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Can the Wave Function in Configuration Space Be Replaced by Single-Particle Wave Functions in Physical Space?

Travis Norsen
Smith College, tnorsen@smith.edu

Damiano Marian
Università di Genova

Xavier Oriols
Universitat Autonoma de Barcelona

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Travis Norsen · Damiano Marian · Xavier Oriols

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Abstract The ontology of Bohmian mechanics includes both the universal wave function (living in 3N-dimensional configuration space) and particles (living in ordinary 3-dimensional physical space). Proposals for understanding the physical significance of the wave function in this theory have included the idea of regarding it as a physically-real field in its 3N-dimensional space, as well as the idea of regarding it as a law of nature. Here we introduce and explore a third possibility in which the configuration space wave function is simply eliminated – replaced by a set of single-particle pilot-wave fields living in ordinary physical space. Such a re-formulation of the Bohmian pilot-wave theory can exactly reproduce the statistical predictions of ordinary quantum theory. But this comes at the rather high ontological price of introducing an infinite network of interacting potential fields (living in 3-dimensional space) which influence the particles' motion through the pilot-wave fields. We thus introduce an alternative approach which aims at achieving empirical adequacy (like that enjoyed by GRW type theories) with a more modest ontological complexity, and provide some preliminary evidence for optimism regarding the (once popular but prematurely-abandoned) program of trying to replace the (philosophically puzzling) configuration space wave function with a (totally unproblematic) set of fields in ordinary physical space.

T. Norsen
Smith College, Northampton, MA 01060, USA
E-mail: tnorsen@smith.edu

D. Marian (corresponding author)
Dipartimento di Fisica dell'Università di Genova and INFN sezione di Genova, Via Dodecaneso 33, 16146 Genova, Italy
Tel: +39-349-3681361
E-mail: damiano.marian@ge.infn.it

X. Oriols
Departament d'Enginyeria Electrònica, Universitat Autònoma de Barcelona, Bellaterra 08193, Barcelona, Spain
E-mail: xavier.oriols@uab.es

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1 Introduction

Questions about how to understand the quantum mechanical wave function are, in this current post-Bell renaissance period for quantum foundations, increasingly popular and increasingly pressing. Some of these questions parallel historical disputes that have arisen in the context of earlier theories proposing novel ontologies – for example, Newtonian gravitation (Hesse 2005; McMullin 1989; Cohen et al. 1999; Janiak 2009) and electromagnetism (Hunt 1994; Darrigol 2000).

But there are also some ways in which the debates about the status of the wave function are somewhat unprecedented. For example, it is unusual in the history of science for there to exist several so radically different (but arguably empirically equivalent) theories, such as we have with the Copenhagen, de Broglie - Bohm, spontaneous collapse, many-worlds, and quantum Bayesian approaches to Quantum Mechanics (to name just a few). So there is not just one question about how to understand the wave function, but instead many questions, about how to understand the wave function in the context of each particular candidate theory.

For the many-worlds theory, for example, the wave function is the only dynamical object in the picture, so to whatever extent the theory is able to extract and explain the elementary physical facts of everyday perceptual experience, it will have to do so exclusively on the basis of the wave function. For the quantum Bayesian, the wave function represents not something physical but instead something mental, subjective, informational; how such an approach might account for elementary physical facts thus remains obscure. Whereas for the Bohmian, elementary physical facts are accounted for in terms of something (namely, particles with definite positions, and/or perhaps fields with definite configurations) that the theory says exists *in addition to* the wave function, which thus in some sense plays a crucial (but background) role.

We will focus here on this last possibility, the de Broglie - Bohm pilot-wave theory, aka Bohmian mechanics. Our central concern here is developing the somewhat new idea that (despite some initial appearances to the contrary) Bohmian mechanics perhaps allows a uniquely appealing possible avenue for addressing questions about the physical significance of the wave function.

We begin by explaining the initial appearance to the contrary. As mentioned above, the wave function plays a somewhat background role in the pilot-wave theory: in the terminology of Allori, Goldstein, Tumulka and Zanghì (2013), for example, it is not part of the *primitive* ontology of the theory. But it is still *there*. Bell for example stressed that (compared to some other interpretations in which the role of the wave function is perhaps even less clear) it must really be taken seriously, as corresponding to something physically real, in the context of the pilot-wave theory:

> "Note that in this [theory] the wave is supposed to be just as 'real' and 'objective' as say the fields of classical Maxwell theory – although its action on the particles ... is rather original. *No one can understand this theory until he is willing to think of ψ as a real objective field rather than just a 'probability amplitude'. Even though it propagates not in 3-space but in 3N-space.*" (Bell 1994, p. 128, emphasis in original).

We will focus in particular on this last point that for a general Bohmian system of N particles (including in principle, indeed especially, the universe as a whole) the wave function is, mathematically, a function not on the usual 3-dimensional physical space, but is instead a function on the 3N-dimensional configuration space. But how can a function on an abstract space like this possibly correspond to an objectively real field?

Critics of the pilot-wave theory have often asked precisely this question. For example, N. David Mermin recently suggested that advocates of the pilot-wave theory must (implausibly in his view) give the "3N-dimensional configuration space ... just as much physical reality as the rest of us ascribe to ordinary 3-dimensional space." (Mermin 2009). Less recently, Heisenberg made essentially the same criticism:

> "For [de Broglie and] Bohm, the particles are 'objectively real' structures, like the point masses of classical mechanics. The waves in configuration space also are objective real fields, like electric fields.... [But] what does it mean to call waves in configuration space 'real'? This space is a very abstract space. The word 'real' goes back to the Latin word 'res', which means 'thing'; but things are in the ordinary 3-dimensional space, not in an abstract configuration space." (Heisenberg 1955).[1]

Proponents of the pilot-wave picture may find an easy rationalization for dismissing such criticism, from the likes of Mermin and Heisenberg, in the fact that their preferred alternatives are, to put it bluntly, non-sensical.

But although he was not speaking of the pilot-wave theory specifically, even the eminently-sensible Einstein seemed to share this same concern, expressing grave doubts about the coherence of an objectively real wave in configuration space. He remarks, for example, that "Schrödinger's works are wonderful – but even so one nevertheless hardly comes closer to a real understanding. The field in a many-dimensional coordinate space does not smell like something real." And similarly: "Schrödinger is, in the beginning, very captivating. But

[1] See also Chapter 8 of Heisenberg (1958)

the waves in n-dimensional coordinate space are indigestible..." (quoted in Howard (1990)) It has been suggested that Einstein's negative reaction to Bohm's 1952 proposal – which reaction almost everyone finds quite puzzling[2] – could be understood in this way: although the theory improved significantly on a number of intolerable aspects of the Copenhagen approach, it simply retained this "indigestible" feature (Norsen 2010).

Indeed, even Bohm himself seems to have found this aspect of his theory (the idea of a physically real field living in an abstract configuration space) to be somewhat indigestible:

> "...a serious problem confronts us when we extend the theory ... to the treatment of more than one electron. This difficulty arises in the circumstance that, for this case, Schrödinger's equation (and also Dirac's equation) do not describe a wave in ordinary 3-dimensional space, but instead they describe a wave in an abstract 3N-dimensional space, where N is the number of particles. While our theory can be extended formally in a logically consistent way by introducing the concept of a wave in a 3N-dimensional space, it is evident that this procedure is not really acceptable in a physical theory, and should at least be regarded as an artifice that one uses provisionally until one obtains a better theory in which everything is expressed once more in ordinary 3-dimensional space." (Bohm 1987, p. 117).

Stepping back, there would seem to be three possible ways to understand the wave function in the context of the Bohmian theory:

View 1: The universal wave function as a physically-real field in a physically-real configuration space

The first possibility is to simply bite the bullet and accept that the 3N-dimensional space in which the wave function lives must, if the wave function is to correspond to a physically-objective (Maxwell-like) field, be regarded as a (or perhaps the) physical space in its own right. For example, one could understand the theory as positing a 3-dimensional physical space (in which the particles move around) and, in addition, a separate 3N-dimensional physical space (in which the wave function lives).[3] Such a view would evidently have to face several obvious and pressing questions about the nature of the physical process through which the wave function affects the particles. To avoid such

[2] In response to Einstein's remark that Bohm's way "seems too cheap to me," for example, Max Born wrote that he thought "this theory was quite in line with [Einstein's] own ideas, to interpret the quantum mechanical formulae in a simple, deterministic way..." (Born 1971, pp. 192-193)

[3] It is not entirely clear, but Peter Holland may endorse this kind of view: "a complete and accurate account of the motions of particles moving in accordance with the laws of quantum mechanics *must* be directly connected with multidimensional waves dynamically evolving in configuration space." (Holland 1995, p. 321). Antony Valentini has also made comments suggesting that the main innovation of quantum mechanics is the need to accept the wave function as a new kind of causal agent which physically affects particles despite its living in a high-dimensional space.

questions, one might follow David Albert in proposing to move the particles also into the high-dimensional space: instead of N points moving in 3-space, one might instead posit a single point (sometimes referred to half-deprecatingly as "the marvellous point") which moves under the influence of the wave function in 3N-space (Albert 1996). It is hardly clear that this is an improvement, however, since one of the primary virtues of the original pilot-wave theory (namely its ability to account for elementary physical facts about the everyday macroscopic world in terms of the particles qua local beables) is lost (Maudlin 2007).

In any case, many people – such as Mermin, Heisenberg, Einstein, and Bohm – find themselves unable to stomach the non-local[4] beables posited in this first view.

View 2: The universal wave function as a law

A second possible view of the wave function is to regard it as *real*, but somehow not *physical* in the sense of *matter* or *stuff*. In particular, it has been suggested that the wave function should be thought of as playing a role, for the pilot-wave theory, like the role that the Hamiltonian plays in the context of classical mechanics. The suggestion is thus that the wave function should be regarded, not as a (Maxwell-like) field, but instead as a *law*.[5] This is an interesting view that deserves serious consideration. However, its plausibility seems to rest on the speculative idea that the wave function of the universe might be static. (It is somehow hard to swallow the idea of a law which evolves non-trivially in time... with the evolution evidently being governed by some further law. Indeed, it is hardly clear that the unusual notion of a law being governed by another law is avoided even in the case of a static universal wave function, for this too is supposed to be for example the solution of the Wheeler - de Witt equation.) The structural similarity between Bohmian mechanics and electrodynamics also raises questions such as: if the wave function should be understood as a law in Bohmian mechanics, why not also interpret the electric and magnetic fields, in classical electrodynamics, as laws? This, of course, would be very strange and nobody has ever suggested it. But why not? Evidently the answer is that those fields, living on ordinary 3-dimensional physical space, *can* be unproblematically understood as material fields. With

[4] We emphasize that the adjective "non-local" has two different (but related) meanings. First, we use "non-local" in the phrase "non-local beable" to denote an object (posited as physically real in some candidate theory) to stress that the object does not assign values to regions in 3-dimentional space (or 3+1 spacetime). And second, we use the word "non-local" to describe the special, faster-than-light, type of causal influence that Bell's theorem shows must exist Bell (1994). In particular, we stress that despite proving the existence of non-locality (in the second sense) Bell's theorem does *not* show that empirically viable theories must include beables that are non-local (in the first sense). Indeed, one of the key points of our paper is a demonstration that the non-locality required by Bell's theorem can actually be embedded in a theory of exclusively local beables.

[5] For example, Dürr, Goldstein and Zanghì (1996) write: "We propose that the wave function belongs to an altogether different category of existence than that of substantive physical entities [–] that the wave function is a component of physical law rather than of the reality described by the law."

no such option apparently available for the quantum mechanical wave function, one begins to get the feeling that View 2 is merely a convenient escape hatch to avoid View 1.

But perhaps after all there could be an option of interpreting the quantum mechanical wave function in terms of material fields in ordinary physical space. We thus introduce:

View 3: The universal wave function as an abstract and indirect description of a set of single-particle wave functions in physical space

In our proposed view, the wave function in a 3N-dimensional space might be understood as a kind of indirect description of some more mundane sort of physical stuff that lives in the ordinary, 3-dimensional physical space and hence avoids the kinds of objections we have seen to View 1.

The remainder of the paper will attempt to provide a case for the viability of View 3 by showing how a concrete model (in which the universal wave function is replaced by a set of one-particle wave functions in physical space) can be devloped from Bohmian mechanics.

As a final point in this motivational overview, however, we note that View 3 has a kind of obvious plausibility to it, when we consider historical examples of pre-quantum theories. Indeed, functions on high-dimensional, abstract (configuration and/or phase) spaces are hardly unique to quantum mechanics. For example, the classical mechanics of N particles (usually conceived in terms of N particles interacting through forces in 3-dimensional space) can be mathematically reformulated as a single (marvellous?) configuration-space point $\mathbf{X} = \{X_1, X_2, ..., X_N\}$ moving under the influence of a configuration-space potential $V(x_1, x_2, ..., x_N)$ according to

$$m_i \frac{d^2 X_i}{dt^2} = -\nabla_i V|_{\mathbf{x}=\mathbf{X}(t)} \tag{1}$$

where m_i is the mass of the i^{th} particle.[6] Contemplating Equation (1), one might begin to worry that classical mechanics implies the existence of a kind of physical field on configuration space, namely $V(\mathbf{x})$, whose gradient controls the motions of the individual particles. Of course, the function V being (at least if one contemplates describing the entire universe) time-independent, it is natural to think of it as having a law-like (as opposed to matter-like) character.

But the dynamics for these N classical particles can also be re-expressed, as in the Hamilton-Jacobi formalism, in terms of a non-trivially time-dependent

[6] In this paper we will use bold letters to indicate a point in the configuration space, while a variable without arrow indicates a point in the physical space. Capital letters denote the actual positions of particles, while lowercase letters denote generic positions. In principle, positions in physical space have three coordinates; however, for simplicity only, we will often consider a 1D physical space. The symbol ∇_i accounts for the gradient in a 3D physical space or $\nabla_i = \partial/\partial x_i$ in a 1D physical space.

function $S(x_1, x_2, ..., x_N, t)$ on the 3N-dimensional configuration space, evolving according to

$$\frac{\partial S}{\partial t} + \sum_i \frac{(\nabla_i S)^2}{2m_i} + V = 0 \tag{2}$$

and guiding the motion of the particles through

$$\frac{dX_i}{dt} = \frac{1}{m_i} \nabla_i S\big|_{\mathbf{x}=\mathbf{X}(t)}. \tag{3}$$

To whatever extent one is required to think of $S(\mathbf{x}, t)$ as physically real (and to whatever extent one dismisses the possibility that, if real, it might be regarded as law-like rather than matter-like) the situation would seem to be exactly parallel to the one in Bohmian mechanics: the theory involves a curious mathematical function on configuration space, whose role is apparently to guide the particles in physical space.

Why, then, were not people already worrying, in the 19th century, about "indigestible" physically-real fields on configuration space? Evidently, the idea of interpreting (for example) Hamilton's principal function, $S(x_1, x_2, ..., x_N, t)$, as an objective, real, Maxwell-like field simply never even occured to anyone. Instead it was always regarded as obviously only an abstract mathematical reformulation, useful perhaps for certain calculations, but not to be taken seriously as a direct description of some piece of physical ontology.

The crucial point here is that this relaxed attitude is available, in the case of $S(x_1, x_2, ..., x_N, t)$, precisely because there exist *also* the alternative formulations, for example in terms of Newtonian forces, giving rise to the *same* dynamical evolution for the N particles, but in which everything in the mathematics can be readily understood as corresponding to some physically real stuff in ordinary 3-dimensional space.

And so the thought is: maybe this is a clue to how we should try to understand $\Psi(x_1, x_2, ..., x_N, t)$ in quantum mechanics. Maybe, that is, the quantum Ψ has the same status as the classical S – the only difference being that, in the quantum case, we happened for whatever reason to stumble first onto the abstract configuration-space formulation of the theory and have not yet managed to find the more directly physically interpretable alternative mathematical re-formulation in terms of something like Newtonian forces, or (genuinely) Maxwell-like fields (i.e., fields in 3-space rather than 3N-space), or perhaps some other, wholly new, as-yet-uncontemplated type of local beables.

Consideration of classical theories thus suggests that View 3 is at least worth looking into. In the rest of the paper we explain how the Bohmian pilot-wave theory provides a particularly promising starting point for this project via the so-called *conditional wave function* which, unlike the universal wave function Ψ, can be understood as a field living on ordinary 3-dimensional physical space. We begin, in Section 2, with an overview of Bohmian mechanics and, in particular, the conditional wave function. In Section 3 we lay out an important result for this work (based on an idea first presented in Norsen (2010)): the usual Bohmian particle trajectories can be reproduced in a theory

in which the configuration space wave function is replaced by single-particle wave functions, if one introduces appropriate time-dependent single-particle potentials. Section 4 elaborates the concern that these novel potentials imply a kind of infinite ontological complexity (perhaps even more problematic than the questions about configuration space that the proposal is designed to avoid) and then proposes a novel reformulation with a more modest ontological complexity. This novel formulation is clearly preferable, on Occam's razor type grounds, but also implies some disagreements with the usual quantum mechanical predictions. We thus present preliminary evidence, in the form of numerical simulations, that the theory can nevertheless be made empirically adequate (much like GRW type theories, whose disagreements with ordinary quantum predictions are confined to exotic situations not yet amenable to experimental test). Section 5 then summarizes the results and indicates some possible directions for future work on the recommended "View 3".

2 Bohmian Mechanics and the Conditional Wave Function

We are primarily interested in the possibility, afforded especially by Bohmian mechanics, of replacing the usual many-particle wave function (in configuration space) by a set of one-particle wave functions (in physical space). To facilitate our discussion, we take the simplest possible multi-particle system: two spinless particles moving in one spatial dimension.[7]

2.1 Bohmian mechanics

The description of this model system, according to Bohmian mechanics, involves a wave function $\Psi(x_1, x_2, t)$ obeying the usual Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m_1}\nabla_1^2\Psi - \frac{\hbar^2}{2m_2}\nabla_2^2\Psi + V(x_1, x_2, t)\Psi \tag{4}$$

as well as positions $X_1(t)$ and $X_2(t)$ for the two particles. These positions evolve according to the guidance formula

$$\frac{dX_i}{dt} = v_i\big|_{\mathbf{x}=\mathbf{X}(t)} = \frac{\hbar}{m_i}\ \mathrm{Im}\frac{\nabla_i\Psi}{\Psi}\bigg|_{\mathbf{x}=\mathbf{X}(t)}. \tag{5}$$

Note that the right hand side can also be understood as $v_i = j_i/\rho$, evaluated at the actual configuration point $\mathbf{X}(t) = \{X_1(t), X_2(t)\}$, where j_i is the standard

[7] The appropriate generalization for $N > 2$ particles moving in 3 spatial dimensions is trivial; dealing with systems of particles with spin in a fully general way probably requires working instead in terms of the Bohmian *conditional density matrices* defined by Dürr, Goldstein, Tumulka and Zanghì (2005), a possibility we set aside for future work.

probability current associated with particle i and ρ is the standard *probability density*. The probability currents and density obey, as a consequence of Equation (4), the continuity equation:

$$\frac{\partial \rho}{\partial t} + \sum_i \nabla_i j_i = 0. \tag{6}$$

If we consider an ensemble of systems with the same wave function Ψ but random configurations $\mathbf{X}$ moving with configuration-space velocity $\mathbf{v}$, it is easy to see that the probability distribution $P(\mathbf{X}, t)$ over the ensemble should evolve according to

$$\frac{\partial P}{\partial t} + \sum_i \nabla_i (P v_i) = 0. \tag{7}$$

Comparison with Equation (6) shows that, since $v_i = j_i/\rho$, $P = \rho$ is a solution. Thus, if the initial configurations $\mathbf{X}(0)$ in the ensemble are chosen with distribution $P[\mathbf{X}(0) = \mathbf{x}] = \rho(\mathbf{x}, 0) = \Psi^*(\mathbf{x}, 0)\Psi(\mathbf{x}, 0)$ – the so-called *quantum equilibrium hypothesis* (QEH) – it is then a consequence of Equations (4) and (5) jointly that

$$P[\mathbf{X}(t) = \mathbf{x}] = \rho(\mathbf{x}, t) = \Psi^*(\mathbf{x}, t)\Psi(\mathbf{x}, t) \tag{8}$$

for all t. It is thus clear that Bohmian mechanics reproduces the statistical predictions of ordinary quantum theory for position measurements, and hence also for any other type of measurement whose outcome is ultimately registered in the position of some *pointer*.

2.2 The conditional wave function

Ordinary quantum theory offers no way to define the wave function for a single part of a larger quantum mechanical system, and it offers only the notoriously vague and problematic measurement axioms to describe how quantum mechanical systems interact with external, separately postulated, classical systems such as measuring instruments. Bohmian mechanics, by contrast, treats the whole universe in a uniform and fully quantum way. It allows the usual quantum measurement formalism (including for example the collapse postulate, which of course has a dubious *ad hoc* status in the context of ordinary quantum theory) to be *derived*, from the fundamental dynamical postulates outlined above. A key role in this analysis (which was systematically presented by Dürr, Goldstein and Zanghì (1992, 2004)) is played by the *conditional wave function*, which is the natural Bohmian notion of the wave function for a subsystem.

Here we will be particularly interested in treating each individual particle as a sub-system of the universe. The conditional wave function for particle i is defined as the universal wave function Ψ evaluated at the actual position(s)

of the other particle(s). For our toy two-particle universe, the two conditional wave functions are:

$$\psi_1(x,t) = \Psi(x, x_2, t)\big|_{x_2=X_2(t)} \tag{9}$$

and

$$\psi_2(x,t) = \Psi(x_1, x, t)\big|_{x_1=X_1(t)}. \tag{10}$$

Each conditional wave function, because it depends only on the spatial coordinate associated with the single-particle in question, can be regarded as a wave that propagates in physical space. The conditional wave functions are thus a natural candidate for the status of centerpiece in a proposal to re-interpret the configuration-space wave function Ψ in terms of many fields (and perhaps other sorts of local beables) living exclusively in physical space.

Note that the guidance formula, Equation (5), may be re-written, for each particle, in terms of its associated conditional wave function:

$$\frac{dX_i(t)}{dt} = \frac{\hbar}{m_i}\ \mathrm{Im}\frac{\nabla_i\Psi}{\Psi}\bigg|_{\mathbf{x}=\mathbf{X}(t)} \equiv \frac{\hbar}{m_i}\,\mathrm{Im}\,\frac{\nabla\psi_i}{\psi_i}\bigg|_{x=X_i(t)}. \tag{11}$$

The conditional wave function may thus be regarded as single-particle pilot-waves (propagating in physical space) which guide the motions of the affiliated particles.[8]

2.3 Conditional wave function evolution for non-entangled particles

The conditional wave function obeys the usual one-particle Schrödinger equation in the expected type of situation in which the particle in question is suitably isolated from its environment. To see this, consider our simple two-particle toy universe and suppose that the particles are initially not entangled:

$$\Psi(x_1, x_2, 0) = \alpha(x_1)\beta(x_2). \tag{12}$$

Furthermore, suppose that the particles do not interact:

$$V(x_1, x_2, t) = V_1(x_1, t) + V_2(x_2, t). \tag{13}$$

One can then show that

$$\Psi(x_1, x_2, t) = \alpha(x_1, t)\beta(x_2, t) \tag{14}$$

solves the Schrödinger equation, our Equation (4) above, if $\alpha(x,t)$ satisfies the one-particle Schrödinger equation

$$i\hbar\frac{\partial\alpha}{\partial t} = -\frac{\hbar^2}{2m_1}\frac{\partial^2\alpha}{\partial x^2} + V_1(x,t)\alpha \tag{15}$$

[8] Note that, as a result of Equations (9) and (10), the conditional wave functions are not normalized in the usual way. It would be easy enough to adjust the definition to yield normalized conditional wave functions; but since any overall multiplicative factor cancels out anyway in Equation (11), we set aside this needless complication.

and similarly for $\beta(x,t)$.

Thus, non-interacting particles which are not initially entangled will remain unentangled for all time. But the individual factors in the factorizable (unentangled) wave function Ψ are just (up to a meaningless overall multiplicative constant; see footnote 8) the conditional wave functions of the respective particles. That is, with $\Psi(x_1,x_2,t)=\alpha(x_1,t)\beta(x_2,t)$, we have

$$\psi_1(x,t)=\Psi(x,x_2,t)|_{x_2=X_2(t)}=\alpha(x,t)\beta(X_2(t),t)\sim\alpha(x,t) \tag{16}$$

and similarly $\psi_2(x,t)\sim\beta(x,t)$. Thus, in the kind of situation where ordinary quantum theory allows us to meaningfully talk of the wave function of a single-particle, that wave function is identical to the Bohmian conditional wave function and evolves in time according to the expected, single-particle Schrödinger equation.

In general, though – when there is non-trivial interaction and hence entanglement among particles – the conditional wave functions will evolve in an unusual (non-linear and non-unitary) way. To give a sense of the possible behavior and also to provide a sense of how the measurement axioms of ordinary quantum mechanics can instead be derived, from a careful analysis of appropriate kinds of interactions, in Bohmian mechanics, we will explain how the one-particle wave function associated with a quantum system *collapses* when there is a suitable interaction with another system such as a *measuring device*. We illustrate this in the following subsection with a simple analytically-solvable toy model.

2.4 Example of the non-unitary evolution of the conditional wave function

Suppose that particle 1 is "the quantum system to be measured" and particle 2 represents the center of mass coordinate of the macroscopic pointer on a measuring device associated with quantum mechanical operator $\hat{A}$. Suppose the full system is initially in a product state

$$\Psi(x_1,x_2,0^-)=\alpha(x_1)\beta(x_2) \tag{17}$$

where $\alpha(x_1)$ is an arbitrary linear combination of $\hat{A}$ eigenstates

$$\alpha(x_1)=\sum_n c_n\alpha_n(x_1)\ \text{ with }\ \hat{A}\alpha_n(x_1)=a_n\alpha_n(x_1) \tag{18}$$

and $\beta(x_2)$ is (say) a narrow gaussian packet centered at $x_2=0$ representing the measuring device in its *ready* state.

In a fully realistic description of a measurement, the "quantum system" and the "measuring device" would need to interact in a way that drives the wave function Ψ into a set of macroscopically-disjoint channels in the configuration space, with each channel corresponding to a distinct perceivable possible

outcome of the measurement. In the context of our simplified two-particle toy-model, the overall idea can be adequately captured by positing, for example, an impulsive interaction Hamiltonian like

$$\hat{H} = \lambda\, \delta(t) \hat{A} \hat{p}_{x_2} \tag{19}$$

where $\hat{p}_{x_2}$ is the momentum operator for the pointer.

In the special case that the "quantum system" happens already to be in an eigenstate α_m of $\hat{A}$, this interaction leaves the two-particle system in the state $\Psi(x_1, x_2, 0^+) = \alpha_m(x_1)\beta(x_2 - \lambda a_m)$. In light of the quantum equilibrium hypothesis, it thus follows that the the actual position X_2 of the pointer will, with unit probability, lie near the value λa_m indicating the pre-measurement (eigen)value a_m of $\hat{A}$. The pointer, in short, displays the value normally associated with $\hat{A}$ in this situation.

In the general case, however, the interaction Hamiltonian takes the state in Equation (17) to

$$\Psi(x_1, x_2, 0^+) = \sum_n c_n \alpha_n(x_1)\beta(x_2 - \lambda a_n) \tag{20}$$

in which the pointer is in an entangled superposition with the system. This final state reflects the notorious measurement problem of ordinary quantum mechanics. But for the Bohmian pilot-wave theory, there is no problem: the empirically observed outcome is not to be found in the wave function, but instead in the actual final pointer position $X_2(0^+)$. Again in light of the quantum equilibrium hypothesis, this will, with probability $|c_n|^2$, be near the value λa_n indicating that the outcome of the measurement was a_n.[9]

Now consider how the conditional wave function for the "quantum system" evolves during the measurement. Prior to the interaction, when the joint two-particle wave function is still given by Equation (17), the conditional wave function for particle 1 is

$$\psi_1(x, 0^-) = \alpha(x)\beta(x_2)\big|_{x_2 = X_2(t)} \sim \sum_n c_n \alpha_n(x). \tag{21}$$

This corresponds to the initial superposed wave function that would be attributed to the "quantum system" in ordinary Quantum Mechanics. But the post-interaction Ψ, given by Equation (20), involves disjoint channels of support in the configuration space. The final pointer position, $X_2(0^+)$, will randomly (depending on initial conditions) end up in the support of *just one* of these channels. That is, $\beta(x_2 - \lambda a_n)|_{x_2 = X_2(0^+)}$ will (approximately) *vanish* for all n except the particular value, n', satisfying $X_2(0^+) \approx \lambda a_{n'}$, which corresponds to the actual result of the measurement. And so the post-interaction conditional wave function for particle 1 will be

$$\begin{aligned} \psi_1(x, 0^+) &= \sum_n c_n \alpha_n(x)\beta(x_2 - \lambda a_n)\big|_{x_2 = X_2(0^+)} \\ &\sim \alpha_{n'}(x). \end{aligned} \tag{22}$$

[9] We assume here that the spacing, $|\lambda(a_n - a_{n+1})|$, between adjacent possible pointer positions is small compared to the width, w, of β.

That is, the interaction causes the initially superposed wave function for particle 1 to *collapse* to the appropriate eigenfunction (corresponding to the realized outcome of the measurement) even though the wave function of the joint system evolves exclusively unitarily according to the two-particle Schrödinger equation. Bohmian mechanics thus *explains*, from the point of view of a theory in which all particles are treated in a fully uniform and fully quantum way, how the wave functions of sub-systems may evolve exactly as described (at the price of non-uniform treatment and additional *ad hoc* postulates) in ordinary quantum mechanics.

3 Wave equation for the single-particle wave functions

In the last Section we described the conditional wave function as playing a certain role in the standard formulation of Bohmian mechanics. From this perspective, one crucial feature is that the conditional wave function automatically evolves as sub-system wave functions should, according to ordinary quantum mechanics. (It is the fact that this happens *automatically* – i.e., as a result of the fundamental dynamical postulates, with no hand-waving and additional *ad hoc* "measurement axioms" – that is crucial and noteworthy here.) But although the motion of the Bohmian particles can be expressed exclusively in terms of their associated conditional wave functions, the conditional wave functions are not usually thought of as having an independent existence. They are, after all, *defined* in terms of the universal (configuration space) wave function, and they do not (in general) evolve autonomously. So they have a status like, for example, the center of mass of a collection of particles in classical mechanics: they are a useful theoretical construct for understanding certain features of the theory, but they do not have any *ontological* significance beyond that of the objects they are defined in terms of.

Our proposal – a concrete implementation of "View 3" – is to reverse the ontological statuses usually assigned to the universal and conditional wave functions. That is, instead of regarding the universal (configuration space) wave function Ψ as "physically real" (with the conditional wave functions being mere theoretical constructs), we propose that the set of one-particle conditional wave functions can be invested with that primary ontological status. To explain this possibility, let us develop the Schrödinger-type equations that can be understood as governing an (almost) autonomous time-evolution for the conditional wave functions.

3.1 General Schrödinger-type equation for the single-particle wave functions

A crucial point, underlying the behavior discussed in the last Section, is that the conditional wave function (for, say, particle 1)

$$\psi_1(x,t) = \Psi\big(x, X_2(t), t\big) \tag{23}$$

depends on time in two ways: through the Schrödinger time-evolution of Ψ, and also through the time-evolution of X_2. We may thus develop a Schrödinger-type equation for the one-particle wave function of particle 1 as follows:

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\psi_1(x,t) &= i\hbar\frac{\partial\Psi(x,x_2,t)}{\partial t}\Big|_{x_2=X_2(t)} + i\hbar\frac{dX_2}{dt}\frac{\partial\Psi(x,x_2,t)}{\partial x_2}\Big|_{x_2=X_2(t)} \\ &= -\frac{\hbar^2}{2m_1}\frac{\partial^2\psi_1(x,t)}{\partial x^2} + V\left[x,X_2(t),t\right]\psi_1(x,t) \\ &\quad +i\hbar\frac{dX_2}{dt}\psi_1'(x,t) - \frac{\hbar^2}{2m_2}\psi_1''(x,t) \end{aligned} \tag{24}$$

where we have defined

$$\psi_1'(x,t) = \frac{\partial\Psi(x,x_2,t)}{\partial x_2}\Big|_{x_2=X_2(t)} \tag{25}$$

and

$$\psi_1''(x,t) = \frac{\partial^2\Psi(x,x_2,t)}{\partial x_2^2}\Big|_{x_2=X_2(t)}. \tag{26}$$

The Schrödinger-type equation for ψ_1 can thus be re-written as

$$i\hbar\frac{\partial\psi_1}{\partial t} = -\frac{\hbar^2}{2m_1}\frac{\partial^2\psi_1}{\partial x^2} + V_1^{\text{eff}}(x,t)\psi_1 \tag{27}$$

where

$$V_1^{\text{eff}}(x,t) = V[x,X_2(t),t] + A_1(x,t) + B_1(x,t). \tag{28}$$

This effective potential includes the *conditional potential* $V[x,X_2(t),t]$ (which is the usual two-particle potential evaluated at the actual Bohmian location of the other particle) plus some additional terms:

$$A_1(x,t) = i\hbar\frac{dX_2}{dt}\frac{\psi_1'(x,t)}{\psi_1(x,t)} \tag{29}$$

and

$$B_1(x,t) = -\frac{\hbar^2}{2m_2}\frac{\psi_1''(x,t)}{\psi_1(x,t)}. \tag{30}$$

It is important that the terms A_1 and B_1 in (29)-(30) depend on $\psi_1(x,t)$ itself, making the whole equation non-linear. (A different path for the same deduction of the $A_1(x,t)$ and $B_1(x,t)$ terms can be found in Oriols (2007).) In addition, these terms can be complex, so the time-evolution of the conditional wave function need not be unitary. This explains how the conditional wave functions are able to exhibit wave-function *collapse*, as we saw in the previous Section.

It is interesting to note that the position $X_2(t)$ of particle 2 has a direct influence on the time-evolution of particle 1's conditional wave function, through the appearance of dX_2/dt in the term A_1. And of course particle 1 likewise has a direct influence on the evolution of particle 2's conditional wave function. This is in contrast to the usual formulation of Bohmian mechancs, in terms of

the configuration space wave function Ψ, in which the wave function evolves completely independently of the particle positions. Note that this implies, for example, that in an ensemble of identically-prepared systems (with identical initial wave functions, but a distribution of initial particle positions) the conditional wave functions will evolve *differently* for the different members of the ensemble.

The *non-local* character of the dynamics can also be seen here (we remind the reader about the two meanings of the adjective "non-local" mentioned in footnote 4). The dependence, for example, of $V_1^{\rm eff}(x,t)$ on $X_2(t)$ means that the pilot-wave field for particle 1 – and hence the motion of particle 1 itself – can be influenced by interventions which alter the trajectory $X_2(t)$ of the other (perhaps quite distant) particle. Our theory thus inherits the dynamical non-locality of standard Bohmian mechanics, i.e., the sort of non-locality that we know is required if one wants to account for the empirically observed violations of Bell's inequalities (Bell 1994). The interesting and important novelty here is that our proposed theory is a (dynamically non-local) theory of exclusively *local beables*: the particles and pilot-wave fields *live* in ordinary physical space, but the effective potentials $V_i^{\rm eff}$ which mediate their interactions imply instantaneous actions at a distance.[10] This is in contrast to standard Bohmian mechanics, in which the non-locality is in some sense mediated by the universal wave function, which of course lives in configuration space (and is hence a *non-local beable*, if it is a beable at all).

We stress, therefore, that the proposed theory (in which each particle's motion is guided by an associated single-particle wave function living in ordinary physical space) really does reproduce the particle trajectories of standard Bohmian mechanics and hence the exact statistical predictions of ordinary quantum theory. In particular, the dynamical non-locality that is manifest in the above expressions for the (single-particle) effective potentials would allow a (suitably generalized) theory of the type proposed here to account for Bell inequality violations, quantum teleportation, and the various other quantum phenomena which are sometimes erroneously thought to *require* a configuration space wave function.

3.2 Defining the effective potential

By construction, ψ_1 obeying Equation (27) exactly reproduces the usual Bohmian trajectory $X_1(t)$, and similarly for other particles. Hence, a proper ensem-

[10] On one hand, the potential $V[x, X_2(t), t]$ in (28) produces correlations between the two particles $X_1(t)$ and $X_2(t)$. The dependence of $V[x, X_2(t), t]$ on x and $X_2(t)$ imposes a restriction on the speed of such interaction. For example, the *retarded* electromagnetic potentials ensures that there is no superluminal electromagnetic influence between particles due to $V[x, X_2(t), t]$. On the other hand, such restriction on the *speed* of the interaction between particles is not present in the new potentials $A_1(x, X_2(t), t)$ and $B_1(x, X_2(t), t)$ in (28). Thus, the particle $X_1(t)$ have an *instantaneous* (non-local) interaction with $X_2(t)$ due to the potentials $A_1(x, X_2(t), t)$ and $B_1(x, X_2(t), t)$. Of course, insofar as the usual Bohmian particle trajectories, and hence the usual statistical predictions of quantum mechanics, are reproduced, we know that this non-locality will not support superluminal communication.

ble will exactly reproduce all statistical predictions of ordinary quantum theory – provided the one-particle effective potentials $V_i^{\text{eff}}(x,t)$ are defined appropriately. There is no difficulty with the conditional potential terms, e.g., $V[x, X_2(t), t]$. But our above definitions of A_1 and B_1 involve ψ_1' and ψ_1'' which are, in turn, defined in terms of the configuration space wave function in Equations (25) and (26). So our proposed implementation of "View 3" – formulating an empirically adequate quantum theory in which the configuration space wave function is not present – requires finding a different way to define the single-particle effective potentials. Below, we discuss two different (but related) possibilities.

One possibility for defining the terms A_1 and B_1 (without reference to Ψ) can by found by adapting a proposal of Norsen (2010). For simplicity, let us define the particle-1-associated potential fields $a(x,t)$, $b(x,t)$, etc., as follows:

$$a(x,t) = \frac{\psi_1'(x,t)}{\psi_1(x,t)}, \tag{31}$$

$$b(x,t) = \frac{\psi_1''(x,t)}{\psi_1(x,t)}, \tag{32}$$

$$c(x,t) = \frac{\psi_1'''(x,t)}{\psi_1(x,t)}, \tag{33}$$

and so on. Then Equation (28) for the full effective potential which drives the evolution of particle 1's single-particle wave function ψ_1 can be re-written as

$$V_1^{\text{eff}}(x,t) = V(x, X_2(t), t) + i\hbar\frac{dX_2}{dt}a(x,t) - \frac{\hbar^2}{2m_2}b(x,t). \tag{34}$$

We may then use the full configuration-space Schrödinger equation to find out how $a(x,t)$ and $b(x,t)$ must evolve in order to exactly reproduce the standard Bohmian trajectories. The important thing here is that the results can be written exclusively in terms of this infinite network of potential fields. For example, the field a should satisfy its own partial differential equation of the form

$$\begin{aligned}\frac{\partial a}{\partial t} = \frac{i\hbar}{2m_1}\left[\frac{\partial^2 a}{\partial x^2} + 2\frac{\partial a}{\partial x}\frac{(\partial\psi_1/\partial x)}{\psi_1}\right] + \\ + \frac{i\hbar}{2m_2}\left[c - ab\right] + \frac{dX_2}{dt}\left[b - a^2\right] - \frac{i}{\hbar}\frac{\partial V}{\partial x_2}\Big|_{x_2=X_2(t)}.\end{aligned} \tag{35}$$

And similarly, b will satisfy an evolution equation of the form

$$\begin{aligned}\frac{\partial b}{\partial t} = \frac{i\hbar}{2m_1}\left[\frac{\partial^2 b}{\partial x^2} + 2\frac{\partial b}{\partial x}\frac{(\partial\psi_1/\partial x)}{\psi_1}\right] + \frac{i\hbar}{2m_2}\left[d - b^2\right] + \frac{dX_2}{dt}\left[c - ab\right] \\ - \frac{2i}{\hbar}a\frac{\partial V}{\partial x_2}\Big|_{x_2=X_2(t)} - \frac{i}{\hbar}\frac{\partial^2 V}{\partial x_2^2}\Big|_{x_2=X_2(t)}.\end{aligned} \tag{36}$$

The c and d which appear here need their own time-evolution equations (which will in turn involve further potentials e and f), and so on. The result is a

countably infinite network of potential fields obeying coupled time-evolution equations. These potentials then of course appear in the Schrödinger-type equations governing the single-particle pilot-wave fields which guide the particles. The exact statistical predictions of quantum theory are reproduced, but the configuration space wave function Ψ is nowhere to be found. We used Ψ, of course, to find out how the potentials a, b, c, etc., must interact and evolve in order to reproduce the usual Bohmian particle trajectories. But once we have Equations (34) as well as (35), (36), etc., the universal wave function Ψ can, like the proverbial ladder, be simply kicked away.

Thus, the answer to the question posted in the paper's title is: yes. It *is* possible to reproduce the exact particle trajectories of Bohmian mechanics, and hence the exact statistical predictions of ordinary quantum mechanics, in a theory in which the configuration space wave function Ψ is replaced with single-particle wave functions (one for each particle) in ordinary physical space. One merely needs to introduce appropriate one-particle effective potentials.

But the price of defining these potentials in the way we have done here – with an *infinite* number of interacting potential fields associated with each particle – seems rather high. We thus turn to attempting to develop an alternative, less ontologically complex, approach to defining appropriate one-particle potentials.

4 An empirically viable theory in 3d physical space with modest ontological complexity

It is not out of the question that some radical new perspective on the problem will allow, in the future, a completely simple and straightforward definition of the one-particle effective potentials needed to reproduce the quantum predictions in the context of a Bohm-inspired theory of particles being guided by one-particle pilot-waves in physical space. Such an innovation might be analogized to the simplification that was afforded in the description of planetary orbits when Kepler abandoned the axiom of explaining the orbits in terms of exclusively circular motion.

Unfortunately, such a development remains a speculative fantasy. But we can develop a more realistic, if also more mundane, alternative definition for the one-particle effective potentials by considering an ontological simplification arrived at by approximating the infinite network introduced in the previous Section. After all, as illustrated by the dynamical collapse (GRW) type theories (Ghirardi et al. 1986; Bassi et al. 2013), perfect agreement with quantum predictions is not sacrosanct. What matters is instead agreement with experimental data. Disagreements with the quantum predictions are perfectly tolerable if they can be confined to situations where no experimental data is yet available, and such options are particularly welcome if they help to address foundational and/or philosophical problems.

That, then, indicates the nature of our proposal: if the price of eliminating the configuration space wave function Ψ is the introduction of an infinitely

complex network of interacting single-particle potential fields, it is not clear that much is gained. However, if the network of single-particle potential fields can be made reasonably simple, while still maintaining empirical adequacy, this would be a very compelling argument that the configuration space wave function (and hence the torturous philosophical conundrums its presence raises) might be simply eliminated. In the remainder of this Section we will present preliminary evidence in support of this possibility, starting with another look at the case of non-entangled particles.

4.1 The case of no entanglement

We argued in Section 2.3 that when the universal wave function factorizes, the two conditional wave functions are given by α and β, respectively, and these should obey the expected one-particle Schrödinger equations. Let us now see how this same conclusion arises from the general Schrödinger-type equation governing the time evolution of the conditional wave functions.

The crucial point is that with $\Psi(x_1, x_2, t) = \alpha(x_1, t)\beta(x_2, t)$, we have

$$\psi_1'(x,t) = \alpha(x,t)\frac{\partial\beta}{\partial x_2}\Big|_{x_2=X_2(t)} \tag{37}$$

so that

$$A_1(t) = i\hbar\frac{dX_2}{dt}\frac{\partial\beta/\partial x_2}{\beta}\Big|_{x_2=X_2(t)} \tag{38}$$

is *independent of* x. (This happened because, when there is no entanglement, the strange object $\psi_1'(x,t)$ is *proportional to* the conditional wave function, $\psi_1(x,t)$. And so the potential A_1, which depends on the *ratio*, has no x-dependence.)

Similarly,

$$\psi_1''(x,t) = \alpha(x,t)\frac{\partial^2\beta}{\partial x_2^2}\Big|_{x_2=X_2(t)} \tag{39}$$

so that

$$B_1(t) = -\frac{\hbar^2}{2m_2}\frac{\partial^2\beta/\partial x_2^2}{\beta}\Big|_{x_2=X_2(t)} \tag{40}$$

is also *independent of* x.

Thus, whenever two particles are un-entangled, the only x-dependence in the effective potential V_1^{eff} arises from the conditional potential, $V[x, X_2(t), t]$. And of course, for non-interacting particles, this conditional potential is simply $V_1(x,t)$ (the external potential experienced by particle 1) plus a (perhaps time-dependent) constant, $V_2(X_2(t), t)$, which simply introduces an uninteresting overall time-dependent phase into $\psi_1(x,t)$.

The important point here is that the mysterious potential energy terms, A_1 and B_1, do not depend on x and hence also have no meaningful influence on $\psi_1(x,t)$, so long as the two-particle state remains unentangled.

4.2 The small entanglement approximation

We saw in the last subsection that, as long as our two particles remain unentangled, the terms A_1 and B_1 are functions of t only and hence affect the single-particle wave function only by giving it an overall time-dependent multiplicative factor which cancels out in the guidance formula, Equation (11). The terms A_1 and B_1, in other words, might as well be set to zero. In this situation, the meaningful and important contribution to V_1^{eff} comes exclusively from the conditional potential, $V[x, X_2(t), t]$.

This suggests a kind of first-order approximation to the infinitely complex network of single-particle potentials developed at the end of the previous Section: we retain, for each particle, the conditional potential term in V_i^{eff} and set the other terms to zero:

$$V_i^{\text{eff}} \approx V(x_1, x_2, t)\Big|_{x_i = x \text{ and } x_n = X_n(t) \forall n \neq i}. \tag{41}$$

On the grounds that this should work perfectly when there is no entanglement, we call this the "small entanglement approximation" or SEA. But note that there is no clear *a priori* reason to expect the SEA to work well when there is even just a little entanglement. It could be, for example, that as soon as any entanglement develops, the small entanglement approximation breaks down completely, giving particle trajectories that are wildly unphysical and/or blatantly different from those of ordinary Bohmian mechanics.

In principle, a gifted theoretician could perhaps intuit or calculate the accuracy of the small entanglement approximation in various kinds of situations. Unfortunately, the present authors lack the requisite intuition. So we have turned instead to solving the relevant equations numerically, and comparing the results of the small entanglement approximation to the results obtained by solving the configuration space Schrödinger equation for Ψ.

In the next subsection we present a numerical example of two particles continuously interacting through a non-separable potential.

4.3 Numerical example for two interacting particles

In the following example, we show a scenario where the *small entanglement approximation* works quite well for a system of continuously interacting particles. In particular, we consider two particles, each moving in the presence of a harmonic external potential, and also interacting:

$$V(x_1, x_2) = F\left(x_1^2 + x_2^2\right) + C x_1 x_2 \tag{42}$$

with $F = 10^{12}\ eV \cdot m^{-2}$. Since we consider an initial product state wave function $\Psi(x_1, x_2, 0) = \alpha(x_1)\beta(x_2)$, entanglement will develop as a result of the interaction term Cx_1x_2. The parameter C may therefore be used to quantify the *amount of entanglement* between the particles. We anticipate that the

small entanglement approximation will be increasingly accurate for smaller values of C.

According to the SEA, the Schrödinger equation satisfied by the single-particle wave function $\psi_1(x,t)$ of particle 1 is

$$i\hbar\frac{\partial\psi_1(x,t)}{\partial t} = -\frac{\hbar^2}{2m_1}\frac{\partial^2\psi_1(x,t)}{\partial x^2} + V[x, X_2(t)]\psi_1(x,t) \tag{43}$$

while the corresponding equation for $\psi_2(x,t)$ is

$$i\hbar\frac{\partial\psi_2(x,t)}{\partial t} = -\frac{\hbar^2}{2m_2}\frac{\partial^2\psi_2(x,t)}{\partial x^2} + V[X_1(t), x]\psi_2(x,t). \tag{44}$$

It is very important to realize that Equations (43) and (44) are coupled through the Bohmian trajectories. This coupling is responsible for the interaction between the two degrees of freedom. The trajectories $X_1(t)$ and $X_2(t)$ are computed from the associated conditional wave functions at each time step. There is no need to track (or even mention) the configuration space wave function $\Psi(x_1, x_2)$.

In order to check the accuracy of the small entanglement approximation, however, the results obtained from Equations (43)-(44) will be compared with the solution of the following two-particle Schrödinger equation:

$$i\hbar\frac{\partial\Psi(x_1,x_2,t)}{\partial t} = \left[-\frac{\hbar^2}{2m_1}\frac{\partial^2}{\partial x_1^2} - \frac{\hbar^2}{2m_2}\frac{\partial^2}{\partial x_2^2} + V(x_1,x_2)\right]\Psi(x_1,x_2,t). \tag{45}$$

We consider $m_1 = m_2 = m_0$ with m_0 the free electron mass. In Fig. 1 we plot the trajectories computed from the 2D solution of the Schrödinger equation (45) and from the 1D small entanglement approximation, Equations (43)-(44). In this particular example, we use $C = -1 \cdot 10^{12}\ eV \cdot m^{-2}$ in Equation (42). We see that there are no significant differences between the 2D and 1D computation of the trajectories $X_1(t)$ and $X_2(t)$.

In Fig. 2 we plot the Bohmian velocity and the kinetic energy for the two particles comparing the standard Bohmian mechanics solution based on Ψ ("2D") with the small entanglement approximation to the theory developed here ("1D"). No significant differences are present. We emphasize that there is an interchange of energy between the first and second particle, showing the non-separable quantum nature of the studied system.

We repeat the same analysis with a different value of the interaction parameter $C = -2 \cdot 10^{12}\ eV \cdot m^{-2}$. In Fig. 3, we see that the difference between the two cases are quite small but detectable. In order to confirm this fact we report in Fig. 4 the velocity and kinetic energy for particle 1 and 2. We clearly see a difference in the two solutions.

In order to show quantitatively what is the deviation when the entanglement between the two particles grows, we compute the absolute error of the 2D, $X_i^{2D}(t)$, and the 1D, $X_i^{1D}(t)$, trajectories as:

$$\text{Deviation}(t) = \sqrt{\left(X_1^{2D}(t) - X_1^{1D}(t)\right)^2 + \left(X_2^{2D}(t) - X_2^{1D}(t)\right)^2}. \tag{46}$$

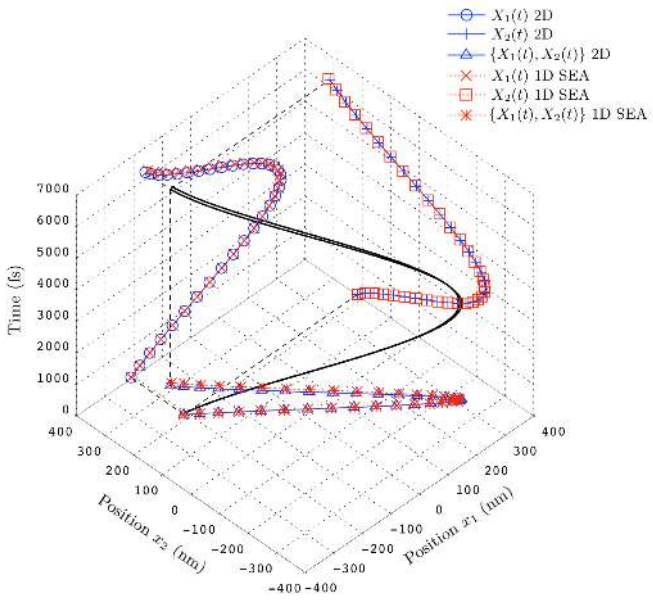

Fig. 1 Comparison between the Bohmian trajectories calculated from the 2D solution (with $\odot$, $+$ and $\triangle$ in blue) of the Schrödinger equation (Equation (45)) and the 1D solution (with $\times$, $\square$ and $*$ in red) of the small entanglement approximation (Equations (43)-(44)) for $C = -1 \cdot 10^{12}\ eV \cdot m^{-2}$. We also plot with black solid lines the trajectories in the configuration space for both solutions

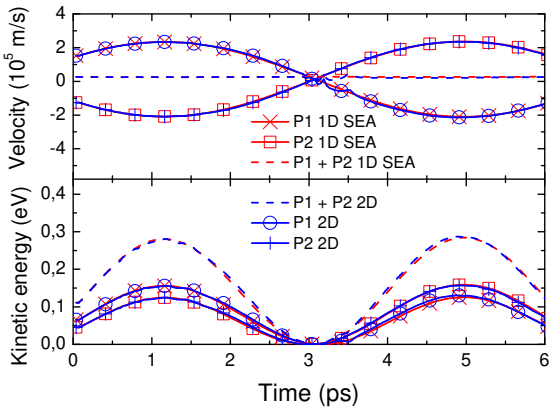

Fig. 2 Bohmian velocity and kinetic energy for particle 1 (P1) and particle 2 (P2) as function of time for $C = -1 \cdot 10^{12}\ eV \cdot m^{-2}$. Dashed lines represent the sum of the velocity and the kinetic energy for particle 1 plus particle 2 (P1 + P2)

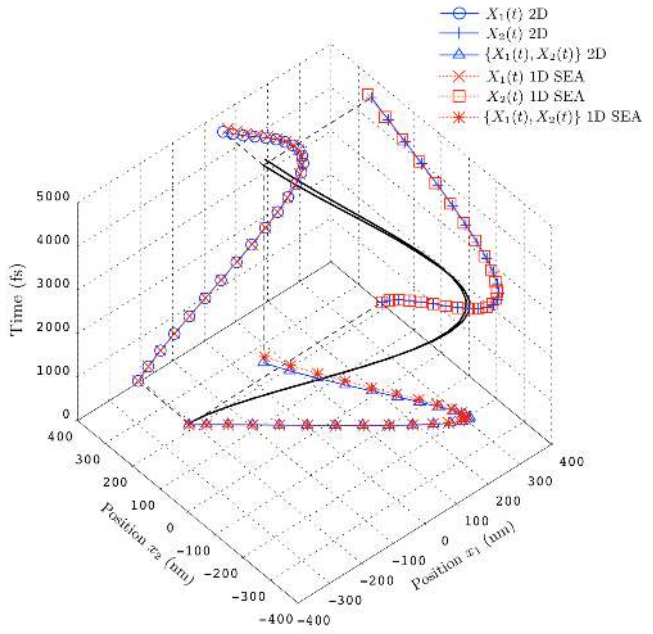

Fig. 3 Comparison between the trajectories calculated from the 2D solution of the Schrödinger equation (Equation (45)) and the 1D solution of the small entanglement approximation (Equations (43)-(44)) for $C = -2 \cdot 10^{12}\ eV \cdot m^{-2}$. Symbols are the same as in Fig. 1

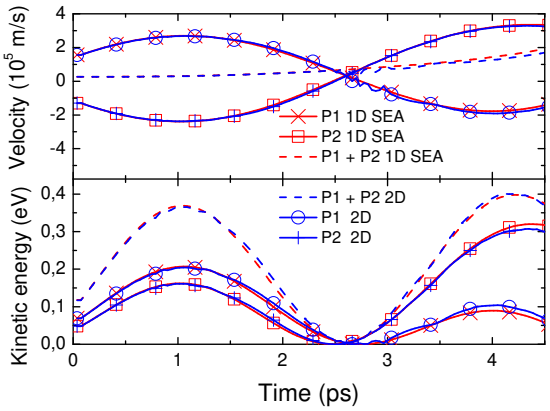

Fig. 4 Bohmian velocity and kinetic energy for the two particles in function of time for $C = -2 \cdot 10^{12}\ eV \cdot m^{-2}$. Symbols are the same as in Fig. 2

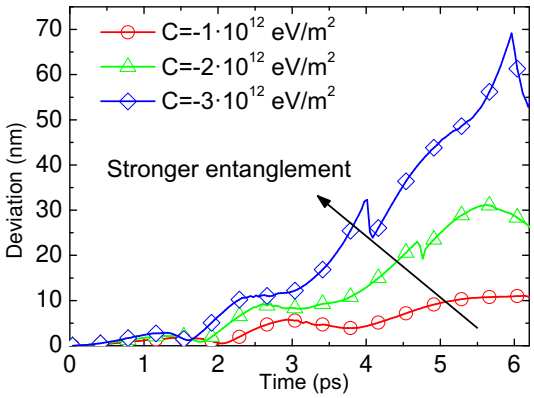

Fig. 5 Deviation computed from Equation (46) as function of time for different values of C

In Fig. 5 we plot Equation (46) for three value of C. First, we see that the deviation increases as time grows. Secondly, the deviation increase for larger absolute values of C confirming that our SEA is increasingly accurate for smaller entanglements.

We conclude that the computation of one-particle wave functions from Equation (27) with the small entanglement approximation works quite well in the scenario governed by Equation (42). This type of computation has already been used to study quantum transport in nano electronic devices (Albareda et al. 2013; Oriols et al. 2012; Albareda et al. 2009, 2010; Traversa et al. 2011; Alarcón et al. 2013). A commercial software named BITLLES[11] has been developed following these ideas. Finally, to be fair, let us mention that there are many other shapes of potentials where this simplest approximation for the conditional wave function does not work as well. In any case, we emphasize that the approximation just presented is only the first step (although quite accurate and sufficient in some cases) towards a method able to tackle quantum phenomena with appreciable entanglement using only one-particle wave functions in the physical 1D (or 3D) space.

The previous computations in the small entanglement approximation (SEA) strongly suggest that the type of theory sketched at the beginning of this Section can be made empirically viable. As already noted, perfect agreement with the quantum predictions is not required if the disagreement is limited to cases where no experimental data is yet available. The fact that the most brute approximation (what we call the SEA) already works quite well in some situations provides strong evidence that a more seriously realistic approximation (in which the network of potentials is cut, for example, between b and c) could

[11] See the website *http://europe.uab.es/bitlles*

adequately reproduce the effects (such as non-locality and non-unitarity) of the full effective potentials.

The question of where, exactly, the tower of potentials should be cut is the subject of ongoing work by the authors. To be clear, though, this is not something that would be adjusted on a case-by-case basis for purely calculational purposes. Instead the idea is to propose that the ontology involves some large number N of sets composed of: a point particle, an associated pilot-wave field, and – say – 5 dynamical potential fields (a, b, c, d, and e). The choice of where to cut the tower, that is, is like the choice of values for the parameters (which define the length scale and frequency of the spontaneous collapses) in GRW-type theories. It is made once and for all, in order to try to achieve empirical adequacy, and then the theory says what it says. What it says will include some empirically testable deviations from the predictions of ordinary quantum mechanics, in situations (for example) involving spatially and/or temporally long-range entanglement.

5 Summary and Future Prospects

The fundamental lesson of our paper is that it is possible to describe quantum phenomena with a set of single-particle pilot-wave fields in ordinary 3-dimensional physical space instead of the usual (ontologically-puzzling) configuration space wave function. Here we summarize the main sub-points:

- *The conditional wave function:* The ordinary quantum theory offers no way to define the wave function for a single part of a larger quantum mechanical system, and it offers only the notoriously vague and problematic measurement axioms to describe how quantum mechanical systems interact with measuring instruments. In Section 2, we briefly reviewed Bohmian mechanics showing how it provides a natural notion of the (conditional) wave function of a subsystem (Dürr, Goldstein and Zanghì 1992, 2004) and allows the usual quantum measurement formalism (including for example the collapse postulate) to be *derived* from the fundamental dynamical postulates of the Bohmian theory. The (Bohmian) conditional wave function offers a very relevant and unique starting point to formulate quantum mechanics in terms of single-particle wave functions in a 3-dimensional physical space.
- *Single-particle wave functions with an infinite network of interacting potentials fields:* In Section 3, we have shown that it is indeed possible to formulate an empirically adequate Bohmian-type quantum theory in which the usual wave function Ψ (the wave function of the universe, living in $3N$-dimensional configuration space) is replaced by N single-particle wave functions that live in 3-dimensional physical space. In particular, we have shown explicitly how to define a (countably infinite) network of interacting potential fields which (together with the usual classical potential function V) drive the single-particle wave functions in such a way that the particle trajectories exactly reproduce those of standard Bohmian mechanics.

The exact empirical predictions of quantum mechanics (including phenomena involving entanglement and non-locality) can therefore in principle be reproduced by a theory in which the philosophically-puzzling wave function Ψ plays no role. This result, reformulated here in terms of interacting potentials fields, was already anticipated in Norsen (2010). We emphasize that the *non-locality* required by Bell's theorem can actually be embedded in a theory of exclusively *local* beables.

This result may initially seem surprising in light of the recent "PBR" theorem of (Pusey et al. 2012), according to which the wave function Ψ must be regarded as physically real in any theory that shares the exact predictions of quantum mechanics. The main point is that "physically real" (for PBR) means that the mathematical object in question is a function of the ontic state posited by the theory. For the kind of theory proposed here, the full ontic state includes not only the particle positions and the states of each single-particle pilot-wave field, but also the network of interacting single-particle potentials. We have shown explicitly how the single-particle potentials can be defined in terms of the universal wave function Ψ; it is hardly surprising, then, that Ψ is determined by the complete ontic state.[12] Thus, for the type of theory proposed here, Ψ is indeed "physically real" in the sense of (Pusey et al. 2012). Superficial appearances to the contrary notwithstanding, however, this is perfectly compatible with our main point, namely, that one need not regard Ψ as "physically real" in the straightforward, direct sense of "View 1" and "View 2" from the Introduction. We have shown, instead, how Ψ can instead be viewed as an indirect and abstract characterization of the state of a certain constellation of interacting *local beables*. That is, we have shown how Ψ can be regarded as having the same (totally unproblematic) status that is usually assigned to, say, Hamilton's principal function $S(x_1, x_2, ..., x_N, t)$ in classical mechanics.

– *Single-particle wave functions with reduced ontological complexity:* In Section 4, in addition to the demonstrations of the plausibility of an explanation of the quantum theory with local beables, we also took a preliminary step toward an ontologically simpler theory (of the general sort proposed above), in which the infinite network of interacting potentials is reduced to a more reasonable size at the price of introducing some disagreements with the predictions of ordinary quantum theory. We showed explicitly that, even making the most draconian imaginable cuts to the network, one still gets reasonable behavior and indeed rather surprisingly good agreement with the predictions of quantum mechanics in certain simple situations. Of course, certain important effects like non-locality and non-unitary evolution ("collapse") cannot be reproduced in this so-called "small entanglement

[12] In fact, Ψ is determined by the one-particle wave function and the associated potentials *for any one particle*. The *complete* ontic state – comprising the one-particle wave function and associated potentials for *all* particles, plus the particle positions themselves – thus contains a tremendous amount of redundancy. This is yet another strong piece of evidence suggesting that empirical viability should be able to be achieved, even with a greatly reduced ontic complexity.

approximation". But our results provide a basis for optimism that such phenomena (and ultimately all currently available empirical data) might be reproduced in a theory that goes beyond the SEA, i.e., a theory with a moderate and tolerable degree of ontological complexity. Such a theory would then have a status comparable to that already enjoyed by GRW-type theories: it would be empirically viable despite making predictions that are (in certain "exotic," as-yet-untested cases) different from those of ordinary quantum mechanics. (And note that, as soon as deviations from quantum predictions are contemplated, the PBR theorem will no longer entail that Ψ be "physically real", even in PBR's somewhat misleading sense. So there is no contradiction in the idea of constructing an empirically viable theory using some restricted subset of the network of potential fields, even though Ψ itself could not be computed from the single-particle wave functions and the restricted subset.)

While we have shown in principle that the wave function in configuration space *can* be replaced by single-particle wave functions in physical space, the details of how best to accomplish "View 3" remain unsettled. Ongoing work by the authors will explore the effects of cutting the network of potentials between b and c –instead of, as was done here in our preliminary SEA, cutting it before a. This should be particularly interesting in that certain key effects of the full effective potential (such as non-locality and non-unitarity) should be included.

But in general "View 3" remains a young and as-yet unproven research program. Whether it will bear important fruit remains to be seen. But it should already be clear that it is simply premature to debate, for example, whether the universal wave function Ψ is better regarded as a physically-real field in a physically-real configuration space or instead as a new and unusual (perhaps time-dependent) type of natural law. *There are other possibilities.* In particular, one alternative possibility is the one suggested already by certain abstract reformulations of classical mechanics, in which the unusual time-dependent function on configuration space is regarded as an indirect description of some more mundane and more familiar and less puzzling pieces of physical ontology.

Let us close by returning to the historical considerations with which we began in the introduction. Shortly after inventing wave mechanics and his eponymous equation, Schrödinger noted that "the use of the [configuration] space is to be seen only as a mathematical tool, as it is often applied also in the old mechanics; ultimately ... the process to be described is one in space and time." (Bacciagaluppi and Valentini 2009, p. 447) This overall sentiment – that the wave function in configuration space cannot be taken seriously as corresponding directly to a physically-real wave in a physically-real, high-dimensional space – seems to have been shared by most of the prominent realist-leaning physicists at the time: not just Schrödinger, but (as we saw in the Introduction) Einstein, and also de Broglie, who explained in his address at the 1927 Solvay conference that "if one wants to *physically* represent the evolution of a system of corpuscles, one must consider the propagation of

N waves in space..." (Bacciagaluppi and Valentini 2009, p. 79) Indeed, as summarized by Linda Wessels based on a 1962 interview with Carl Eckart (conducted by John Heilbron), it would be desirable "to rewrite the equations of wave mechanics so that even for a system of several 'particles', only 3-dimensional wave functions would be determined. C. Eckart has reported that at one time he attempted this and remarked that it was something that initially 'everybody' was trying to do." (Wessels 1979)

Of course, eventually the more positivist/instrumentalist outlook of Bohr and Heisenberg won out, and this concern about physically interpreting the nature of the quantum wave function largely died away. It is a very positive development that, in recent decades, the shortcomings of the positivist approach are being increasingly recognized and interpretational questions are again being taken seriously by physicists and philosophers. Our main message can then be summarized as follows: as we re-open the discussion about the physical meaning and significance of the quantum mechanical wave function, let us not forget about the original view of Einstein, Schrödinger, de Broglie, Eckart, and others – that the configuration space wave function must be some kind of abstract, indirect description of physical processes in ordinary physical space. As we have shown, this prematurely-abandoned program remains viable, and the pilot-wave theory of de Broglie and Bohm (with its natural way of defining one-particle wave functions in physical space) provides an especially promising starting point. It may yet vindicate Schrödinger, who expressed the hope, already in 1927, that "in the end everything will indeed become intelligible in three dimensions again." (Bacciagaluppi and Valentini 2009, p. 461)

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