Progress of Theoretical Physics, Vol. 76, No. 6, December 1986

Can the Membrane Be a Unification Model?*)

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(Received August 15, 1986)

Whether a membrane model can generate massless particles in the spectrum is studied by calculating the Casimir energy.

§ 1. Introduction

String theories have opened a new area in quantum field theory, i.e., the field theory of extended objects. Although the notion of the field for non-locally extended object can be traced back to the article by Yukawa in 1950,[1] the theory has been known to have a number of difficulties[2] such as the causality trouble, the existence of ghosts, distresses in introducing interactions, etc.

The success of string field theories,[3] however, taught us how to overcome these problems. First, the theory must be so made that the model represents a continuously extended matter which is causal as a classical theory. Second, the model should have high enough local symmetry to eliminate all negative norm components of coordinates. Third, when one introduces interactions between extended objects, the continuity condition of the world manifold should be imposed. The reparametrization invariance of the world manifold swept out by the extended body seems to be necessary at least. Fourth, when the model is quantized, there occur a number of quantum mechanical anomalies.[4] These must be eliminated by adjusting some parameters, such as dimensions of space-time.

However, when it comes to the question whether the extended object model can be a unification theory of fundamental interactions, it is another problem. In string theories we know that there appear always massless spin one and two particles which play the roles of gauge particle and graviton in certain critical dimensions.[5] In this article we address ourselves to this problem and ask whether massless particles are able to be expected[6] in the membrane model, for instance.

This problem is also related with the speculation that the superstring might be a thin limit of a membrane.[7] As will be discussed below, we can answer this question by calculating the Casimir energy of a membrane.

In the following we will first make simple observations about classical and quantum mechanics of extended objects. Then we will point out that a spin-mass relationship (the Regge trajectory) can be calculable in an infinite limit of string tension. This enables us to judge whether massless particles appear or not. In a simple membrane model we show our calculations.

*) Work supported in part by the Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture (# 61540202) and the Grant-in-Aid for Co-operative Research (# 61306010).

§ 2. Spin-mass relation of extended objects

Let us consider an n-dimensinally extended object which is governed by the action[8]

$$S_n = -\frac{\varkappa_n}{2\pi}\int d^n\sigma d\tau [\det g_{ab}]^{1/2}, \qquad (2\cdot1)$$

where

$$g_{ab} = \partial_a X^\mu \partial_b X_\mu$$

and $\varkappa_n$ is a tension parameter. The coordinate $X^\mu(\tau, \sigma_1, \cdots, \sigma_n)$ represents a point of the object parametrized by $\sigma_a (a=0, 1, \cdots, n)$ and τ. The dimensions of space-time are assumed to be D. The right-hand side of (2·1) is nothing but the volume of world body swept out by the object, and reduces to the Nambu-Goto action[9] for $n=1$.

The dimension of the parameter $\varkappa_n$ is $[ML^{1-n}T^{-1}]$ where M, L and T represent mass, length and time, respectively. We try to obtain a relationship between the spin angular momentum J and the rest mass of rotating object. Possible parameters which participate in the relationship are the light velocity c, $\varkappa_n$, J, m and the Planck constant $\hbar$. Dimensional analysis at once leads us to

$$J = A(\varkappa_n)^{-1/n}(cm)^{(n+1)/n} + B\hbar, \qquad (2\cdot2)$$

where A and B are dimensionless numbers.[10] In the classical theory only the first term on the r.h.s. of (2·2) survives. It may be worth while to note that (2·2) is valid not only for the particular model (2·1) but also for any model which is characterized by a single constant $\varkappa_n$. In what follows we adopt the natural unit convention where $c=\hbar=1$.

In quantum theory the second term on the r.h.s. of (2·2) arises due to the Casimir effect of the extended object, i.e., the sum of zero-point energies. In the bosonic string model, for example, the mass squared is shifted by

$$\frac{1}{\alpha'}\frac{D-2}{2}\left(\sum_{n=1}^{\infty} n\right), \qquad (2\cdot3)$$

where $D-2$ is the number of independent oscillation modes. Although (2·3) is infinite Brink and Nielsen[11] argued that a physically meaningful number can be extracted out by some regularizations. One simple method is to use Riemann's zeta function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}. \qquad (2\cdot4)$$

The quantity of our interest is now obtained by making an analytic continuation of $\zeta(s)$ in s from $s>1$ to negative region and is

$$\sum_{n=1}^{\infty} n = \zeta(-1) = -\frac{1}{12}. \qquad (2\cdot5)$$

Using (2·5) and (2·3), we arrive at the spin-mass relation

$$J = \frac{1}{p}M^2 + \frac{D-2}{24}p, \qquad (2\cdot6)$$

where M^2 is the mass operator of string and $p=1$ for open string and $p=2$ for closed string. To get a massless particle ($M^2=0$) with an integer spin, the dimensions of space-time has to be either 2 or 26 [$D>26$ is excluded by other reason]. It is a miracle that the Casimir term (2·5) in the string model has happened to be a *rational number* so that J is made an integer by adjusting dimensions of space-time.

What would happen in other extended models? Note, first of all, that the constant B in (2·2) is independent of $\varkappa_n$, so one can calculate B even in $\varkappa_n \to \infty$ limit. The partition function

$$Z_n = \lim_{\varkappa_n \to \infty} \int \mathcal{D} X^\mu \exp\left(-\frac{\varkappa_n}{2\pi} \int d^n\sigma d\tau \sqrt{-g} \right) \tag{2·7}$$

implies that a semi-classical approximation becomes exact in the infinite limit of $\varkappa_n$.

In the next section we will first find a classical solution to the equation of motion

$$\partial_a (\sqrt{-g}\, g^{ab} \partial_b X^\mu) = 0\,. \tag{2·8}$$

Then we make an expansion of X^μ around the classical solution as

$$X^\mu = X^\mu_{\rm cl} + \frac{1}{\sqrt{\varkappa_n}} Y^\mu \tag{2·9}$$

and substitute it into S_n and keep the terms up to the second order in quantum fluctuation Y^μ,

$$S_n = S_n^{(\rm cl)} + \int d^n\sigma d\tau \sqrt{-g_{\rm cl}}\, \partial_a Y^\mu K^{ab}_{\mu\nu} \partial_b Y_\mu\,. \tag{2·10}$$

The Casimir term is able to be obtained from the second term in (2·10).

Before closing this section let us make a comment for the validity of our arguments. The action S_n is in general unrenormalizable when it is regarded as a field theory in $n+1$ dimensions for $n>1$. To make our arguments well defined we assume that the theory is effectively described by the action (2·1) and the ultraviolet behavior is regularized by a cutoff parameter. Since the Casimir effect is known to be cutoff independent as demonstrated by Brink and Nielsen[11] in the string model, we assume that it works also for higher dimensional models. As is well known the Casimir energy is always finite in the semi-classical approximation if n is even.[12]

§ 3. Membrane model

The classical solution to (2·7) which corresponds to the leading trajectory may be found among rigid rotator solutions, because all kinematical motions so contribute to generating angular momentum that the solution attains the highest spin.[13] For a membrane to keep the form extended in two dimensions we need the body rotating in the X^1-X^2 plane as well as in the X^3-X^4 plane. In a synchronous gauge where $X^0=\tau$ and $\dot{\boldsymbol{X}}\partial_i \boldsymbol{X}=0$ ($i=1, 2$), a solution is given by

$$X^1_{\rm cl} = f(\sigma_1, \sigma_2)\cos\omega_1\tau\,,$$

$$X^2_{\rm cl} = f(\sigma_1, \sigma_2)\sin\omega_1\tau\,,$$

$$X^3_{\rm cl}=g(\sigma_1,\sigma_2)\cos\omega_2\tau\,,\qquad(3\cdot1)$$
$$X^4_{\rm cl}=g(\sigma_1,\sigma_2)\sin\omega_2\tau\,,$$
$$X^i_{\rm cl}=0\qquad \text{for}\quad 5\leqq i\leqq D-1\,,$$

where f and g are arbitrary functions satisfying the condition

$$1-\omega_1{}^2f^2-\omega_2{}^2g^2=0\qquad(3\cdot2)$$

at the boundary of membrane.*)

The classical action for the solution (3·1) is

$$S_{\rm cl}=-\frac{\varkappa_2 T}{3\omega_1\omega_2}\,,\qquad(3\cdot3)$$

where $T=\tau_2-\tau_1$.

The semiclassical quantization of periodic motions has been explored by Dashen, Hasslacher and Neveu[14] and applied to the rotating string by Sato, Uehara and Kikkawa.[15] Provided that the center of mass is at rest we find a set of classical periodic solutions with the energy (rest mass) E and the period τ_0 is given by (3·1) with

$$\frac{\omega_1}{\omega_2}=\frac{p}{q}\,,\qquad \omega_1=\sqrt{p/2q}\left(\frac{\varkappa_2}{E}\right)^{1/2}\equiv\omega p\,,\qquad(3\cdot4)$$

where (p, q) are relatively prime integers. For given (p, q) the membrane revolves p and q times in the X^1-X^2 and X^3-X^4 planes, respectively, in a period τ_0.

The propagator of the membrane is given by[15]

$$\left\langle j\left|\frac{1}{E-H}\right|j\right\rangle={\rm Tr}\left[\delta(j-\mathbf{J})\frac{1}{E-H}\right]$$
$$=\int\frac{d(\Delta\theta)}{2\pi}\exp(i\Delta\theta j)D_E(\Delta\theta)\qquad(3\cdot5)$$

and

$$D_E(\Delta\theta)=-i\sum_l\int d\tau_0\tau_0\sqrt{1+\Delta\theta/2\pi}\left|\frac{i\partial^2S_{\rm cl}}{2\pi\partial\theta(0)\partial\theta(T)}\right|^{1/2}$$
$$\times{\rm Tr}^{(Q)}[\exp(i\tau_0(l+\Delta\theta/2\pi)(E-H^{(Q)})+iS_{\rm cl})]$$
$$={\rm Tr}^{(Q)}\left[2\pi\frac{d\alpha(E)}{dE}\cdot\frac{\exp(i\Delta\theta\alpha(E))}{1-\exp(2\pi i\alpha(E))}\right],\qquad(3\cdot6)$$

where

$$\alpha(E)=\frac{2}{3}\sqrt{\frac{2pq}{\varkappa_2}}\left(E^{3/2}-\frac{3}{2}H^{(Q)}E^{1/2}\right).\qquad(3\cdot7)$$

In the above expressions, $H^{(Q)}$ represents the operator Hamiltonian associated with the second term on the r.h.s. of (2·10). The integer j is defined as an eigenvalue of the angular momentum operator

*) Our classical solutions are acceptable for $D\geqq5$. For $D=4$, one has to look for more complicated, probably non-rigid rotator, solutions. Following arguments are valid for $D\geqq5$.

$$\mathbf{J}=p\mathbf{J}^{12}+q\mathbf{J}^{34}, \qquad (3\cdot 8)$$

where $\mathbf{J}^{12}$ and $\mathbf{J}^{34}$ are generators of X^1-X^2 and X^3-X^4 rotations. The summation over l in (3·6) extends over the number of revolutions within the time interval T. In performing the integral of τ_0 in (3·6), we have made the stationary phase approximation with respect to τ_0.

The mass spectrum of the membrane is given by the pole of the propagator, i.e.,

$$j=\alpha(E). \qquad (3\cdot 9)$$

The leading trajectory is the solution having the maximum angular momentum for a given value of E. An average angular momentum $(p\mathbf{J}^{12}+q\mathbf{J}^{34})/(p+q)=\alpha/(p+q)$ takes maximum for $p=q$ since α is proportional to $\sqrt{pq}$. The minimum value of $H^{(Q)}$ is equal to the ground state expectation value $\langle 0|H^{(Q)}|0\rangle$ which is nothing but the Casimir energy. $H^{(Q)}$ has the following form:

$$H^{(Q)}=\sum_{n,l}\omega\lambda_{n,l}\left(N_{n,l}+\frac{1}{2}\right), \qquad (3\cdot 10)$$

where the suffix (n, l) represents a quantum oscillation, $N_{n,l}$ the number operator of the mode, and ω is the angular frequency defined in (3·4). [Details will be found in Appendix A.] The Casimir energy is now given by

$$\frac{2}{\omega}\langle 0|H^{(Q)}|0\rangle=(D-5)A^{\perp}_{p,q}+A^{(1)}_{p,q}+A^{(2)}_{p,q}, \qquad (3\cdot 11)$$

where $A^{\perp}$ and $A^{(i)}(i=1, 2)$ are contributions from transverse modes $Y^i(i=5, 6, \cdots, D-1)$ and other two modes.

Combining (3·11) with (3·7) and (3·9), and taking the limit $E\to 0$, we finally obtain the intercept of the spin-mass trajectory,

$$j=-\frac{1}{2}[(D-5)A^{\perp}_{p,q}+A^{(1)}_{p,q}+A^{(2)}_{p,q}]. \qquad (3\cdot 12)$$

Now the question is whether the r.h.s. of (3·12) is able to be made an integer by adjusting an integer parameter D.

For the leading trajectory $(p=q=1)$ we have calculated A's [Appendix A], and obtained

$$A^{\perp}_{1,1}=A^{(1)}_{1,1}=\sum_{n=0}^{\infty}\sum_{l=-\infty}^{\infty}2\sqrt{(n+|l|+1/4)(n+1/4)-1/16}, \qquad (3\cdot 13)$$

$$A^{(2)}_{1,1}=\sum_{n=0}^{\infty}\sum_{l=-\infty}^{\infty}2\sqrt{(n+|l|+3/4)(n+3/4)+7/16}. \qquad (3\cdot 14)$$

We carry out these summations by using a generalized ζ-function method. Let us define $Z(\nu; b, c)$ by

$$Z(\nu; b, c)=\sum_{n=0}^{\infty}\sum_{l=-\infty}^{\infty}[(n+|l|+c)(n+c)+b]^{-\nu/2}$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}[(n+c)(m+c)+b]^{-\nu/2}. \qquad (3\cdot 15)$$

As will be shown in Appendix B we have succeeded in getting an integral representation of the r.h.s. of (3·15). Then we have made an analytic continuation of Z in ν from $\nu>2$ to $\nu=-1$ to get the values of A's.

It is very difficult to prove in general that (3·12) can be or cannot be an integer. For practical purpose, however, massless particles have to have either $j=1$ or 2. We obtained the values by numerical calculation, namely,

$$\frac{1}{2}A_{1,1}^{\perp}=\frac{1}{2}A_{1,1}^{(1)}=-0.1392569\cdots, \tag{3·16}$$

$$\frac{1}{2}A_{1,1}^{(2)}=0.028828483\cdots. \tag{3·17}$$

To get the spin 2 massless particle (the graviton) we find from (3·12)

$$D=18.568962\cdots \tag{3·18}$$

and to get the spin 1 (the gauge boson)

$$D=11.387989\cdots. \tag{3·19}$$

In either case the dimensions of space-time have to be far from an integer. We are confident in these numbers at least down to six places of decimals, hence the membrane does not provide massless particles in any integer dimensions.

§ 4. Conclusion and discussion

We have discussed possibility that higher dimensionally extended object models can generate massless particles. Unless massless particles are generated, the model cannot be a candidate for a unification theory. It has been shown that the massless criterion can be exactly calculable in the semiclassical approximation provided that the model is meaningful.

For a simple membrane model we have demonstrated that the massless particle is unable to be generated in integer dimensions. This also excludes that the superstring may be a certain limit of a membrane model.[7] Classically one can imagine a thin strip limit of membrane which may behave like a string with polarizations distributed on it, but the Casimir term is different depending on whether the object is a membrane or a string because their excitation spectra are distinct from one another.

Considering the behavior of Z-functions, we are skeptical that any other model than string can generate massless particles. We, however, have not considered the role of supersymmetry. A possible model which deserves being considered may be the model,

$$S=-\frac{\varkappa_n}{4\pi}\int d\tau d^n\sigma[eg^{ab}(\partial_a X^\mu\partial_b X_\mu+\partial_a Y^\mu\partial_b Y_\mu)+e\bar{\lambda}^\mu\gamma^a D_a\lambda_\mu-e(F_\mu{}^2+G_\mu{}^2)], \tag{4·1}$$

which will be discussed elsewhere.

Appendix A

We define the quantum fluctuation Z^μ around the classical solution (3·1) as follows:

$$X^{\mu}=X_{cl}^{\mu}+\mathbf{R}_{12}(\omega_1\tau)\mathbf{R}_{34}(\omega_2\tau)Z^{\mu}\,, \tag{A·1}$$

where $\mathbf{R}_{ij}(\theta)$ is a rotation operator in the i-j plane by θ. To the second order of Z^{μ}, the action (2·10) for $n=2$ becomes

$$S[X]=S_{\rm cl}[X_{\rm cl}^{\mu}]+S^{(Q)}[Z^{\mu};X_{\rm cl}^{\mu}]\,, \tag{A·2}$$

$$S^{(Q)}=S_{/\!/}{}^{(Q)}+S_{\perp}{}^{(Q)}\,, \tag{A·3}$$

$$\begin{aligned}S_{/\!/}{}^{(Q)}=-\frac{\varkappa_2}{4\pi}\int d\tau dfdg\sqrt{A}\Big[&-\frac{2\omega_1\omega_2 fg}{A^2}(\dot{Z}^2+\omega_1 Z^1)(\dot{Z}^4+\omega_2 Z^3)\\
&-\frac{1-\omega_2{}^2g^2}{A^2}(\dot{Z}^2+\omega_1 Z^1)^2-\frac{1-\omega_2{}^2f^2}{A^2}(\dot{Z}^4+\omega_2 Z^3)^2\\
&+\frac{2}{A}(\dot{Z}^1-\omega_1 Z^2)\Big\{\omega_1 f\frac{\partial}{\partial f}Z^2+\omega_2 g\frac{\partial}{\partial f}Z^4\Big\}\\
&+\frac{2}{A}(\dot{Z}^3-\omega_2 Z^4)\Big\{\omega_1 f\frac{\partial}{\partial g}Z^2+\omega_2 g\frac{\partial}{\partial g}Z^4\Big\}\\
&-\frac{2}{A}(\dot{Z}^2+\omega_1 Z^1)\Big\{\omega_1 f\frac{\partial}{\partial f}Z^1+\omega_1 f\frac{\partial}{\partial g}Z^3\Big\}\\
&-\frac{2}{A}(\dot{Z}^4+\omega_2 Z^3)\Big\{\omega_2 g\frac{\partial}{\partial f}Z^1+\omega_2 g\frac{\partial}{\partial g}Z^3\Big\}\\
&+\frac{1-\omega_2{}^2g^2}{A}\Big\{\Big(\frac{\partial Z^2}{\partial f}\Big)^2+\Big(\frac{\partial Z^2}{\partial g}\Big)^2\Big\}+\frac{1-\omega_1{}^2g^2}{A}\Big\{\Big(\frac{\partial Z^4}{\partial f}\Big)^2+\Big(\frac{\partial Z^4}{\partial g}\Big)^2\Big\}\\
&+\frac{2\omega_1\omega_2 fg}{Z}\Big\{\Big(\frac{\partial Z^2}{\partial f}\Big)\Big(\frac{\partial Z^4}{\partial f}\Big)+\Big(\frac{\partial Z^2}{\partial g}\Big)\Big(\frac{\partial Z^4}{\partial g}\Big)\Big\}\\
&+2\Big\{\Big(\frac{\partial Z^1}{\partial f}\Big)\Big(\frac{\partial Z^3}{\partial g}\Big)-\Big(\frac{\partial Z^1}{\partial g}\Big)\Big(\frac{\partial Z^3}{\partial f}\Big)\Big\}\Big]\,,\end{aligned} \tag{A·4}$$

$$S_{\perp}{}^{(Q)}=-\frac{\varkappa_2}{4\pi}\int d\tau dfdg\sqrt{A}\sum_{i=5}^{D-1}\Big\{-\frac{1}{A}(\dot{Z}^i)^2+\Big(\frac{\partial Z^i}{\partial f}\Big)^2+\Big(\frac{\partial Z^i}{\partial g}\Big)^2\Big\}\,, \tag{A·5}$$

$$A=1-\omega_1{}^2f^2-\omega_2{}^2g^2\,, \tag{A·6}$$

where the integral variables f and g are what appear in the classical solution (3·1), and they are in the range,

$$0\leqq 1-\omega_1{}^2f^2-\omega_2{}^2g^2\leqq 1\,. \tag{A·7}$$

The variables (σ_1, σ_2) on a membrane have been rewritten in (f, g) as new variables. In the following, we consider only when $\omega_1=\omega_2=\omega$, since this case provides the leading trajectory.

The equations of motion for the transverse components $Z^i (5\leqq i\leqq D-1)$ are

$$\Big(\frac{\partial^2}{\partial\tau^2}-\mathbf{D}\Big)Z^i=0\,, \tag{A·8}$$

where

$$\mathbf{D}\equiv(1-\omega^2 f^2-\omega^2 g^2)\left(\frac{\partial^2}{\partial f^2}+\frac{\partial^2}{\partial g^2}\right)-\omega^2\left(f\frac{\partial}{\partial f}+g\frac{\partial}{\partial g}\right) \tag{A·9}$$

and

$$\sqrt{1-\omega^2 f^2-\omega^2 g^2}\left(f\frac{\partial}{\partial f}+g\frac{\partial}{\partial g}\right)Z^i=0 \tag{A·10}$$

on the boundary. Using polar variables (θ, r) in the (f, g) plane, and using Z^i defined as

$$Z^i=e^{iE^i\tau}e^{il\theta}R^i(r)\,, \tag{A·11}$$

we obtain the following ordinary differential equation for $R(r)$,

$$(1-\omega^2 r^2)\left(\frac{d^2}{dr^2}+(1-\omega^2 r^2)\frac{d}{rdr}-\frac{l^2}{r^2}\right)R^i(r)=-(E^i)^2 R^i(r)\,. \tag{A·12}$$

To satisfy the boundary condition (A·10), $R^i(r)$ has to be finite at $r=0$ and $r=1/\omega$. Using power series expansion of r, we obtain the energy spectra,

$$E_\perp=2\omega\sqrt{(n+|l|+1/4)(n+1/4)-1/16}\,.\quad (n=0, 1, 2, \cdots,\quad l=0, \pm1, \pm2, \cdots) \tag{A·13}$$

Next, we consider the longitudinal components Z^1, Z^2, Z^3 and Z^4. The variation of $S^{(Q)}$ by Z^1 provides,

$$\frac{2}{A}(g-f)\frac{\partial Z^3}{\partial f}=0 \tag{A·14}$$

and

$$\frac{1}{\sqrt{A}}\{f(\dot{Z}^{12}+\omega Z^{12})+g(\dot{Z}^{42}+\omega Z^{32})\}=0 \tag{A·15}$$

on the boundary. Similarly the variation by Z^3 provides the same equations but in which Z^1 is replaced by Z^3, Z^2 by Z^4, and f by g. Z^1 and Z^3 have no dynamical mode due to reparametrization invariance of f and g.

On the other hand, the equation of motion for Z^2 and Z^4 are as follows:

$$\left\{\left(\frac{\partial^2}{\partial\tau^2}+\omega^2-\mathbf{D}\right)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}-2\omega^2\begin{bmatrix} f\frac{\partial}{\partial f} & g\frac{\partial}{\partial f}\\ f\frac{\partial}{\partial g} & g\frac{\partial}{\partial g}\end{bmatrix}\right\}\begin{bmatrix}Z^2\\ Z^4\end{bmatrix}=0 \tag{A·16}$$

with the boundary condition,

$$\frac{1}{\sqrt{A}}\Big\{\omega f^2(\dot{Z}^1-\omega Z^2)+\omega fg(\dot{Z}^3-\omega Z^4)$$
$$+(1-\omega^2 g^2)\left(f\frac{\partial}{\partial f}+g\frac{\partial}{\partial g}\right)Z^2+\omega^2 fg\left(f\frac{\partial}{\partial f}+g\frac{\partial}{\partial g}\right)Z^4\Big\}=0 \tag{A·17}$$

and similar equations with Z^1 replaced by Z^3, Z^2 by Z^4, and f by g. Combining (A·13) and (A·15), we find that Z^2 and Z^4 have to be regular on the boundary. Let Z^2 and Z^4 be as follows:

$$\begin{bmatrix} Z^2 \\ Z^4 \end{bmatrix} = e^{iE_{/\!/}\tau}\begin{bmatrix} A_{m,k}(r)\chi^{(\pm)}_{2m,k}(\theta)+B_{m,k}(r)\chi^{(\pm)}_{2m-1,k}(\theta) \\ \pm A_{m,k}(r)\chi^{(\pm)}_{2m,k}(\pi/2-\theta)\pm B_{m,k}(r)\chi^{(\pm)}_{2m-1,k}(\pi/2-\theta) \end{bmatrix}, \tag{A·18}$$

$$(m=0, 1, 2, \cdots, \quad k=0, 1, 2, 3)$$

$$\chi^{(\pm)}_{n,k}(\theta)=\cos\left((2n+k)\theta+\frac{3}{4}\pi k\pm\frac{\pi}{4}\right). \tag{A·19}$$

Then we obtain the following coupled ordinary differential equations:

$$\left[(1-\omega^2r^2)\left\{\frac{d^2}{dr^2}+\frac{d}{rdr}-\frac{1}{r^2}(4m+k)\right\}+E_{/\!/}{}^2-\omega^2(4m+k+1)\right]A_{m,k}(r)$$

$$+\omega^2\left(r\frac{d}{dr}-4m-k-2\right)B_{m,k}(r)=0\,, \tag{A·20}$$

$$\left[(1-\omega^2r^2)\left\{\frac{d^2}{dr^2}+\frac{d}{rdr}-\frac{1}{r^2}(4m+k-2)\right\}+E_{/\!/}{}^2+\omega^2(4m+k-3)\right]B_{m,k}(r)$$

$$+\omega^2\left(r\frac{d}{dr}+4m+k\right)A_{m,k}(r)=0\,. \tag{A·21}$$

Imposing $A_{m,k}(r)$ and $B_{m,k}(r)$ to be regular at $r=0$ and $r=1/\omega$, we look for eigenvalues by well-known power expansion method and find the eigenvalues,

$$E_{/\!/}=2\omega\sqrt{(n+|l|+3/4)(n+3/4)+7/16} \tag{A·22}$$

and

$$E_{/\!/}=2\omega\sqrt{(n+|l|+1/4)(n+1/4)-1/16}\,, \tag{A·23}$$

where $n=0, 1, 2, \cdots$ and $l=0, \pm1, \pm2, \cdots$.

Appendix B

We perform the analytic continuation of $Z(\nu;\, b,\, c)$ defined by (3·15) with respect to ν. Note first the summand is expressed as

$$\{(n+c)(m+c)+b\}^{-\nu/2}=\frac{1}{\Gamma\left(\frac{\nu}{2}\right)}\int dt t^{(\nu/2)-1}e^{-t(m+c)}e^{-bt/(n+c)}(n+c)^{-\nu/2}\,. \tag{B·1}$$

Performing the sum with respect to n, then we obtain

$$\sum_{n=0}^{\infty}e^{-bt/(n+c)}(n+c)^{-\nu/2}=\sum_{k=0}^{\infty}\frac{1}{k!}(-bt)^k\sum_{n=0}^{\infty}(n+c)^{-(\nu/2)-k}$$

$$=(tb)^{-(\nu/4)+(1/2)}\int_0^{\infty}ds s^{(\nu/4)-(1/2)}\frac{e^{-sc}}{1-e^{-s}}J_{(\nu/2)-1}(2\sqrt{tbs})\,. \tag{B·2}$$

Here we have used the following parameter integral formula:

$$\sum_{n=0}^{\infty}(n+c)^{-\alpha}=\sum_{n=0}^{\infty}\frac{1}{\Gamma(\alpha)}\int_0^{\infty}ds s^{\alpha-1}e^{-s(n+c)}$$

$$=\frac{1}{\Gamma(\alpha)}\int_0^\infty dss^{\alpha-1}\frac{e^{-sc}}{1-e^{-s}}, \tag{B·3}$$

$$J_\nu(z)=\left(\frac{z}{2}\right)^\nu\sum_{k=0}^\infty\frac{(-1)^k(z/2)^{2k}}{k!\Gamma(k+\nu+1)}, \tag{B·4}$$

where $J_\nu(z)$ is the ν-th Bessel function. Using (B·1) and (B·2), we obtain

$$Z(\nu;b,c)=\frac{b^{-(\nu/4)+(1/2)}}{\Gamma\left(\frac{\nu}{2}\right)}\int_0^\infty dt\int_0^\infty ds(ts)^{(\nu/4)-(1/2)}\frac{e^{-tc}}{1-e^{-t}}\frac{e^{-sc}}{1-e^{-s}}J_{(\nu/2)-1}(2\sqrt{tbs}) \tag{B·5}$$

$$=\frac{2b^{-(\nu/4)+(1/2)}}{\Gamma\left(\frac{\nu}{2}\right)}\int_0^\infty dyy^{\nu-3}f_\nu(y), \tag{B·6}$$

where

$$f_\nu(y)=\int_0^1 dz(1-z)^{-1/2}z^{(\nu/2)-(1/2)}g_\nu(z,y), \tag{B·7}$$

$$g_\nu(z,y)=(\sqrt{z}\,y)^{-(\nu/2)+1}J_{(\nu/2)-1}(2\sqrt{bzy^2})\frac{zy^2e^{2(1-c)y}}{(e^{y(1+\sqrt{1-z})}-1)(e^{y(1-\sqrt{1-z})}-1)}. \tag{B·8}$$

Equation (B·6) is well defined at $\nu>2$. Performing the analytic continuation with respect to ν down to around $\nu=-1$, we obtain

$$f_\nu(y)=\frac{(\nu-1)(\nu+1)}{(\nu-2)\nu}\int_0^1 dzz^{\nu/2}(1-z)^{-1/2}g_\nu(z,y)$$
$$-\frac{4(\nu+1)}{(\nu-2)\nu}\int_0^1 dzz^{\nu/2}(1-z)^{1/2}\frac{d}{dz}g_\nu(z,y)$$
$$+\frac{4}{(\nu-2)\nu}\int_0^1 dzz^{\nu/2}(1-z)^{3/2}\frac{d^2}{dz^2}g_\nu(z,y), \tag{B·9}$$

$$Z(\nu;b,c)=\frac{1}{(\nu-2)(\nu-1)\nu(\nu+1)}\int dyy^{\nu+1}\frac{d^4}{dy^4}f_\nu(y). \tag{B·10}$$

Let $\nu=-1+\varepsilon$, then

$$Z(-1+\varepsilon;b,c)=\frac{b^{3/4}}{6\sqrt{\pi}}\left[\frac{A_0}{\varepsilon}+A_1+O(\varepsilon)\right], \tag{B·11}$$

$$A_0=-\left[\frac{d^3}{dy^3}f_{-1}(y)\right]_{y=0}, \tag{B·12}$$

$$A_1=\int_0^\infty dy\,\log(y)\frac{d^4}{dy^4}f_{-1}(y)-\left[\frac{\partial^3}{\partial y^3}\frac{\partial}{\partial\nu}f_\nu(y)\right]_{\substack{y=0\\ \nu=-1}}. \tag{B·13}$$

Using (B·8) and (B·9), we have found that A_0 vanishes and therefore $Z(-1;b,c)$ is finite. In evaluation of A_1 in Eq. (B·13), the second term is analytically calculable, but the first term is not. So we have computed the first term numerically. The following is the result. When $c=5/4$ and $b=-1/16$,

$$A_1 = -0.3647609 + \frac{7}{2}\sqrt{\pi} = 5.838828\,. \qquad (\text{B}\cdot 14)$$

When $c=3/4$ and $b=7/16$,

$$A_1 = -3.754627 + \frac{3}{2}7^{1/4}\sqrt{\pi} = 0.5699207\,. \qquad (\text{B}\cdot 15)$$

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