

# Can you do quantum mechanics without Einstein?

Y. S. Kim\* and Marilyn E. Noz†

*\*Department of Physics, University of Maryland,  
College Park, Maryland 20742, USA*

*†Department of Radiology, New York University,  
New York, New York 10016, USA*

**Abstract.** The present form of quantum mechanics is based on the Copenhagen school of interpretation. Einstein did not belong to the Copenhagen school, because he did not believe in probabilistic interpretation of fundamental physical laws. This is the reason why we are still debating whether there is a more deterministic theory. One cause of this separation between Einstein and the Copenhagen school could have been that the Copenhagen physicists thoroughly ignored Einstein's main concern: the principle of relativity. Paul A. M. Dirac was the first one to realize this problem. Indeed, from 1927 to 1963, Paul A. M. Dirac published at least four papers to study the problem of making the uncertainty relation consistent with Einstein's Lorentz covariance. It is interesting to combine those papers by Dirac to make the uncertainty relation consistent with relativity. It is shown that the mathematics of two coupled oscillators enables us to carry out this job. We are then led to the question of whether the concept of localized probability distribution is consistent with Lorentz covariance.

**Keywords:** lorentz invariance, time separation variable

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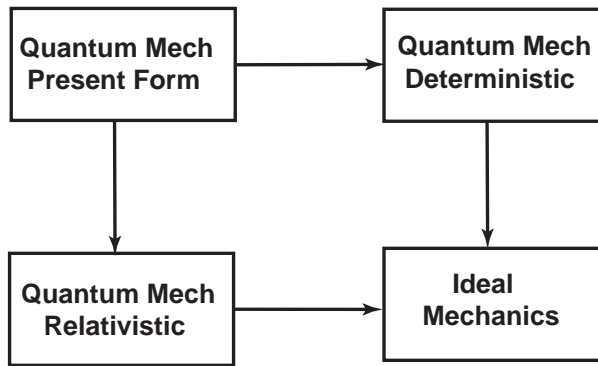
## INTRODUCTION

Einstein was against the Copenhagen interpretation of quantum mechanics. Why was he so against it? The present form of quantum mechanics is regarded as unsatisfactory because of its probabilistic interpretation. At the same time, it is unsatisfactory because it does not appear to be Lorentz-covariant. We still do not know how the hydrogen atom appears to a moving observer. Indeed, we have to go through two-track routes to reach the ideal mechanics, as illustrated in Fig. 1.

While relativity was Einstein's main domain of interest, why did he not complain about the lack of Lorentz covariance? It is possible that Einstein was too modest to mention relativity, and instead concentrated his complaint against its probabilistic interpretation. It is also possible that Einstein did not want to send his most valuable physics asset to a battle ground. We cannot find a definite answer to this question, but it is gratifying to note that the present authors are not the first ones to question whether the Copenhagen school of thought is consistent with the concept of relativity.

Paul A. M. Dirac was never completely happy with the Copenhagen interpretation of quantum mechanics, but he thought it was a necessary temporary step. In that case, he thought we should examine whether quantum mechanics is consistent with special relativity.

As for combining quantum mechanics with special relativity, there was a giant step



**FIGURE 1.** Toward ideal mechanics. The ideal mechanics should be both deterministic and relativistic. Einstein had enough reason to complain about the lack of Lorentz covariance in the Copenhagen interpretation, but he was reticent about it. Instead, Einstein concentrated his opposition on the probabilistic interpretation. Why?

of constructing the present form of quantum field theory. It leads to a Lorentz covariant S-matrix which enables us to calculate scattering amplitudes using Feynman diagrams. However, we cannot solve bound-state problems or localized probability distributions using Feynman diagrams [1]. We have to construct a separate theoretical device to address this issue, as illustrated in Fig. 2.

Dirac was never happy with the present form of field theory [2], particularly with infinite quantities in its renormalization processes. Furthermore, field theory never addresses the issue of localized probability. Indeed, Dirac concentrated his efforts in seeing whether localized probability distribution is consistent with Lorentz covariance.


In 1927 [3], Dirac noted that there is a time-energy uncertainty relation without time-like excitations. He pointed out that this space-time asymmetry causes a difficulty in combining quantum mechanics with special relativity.

In 1945 [4], Dirac constructed four-dimensional harmonic oscillator wave functions including the time variable. His oscillator wave functions took normalizable Gaussian form, but he did not attempt to give a physical interpretation to this mathematical device.

In 1949 [5], Dirac emphasized that the task of building a relativistic quantum mechanics is equivalent to constructing a representation of the Poincaré group. He then pointed out difficulties in constructing such a representation. He also introduced the light-cone coordinate system.

In 1963 [6], Dirac used two coupled oscillators to construct a representation of the  $O(3,2)$  deSitter group which later became the basic mathematical base for two-photon coherent states known as squeezed states of light [7].

In this report, we combine all of these works by Dirac to make the present form of uncertainty relations consistent with special relativity. Once this task is complete, we can start examining whether the probability interpretation is ultimately valid for quantum

Scattering	Bound States	Space/Time
COMETS	PLANETS	GALILEI
NEWTON		
	BOHR	
HEISENBERG		
FEYNMAN	STEP 1	EINSTEIN
STEP 2		

**FIGURE 2.** History of dynamical and kinematical developments. It is important to note that mankind's unified understanding of scattering and bound states has been very brief. It is therefore not unusual to expect that separate theoretical models be developed for scattering and for bound states. The successes and limitations of the Feynman diagram are well known. If we cannot build a covariant quantum mechanics, it is worthwhile to see whether we can construct a relativistic theory of bound states to supplement quantum field theory, as Step 1 before attempting to construct a Lorentz-covariant theory applicable to both in Step 2.

mechanics.

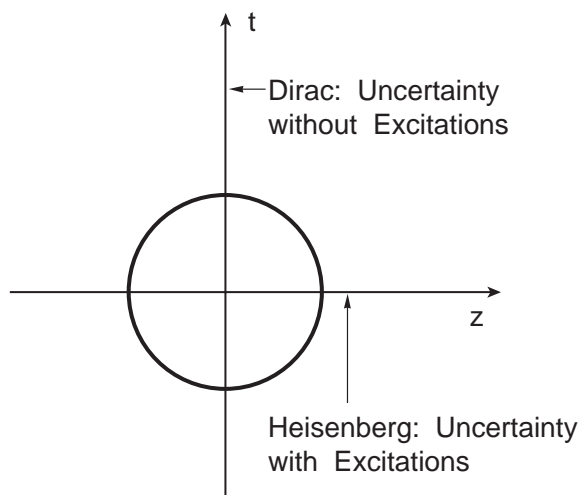
In Secs. 2 – 5, we examine each of the above-mentioned papers of Dirac. In Sec. 6, we combine these four papers into one paper using the language of coupled harmonic oscillators.

## **DIRAC'S C-NUMBER TIME-ENERGY UNCERTAINTY RELATION**

The time-energy uncertainty relation was known before 1927 from the transition time and line broadening in atomic spectroscopy. As soon as Heisenberg formulated his uncertainty, Dirac considered whether this uncertainty can be combined with the position momentum uncertainty to form a Lorentz covariant uncertainty relation [3].

He noted one major difficulty. There are excitations along the space-like longitudinal direction starting from the position-momentum uncertainty, while there are no excitations along the time-like direction. The time variable is a c-number. How then can this space-time asymmetry be made consistent with Lorentz covariance, where space and time coordinate are mixed up for moving observers.

On the other hand, Dirac forgot to consider Heisenberg's uncertainty relation is applicable to space separation variables. For instance, the Bohr radius measures the difference between the proton and electron. Dirac never addressed the question of separation in time variable or time interval even in his later papers.



**FIGURE 3.** Space-time picture of quantum mechanics. There are quantum excitations along the space-like longitudinal direction, but there are no excitations along the time-like direction. The time-energy relation is a c-number uncertainty relation.

As for the space-time asymmetry, Dirac came back to this question in his 1949 paper [5] where he discusses the “instant form” of relativistic dynamics. He talks about indirectly freezing the possibility of three of the six parameters parameters of the Lorentz group, and thus working only with three free parameters. This idea was presented earlier by Wigner [8, 9] who observed that the internal space-time symmetries of particles are dictated by his little groups with three independent parameters.

### DIRAC'S FOUR-DIMENSIONAL OSCILLATORS

During World War II, Dirac was looking into the possibility of constructing representations of the Lorentz group using harmonic oscillator wave functions [4]. The Lorentz group is the language of special relativity, and the present form of quantum mechanics starts with harmonic oscillators. Therefore, he was interested in making quantum mechanics Lorentz-covariant by constructing representations of the Lorentz group using harmonic oscillators.

In his 1945 paper [4], Dirac considers the Gaussian form

$$\exp \left\{ -\frac{1}{2} (x^2 + y^2 + z^2 + t^2) \right\}. \quad (1)$$

We note that this Gaussian form is in the  $(x, y, z, t)$  coordinate variables. Thus, if we consider a Lorentz boost along the  $z$  direction, we can drop the  $x$  and  $y$  variables, and write the above equation as

$$\exp \left\{ -\frac{1}{2} (z^2 + t^2) \right\}. \quad (2)$$

This is a strange expression for those who believe in Lorentz invariance where  $(z^2 - t^2)$  is an invariant quantity.

On the other hand, this expression is consistent with his earlier papers on the time-energy uncertainty relation [3]. In those papers, Dirac observed that there is a time-energy uncertainty relation, while there are no excitations along the time axis.

Let us look at Fig. 3 carefully. This figure is a pictorial representation of Dirac's Eq.(2), with localization in both space and time coordinates. Then Dirac's fundamental question would be how to make this figure covariant? This is where Dirac stops. However, this is not the end of the Dirac story.

## DIRAC'S LIGHT-CONE COORDINATE SYSTEM

In 1949, the Reviews of Modern Physics published a special issue to celebrate Einstein's 70th birthday. This issue contains Dirac paper entitled "Forms of Relativistic Dynamics" [5]. In this paper, he introduced his light-cone coordinate system, in which a Lorentz boost becomes a squeeze transformation.

When the system is boosted along the  $z$  direction, the transformation takes the form

$$\begin{pmatrix} z' \\ t' \end{pmatrix} = \begin{pmatrix} \cosh(\eta/2) & \sinh(\eta/2) \\ \sinh(\eta/2) & \cosh(\eta/2) \end{pmatrix} \begin{pmatrix} z \\ t \end{pmatrix}. \quad (3)$$

This is not a rotation, and people still feel strange about this form of transformation. In 1949 [5], Dirac introduced his light-cone variables defined as [5]

$$u = (z+t)/\sqrt{2}, \quad v = (z-t)/\sqrt{2}, \quad (4)$$

the boost transformation of Eq.(3) takes the form

$$u' = e^{\eta/2}u, \quad v' = e^{-\eta/2}v. \quad (5)$$

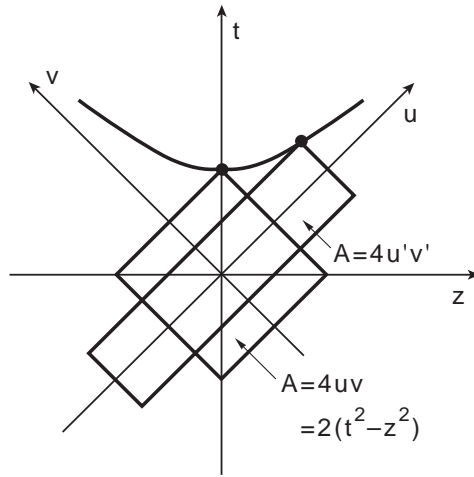
The  $u$  variable becomes expanded while the  $v$  variable becomes contracted, as is illustrated in Fig. 4. Their product

$$uv = \frac{1}{2}(z+t)(z-t) = \frac{1}{2}(z^2 - t^2) \quad (6)$$

remains invariant. In Dirac's picture, the Lorentz boost is a squeeze transformation.

If we combine Fig. 3 and Fig. 4, then we end up with Fig. 5. In mathematical formula, this transformation changes the Gaussian form of Eq.(2) into

$$\psi_{\eta}(z, t) = \left( \frac{1}{\pi} \right)^{1/2} \exp \left\{ -\frac{1}{4} [e^{-\eta}(z+t)^2 + e^{\eta}(z-t)^2] \right\}. \quad (7)$$



**FIGURE 4.** Lorentz boost in the light-cone coordinate system. The boost traces a point along the hyperbola. The boost also squeezes the square into a rectangle.

This formula together with Fig. 5 is known to describe all essential high-energy features observed in high-energy laboratories [10, 11, 9, 12].

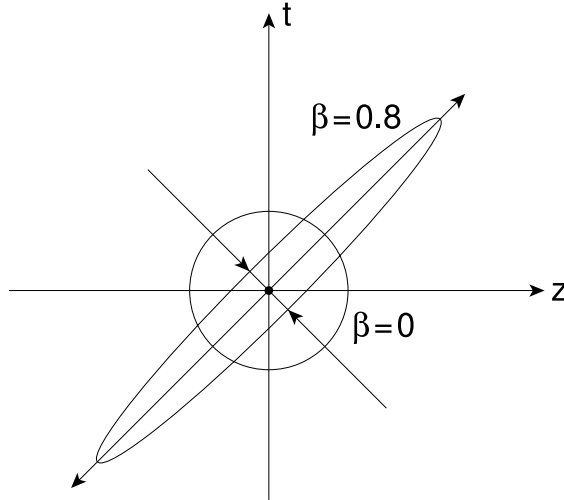
Indeed, this elliptic deformation explains one of the most controversial issues in high-energy physics. Hadrons are known to be bound states of quarks. Its bound-state quantum mechanics is assumed to be the same as that of the hydrogen atom. The question is how the hadron would look to an observer on a train. If the train moves with a speed close to that of light, the hadron appears like a collection of partons, according to Feynman [10]. Feynman's partons have properties quite different from those of the quarks. For instance, they interact incoherently with external signals. The elliptic deformation property described in Fig. 5 explains the quark and parton models are two different manifestations of the same covariant entity.

## DIRAC'S COUPLED OSCILLATORS

Dirac's interest in harmonic oscillators did not stop with his 1945 paper on the representations of the Lorentz group. In his 1963 [6] paper, he constructed a representation of the  $O(3, 2)$  deSitter group using two coupled harmonic oscillators. He starts with two sets of oscillator step-up and step-down operators. He then ends up with ten operators which act like the generators of of the  $O(3, 2)$  deSitter group. In so doing he constructed the scientific language of two-photon coherent states or squeezed states of light which became an important branch of physics 20 years later [7].

The  $O(3, 1)$  Lorentz group is a subgroup of  $O(3, 2)$ . Therefore, we are led to suspect that there is a symmetry of Lorentz group in two coupled harmonic oscillators. We are particularly interested in the Lorentz boost property shown in Sec. 4 and Fig. 4.

Let us see how these Lorentz-covariant properties are contained in Dirac's study of



**FIGURE 5.** Effect of the Lorentz boost on the space-time wave function. The circular space-time distribution in the rest frame becomes Lorentz-squeezed to become an elliptic distribution.

the Lorentz group using the two coupled oscillators. We start with a simple problem of two oscillators with equal mass. Then the Hamiltonian takes the form

$$H = \frac{1}{2} \left\{ \frac{1}{m} p_1^2 + \frac{1}{m} p_2^2 + A x_1^2 + A x_2^2 + 2C x_1 x_2 \right\}. \quad (8)$$

This Hamiltonian can be written as

$$H = \frac{1}{2m} \{ p_1^2 + p_2^2 \} + \frac{K}{4} \left\{ e^{-2\eta} (x_1 + x_2)^2 + e^{2\eta} (x_1 - x_2)^2 \right\}, \quad (9)$$

where

$$K = \sqrt{A^2 - C^2}, \quad \exp(2\eta) = \sqrt{\frac{A-C}{A+C}}. \quad (10)$$

The wave function then becomes [13]

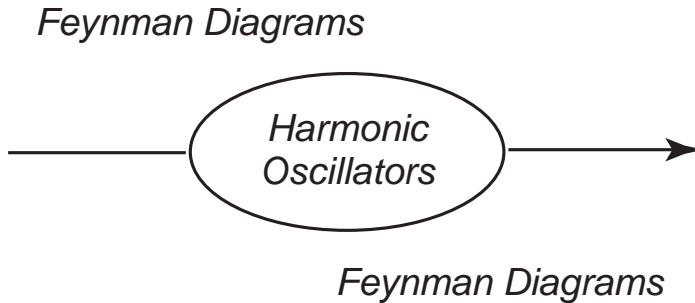
$$\psi_\eta(x_1, x_2) = \frac{1}{\sqrt{\pi}} \exp \left\{ -\frac{1}{4} \left[ e^{-\eta} (x_1 + x_2)^2 + e^{\eta} (x_1 - x_2)^2 \right] \right\}. \quad (11)$$

This expression is strikingly similar to the wave function given in Eq.(7). It becomes the same if we replace  $x_1$  and  $x_2$  by  $z$  and  $t$  respectively.

It is indeed remarkable that the Lorentz boost shares the same geometry as the coupled harmonic oscillators. It can be seen from the light-cone view of the Lorentz boost illustrated in Fig. 4, while the geometry of the coupled oscillator is basically that of squeezing a circle into ellipse.

## ONE MISSING COMPONENT IN DIRAC'S PAPERS

Quantum field theory has been quite successful in terms of Feynman diagrams based on the S-matrix formalism, but is useful only for physical processes where a set of free particles becomes another set of free particles after interaction. Quantum field theory does not address the question of localized probability distributions and their covariance under Lorentz transformations. In order to address this question, Feynman *et al.* suggested harmonic oscillators to tackle the problem [1]. Their idea is indicated in Fig. 6, and also in Fig. 2.



**FIGURE 6.** Feynman's roadmap for combining quantum mechanics with special relativity. Feynman diagrams work for running waves, and they provide a satisfactory resolution for scattering states in Einstein's world. For standing waves trapped inside an extended hadron, Feynman suggested harmonic oscillators as the first step.

In this report, we are concerned with quantum bound system, and we have examined the four-papers of Dirac on the question of making the uncertainty relations consistent with special relativity. Indeed, Dirac discussed this fundamental problem with mathematical devices which are both elegant and transparent.

Dirac of course noted that the time variable plays the essential role in the Lorentz-covariant world. On the other hand, he did not take into consideration the concept of time separation. When we talk about the hydrogen atom, we are concerned with the distance between the proton and electron. To a moving observer, there is also a time-separation between the two particles.

Instead of the hydrogen atom, we use these days the hadron consisting of two quarks bound together with an attractive force, and consider their space-time positions  $x_a$  and  $x_b$ , and use the variables [1]

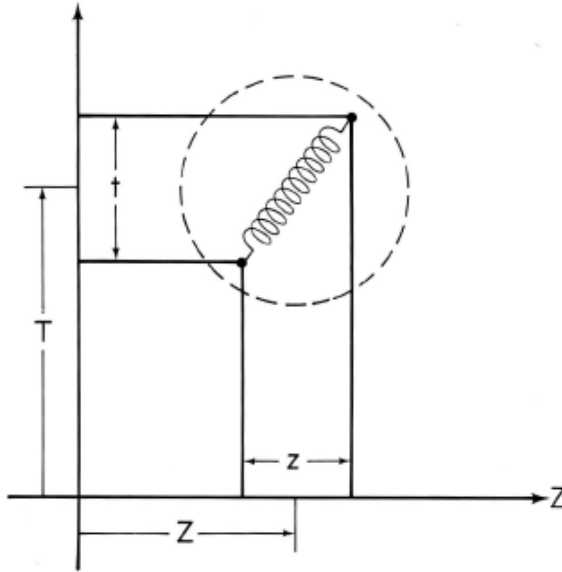
$$X = (x_a + x_b)/2, \quad x = (x_a - x_b)/2\sqrt{2}. \quad (12)$$

The four-vector  $X$  specifies where the hadron is located in space and time, while the variable  $x$  measures the space-time separation between the quarks. Let us call their time components  $T$  and  $t$  as illustrated in Fig. 7. These variables actively participate in Lorentz transformations. The existence of the  $T$  variable is known, but the Copenhagen school was not able to see the existence of this  $t$  variable.

Paul A. M. Dirac was concerned with time variable throughout his four papers discussed in this report. However, he did not make a distinction between the  $T$  and  $t$  vari-



ables. The  $T$  variable ranges from  $-\infty$  to  $+\infty$ , and is constantly increasing. On the other hand, the  $t$  variable is the time interval, and remains unchanged in a given Lorentz frame.



**FIGURE 7.** Space and time separations in the Lorentz-covariant world. Wherever there is a space-separation, there is a time-separation. Two simultaneous events separated by a distance are not simultaneous for moving observers.

Indeed, when Feynman *et al.* wrote down the Lorentz-invariant differential equation [1]

$$\frac{1}{2} \left\{ x_{\mu}^2 - \frac{\partial^2}{\partial x_{\mu}^2} \right\} \psi(x) = \lambda \psi(x), \quad (13)$$

$x_{\mu}$  was for the space-time separation between the quarks.

This four-dimensional differential equation has more than 200 forms of solutions depending on boundary conditions. However, there is only one set of solutions to which we can give a physical interpretation. Indeed, the Gaussian form of Eq.(1) is a solution of above differential equation. If we boost the system along the  $z$  direction, we can separate away the  $x$  and  $y$  components in the Gaussian form and write the wave function in the form of Eq.(2).

It is then possible to construct a representation of the Poincaré group from the solutions of the above differential equation [9]. If the system is boosted, the wave function becomes the Gaussian form given in Eq.(7), which becomes Eq.(2) if  $\eta$  becomes zero. This wave function is also a solution of the Lorentz-invariant differential equation of Eq.(13). The transition from Eq.(2) to Eq.(7) is illustrated in Fig. 5.

## CONCLUDING REMARKS

The easiest way to build a canal is to link up existing lakes. Paul A. M. Dirac indeed dug four big lakes. It is a pleasure to link them up. Dirac constructed those lakes in order to study whether the Copenhagen school of quantum mechanics can be made consistent with Einstein's Lorentz-covariant world.

After studying Dirac's papers, we arrived at the conclusion that the Copenhagen school completely forgot to take into account the question of simultaneity and time separation [14]. The question then is whether the localized probability distribution can be made consistent with Einstein's Lorentz covariance.

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