). \quad (3.40)$$

For future use, let $\phi_{\pm}(\hat{x})$ denote the above asymptotic values, that is,

$$\varphi_{\mathcal{H}^o}(\tau, \hat{x}) \xrightarrow{\tau \rightarrow \pm\infty} \frac{1}{|\tau|} \phi_{\pm}(\hat{x}). \quad (3.41)$$

In Appendix B 1 it is shown these asymptotic values satisfy

$$\phi_-(\hat{x}) = \phi_+(-\hat{x}), \quad (3.42)$$

as a consequence of the wave equation (3.38). This in turn implies Eq. (3.27) and hence the charge conservation $Q_{\hat{q}}^+ = Q_{\hat{q}}^-$.

C. Smeared charges

In analogy to the gauge and gravity cases we now consider smeared version of the charges,

$$Q^+[\lambda] := \frac{1}{4\pi} \int_{S^2} d^2\hat{q} \lambda(\hat{q}) Q_{\hat{q}}^+, \quad (3.43)$$

where $\lambda(\hat{q})$ is an arbitrary function on the sphere. Using the splitting $Q_{\hat{q}}^+ = Q_{\hat{q}}^{+\text{soft}} + Q_{\hat{q}}^{+\text{hard}}$ and Eq. (3.5), (3.21) we write $Q^+[\lambda]$ as a sum of soft and hard pieces with

$$Q^{+\text{soft}}[\lambda] = - \int_{\mathcal{I}^+} du d^2\hat{x} \lambda(\hat{x}) \partial_u \varphi_{\mathcal{I}^+}(u, \hat{x}), \quad (3.44)$$

$$Q^{+\text{hard}}[\lambda] = - \int_{\mathcal{H}^+} d^3V \Lambda_{\mathcal{H}^+}(y) j(y), \quad (3.45)$$

where we defined:

²See [77] for a discussion in the context of electrodynamics.

$$\Lambda_{\mathcal{H}^+}(y) := -\frac{1}{4\pi} \int d^2\hat{q} \frac{\lambda(\hat{q})}{q \cdot Y}. \quad (3.46)$$

As described in Appendix B 1, definition (3.46) implies $\Lambda_{\mathcal{H}^+}$ is the function on \mathcal{H}^+ that satisfies the Laplace-type equation

$$(D^2 + 1)\Lambda_{\mathcal{H}^+} = 0 \quad (3.47)$$

and has the $\rho \rightarrow \infty$ behavior $\Lambda_{\mathcal{H}^+}(\rho, \hat{x}) \sim \frac{\ln \rho}{\rho} \lambda(\hat{x})$. Equation (3.47) suggests $\Lambda_{\mathcal{H}^+}$ should be interpreted as the time-infinity asymptotic value of a spacetime field

$$\Lambda(x) = \Lambda_{\mathcal{H}^+}(y)/\tau + \dots \quad (3.48)$$

that satisfies (asymptotically) the free wave equation $\square\Lambda = 0$.³ A simple expression for a spacetime field satisfying these conditions is obtained by diving Eq. (3.46) by τ :

$$\Lambda(x) := -\frac{1}{4\pi} \int d^2\hat{q} \frac{\lambda(\hat{q})}{q \cdot x}. \quad (3.49)$$

Indeed one can verify such expressions satisfies $\square\Lambda = 0$ and Eq. (3.48) with no subleading terms. We leave for next section further discussion on this spacetime perspective.

We now go to the spatial infinity description of (3.43). The spacetime field $\Lambda(x)$ introduced above has a spatial infinity expansion,

$$\Lambda(x) = \Lambda_{\mathcal{H}^0}(y)/\rho + \dots \quad (3.50)$$

with $\Lambda_{\mathcal{H}^0}$ satisfying the wave equation

$$(D^2 - 1)\Lambda_{\mathcal{H}^0} = 0. \quad (3.51)$$

From this perspective, λ arises as the asymptotic value of $\Lambda_{\mathcal{H}^0}$ on \mathcal{H}^0 ,

$$\Lambda_{\mathcal{H}^0} \stackrel{\tau \rightarrow \infty}{\sim} \frac{\ln \tau}{\tau} \lambda(\hat{x}). \quad (3.52)$$

The structure is analogous to the gauge and gravity cases. There, the symmetry parameter on \mathcal{H}^0 ($\Lambda_{\mathcal{H}^0}(y)$ in the present case) satisfies the same differential equation as the field associated to the unsmeared charge ($\varphi_{\mathcal{H}^0}(y)$ in the present case). The charge is then written as a Klein-Gordon symplectic product between the two fields and is thus conserved. In the present case, this amounts to define the following conserved charge:

³Another possibility leading to (3.47) would be $\square\Lambda = 0$ with $\Lambda(x) \sim \ln \tau / \tau \Lambda_{\mathcal{H}^+}(y)$. However, this option does not seem to yield any sensible spacetime picture for the smeared charges.

$$Q_\tau = \int_{\tau=\text{const}} dS_\alpha \sqrt{h} h^{\alpha\beta} (\partial_\beta \varphi_{\mathcal{H}^0} \Lambda_{\mathcal{H}^0} - \partial_\beta \Lambda_{\mathcal{H}^0} \varphi_{\mathcal{H}^0}). \quad (3.53)$$

The integral (3.53) can be evaluated in the $\tau \rightarrow \infty$ limit by using the fall-offs (3.41) and (3.52). Doing so one finds:

$$Q_{\tau=\infty} = \int_{S^2} d^2V \lambda(\hat{x}) \phi_+(\hat{x}), \quad (3.54)$$

which precisely coincides with the charge (3.43). By taking the $\tau \rightarrow -\infty$ limit one obtains charge conservation between the asymptotic past and future smeared charges.

IV. SYMMETRY INTERPRETATION OF CHARGES?

In the previous sections we expressed the scalar soft theorem (1.2) as the conservation of certain charges defined at the asymptotic past and future, $Q^-[\lambda] = Q^+[\lambda]$. These charges are spin zero analogues of the charges associated to spin 1 and 2 soft theorems, schematically:

$$Q_{s=0} \sim \int_{\mathcal{I}} \lambda \partial_u \phi + \dots \quad (4.1)$$

$$Q_{s=1} \sim \int_{\mathcal{I}} D^A \lambda \partial_u A_A + \dots \quad (4.2)$$

$$Q_{s=2} \sim \int_{\mathcal{I}} D^A D^B \lambda \partial_u C_{AB} + \dots, \quad (4.3)$$

where we are only displaying the soft part of the charges and ϕ , A_A , C_{AB} represent the scalar, photon and graviton field at null infinity. The analogy between the $s = 0$ and $s = 1, 2$ charges appears to stop when it comes to symmetries. Expressions (4.2) and (4.3) can be understood as canonical charges associated to “large” $U(1)$ gauge symmetries $\lambda(\hat{x})$ and “supertranslation” diffeomorphisms $\xi_\lambda = \lambda(\hat{x}) \partial_u$ respectively, but there is no obvious symmetry interpretation for (4.1). A related point is the fact that for $s = 1, 2$, Q_s includes “global” charges: total electric charge and total linear momentum respectively, whereas there is no global charge associated to (4.1).

In the spin 1 and 2 cases there are in fact a second class of charges associated to soft theorems: These are the “magnetic” versions of (4.2) and (4.3). For instance, in the $s = 1$ case it is given by [56,82]

$$\tilde{Q}_{s=1} \sim \int_{\mathcal{I}} \epsilon^{AB} D_A \lambda \partial_u A_B, \quad (4.4)$$

where ϵ^{AB} is the area 2-form on the sphere. These magnetic charges have a closer analogy with the $s = 0$ charges in that they do not include global charges⁴ and their symmetry

⁴Unless one allows for magnetic monopoles [82].

description is subtle. A simple symmetry interpretation of (4.4) can be given if one works with ‘dual’ variables \tilde{A}_A describing the potential of the dual field strength $\tilde{F}_{\mu\nu} \sim \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$ [82,83].⁵ In the scalar case, we do not know of an alternative description that would make the symmetry associated to (4.1) transparent. As a first step towards understanding this symmetry, in subsection IV A we compute the action of the charges on the asymptotic data.

Another aspect of the charges associated to soft theorems is that they can be written as spatial integrals of total derivative currents (see e.g., [43]),

$$j^a = \partial_b k^{ab} \quad (4.5)$$

where k^{ab} is a densitized antisymmetric tensor. For instance, in the electrodynamics case we have $k^{ab} = \sqrt{\eta} \Lambda F^{ab}$ and $k^{ab} = \sqrt{\eta} \Lambda \tilde{F}^{ab}$ for the electric and magnetic charges respectively, where $\Lambda(x)$ is the ‘‘large gauge’’ parameter such that $\Lambda(x) \rightarrow \lambda(\hat{x})$ at null infinity. In subsection IV B we describe a current (4.5) that reproduces the charge (4.1).

A. Action of charges on asymptotic fields

From the perspective of the asymptotic future, the smeared charge $Q^+[\lambda]$ (3.43) is defined on the space of asymptotic free fields. One can thus compute its action on the asymptotic fields by Poisson brackets/commutators. For the field φ the relevant term is the soft charge (3.44) from which one finds,

$$\delta_\lambda \varphi_{\mathcal{I}^+}(u, \hat{x}) = -\lambda(\hat{x}), \quad (4.6)$$

Thus, asymptotically the field φ transforms by a ‘‘shift’’

$$\delta_\lambda \varphi = -\lambda(\hat{x})/r + \dots \quad (4.7)$$

where the dots denote terms subleading in $1/r$.

For the field ψ , it is easier to first study the transformation on the asymptotic Fock operator $b(\vec{p})$. The relevant piece is now the hard charge (3.45) from which one concludes

$$\delta_\lambda b(\vec{p}) = -i \frac{g}{2m} \Lambda_{\mathcal{H}^+}(\vec{p}/m) b(\vec{p}). \quad (4.8)$$

Given the asymptotic form of ψ in terms of b (3.12), we can infer the asymptotic action on ψ . We seek an expression involving only spacetime quantities. Let us interpret $\Lambda_{\mathcal{H}^+}$ in (4.8) as the τ^{-1} coefficient of the spacetime field $\Lambda(x)$ (3.48). The appearance of an i in (4.8) suggests a time-derivative action on ψ . In order to compensate for the τ^{-1} term, we are lead to consider the time-derivative vector field:

⁵We thank Beatrice Bongia, Laurent Freidel and Ali Seraj for discussions on this point.

$$X^a = \tau \partial_\tau, \quad (4.9)$$

which is just the dilatation vector field $x^\mu \partial_\mu$ expressed in hyperbolic coordinates. With these ingredients one concludes:

$$\delta_\lambda \psi = \frac{g}{2m^2} \Lambda X^a \partial_a \psi + \dots \quad (4.10)$$

where the dots denote terms subleading in $1/\tau$.

As stated in the beginning of the section, we do not have a geometric understanding for the asymptotic transformation (4.6), (4.10). The transformation for the field φ suggests a shift transformation analogous to the one occurring in electromagnetic or gravity cases. The analogy however fails in describing the ‘‘total derivative’’ nature of the current (4.5), see Appendix D for details. The appearance of the dilatation vector field in (4.10) suggests a relation to scale symmetry but we have not been able to establish any concrete connection.⁶

Regardless of the interpretation, the above transformation suffers a basic problem we now describe. If we think of the future asymptotic data as given by $\varphi_{\mathcal{I}^+}(u, \hat{x})$ and $b(\vec{p})$, it is in fact not entirely free but must satisfy Eq. (3.28) as described in Sec. III B 1. This condition is violated by the transformation (4.6), (4.8), since it changes $\varphi_{\mathcal{I}^+}(u = \infty, \hat{x})$ while leaving $Q_{\hat{q}}^{+\text{hard}}$ unchanged. This problem hints at the need to include boundary terms in the symplectic product. We note that similar problem occurs for the magnetic shifts generated by (4.4), which violates the condition $\epsilon^{AB} \partial_A A_B(u = \infty, \hat{x}) = 0$ that electromagnetic fields satisfy in the absence of magnetic monopoles [84].⁵

B. Spacetime current

Our aim is to find an antisymmetric k^{ab} whose current $j^a = \partial_b k^{ab}$ reproduces the smeared charges of Sec. III C. Note that this requirement only determines k^{ab} asymptotically but otherwise leaves it arbitrary in the bulk. Below we provide a k^{ab} that has a particularly simple spacetime form.

As in the gauge and gravity case, we seek for a k^{ab} that depends on the massless field $\varphi(x)$ and on the ‘‘symmetry parameter’’ field $\Lambda(x)$ introduced in the previous section. We will also need a third ingredient, the dilatation vector field

$$X^a = x^\mu \partial_\mu. \quad (4.11)$$

In terms of φ , Λ and X^a we define k^{ab} by⁷:

$$k^{ab} = \sqrt{\eta} ((\nabla^a \varphi \Lambda - \nabla^a \Lambda \varphi) X^b - (a \leftrightarrow b)). \quad (4.12)$$

We now show this current reproduces the smeared charges. Let us start by discussing the charges from the spatial infinity perspective. Consider the hyperbolic

⁶This point could perhaps be clarified in an a setting where φ is a dilaton field. See the discussion section for further comments.

⁷An equivalent form is $k^{ab} = 2\sqrt{\eta} (\Lambda \nabla^{[a} \varphi X^{b]} - \varphi \nabla^{[a} \Lambda X^{b]})$ where we used that $\nabla^{[a} X^{b]} = 0$.

coordinates (ρ, τ, \hat{x}) adapted to spatial infinity. In these coordinates, the dilatation vector field takes the form

$$X^a = \rho \partial_\rho. \quad (4.13)$$

Take a spacetime Cauchy slice Σ_τ that approaches a $\tau = \text{const}$ slice as $\rho \rightarrow \infty$. The integral of j^a over Σ_τ is then given by

$$\int_{\Sigma_\tau} dS_a \partial_b k^{ab} = \lim_{\rho \rightarrow \infty} \int_{\tau=\text{const}} d^2 \hat{x} k^{\tau\rho}. \quad (4.14)$$

Using the $\rho \rightarrow \infty$ expansion of the fields one finds,

$$k^{\tau\rho} = \sqrt{h} h^{\tau\tau} (\partial_\tau \varphi_{\mathcal{H}^0} \Lambda_{\mathcal{H}^0} - \partial_\tau \Lambda_{\mathcal{H}^0} \varphi_{\mathcal{H}^0}) + O(\rho^{-1}). \quad (4.15)$$

and one recovers the charge Q_τ given in Eq. (3.53).

We now discuss the charge from the perspective of the asymptotic future. Consider a family of Minkowski time $t = \text{constant}$ slices and evaluate the integral of j^a in a $t \rightarrow \infty$ limit:

$$Q^\infty := \lim_{t \rightarrow \infty} \int_{t=\text{const}} dS_a j^a. \quad (4.16)$$

Since j^a is built out of both massless and massive fields, there appears two contributions to (4.16) one associated to null infinity and the other to time-like infinity [61]. To study the null infinity contribution we go to retarded (r, u) coordinates. In these coordinates the dilatation vector field reads

$$X^a = r \partial_r + u \partial_u, \quad (4.17)$$

φ has the expansion (3.3) and $\Lambda(x)$ the expansion (see Appendix C)

$$\Lambda(r, u, \hat{x}) = \frac{\ln r}{2r} \lambda(\hat{x}) - \frac{1}{2r} \ln |2u| \lambda(\hat{x}) + O(r^{-1-\epsilon}). \quad (4.18)$$

From these expressions one can evaluate (4.16) in the $t \rightarrow \infty$, $u = \text{const}$ limit to obtain the null infinity contribution to the charge:

$$Q^{\mathcal{I}^+} := \lim_{t \rightarrow \infty} \int dud^2 \hat{x} \partial_u k^{ru} \quad (4.19)$$

$$= - \int dud^2 \hat{x} \lambda(\hat{x}) \partial_u \varphi_{\mathcal{I}^+}(u, \hat{x}), \quad (4.20)$$

which recovers (3.44). Note that the result is not entirely obvious since there are potentially logarithmic divergent terms in (4.19) that either cancel out or integrate to zero. See Appendix C for details.

The time-like infinity contribution to (4.16) is given by evaluating the integral in the limit where $t \rightarrow \infty$ with

$r/t = \text{const}$. As before, this limit is most conveniently described in hyperbolic coordinates adapted to time infinity. In this coordinates the dilatation vector field is given by Eq. (4.9). The timelike infinity contribution is found to be given by:

$$Q^{\mathcal{H}^+} := \lim_{\tau \rightarrow \infty} \int_{\tau=\text{const}} \partial_\alpha k^{\tau\alpha} \quad (4.21)$$

$$= - \int_{\mathcal{H}^+} \partial_\alpha (\sqrt{h} h^{\alpha\beta} (\partial_\beta \varphi_{\mathcal{H}^+} \Lambda_{\mathcal{H}^+} - \partial_\beta \Lambda_{\mathcal{H}^+} \varphi_{\mathcal{H}^+})) \quad (4.22)$$

$$= - \int_{\mathcal{H}^+} d^3 V \Lambda_{\mathcal{H}^+} j. \quad (4.23)$$

The second line follows from the time-infinity fall-offs and the third line from the field equations at time-infinity. Expression (4.23) recovers (3.45).⁸

In this way we see how the current $j^a = \partial_b k^{ab}$ reproduces the expected charge in the asymptotic future. A similar result applies to the asymptotic past charge.

V. DISCUSSION

In this work we studied a family of soft factorization theorems for scalar particles and showed their equivalence to Ward identities of an infinite number of new asymptotic charges. We identified the charges in terms of asymptotic fields and proved their conservation as a consequence of the field equations. As in other soft theorems, we showed the charges can be written in terms of a total derivative current.

The main open question regards the asymptotic symmetry interpretation of the charges. As a first step in answering this question, we computed the action of the charges on the asymptotic fields. However, we found a conflict between the resulting transformations and a constraint occurring in the asymptotic fields. We hope to resolve this issue in future work.

An interesting setup where to explore the question of symmetry interpretation could be to realize the massless scalar as the dilaton field of spontaneously broken conformal symmetry. As shown in [12], the Ward identities of this broken symmetry imply a soft theorem of the type studied here.⁹ It would also be interesting to study the dilaton subleading soft theorems [12] from the perspective of asymptotic symmetries.

⁸One can also get this result by evaluating (4.22) in terms of the $\rho \rightarrow \infty$ asymptotics of $\varphi_{\mathcal{H}^+}$ and $\Lambda_{\mathcal{H}^+}$.

⁹An analogous derivation of the soft photon theorem is given in [85,86]. See [33,34] for recent discussions on deriving soft theorems from Ward identities of spontaneously broken symmetries.

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APPENDIX A: SOFT SCALAR THEOREMS

In this Appendix we derive the soft theorems discussed throughout this work. We follow the classic derivation due to Weinberg [1,87] using a Feynman diagram argument. For any local theory, an external soft particle can couple to the remaining ones in two different ways, see. Fig. 1.

Theories considered in this work have a massless field φ coupled via a three-valent vertex to a massive field.¹⁰ In these cases, the first class of diagrams behaves as $\mathcal{O}(q^{-1})$, yielding the leading contribution to the soft factor. Let us consider in turn theories with different spin of the massive field.

First, we study the spin-zero case given by the interaction Lagrangian:

$$\mathcal{L}_{\text{int}} = \frac{1}{2} g \varphi \psi^2, \quad (\text{A1})$$

where ψ is a massive real scalar field. Let us study a tree-level scattering amplitude with $n + 1$ particles, where one massless scalar with incoming momentum ωq becomes soft, i.e., $w \rightarrow 0$. In this limit, the contribution from coupling to an external massive particle ψ with incoming momentum p_i is

$$ig \frac{-i}{(p_i + \omega q)^2 + m^2} \tilde{\mathcal{A}}_n(p_1, \dots, p_i + \omega q, \dots, p_n), \quad (\text{A2})$$

where the intermediate propagator corresponds to a massive particle due to the nature of the interaction term (A1). Here, $\tilde{\mathcal{A}}_n$ represents the rest of the amplitude. In the soft limit, $\omega \rightarrow 0$, we obtain

¹⁰Note that we do not have to make any assumptions about self-interactions of the massive field, and we allow self-interactions of the massless field φ starting at quartic order.

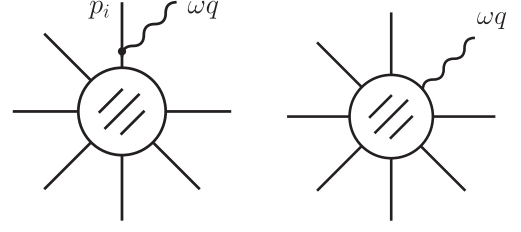


FIG. 1. Two topologies contributing in the soft limit. Wavy line denotes the soft particle, while straight lines denote the hard ones. All momenta are incoming. Only the left topology contributes to the leading soft behavior.

$$\frac{g}{2\omega} \frac{1}{p_i \cdot q} \tilde{\mathcal{A}}_n(p_1, \dots, p_i, \dots, p_n), \quad (\text{A3})$$

where now $\tilde{\mathcal{A}}_n$ becomes an on-shell amplitude for all particles except for the soft one. Summing over all diagrams contributing to this process gives us the soft factorization:

$$\mathcal{A}_{n+1}(p_1, \dots, p_n; \omega q) \stackrel{\omega \rightarrow 0}{=} \frac{g}{2\omega} \sum_{i \in \psi} \frac{1}{p_i \cdot q} \times \mathcal{A}_n(p_1, \dots, p_n) + \mathcal{O}(1), \quad (\text{A4})$$

where the sum proceeds over all external massive particles ψ , and the subleading terms come from diagrams of different topology.

Having outlined a general strategy, we can now generalize to a spin-half case. We study Dirac fermions in four dimensions coupled to the massless scalar via Yukawa interaction,

$$\mathcal{L}_{\text{int}} = g \varphi \bar{\Psi} \Psi. \quad (\text{A5})$$

A contribution from an incoming particle Ψ becomes

$$\tilde{\mathcal{A}}_n(p_1, \dots, p_i + \omega q, \dots, p_n) ig \frac{-i(-\not{p}_i - \omega \not{q} + m)}{(p_i + \omega q)^2 + m^2} u_{s_i}(\vec{p}_i). \quad (\text{A6})$$

In the soft limit $\omega \rightarrow 0$ we find:

$$\tilde{\mathcal{A}}_n(p_1, \dots, p_i, \dots, p_n) \frac{g}{2\omega} \frac{-\not{p}_i + m}{p_i \cdot q} u_{s_i}(\vec{p}_i). \quad (\text{A7})$$

We now use the spinor identities,

$$\sum_{s'} u_{s'}(\vec{p}_i) \bar{u}_{s'}(\vec{p}_i) = -\not{p}_i + m, \quad (\text{A8})$$

$$\bar{u}_{s'}(\vec{p}_i) u_{s_i}(\vec{p}_i) = 2m \delta_{s' s_i},$$

to obtain the soft contribution:

$$\frac{gm}{\omega} \frac{1}{p_i \cdot q} \tilde{\mathcal{A}}_n(p_1, \dots, p_i, \dots, p_n) u_{s_i}(\vec{p}_i), \quad (\text{A9})$$

where the spinor $u_{s_i}(\vec{p}_i)$ is now decorating the particle which used to give the propagator. Note that it has the same spin and momentum as the original particle p_i . Similar expression can be obtained for an incoming antiparticle $\tilde{\Psi}$, and outgoing particles using crossing symmetry. After summing over all contributions, we find the soft factorization:

$$\begin{aligned} \mathcal{A}_{n+1}(p_1, \dots, p_n; \omega q) &\stackrel{\omega \rightarrow 0}{\equiv} \frac{gm}{\omega} \sum_{i \in \{\Psi, \tilde{\Psi}\}} \frac{1}{p_i \cdot q} \\ &\times \mathcal{A}_n(p_1, \dots, p_n) + \mathcal{O}(1). \end{aligned} \quad (\text{A10})$$

The leading soft factor is proportional to the mass of the fermion. Since the limit $m \rightarrow 0$ is smooth, we conclude that the leading soft factor in a massless Yukawa theory vanishes.

Finally, let us consider the spin-one case. We couple the scalar to massive Z-bosons with the interaction term

$$\mathcal{L}_{\text{int}} = \frac{1}{2} g \varphi Z_\mu Z^\mu. \quad (\text{A11})$$

Contribution from a single diagram for an incoming Z-boson with momentum p_i and polarization λ_i is

$$\begin{aligned} &ig \frac{-i(\eta_{\mu\nu} + (p_i + \omega q)_\mu (p_i + \omega q)_\nu / m^2)}{(p_i + \omega q)^2 + m^2} \\ &\times \epsilon_{\lambda_i}^\mu(p_i) \tilde{\mathcal{A}}_n^\nu(p_1, \dots, p_i + \omega q, \dots, p_n). \end{aligned} \quad (\text{A12})$$

Using the identities,

$$\begin{aligned} \sum_{\lambda' = \pm, 0} \epsilon_{\lambda'}^{\mu*}(p_i) \epsilon_{\lambda'}^\nu(p_i) &= \eta^{\mu\nu} + p_i^\mu p_i^\nu / m^2, \\ \epsilon_{\lambda'}^*(p_i) \cdot \epsilon_{\lambda_i}(p_i) &= \delta_{\lambda' \lambda_i}, \end{aligned} \quad (\text{A13})$$

and a derivation analogous to the previous cases, we find that in the soft limit,

$$\begin{aligned} \mathcal{A}_{n+1}(p_1, \dots, p_n; \omega q) &\stackrel{\omega \rightarrow 0}{\equiv} \frac{g}{2\omega} \sum_{i \in Z} \frac{1}{p_i \cdot q} \\ &\times \mathcal{A}_n(p_1, \dots, p_n) + \mathcal{O}(1). \end{aligned} \quad (\text{A14})$$

The discussion in this appendix has been restricted only to tree-level amplitudes. Unlike in the case of soft theorems studied so far [47,49,50,63–65], at loop level the massless scalar considered in this work acquires mass. In this case, one cannot take a soft limit, meaning that the asymptotic symmetries identified in this work must be broken on a quantum level. However, it might be possible to introduce

extra fields into the theories under considerations, such that an enhanced symmetry would protect masslessness of φ at loop level.

APPENDIX B: GREEN'S FUNCTIONS ON \mathcal{H}^+ , \mathcal{H}^o

In this appendix we describe the differential operators and associated Green's functions that arise when expanding the fields at time and spatial infinity. Most of the considerations are particular cases of the general analysis given in [67]. But there is also some new (to us) material, for instance Secs. B 2 a and B 2 c.

1. (Future) Time-like infinity \mathcal{H}^+

Here we work in hyperbolic coordinates adapted to future time infinity

$$\tau = \sqrt{t^2 - r^2}, \quad \rho = \frac{r}{\sqrt{t^2 - r^2}}, \quad (\text{B1})$$

in terms of which the Minkowski line element reads:

$$ds^2 = -d\tau^2 + \tau^2 d\sigma^2 \quad (\text{B2})$$

where

$$d\sigma^2 = \frac{d\rho^2}{1 + \rho^2} + \rho^2 q_{AB} dx^A dx^B \equiv h_{\alpha\beta} dy^\alpha dy^\beta \quad (\text{B3})$$

is the metric of the future unit hyperboloid \mathcal{H}^+ .

For a field of the form

$$\varphi(x) = \tau^{-n} \overset{(n)}{\varphi}(y) \quad (\text{B4})$$

one has

$$\square \varphi(x) = \tau^{-n-2} (D^2 \overset{(n)}{\varphi} + n(2-n) \overset{(n)}{\varphi}), \quad (\text{B5})$$

where $D^2 \equiv h^{\alpha\beta} D_\alpha D_\beta$ is the Laplacian on \mathcal{H}^+ . By studying the differential operator on \mathcal{H}^+ defined by (B5) one concludes:

$$(D^2 + n(2-n)) \overset{(n)}{\varphi} = 0 \Rightarrow \overset{(n)}{\varphi} \sim 1/\rho^n \quad \text{or} \quad \overset{(n)}{\varphi} \sim 1/\rho^{2-n} \quad (\text{B6})$$

when $\rho \rightarrow \infty$. For the present paper we are interested in the $n = 1$ case. This happens to be the special case where the two independent asymptotic solutions (B6) coincide. What occurs then is that a second independent solution appears with a $\ln \rho$ dependence [67]:

$$(D^2 + 1) \overset{(1)}{\varphi} = 0 \Rightarrow \overset{(1)}{\varphi} \sim 1/\rho \quad \text{or} \quad \overset{(1)}{\varphi} \sim \ln \rho / \rho. \quad (\text{B7})$$

The PDE problems appearing in our discussion are¹¹:

$$(D^2 + 1)\varphi^{(1)} = j \quad \text{with} \quad \varphi^{(1)} \sim 1/\rho \quad (\text{B8})$$

$$(D^2 + 1)\Lambda^{(1)} = 0 \quad \text{with} \quad \Lambda^{(1)} \sim \frac{\ln \rho}{\rho} \lambda(\hat{x}). \quad (\text{B9})$$

We now describe the Green's function associated to each equation.

a. Green's function for Eq. (B8)

We want to find the solution to

$$(D^2 + 1)\mathcal{G}(y; y') = \delta^{(3)}(y, y'). \quad (\text{B10})$$

Due to the $SO(3, 1)$ symmetry it has to be of the form

$$\mathcal{G}(y; y') = g(Y \cdot Y') \quad (\text{B11})$$

for some function g . We find the general solution for $y \neq y'$ and then get the overall coefficient by requiring the appropriate $y \rightarrow y'$ behavior. Taking $\rho' = 0$ so that $Y \cdot Y' = -\sqrt{1 + \rho^2}$ one finds the general solution to $(D^2 + 1)f = 0$ is given by

$$g = \frac{A}{\rho} + B \frac{\text{arcsinh} \rho}{\rho}. \quad (\text{B12})$$

The fall-off requirement in (B8) implies $B = 0$. The coefficient A can be obtained by demanding that for $\rho \rightarrow 0$ one recovers the flat space Green's function. This sets $A = -1/(4\pi)$. The Green's function is finally obtained by expressing ρ in terms of $-Y \cdot Y'$ resulting in

$$\mathcal{G}(y; y') = -\frac{1}{4\pi} \frac{1}{\sqrt{(Y \cdot Y')^2 - 1}} \quad (\text{B13})$$

b. Green's function for Eq. (B9)

We now seek for $G(y; \hat{q})$ such that

$$(D^2 + 1)G = 0, \quad G(\rho, \hat{x}; \hat{q}) \sim \frac{\ln \rho}{\rho} \delta^{(2)}(\hat{x}, \hat{q}). \quad (\text{B14})$$

We just state the solution and verify it satisfies the desired conditions:

¹¹The source $j(y)$ decays fast enough at $\rho \rightarrow \infty$ so that the asymptotic form of $\varphi^{(1)}$ is still dictated by the sourcefree equation (B7). The absence of $\ln \rho / \rho$ term in $\varphi^{(1)}$ is due to the regular behavior of φ at null infinity. It is also interesting to note that fields f such that $(D^2 + 1)f = 0$ with $f = O(1/\rho)$ are always singular at $\rho = 0$ (this can be established by writing the explicit solution in terms of spherical harmonics).

$$G(y; \hat{q}) = -\frac{1}{4\pi} \frac{1}{Y \cdot q} \quad (\text{B15})$$

where

$$Y \cdot q = -\sqrt{1 + \rho^2} + \rho \hat{x} \cdot \hat{q} \quad (\text{B16})$$

is the Minkowski product between Y^μ (3.19) and q^μ (2.4). First, one can verify that

$$D^2 f(Y \cdot q) = (Y \cdot q)^2 f''(Y \cdot q) + 3(Y \cdot q) f'(Y \cdot q) \quad (\text{B17})$$

from which it follows that $(Y \cdot q)^{-1}$ satisfies the desired equation. Next we note that

$$G(y; \hat{q}) \sim \begin{cases} 1/\rho & \text{for } \hat{x} \neq \hat{q} \\ \rho & \text{for } \hat{x} = \hat{q} \end{cases} \quad (\text{B18})$$

and

$$\int d^2 \hat{q} G(y; \hat{q}) = \frac{\text{arcsinh} \rho}{\rho} \sim \frac{\ln \rho}{\rho} \quad (\text{B19})$$

from which it follows that G satisfies the asymptotic behavior given in (B14).¹²

c. Relation between Green's functions and Eq. (3.28)

From the expressions above one can verify that the large ρ behavior of \mathcal{G} is dictated by G according to¹³

$$\mathcal{G}(y; y') = -\frac{1}{\rho} G(y'; \hat{x}) + O(\rho^{-3}) \quad (\text{B20})$$

We now have all elements to show Eq. (3.28):

$$\phi(u = +\infty, \hat{q}) = \lim_{\rho \rightarrow \infty} \rho^{(1)} \varphi(\rho, \hat{q}) \quad (\text{B21})$$

$$= \lim_{\rho \rightarrow \infty} \rho \int d^3 V' \mathcal{G}(\rho, \hat{q}; y') j(y') \quad (\text{B22})$$

$$= - \int d^3 V' G(y'; \hat{q}) j(y') \quad (\text{B23})$$

$$= \frac{1}{4\pi} \mathcal{Q}_{\hat{q}}^{\text{hard}}. \quad (\text{B24})$$

The first equality arises from consistency of the null and time-infinity expansions. The second equality uses (3.31)

¹²The reasoning here is the same as the one given in [88]. The present case extends the analysis of [88] from $n > 1$ to $n = 1$.

¹³This is just a particular instance of more general relations between “bulk-bulk” and “bulk-boundary” Green's functions that are well known in the AdS/CFT literature.

and the third (B20). In the last equality we used the expressions (3.21) and (B15).

2. Spatial infinity \mathcal{H}^o

Hyperbolic coordinates adapted to spatial infinity are

$$\rho := \sqrt{r^2 - t^2}, \quad \tau := \frac{t}{\sqrt{r^2 - t^2}} \quad (\text{B25})$$

in terms of which the Minkowski line element reads:

$$ds^2 = d\rho^2 + \rho^2 d\sigma^2 \quad (\text{B26})$$

with

$$d\sigma^2 = -\frac{d\tau^2}{1 + \tau^2} + (1 + \tau^2)q_{AB}dx^A dx^B =: h_{\alpha\beta}dy^\alpha dy^\beta \quad (\text{B27})$$

the line element of the unit hyperboloid \mathcal{H}^o .

For a field of the form

$$\varphi(x) = \rho^{-n} \varphi^{(n)}(y) \quad (\text{B28})$$

one has

$$\square\varphi(x) = \rho^{-n-2}(D^2\varphi^{(n)} - n(2-n)\varphi^{(n)}), \quad (\text{B29})$$

where $D^2 \equiv h^{\alpha\beta}D_\alpha D_\beta$ is the wave operator on \mathcal{H}^o . By studying the differential operator on \mathcal{H}^o defined by (B29) one concludes:

$$(D^2 - n(2-n))\varphi^{(n)} = 0 \Rightarrow \varphi^{(n)} \sim 1/\tau^n \quad \text{or} \quad \varphi^{(n)} \sim 1/\tau^{2-n} \quad (\text{B30})$$

when $\tau \rightarrow \pm\infty$. Again, for $n = 1$ these two asymptotic solutions coincide but a new one appears:

$$(D^2 - 1)\varphi^{(1)} = 0 \Rightarrow \varphi^{(1)} \sim 1/\tau \quad \text{or} \quad \varphi^{(1)} \sim \ln \tau/\tau. \quad (\text{B31})$$

For the purposes of the present paper, we need the general solution associated to the each type of fall-offs:

$$(D^2 - 1)\varphi^{(1)} = 0 \quad \text{with} \quad \varphi^{(1)} \underset{\tau \rightarrow +\infty}{\sim} \frac{1}{\tau} \phi_+(\hat{x}) \quad (\text{B32})$$

$$(D^2 - 1)\Lambda^{(1)} = 0 \quad \text{with} \quad \Lambda^{(1)} \underset{\tau \rightarrow +\infty}{\sim} \frac{\ln \tau}{\tau} \lambda_+(\hat{x}). \quad (\text{B33})$$

For definitiveness we consider the ‘‘backwards’’ evolution problem of solving the wave equation with data on the asymptotic future. Note that, unlike the elliptic problem in \mathcal{H}^+ , the source free equation (B31) has two independent

solutions. This corresponds to the fact that we now have a Cauchy problem and two data are needed to solve the equation (the field and its momentum). Asymptotically the two data correspond to the different fall-offs.

We now describe the Green’s function associated to each equation. We will just state them and verify they satisfy the desired conditions.

a. Solution to Eq. (B32)

The solution to (B32) we claim is given by:

$$\varphi^{(1)}(y) = \frac{1}{2\pi} \int d^2\hat{q} \delta(Y \cdot q) \phi_+(\hat{q}). \quad (\text{B34})$$

First, we note that the analogue of Eq. (B6) in \mathcal{H}^o is

$$D^2 f(Y \cdot q) = -(Y \cdot q)^2 f''(Y \cdot q) - 3(Y \cdot q) f'(Y \cdot q). \quad (\text{B35})$$

Noting the distributional identities $\sigma\delta'(\sigma) = -\delta(\sigma)$ and $\sigma^2\delta''(\sigma) = 2\delta(\sigma)$ one concludes that $D^2\delta(Y \cdot q) = \delta(Y \cdot q)$ and so (B34) satisfies the desired differential equation. To check for the boundary condition, note that the support of the Delta function in (B34) is given by

$$Y \cdot q = 0 \Leftrightarrow \hat{x} \cdot \hat{q} = \frac{\tau}{\sqrt{1 + \tau^2}} \underset{\tau \rightarrow +\infty}{\sim} 1 - \frac{1}{2\tau^2} + O(\tau^{-4}). \quad (\text{B36})$$

If we denote by θ the angle between \hat{x} and \hat{q} then $\hat{x} \cdot \hat{q} \approx 1 - \frac{1}{2}\theta^2$ for small θ . Thus for large τ the support of the Delta is a circle of radius τ^{-1} centered around $\hat{q} = \hat{x}$. It then follows that $\varphi^{(1)}(y)$ defined by (B34) satisfies $\varphi^{(1)}(\tau) \sim \tau^{-1}\phi_+(\hat{x})$ for $\tau \rightarrow +\infty$ as desired. The solution (B34) also allow us to evaluate $\varphi^{(1)}$ in the $\tau \rightarrow -\infty$ limit. In this case $\frac{\tau}{\sqrt{1 + \tau^2}} \sim -(1 - \frac{1}{2\tau^2})$ and the Delta becomes supported in a $|\tau|^{-1}$ radius circle centered at $\hat{q} = -\hat{x}$. This then implies relation (3.42).

b. Solution to Eq. (B33)

The solution to (B33) we claim is given by:

$$\Lambda^{(1)}(y) = -\frac{1}{4\pi} \int d^2\hat{q} \frac{\lambda_+(\hat{q})}{Y \cdot q}, \quad (\text{B37})$$

which is the \mathcal{H}^o version of the analogous expression found in the \mathcal{H}^+ case. The proof then goes parallel to that case. The differential equation can be seen to be satisfied by using Eq. (B35). For the boundary condition we note that

$$\frac{1}{Y \cdot q} \underset{\tau \rightarrow \pm\infty}{\sim} \begin{cases} O(1/\tau) & \text{for } \hat{x} \neq \pm\hat{q} \\ O(\tau) & \text{for } \hat{x} = \pm\hat{q} \end{cases} \quad (\text{B38})$$

and

$$-\frac{1}{4\pi} \int \frac{d^2\hat{q}}{Y \cdot q} = \frac{\operatorname{arcsinh}\tau}{\sqrt{1+\tau^2}} \stackrel{\tau \rightarrow \pm\infty}{\sim} \pm \frac{\ln|\tau|}{|\tau|}. \quad (\text{B39})$$

from which the $\tau \rightarrow +\infty$ condition in (B33) follows. The $\tau \rightarrow -\infty$ limit yields the ‘‘antipodal matching’’ $\lambda_-(\hat{x}) = -\lambda_+(\hat{x})$.

c. Relation between fields (B32) and (B33)

The solutions $\overset{(1)}{\varphi}(y)$ and $\overset{(1)}{\Lambda}(y)$ given in (B34) and (B37) represent two type of solutions of the differential equation

$$(D^2 - 1)f = 0. \quad (\text{B40})$$

A geometrical characterization of each type of solution can be given by considering the inversion map on \mathcal{H}^o :

$$y^\alpha \rightarrow -y^\alpha := (-\tau, -\sqrt{1+\tau^2}\hat{x}) \quad (Y^\mu \rightarrow -Y^\mu). \quad (\text{B41})$$

From the integral representations (B34) and (B37) it follows that each type of field is of definite parity:

$$\overset{(1)}{\varphi}(-y) = \overset{(1)}{\varphi}(y) \quad \text{and} \quad \overset{(1)}{\Lambda}(-y) = -\overset{(1)}{\Lambda}(y). \quad (\text{B42})$$

A general solution to (B40) will in general be a sum of the two type of solutions. Thus if we denote by Γ the space of all fields satisfying (B40) we have:

$$\Gamma = \Gamma^{\text{even}} \oplus \Gamma^{\text{odd}}. \quad (\text{B43})$$

We now show that in fact (B43) corresponds to a ‘‘ $q-p$ ’’ splitting when Γ is regarded as a phase space. Recall that fields satisfying a (massive) Klein-Gordon equation such as (B40) form a symplectic vector space under the usual Klein-Gordon symplectic product

$$\Omega(f, g) = \int_C dS_\alpha \sqrt{h} h^{\alpha\beta} (\partial_\beta f g - \partial_\beta g f), \quad (\text{B44})$$

where C is any Cauchy surface on \mathcal{H}^o . We claim that

$$\Omega(\overset{(1)}{\varphi}, \overset{(1)'}{\varphi}) = \Omega(\overset{(1)}{\Lambda}, \overset{(1)'}{\Lambda}) = 0, \quad (\text{B45})$$

$$\Omega(\overset{(1)}{\varphi}, \overset{(1)}{\Lambda}) = \int_{S^2} d^2V \lambda_+(\hat{x}) \phi_+(\hat{x}). \quad (\text{B46})$$

These results can be obtained by evaluating (B44) in the $\tau \rightarrow \infty$ slice and using the asymptotic forms of the fields. It is interesting however to see how Eq. (B45) arises for the $\tau = 0$ slice (recall Ω is independent of the choice of Cauchy slice).

From the $\tau = 0$ slice perspective, Γ is the space of initial conditions (f, \dot{f}) on the $\tau = 0$ sphere with symplectic product

$$\Omega((f, \dot{f}), (g, \dot{g})) = \int_{S^2} d^2V (\dot{g}f - \dot{f}g). \quad (\text{B47})$$

By evaluating the ϕ and λ integral solutions (and their τ derivatives) at $\tau = 0$ one concludes that

$$(f, \dot{f}) \in \Gamma^{\text{even}} \Leftrightarrow f_{\text{even}}, \dot{f}_{\text{odd}} \quad (\text{B48})$$

$$(f, \dot{f}) \in \Gamma^{\text{odd}} \Leftrightarrow f_{\text{odd}}, \dot{f}_{\text{even}} \quad (\text{B49})$$

where the parity of f, \dot{f} refers to S^2 . Since an odd function on the sphere integrates to zero, it follows that (B47) vanishes if evaluated on either Γ^{even} or Γ^{odd} .

We finally note that the $\tau = 0$ sphere corresponds to the limiting spheres of $t = \text{constant}$ spacetime slices Σ_t ($t = \text{Minkowski time}$). From this perspective, Γ^{even} determines the asymptotic parity of φ -initial data on Σ_t . These are the scalar field analogue of the Regge-Teitelboim parity conditions in gravity [89].

APPENDIX C: ASYMPTOTICS OF $\Lambda(x)$ AT NULL INFINITY

Here we describe the null infinity behavior of Λ as given in Eq. (3.49). Rather than attempting a direct evaluation of (3.49) in the $r \rightarrow \infty, u = \text{const}$ limit, we will make use of the results from the previous Appendix and work in hyperbolic coordinates as an intermediate step. Consider first hyperbolic coordinates adapted to future time infinity. (these only allow us to cover the $u > 0$ region of \mathcal{I}^+). In these coordinates we have:

$$\Lambda(x) = \frac{\Lambda_{\mathcal{H}^+}(y)}{\tau} = \frac{1}{\rho\tau} (\ln \rho \lambda(\hat{x}) + O(\rho^{-1})). \quad (\text{C1})$$

We now write this expression in retarded u, r coordinates. Noting that $\rho\tau = r$ and

$$\rho = \frac{r}{\sqrt{2ur + u^2}} = \sqrt{\frac{r}{2u}} (1 + O(r^{-1})), \quad (\text{C2})$$

$$\ln \rho = \frac{1}{2} \ln r - \frac{1}{2} \ln(2u) + O(r^{-1}) \quad (\text{C3})$$

one finds

$$\Lambda(x) = \frac{1}{2r} \lambda(\hat{x}) (\ln r - \ln(2u) + O(r^{-1/2})) \quad (u > 0). \quad (\text{C4})$$

We recall the above expression is only valid for $u > 0$. To cover the $u < 0$ region we consider the expression of $\Lambda(x)$ in hyperbolic coordinates adapted to spatial infinity:

$$\Lambda(x) = \frac{\Lambda_{\mathcal{H}^0}(y)}{\rho} = \frac{1}{\rho\tau} (\ln \tau \lambda(\hat{x}) + O(\tau^{-1})). \quad (\text{C5})$$

In terms of (r, u) coordinates we now have $\rho\tau = r + u$ and

$$\tau = \frac{r + u}{\sqrt{-2ur - u^2}} = \sqrt{\frac{r}{-2u}} (1 + O(r^{-1})), \quad (\text{C6})$$

leading to

$$\Lambda(x) = \frac{1}{2r} \lambda(\hat{x}) (\ln r - \ln(-2u) + O(r^{-1/2})) \quad (u < 0). \quad (\text{C7})$$

Combining (C4) and (C7) we obtain (4.18).

1. Eq. (4.20)

We concluded that near null infinity $\Lambda(x)$ has an expansion of the form

$$\Lambda(x) = \frac{\ln r}{r} \overset{(-1, \ln)}{\lambda}(u, \hat{x}) + \frac{1}{r} \overset{(-1)}{\lambda}(u, \hat{x}) + O(r^{-1-\epsilon}) \quad (\text{C8})$$

with

$$\overset{(-1, \ln)}{\lambda}(u, \hat{x}) = \frac{1}{2} \lambda(\hat{x}), \quad \overset{(-1)}{\lambda}(u, \hat{x}) = -\frac{1}{2} \ln |2u| \lambda(\hat{x}). \quad (\text{C9})$$

From (C8) one finds the following null infinity expansion for k^{ru} (4.12):

$$\begin{aligned} k^{ru} = & -(\ln r \overset{(-1, \ln)}{\lambda} + \overset{(-1)}{\lambda}) u \partial_u \varphi_{\mathcal{I}^+} \\ & + (u \partial_u \overset{(-1)}{\lambda} - \overset{(-1, \ln)}{\lambda}) \varphi_{\mathcal{I}^+} + O(r^{-\epsilon}) \end{aligned} \quad (\text{C10})$$

The $u \rightarrow \pm\infty$ fall-offs for $\varphi_{\mathcal{I}^+}$ are such that $\partial_u \varphi_{\mathcal{I}^+} = O(u^{-1-\epsilon})$. This implies the first term in (C10) vanishes when $u \rightarrow \pm\infty$ and so it does not contribute to the charge (4.19) (in particular the potential logarithmic divergence is absent). From (C9) one finds the second term is

$$(u \partial_u \overset{(-1)}{\lambda} - \overset{(-1, \ln)}{\lambda}) \varphi_{\mathcal{I}^+} = -\varphi_{\mathcal{I}^+}(u, \hat{x}) \lambda(\hat{x}), \quad (\text{C11})$$

from which (4.20) follows.

APPENDIX D: SHIFT SYMMETRY FOR FREE MASSLESS FIELDS

In this appendix we discuss “shift symmetries” of the free massless scalar theory. We will discuss in parallel the free Maxwell and linearized gravity cases in harmonic gauge to highlight their similarities.

As described below, the naturally occurring “shift” field for the massless spin $s = 0, 1, 2$ theory can be written in the simple form,

$$\Lambda^{(s)}(x) := -\frac{(x \cdot x)^s}{4\pi} \int d^2 \hat{q} \frac{\lambda(\hat{q})}{(q \cdot x)^{s+1}}. \quad (\text{D1})$$

$\Lambda^{(s)}(x)$ depends on a function on the sphere $\lambda(\hat{q})$ which determines its asymptotic value.¹⁴ It satisfies the scaling behavior,

$$t^{(1-s)} \Lambda^{(s)}(tx) = \Lambda^{(s)}(x), \quad (\text{D2})$$

and obeys the free wave equation

$$\square \Lambda^{(s)} = 0. \quad (\text{D3})$$

For $s = 0$ (D1) becomes the expression for the field $\Lambda(x)$ introduced in Eq. (3.49). For $s = 1$ (D1) corresponds to the large $U(1)$ gauge parameter in harmonic gauge given in [61] which satisfies

$$\Lambda^{(1)} \rightarrow \lambda(\hat{x}) \quad (\text{D4})$$

at null infinity. Finally, for $s = 2$ the vector field defined by $\xi_a := \partial_a \Lambda^{(2)}$ corresponds to the supertranslation vector field in harmonic gauge given in [62] which satisfies

$$\xi^a \rightarrow \lambda(\hat{x}) \partial_u \quad (\text{D5})$$

at null infinity. These three parameters yield the following family of “shifted vacua” for the scalar, Maxwell and gravity theories:

$$\varphi = \Lambda^{(0)} \quad (\text{D6})$$

$$A_\mu = \partial_\mu \Lambda^{(1)} \quad (\text{D7})$$

$$h_{\mu\nu} = 2\partial_\mu \partial_\nu \Lambda^{(2)}. \quad (\text{D8})$$

Condition (D2) implies all three solutions are scale invariant (with the standard scaling dimension 1).

So far we have tried to exhibit the similarities between the different massless theories. We now discuss the differences between the $s = 0$ and $s = 1, 2$ cases. First, the configurations (D7), (D8) correspond to nontrivial vacuum configurations associated to the large local symmetry group.¹⁵ In particular, all these solutions have zero energy. One may be tempted to interpret (D6) as a nontrivial vacuum solution. However if one tries to compute the energy associated to this configuration one

¹⁴One can also consider Poincare transformed versions of (D1). These are however redundant from the perspective of infinity, namely $\Lambda_\lambda(L^\mu_\nu x^\nu + b^\mu) \rightarrow \Lambda_{\lambda'}(x)$ with $\lambda^L(\hat{q})$ a Lorentz transformed version of $\lambda(\hat{q})$.

¹⁵To simplify the discussion we are ignoring “superrotated” vacua in the gravity case.

gets infinity!¹⁶ Related to this is the fact that the transformation generated by the “soft theorem charge” does not yield a solution of the type (D6) but rather one of the form given in Eq. (4.7) (which does have zero energy).

As discussed earlier, in the gauge and gravity cases the currents associated to the ‘shift’ transformations leading to (D7) and (D8) are total derivatives as in (4.5). This is not so in the scalar case. The shift transformation

$$\delta_\Lambda \varphi = \Lambda \quad (\text{D9})$$

with $\square\Lambda = 0$ is a symmetry of the free φ Lagrangian with Noether current given by:

$$j^a[\Lambda] = \sqrt{\eta}(\varphi\nabla^a\Lambda - \Lambda\nabla^a\varphi) \quad (\text{D10})$$

which is not of the total derivative form. One may ask if there is any relation between (D10) and the total derivative current (4.12). If we denote by

$$\delta_X\varphi = \varphi + X^a\partial_a\varphi \quad (\text{D11})$$

the action of dilatations on φ then one finds (in the free theory case)

$$\delta_X j^a[\Lambda] = -\partial_b k^{ab}, \quad (\text{D12})$$

with k^{ab} given in Eq. (4.12).

¹⁶This can be achieved, for instance, by evaluating the total energy near null infinity using the expansion (4.18).

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