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CANNON-THURSTON MAPS FOR TREES OF HYPERBOLIC METRIC SPACES

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Abstract

Let (X,d) be a tree (T) of hyperbolic metric spaces satisfying the quasi-isometrically embedded condition. Let v be a vertex of T. Let (X_v, d_v) denote the hyperbolic metric space corresponding to v. Then $i : X_v \to X$ extends continuously to a map $\hat{i} : \widehat{X_v} \to \widehat{X}$. This generalizes and gives a new proof of a Theorem of Cannon and Thurston. The techniques are used to give a different proof of a result of Minsky: Thurston's ending lamination conjecture for certain Kleinian groups. Applications to graphs of hyperbolic groups and local connectivity of limit sets of Kleinian groups are also given.

1. Introduction

Let G be a hyperbolic group in the sense of Gromov [14]. Let H be a hyperbolic subgroup of G. We choose a finite symmetric generating set for H and extend it to a finite symmetric generating set for G. Let Γ_H and Γ_G denote the Cayley graphs of H, G respectively with respect to these generating sets. By adjoining the Gromov boundaries $\partial\Gamma_H$ and $\partial\Gamma_G$ to Γ_H and Γ_G, one obtains their compactifications $\widehat{\Gamma_H}$ and $\widehat{\Gamma_G}$ respectively.

We would like to understand the extrinsic geometry of H in G. Since the objects of study here come under the purview of coarse geometry, asymptotic or 'large-scale' information is of crucial importance. That is to say, one would like to know what happens 'at infinity'. We put this in the more general context of a hyperbolic group H acting freely and properly discontinuously by isometries on a proper hyperbolic metric

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space X. Then there is a natural map $i : \Gamma_H \to X$, sending the vertex set of Γ_H to the orbit of a point under H, and connecting images of adjacent vertices in Γ_H by geodesics in X. Let $\widehat{X}$ denote the Gromov compactification of X.

A natural question seems to be the following:

Question. Does the continuous proper map $i : \Gamma_H \to X$ extend to a continuous map $\hat{i} : \widehat{\Gamma_H} \to \widehat{X}$?

Questions along this line have been raised by Bonahon [4]. Related questions in the context of Kleinian groups have been studied by Cannon and Thurston [7], Bonahon [5], Floyd [10] and Minsky [21]. In, [7], [10] or [21], explicit metrics were used. So though some of their results can be thought of as 'coarse', the techniques of proof are not. In [25], coarse techniques were used to answer the above question affirmatively for $X = \Gamma_G$, where G is a hyperbolic group, and H a normal subgroup of G. In this paper, we cover examples arising from trees of hyperbolic metric spaces satisfying an extra technical condition introduced by Bestvina and Feighn in [2]: the *quasi-isometrically embedded* condition. [See Section 3 of this paper or [2] for definitions.] Much of this work was motivated by Cannon and Thurston's results [7]. In the case of a closed hyperbolic 3-manifold fibering over the circle, we obtain a different proof of Cannon and Thurston's result.

Definition. Let X and Y be hyperbolic metric spaces and $i : Y \to X$ be a proper embedding. A **Cannon-Thurston map** $\hat{i}$ from $\widehat{Y}$ to $\widehat{X}$ is a continuous extension of i. Such a continuous extension will occassionally be called a Cannon-Thurston map for the pair (Y, X). If $Y = \Gamma_H$ and $X = \Gamma_G$ for a hyperbolic subgroup H of a hyperbolic group G, a Cannon-Thurston map for (Γ_H, Γ_G) will occassionally be referred to as a Cannon-Thurston map for (H, G).

It is easy to see that such a continuous extension, if it exists, is unique.

The main theorem of this paper is :

Theorem 3.10. *Let (X,d) be a tree (T) of hyperbolic metric spaces satisfying the quasi-isometrically embedded condition. Let v be a vertex of T. Let (X_v, d_v) denote the hyperbolic metric space corresponding to v. If X is hyperbolic, there is a Cannon-Thurston map for (X_v, X).*

A direct consequence of Theorem 3.10 above is the following:

Corollary 3.11. *Let G be a hyperbolic group acting cocompactly on a simplicial tree T such that all vertex and edge stabilizers are hyperbolic. Also suppose that every inclusion of an edge stabilizer in a vertex stabilizer is a quasi-isometric embedding. Let H be the stabilizer of a vertex or edge of T. Then there exists a Cannon-Thurston map for* (H, G)

In [2], Bestvina and Feighn give sufficient conditions for a graph of hyperbolic groups to be hyperbolic. Vertex and edge subgroups are thus natural examples of hyperbolic subgroups of hyperbolic groups. Essentially all previously known examples of non-quasiconvex hyperbolic subgroups of hyperbolic groups arise this way. Theorem 3.10 shows that these have Cannon-Thurston maps.

Another consequence of Theorem 3.10 above is:

Theorem 4.7. *Let Γ be a freely indecomposable Kleinian group, such that $\mathbb{H}^3/\Gamma = M$ has injectivity radius uniformly bounded below by some $\epsilon > 0$. Then there exists a continuous map from the Gromov boundary of Γ (regarded as an abstract group) to the limit set of Γ in* $\mathbb{S}^2_\infty$.

A different proof of Theorem 4.7 is given by Klarreich [17], where other examples of maps between boundaries of hyperbolic metric spaces are described.

After some further work and using a theorem of Minsky [22], we are able to give a different proof of another result of Minsky [21] : Thurston's Ending Lamination Conjecture for geometrically tame manifolds with freely indecomposable fundamental group and a uniform lower bound on injectivity radius.

Theorem 4.15 [21]. *Let N_1 and N_2 be homeomorphic hyperbolic 3-manifolds with freely indecomposable fundamental group. Suppose there exists a uniform lower bound $\epsilon > 0$ on the injectivity radii of N_1 and N_2. If the end invariants of corresponding ends of N_1 and N_2 are equal, then N_1 and N_2 are isometric.*

In Section 5, we describe examples where existence of a Cannon-Thurston map is not known. Further, certain examples of Minsky [23] are shown to answer a question of Gromov [15].

2. Preliminaries

We start off with some preliminaries about hyperbolic metric spaces in the sense of Gromov [14]. For details, see [8], [12]. Let (X,d) be a hyperbolic metric space. The **Gromov boundary** of X, denoted by ∂X, is the collection of equivalence classes of geodesic rays $r : [0,\infty) \to \Gamma$ with $r(0) = x_0$ for some fixed $x_0 \in X$, where rays r_1 and r_2 are equivalent if $sup\{d(r_1(t), r_2(t))\} < \infty$. Let $\widehat{X}=X \cup \partial X$ denote the natural compactification of X topologized the usual way(cf.[12] pg. 124).

The **Gromov inner product** of elements a and b relative to c is defined by

$$(a,b)_c=1/2[d(a,c) + d(b,c) - d(a,b)].$$

Definitions. A subset Z of X is said to be k**-quasiconvex** if any geodesic joining $a,b \in Z$ lies in a k-neighborhood of Z. A subset Z is **quasiconvex** if it is k-quasiconvex for some k. A map f from one metric space (Y,d_Y) into another metric space (Z,d_Z) is said to be a (K,ϵ)**-quasi-isometric embedding** if

$$\frac{1}{K}(d_Y(y_1,y_2)) - \epsilon \leq d_Z(f(y_1),f(y_2)) \leq Kd_Y(y_1,y_2) + \epsilon,$$

If f is a quasi-isometric embedding, and every point of Z lies at a uniformly bounded distance from some $f(y)$, then f is said to be a **quasi-isometry**. A (K,ϵ)-quasi-isometric embedding that is a quasi-isometry will be called a (K,ϵ)-quasi-isometry.

A (K,ϵ)**-quasigeodesic** is a (K,ϵ)-quasi-isometric embedding of a closed interval in $\mathbb{R}$. A $(K,0)$-quasigeodesic will also be called a K-quasigeodesic.

Let (X,d_X) be a hyperbolic metric space, and Y be a subspace that is hyperbolic with the inherited path metric d_Y. By adjoining the Gromov boundaries ∂X and ∂Y to X and Y, one obtains their compactifications $\widehat{X}$ and $\widehat{Y}$ respectively.

Let $i : Y \to X$ denote inclusion.

Definition. Let X and Y be hyperbolic metric spaces, and $i : Y \to X$ be a proper embedding. A **Cannon-Thurston map** $\hat{i}$ from $\widehat{Y}$ to $\widehat{X}$ is a continuous extension of i.

The following lemma says that a Cannon-Thurston map exists if for all $M > 0$ and $y \in Y$, there exists $N > 0$ such that if λ lies outside an N ball around y in Y then any geodesic in X joining the end-points of λ lies outside the M ball around $i(y)$ in X. For convenience of use later on, we state this somewhat differently. The proof is similar to that of Lemma 2.1 of [25].

Lemma 2.1. *A Cannon-Thurston map from $\widehat{Y}$ to $\widehat{X}$ exists if the following condition is satisfied:*

Given $y_0 \in Y$, there exists a non-negative function $M(N)$, such that $M(N) \to \infty$ as $N \to \infty$ and for all geodesic segments λ in Y lying outside an N-ball around $y_0 \in Y$ any geodesic segment in X joining the end-points of $i(\lambda)$ lies outside the $M(N)$-ball around $i(y_0) \in X$.

Proof. Suppose $i : Y \to X$ does not extend continuously . Since i is proper, there exist sequences x_m, $y_m \in Y$ and $p \in \partial Y$, such that $x_m \to p$ and $y_m \to p$ in $\widehat{Y}$, but $i(x_m) \to u$ and $i(y_m) \to v$ in $\widehat{X}$, where $u, v \in \partial X$ and $u \neq v$.

Since $x_m \to p$ and $y_m \to p$, any geodesic in Y joining x_m and y_m lies outside an N_m-ball $y_0 \in Y$, where $N_m \to \infty$ as $m \to \infty$. Any bi-infinite geodesic in X joining $u, v \in \partial X$ has to pass through some M-ball around $i(y_0)$ in X as $u \neq v$. There exist constants c and L such that for all $m > L$ any geodesic joining $i(x_m)$ and $i(y_m)$ in X passes through an $(M + c)$-neighborhood of $i(y_0)$. Since $(M + c)$ is a constant not depending on the index m this proves the lemma. q.e.d.

The above result can be interpreted as saying that a Cannon-Thurston map exists if the space of geodesic segments in Y embeds properly in the space of geodesic segments in X.

3. Trees of hyperbolic metric spaces

We start with a notion closely related to one introduced in [2].

Definition. A tree (T) of hyperbolic metric spaces satisfying the q(uasi) i(sometrically) embedded condition is a path metric space (X, d) admitting a map $P : X \to T$ onto a simplicial tree T, such that there exist δ,ϵ and $K > 0$ satisfying the following:

1. For all vertices $v \in T$, $X_v = P^{-1}(v) \subset X$ is path connected and a rectifiable subset of X. Equipped with the induced path metric d_v, X_v is a δ-hyperbolic metric space. Further, the inclusions $i_v : X_v \to X$ are uniformly proper, i.e., for all $M > 0$, $v \in T$ and $x, y \in X_v$, there exists $N > 0$ such that $d(i_v(x), i_v(y)) \leq M$ implies $d_v(x, y) \leq N$.

2. Let e be an edge of T with initial and final vertices v_1 and v_2 respectively. Let X_e be the pre-image under P of the mid-point

of e. Then X_e is path connected and a rectifiable subset of X. Equipped with the induced path metric d_e, X_e is δ-hyperbolic.

3. There exist maps $f_e : X_e{\times}[0,1] \to X$, such that $f_e|_{X_e\times(0,1)}$ is an isometry onto the pre-image of the interior of e equipped with the path metric.

4. $f_e|_{X_e\times\{0\}}$ and $f_e|_{X_e\times\{1\}}$ are (K,ϵ)-quasi-isometric embeddings into X_{v_1} and X_{v_2} respectively. $f_e|_{X_e\times\{0\}}$ and $f_e|_{X_e\times\{1\}}$ will occassionally be referred to as f_{v_1} and f_{v_2} respectively.

d_v and d_e will denote path metrics on X_v and X_e respectively. i_v, i_e will denote inclusion of X_v, X_e respectively into X.

The main theorem of this section can now be stated:

Theorem 3.10. *Let (X,d) be a tree (T) of hyperbolic metric spaces satisfying the qi-embedded condition. Let v be a vertex of T. If X is hyperbolic, there exists a Cannon-Thurston map for (X_v, X).*

Some aspects of the proof of the main theorem of this section are similar to the proof of the main theorem of [25]. Given a geodesic segment $\lambda \subset X_v$, we construct a quasi-convex set $B_\lambda \subset X$ containing λ. It follows from the construction that if λ lies outside a large ball around $y_0 \in X_v$, B_λ lies outside a large ball around $i_v(y_0) \in X$, i.e., for all $M \geq 0$ there exists $N \geq 0$ such that if λ lies oustside the N-ball around $y_0 \in X_v$, B_λ lies outside the M-ball around $i_v(y_0) \in X$. Combining this with Lemma 2.1 above, the proof of Theorem 3.10 is completed.

For convenience of exposition, T shall be assumed to be rooted, i.e., equipped with a base vertex v_0. Since this choice is arbitrary, we can choose X_{v_0} to be the vertex space for which we want to construct a Cannon-Thurston map. Let $v \neq v_0$ be a vertex of T. Let v_- be the penultimate vertex on the geodesic edge path from v_0 to v. Let e denote the directed edge from v_- to v. Define

$$\phi_v : f_e(X_e{\times}\{0\}) \to f_e(X_e{\times}\{1\})$$

as follows: If $p{\in}f_e(X_e{\times}\{0\}){\subset}X_{v_-}$, choose $x \in X_e$ such that

$$p = f_e(x{\times}\{0\})$$

and define

$$\phi_v(p) = f_e(x{\times}\{1\}).$$

Note that in the above definition, x is chosen from a set of bounded diameter.

Let μ be a geodesic in X_{v_-}, joining $a, b \in f_e(X_e \times \{0\})$. $\Phi_v(\mu)$ will denote a geodesic in X_v joining $\phi_v(a)$ and $\phi_v(b)$. Let $X_{v_0} = Y$.

For convenience of exposition, we shall modify X, X_v, X_e by quasi-isometric perturbations. Given a complete metric space (Z, d) , choose a maximal disjoint collection $\{N_1(z_\alpha)\}$ of disjoint 1-balls. Then by maximality, for all $z \in Z$ there exists z_α in the collection such that $d(z, z_\alpha) < 2$. Construct a graph Z_1 with vertex set $\{z_\alpha\}$ and edge set consisting of distinct vertices z_α, z_β such that $d(z_\alpha, z_\beta) \leq 4$. Assigning length one to each edge, Z_1 equipped with the path-metric is quasi-isometric to (Z, d). All metric spaces in this section will henceforth be assumed to be graphs of edge length 1, and maps between them will be assumed to be cellular.

We start with a general lemma about hyperbolic metric spaces. This follows easily from the fact that local quasigeodesics in a hyperbolic metric space are quasigeodesics [12]. If x, y are points in a hyperbolic metric space, $[x, y]$ will denote a geodesic joining them.

Lemma 3.1. *Given $\delta > 0$, there exist D, C_1 such that if a, b, c, d are vertices of a δ-hyperbolic metric space (Z, d), with $d(a, [b, c]) = d(a, b)$, $d(d, [b, c]) = d(c, d)$ and $d(b, c) \geq D$, then $[a, b] \cup [b, c] \cup [c, d]$ lies in a C_1-neighborhood of any geodesic joining a, d.*

Given a geodesic segment $\lambda \subset Y$, we now construct a quasi-convex set $B_\lambda \subset X$ containing λ.

Construction of B_λ

Choose $C_2 \geq 0$ such that for all $e \in T$, $f_e(X_e \times \{0\})$ and $f_e(X_e \times \{1\})$ are C_2-quasiconvex in the appropriate vertex spaces. Let $C = C_1 + C_2$, where C_1 is as in Lemma 3.1.

For $Z \subset X_v$, let $N_C(Z)$ denote the C-neighborhood of Z, that is the set of points at distance less than or equal to C from Z.

<u>*Step 1.*</u> Let $\mu \subset X_v$ be a geodesic segment in (X_v, d_v). Then $P(\mu) = v$. For each edge e incident on v, but not lying on the geodesic (in T) from v_0 to v, choose p_e, $q_e \in N_C(\mu) \cap f_v(X_e)$ such that $d_v(p_e, q_e)$ is maximal. Let $v_1, \cdots, v_n$ be terminal vertices of edges e_i for which $d_v(p_{e_i}, q_{e_i}) > D$, where D is as in Lemma 3.1 above. Observe that there are only finitely many v_i's as μ is finite. Define

$$B^1(\mu) = i_v(\mu) \cup \bigcup_{k=1\cdots n} \Phi_{v_i}(\mu_i),$$

where μ_i is a geodesic in X_v joining p_{e_i}, q_{e_i}.

Note that the convex hull of $P(B^1(\mu)) \subset T$ is a finite tree.

The reason for insisting that the edges e do not lie on the geodesic from v_0 to v is to prevent 'backtracking' in Step 2 below.

Step 2. Step 1 above constructs $B^1(\lambda)$ in particular. We proceed inductively. Suppose that $B^m(\lambda)$ has been constructed such that the convex hull of $P(B^m(\lambda)) \subset T$ is a finite tree. Let $\{w_1, \cdots, w_n\} = P(B^m(\lambda)) \backslash P(B^{m-1}(\lambda))$. (Note that n may depend on m, but we avoid repeated indices for notational convenience.) Assume further that

$$P^{-1}(v_k) \cap B^m(\lambda)$$

is a path of the form $i_{v_k}(\lambda_k)$, where λ_k is a geodesic in (X_{v_k}, d_{v_k}). Define

$$B^{m+1}(\lambda) = B^m(\lambda) \cup \bigcup_{k=1\cdots n} (B^1(\lambda_k)),$$

where $B^1(\lambda_k)$ is defined in Step 1 above.

Since each λ_k is a finite geodesic segment in Γ_H, the convex hull of $P(B^{m+1}\lambda)$ is a finite subtree of T. Further, $P^{-1}(v) \cap B^{m+1}(\lambda)$ is of the form $i_v(\lambda_v)$ for all $v \in P(B^{m+1}(\lambda))$. This enables us to continue inductively. Define

$$B_\lambda = \cup_{m \geq 0} B^m \lambda.$$

Note finally that the convex hull of $P(B_\lambda)$ in T is a locally finite tree T_1.

Quasiconvexity of B_λ

We shall now show that there exists $C' \geq 0$ such that for every geodesic segment $\lambda \subset Y$, $B_\lambda \subset X$ is C'-quasiconvex. To do this we construct a retraction Π_λ from (the vertex set of) X onto B_λ and show that there exists $C_0 \geq 0$ such that $d(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C_0 d(x,y)$. Let $\pi_v : X_v \to \lambda_v$ be a nearest point projection of X_v onto λ_v. Π_λ is defined on $\bigcup_{v \in T_1} X_v$ by

$$\Pi_\lambda(x) = i_v \cdot \pi_v(x) \qquad \text{for } x \in X_v.$$

If $x \in P^{-1}(T \setminus T_1)$, choose $x_1 \in P^{-1}(T_1)$ such that $d(x, x_1) = d(x, P^{-1}(T_1))$ and define $\Pi'_\lambda(x) = x_1$. Next define $\Pi_\lambda(x) = \Pi_\lambda \cdot \Pi'_\lambda(x)$.

The following Lemma says nearest point projections in a δ-hyperbolic metric space do not increase distances much.

Lemma 3.2. *Let (Y, d) be a δ-hyperbolic metric space and let $\mu \subset Y$ be a geodesic segment. Let $\pi : Y \to \mu$ map $y \in Y$ to a point on μ nearest to y. Then $d(\pi(x), \pi(y)) \leq C_3 d(x, y)$ for all $x, y \in Y$ where C_3 depends only on δ.*

Proof. Let $[a, b]$ denote a geodesic edge-path joining vertices a, b. Recall that the Gromov inner product

$$(a, b)_c = \tfrac{1}{2}[d(a, c) + d(b, c) - d(a, b)].$$

It suffices by repeated use of the triangle inequality to prove the Lemma when $d(x, y) = 1$. Let u, v, w be points on $[x, \pi(x)]$, $[\pi(x), \pi(y)]$ and $[\pi(y), x]$ respectively such that

$$\begin{aligned} d(u, \pi(x)) &= d(v, \pi(x)), \\ d(v, \pi(y)) &= d(w, \pi(y)) \quad \text{and} \\ d(w, x) &= d(u, x). \end{aligned}$$

Then $(x, \pi(y))_{\pi(x)} = d(u, \pi(x))$. Also, since Y is δ-hyperbolic, the diameter of the inscribed triangle with vertices u, v, w is less than or equal to 2δ (See [35]). Thus

$$\begin{aligned} d(u, x) + d(u, v) &\geq d(x, \pi(x)) = d(u, x) + d(u, \pi(x)) \\ \Rightarrow d(u, \pi(x)) &\leq d(u, v) \leq 2\delta \\ \Rightarrow (x, \pi(y))_{\pi(x)} &\leq 2\delta. \end{aligned}$$

Similarly, $(y, \pi(x))_{\pi(y)} \leq 2\delta$.

$$\begin{aligned} &\text{i.e., } d(x, \pi(x)) + d(\pi(x), \pi(y)) - d(x, \pi(y)) \leq 4\delta, \\ &\text{and } d(y, \pi(y)) + d(\pi(x), \pi(y)) - d(y, \pi(x)) \leq 4\delta. \end{aligned}$$

Therefore,

$$\begin{aligned} &2d(\pi(x), \pi(y)) \\ &\leq 8\delta + d(x, \pi(y)) - d(y, \pi(y)) + d(y, \pi(x)) - d(x, \pi(x)) \\ &\leq 8\delta + d(x, y) + d(x, y) \\ &\leq 8\delta + 2, \end{aligned}$$

which gives $d(\pi(x), \pi(y)) \leq 4\delta + 1$. Choosing $C_3 = 4\delta + 1$, we are through.

q.e.d.

Lemma 3.3. *Let (Y, d) be a δ-hyperbolic metric space. Let μ be a geodesic segment in Y with end-points a, b and let x be any vertex in Y. Let y be a vertex on μ such that $d(x, y) \leq d(x, z)$ for any $z \in \mu$. Then a geodesic path from x to y followed by a geodesic path from y to z is a k-quasigeodesic for some k dependent only on δ.*

Proof. As in Lemma 3.2, let u, v, w be points on edges $[x, y]$, $[y, z]$ and $[z, x]$ respectively such that $d(u, y) = d(v, y)$, $d(v, z) = d(w, z)$ and $d(w, x) = d(u, x)$. Then $d(u, y) = (z, x)_y \leq 2\delta$ and the inscribed triangle with vertices u, v, w has diameter less than or equal to 2δ (See [35]). $[x, y] \cup [y, z]$ is a union of 2 geodesic paths lying in a 4δ neighborhood of a geodesic $[x, z]$. Hence a geodesic path from x to y followed by a geodesic path from y to z is a $k-$quasigeodesic for some k dependent only on δ. q.e.d.

Lemma 3.4. *Suppose (Y, d) is a δ-hyperbolic metric space. If μ is a (k_0, ϵ_0)-quasigeodesic in Y and p, q, r are 3 points in order on μ, then $(p, r)_q \leq k_1$ for some k_1 dependent on k_0, ϵ_0 and δ only.*

Proof. $[a, b]$ will denote a geodesic path joining a, b. Since p, q, r are 3 points in order on μ, $[p, q]$ followed by $[q, r]$ is a (k_0, ϵ_0)-quasigeodesic in the δ-hyperbolic metric space Y. Hence there exists a k_1 dependent on k_0, ϵ_0 and δ alone such that $d(q, [p, r]) \leq k_1$. Let s be a point on $[p, r]$ such that $d(q, s) = d(q, [p, r]) \leq k_1$. Then

$$\begin{aligned} (p, r)_q &= \tfrac{1}{2}(d(p, q) + d(r, q) - d(p, r)) \\ &= \tfrac{1}{2}(d(p, q) + d(r, q) - d(p, s) - d(r, s)) \\ &\leq d(q, s) \leq k_1. \qquad \text{q.e.d.} \end{aligned}$$

The following Lemma says that nearest point projections and quasi-isometries in hyperbolic metric spaces 'almost commute'. (See also [29], [30].)

Lemma 3.5. *Suppose (Y, d) is δ-hyperbolic. Let μ_1 be some geodesic segment in Y joining a, b and let p be any vertex of Y. Also let q be a vertex on μ_1 such that $d(p, q) \leq d(p, x)$ for $x \in \mu_1$. Let ϕ be a (K, ϵ) - quasiisometry from Y to itself. Let μ_2 be a geodesic segment in Y joining $\phi(a)$ to $\phi(b)$. Let r be a point on μ_2 such that $d(\phi(p), r) \leq d(\phi(p), x)$ for $x \in \mu_2$. Then $d(r, \phi(q)) \leq C_4$ for some constant C_4 depending only on K, ϵ and δ.*

Proof. Since $\phi(\mu_1)$ is a $(K, \epsilon)-$ quasigeodesic joining $\phi(a)$ to $\phi(b)$, it lies in a K'-neighborhood of μ_2 where K' depends only on K, ϵ, δ. Let

u be a point in $\phi(\mu_1)$ lying at a distance at most K' from r. Without loss of generality suppose that u lies on $\phi([q,b])$, where $[q,b]$ denotes the geodesic subsegment of μ_1 joining q,b. [See Figure 1 below.]

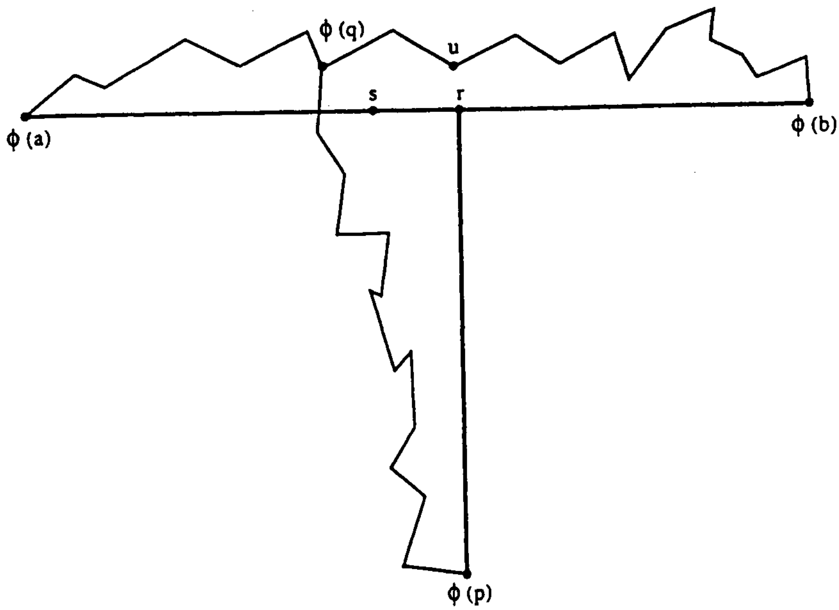

FIGURE 1

Let $[p,q]$ denote a geodesic joining p,q. From Lemma 3.3 $[p,q]\cup[q,b]$ is a k-quasigeodesic, where k depends on δ alone. Therefore $\phi([p,q])\cup\phi([q,b])$ is a (K_0,ϵ_0)-quasigeodesic, where K_0,ϵ_0 depend on K,k,ϵ. Hence, by Lemma 3.4 $(\phi(p),u)_{\phi(q)}\leq K_1$, where K_1 depends on K, k, ϵ and δ alone. Thus,

$$\begin{aligned}
&(\phi(p),r)_{\phi(q)}\\
&= \tfrac{1}{2}[d(\phi(p),\phi(q))+d(r,\phi(q))-d(r,\phi(p))]\\
&\leq \tfrac{1}{2}[d(\phi(p),\phi(q))+d(u,\phi(q))+d(r,u)\\
&\qquad -d(u,\phi(p))+d(r,u)]\\
&= (\phi(p),u)_{\phi(q)}+d(r,u)\\
&\leq K_1+K'.
\end{aligned}$$

There exists $s\in\mu_2$ such that $d(s,\phi(q))\leq K'$ so that

$$\begin{aligned}
(\phi(p),r)_s &= \tfrac{1}{2}[d(\phi(p),s)+d(r,s)-d(r,\phi(p))] \\
&\leq \tfrac{1}{2}[d(\phi(p),\phi(q))+d(r,\phi(q))-d(r,\phi(p))]+K' \\
&= (\phi(p),r)_{\phi(q)}+K' \\
&\leq K_1+K'+K' \\
&= K_1+2K'.
\end{aligned}$$

Also, as in the proof of Lemma 3.2 $(\phi(p),s)_r \leq 2\delta$. Thus

$$\begin{aligned}
d(r,s) &= (\phi(p),s)_r+(\phi(p),r)_s \\
&\leq K_1+2K'+2\delta \\
d(r,\phi(q)) &\leq K_1+2K'+2\delta+d(s,\phi(q)) \\
&\leq K_1+2K'+2\delta+K'.
\end{aligned}$$

Let $C_4=K_1+3K'+2\delta$. Then $d(r,\phi(q)) \leq C_4$, and C_4 is independent of a,b,p. q.e.d.

Let C_1, D be as in Lemma 3.1. Recall that each $f_v(X_e)$ is C_2-quasiconvex and $C = C_1 + C_2$. $[x,y]$ will denote a gedoesic joining x,y.

Lemma 3.6. *Let $\mu_1 = [a,b] \subset X_v$ be a geodesic and let e be an edge of T incident on v. Let $p,q \in N_C(\mu_1) \cap f_v(X_e)$ be such that $d_v(p,q)$ is maximal. Let μ_2 be a geodesic in X_v joining p,q. If $r \in N_C(\mu_1) \cap f_v(X_e)$, then $d_v(r,\mu_2) \leq D_1$ for some constant D_1 depending only on C,D,δ.*

Proof. Let π denote a nearest point projection onto μ_1. Since μ_2 and $[\pi(p),\pi(q)] \subset \mu_1$ are geodesics whose end-points lie at distance at most C apart, there exists C' such that $[\pi(p),\pi(q)] \subset N_{C'}(\mu_2)$. If $\pi(r) \in [\pi(p),\pi(q)]$, then

$$d(r,\mu_2) \leq C + C'.$$

If $\pi(r) \notin [\pi(p),\pi(q)]$, then without loss of generality, assume $\pi(r) \in [a,\pi(p)] \subset [a,\pi(q)]$. Thus

$$\begin{aligned}
d(p,q) &\geq d(r,q)\\
&\geq d(\pi(r),\pi(q)) - 2C\\
&= d(\pi(r),\pi(p)) + d(\pi(p),\pi(q)) - 2C\\
&\geq d(\pi(r),\pi(p)) + d(p,q) - 4C\\
\Rightarrow\ d(\pi(r),\pi(p)) &\leq 4C\\
\Rightarrow\ d(r,p) &\leq 6C\\
\Rightarrow\ d(r,\mu_2) &\leq 6C.
\end{aligned}$$

Choosing $D_1 = max\{C + C', 6C\}$, we are through. q.e.d.

Lemma 3.7. *Let μ_1, μ_2 be as in Lemma 3.6 above. Let π_i denote nearest point projections onto μ_i $(i = 1,2)$. If $p \in f_v(X_e)$, then $d(\pi_1(p), \pi_2(p)) \leq C_6$ for some constant C_6 depending on δ alone.*

Proof. If $d(\pi_1(p), \pi_1{\cdot}\pi_2(p)) \leq D$, then $d(\pi_1(p), \pi_2(p)) \leq C + D$.
If not, there exists $r \in f_v(X_e)$ such that $d(r, \pi_1(p)) \leq C$, by Lemma 3.1.

Thus, by Lemma 3.6 above, there exists $s \in \mu_2$ such that

$$d(s, \pi_1(p)) \leq C + D_1.$$

As in the proof of Lemma 3.2, $(p,s)_{\pi_2(p)} \leq 2\delta$. Hence,

$$(p, \pi_1(p))_{\pi_2(p)} \leq 2\delta + C + D_1.$$

Similarly, $(p, \pi_1{\cdot}\pi_2(p))_{\pi_1(p)} \leq 2\delta$. Thus, $(p, \pi_2(p))_{\pi_1(p)} \leq 2\delta + C$.
Therefore,

$$d(\pi_1(p), \pi_2(p)) \leq (p, \pi_1{\cdot}\pi_2(p))_{\pi_1(p)} + (p, \pi_2(p))_{\pi_1(p)} \leq 4\delta + 2C + D_1.$$

Choosing $C_6 = 4\delta + 2C + D_1$ we are through. q.e.d.

d_T will denote the metric on T. We are now in a position to prove:

Theorem 3.8. *There exists $C_0 \geq 0$ such that*

$$d(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C_0 d(x,y)$$

for x, y vertices of X.

Proof. It suffices to prove the theorem when $d(x, y) = 1$.

Case (a). $x, y \in P^{-1}(v)$ for some $v \in T_1$. From Lemma 3.2, there exists C_3 such that $d_v(\pi_v \cdot i_v^{-1}(x), \pi_v \cdot i_v^{-1}(y)) \leq C_3$. Since embeddings of X_v in X are cellular, $d(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C_3$.

Case (b). $x \in P^{-1}(w)$ and $y \in P^{-1}(v)$ for some $v, w \in T_1$.

Since $d(x, y) = 1$, v and w are adjacent in T_1. Assume, without loss of generality, $w = v_-$.

Recall that

$$B_\lambda \cap P^{-1}(v) = i_v(\lambda_v),$$
$$B_\lambda \cap P^{-1}(w) = i_w(\lambda_w).$$

Also, $\lambda_v = \Phi_v(\mu_w)$, for some geodesic μ_w contained in X_w, such that end-points of μ_w lie in a C-neighborhood of λ_w.

Let $z \in X_w$ denote a nearest point projection of $i_w^{-1}(x)$ onto μ_w. Then, by Lemma 3.5,

$$d(i_w(z), \Pi_\lambda \cdot \phi_v(x)) \leq d(i_w(z), \phi_v \cdot i_w(z)) + d(\phi_v \cdot i_w(z), \Pi_\lambda \cdot \phi_v(x)) \leq 1 + C_4.$$

Since, $d(x, y) = 1 = d(x, \phi_v(x))$ and i_v's are uniformly proper embeddings, there exists $C_5 > 0$ such that $d_v(\phi_v(x), y) \leq C_5$ and $d(\Pi_\lambda(\phi_v(x)), \Pi_\lambda(y)) \leq C_3 C_5$.

Since the end-points of μ_w lie in a C-neighborhood of λ_w, there exists C_6 from Lemma 3.7, depending on δ and C such that $d(z, \Pi_\lambda(x)) \leq C_6$.

Finally, by the triangle inequality,

$$d(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C_6 + 1 + C_4 + C_3 C_5 = C_7(say).$$

Case (c). $P([x, y])$ is not contained in T_1.

Since $d(x, y) = 1$, $P(x)$ and $P(y)$ belong to the closure T_2 of the same component of $T \backslash T_1$. Then $P \cdot \Pi'_\lambda(x) = P \cdot \Pi'_\lambda(y) = v$ for some $v \in T$.

Also $d(\Pi_\lambda(x), \Pi_\lambda(y)) = d(\Pi_\lambda \cdot \Pi'_\lambda(x), \Pi_\lambda \cdot \Pi'_\lambda(y))$.

Let $x_1 = \Pi'_\lambda(x)$ and $y_1 = \Pi'_\lambda(y)$.

Let D and C_1 be as in Lemma 3.1. If $d(\Pi_\lambda(x_1), \Pi_\lambda(y_1)) \geq D$, let

$$\begin{aligned} {i_v}^{-1}(x_1) &= u_1, \\ {i_v}^{-1}(\Pi_\lambda(x_1)) &= u_2, \\ {i_v}^{-1}(y_1) &= v_1, \\ {i_v}^{-1}(\Pi_\lambda(y_1)) &= v_2. \end{aligned}$$

Then by Lemma 3.1 $[u_1, u_2] \cup [u_2, v_2] \cup [v_2, v_1]$ is a quasigeodesic lying in a C_1-neighborhood of $[u_1, v_1]$.

Also, $x_1, y_1 \in i_v(X_v)$. Since the image of an edge space in a vertex space is C_2-quasiconvex, there exist $e \in T$ and $x_2, y_2 \in f_e(X_e \times \{0\})$ such that $d(x_2, \Pi_\lambda(x_1)) \leq C_1 + C_2 = C$ and $d(y_2, \Pi_\lambda(y_1)) \leq C_1 + C_2 = C$.

By construction $d(\Pi_\lambda(x_2), \Pi_\lambda(y_2)) \leq D$. (Else the edge $P([x, y])$ of T would be in T_1.) Therefore,

$$\begin{aligned} d(\Pi_\lambda(x), \Pi_\lambda(y)) &= d(\Pi_\lambda(x_1), \Pi_\lambda(y_1)) \\ &\leq 2C + D + 2C \\ &= 4C + D. \end{aligned}$$

Choosing $C_0 = \max \{C_3, C_7, 4C + D\}$, we are through. q.e.d.

To complete the proof of our main Theorem, we need a final Lemma.

Lemma 3.9. *There exists $A > 0$, such that if $a \in P^{-1}(v) \cap B_\lambda$ for some $v \in T_1$, then there exists $b \in i(\lambda)$ with $d(a, b) \leq A d_T(Pa, Pb)$.*

Proof. Let μ be a geodesic path from v_0 to v in T. Order the vertices on μ so that we have a finite sequence $v_0 = y_0, y_1, \cdots, y_n = v$ such that $d_T(y_i, y_{i+1}) = 1$. and $d_T(v_0, v) = n$. Recall further, $P(B_\lambda) = T_1$. Hence $y_i \in T_1$.

Recall that B_λ is of the form $\bigcup_{v \in T_1} i_v(\lambda_v)$.

It suffices to prove that there exists $A > 0$ independent of v such that if $p \in i_{y_j}(\lambda_{y_j})$, there exists $q \in i_{y_{j-1}}(\lambda_{y_{j-1}})$ with $d(p, q) \leq A$.

By construction, $\lambda_{y_j} = \Phi_{y_j}(\mu)$ for some geodesic μ in $X_{y_{j-1}}$ such that end-points of μ lie in a C-neighborhood of $\lambda_{y_{j-1}}$. Since ϕ_{y_j} is a quasi-isometry, there exists C_1 such that p lies in a C_1 neighborhood of $\phi_{y_j}(q_0)$ for some $q_0 \in \mu$. Therefore, $d(q_0, p) \leq 1 + C$.

Also, since end-points of μ lie in a C-neighborhood of $\lambda_{y_{j-1}}$, there exists $q \in i_{y_j}(\lambda_{y_{j-1}})$ with $d(q_0, q) \leq C_2$ where C_2 depends only on δ and C. Choosing $A = 1 + C + C_2$, we are through. q.e.d.

Note that the hyperbolicity of X has not yet been used. We will apply Lemma 2.1 to derive Theorem 3.10 below. It is only here that the hyperbolicity of X is used. The main theorem of this paper follows:

Theorem 3.10. *Let (X,d) be a tree (T) of hyperbolic metric spaces satisfying the qi-embedded condition. Let v be a vertex of T. If X is hyperbolic, then $i_v : X_v \to X$ extends continuously to $\widehat{i_v} : \widehat{X_v} \to \widehat{X}$.*

Proof. Without loss of generality, let $v_0 = v$ be the base vertex of T. To prove the existence of a Cannon-Thurston map, it suffices to show (from Lemma 2.1) that for all $M \geq 0$ and $x_0 \in X_v$ there exists $N \geq 0$ such that if a geodesic segment λ lies oustside the N-ball around $x_0 \in X_v$, then B_λ lies outside the M-ball around $i_v(x_0) \in X$.

To prove this, we show that if λ lies outside the N-ball around $x_0 \in X_v$, B_λ lies outside a certain $M(N)$-ball around $i_v(x_0) \in X$, where $M(N)$ is a proper function from $\mathbb{N}$ into itself.

Since X_v is properly embedded in X, there exists $f(N)$ such that $i_v(\lambda)$ lies outside the $f(N)-$ball around x_0 in X and $f(N) \to \infty$ as $N \to \infty$.

Let p be any point on B_λ. There exists $y \in i_v(\lambda)$ such that $d(y,p) \leq Ad_T(Py, Pp)$ by Lemma 3.9. Therefore,

$$\begin{aligned} d(x_0,p) &\geq d(x_0,y) - Ad_T(Py,Pp) \\ &\geq f(N) - Ad_T(P(x_0),Pp). \end{aligned}$$

By our choice of metric on X,

$$d(x_0,p) \geq d_T(P(x_0),Pp).$$

Hence

$$\begin{aligned} d(x_0,p) &\geq max(f(N) - Ad_T(P(x_0),Pp), d_T(P(x_0),Pp)) \\ &\geq \frac{f(N)}{A+1}. \end{aligned}$$

From Theorem 3.8 there exists C' independent of λ such that B_λ is a C'-quasiconvex set containing $i_v(\lambda)$. Therefore any geodesic joining the end-points of $i_v(\lambda)$ lies in a C'-neighborhood of B_λ.

Hence any geodesic joining end-points of $i_v(\lambda)$ lies outside a ball of radius $M(N)$ where

$$M(N) = \tfrac{f(N)}{A+1} - C'.$$

Since $f(N) \to \infty$ as $N \to \infty$, so does $M(N)$. q.e.d.

The following is a direct consequence of Theorem 3.10 above.

Corollary 3.11. *Let G be a hyperbolic group acting cocompactly on a simplicial tree T such that all vertex and edge stabilizers are hyperbolic. Also suppose that every inclusion of an edge stabilizer in a vertex stabilizer is a quasi-isometric embedding. Let H be the stabilizer of a vertex or edge of T. Then there exists a Cannon-Thurston map for (H,G).*

Corollary 3.11 above covers all the examples arising from Bestvina and Feighn's work on combination theorems [2] . We note however, that one does not use the main theorem of [2] in the proof of Corollary 3.11.

4. Geometrically tame Kleinian groups

In this section we apply Theorem 3.10 to geometrically tame Kleinian groups.

The *convex core* of a hyperbolic 3-manifold N (without cusps) is the smallest convex submanifold $C(N) \subset N$ for which inclusion is a homotopy equivalence. If an ϵ- neighborhood of $C(N)$ has finite volume for some $\epsilon > 0$, then N is said to be *geometrically finite*. There exists a compact 3-dimensional submanifold $M \subset N$, the *Scott core* [33], whose inclusion is a homotopy equivalence. The ends of N are in one-to-one correspondence with the components of $N - M$ or, equivalently, the components of ∂M. We say that an end of N is *geometrically finite* if it has a neighborhood missing $C(N)$. An end of N is *simply degenerate* if it has a neighborhood homeomorphic to $S \times \mathbb{R}$, where S is the corresponding component of ∂M, and if there is a sequence of pleated surfaces homotopic in this neighborhood to the inclusion of S, and exiting every compact set. N is called *geometrically tame* if all of its ends are either geometrically finite or simply degenerate. In particular, N is homeomorphic to the interior of M. For a more detailed discussion of pleated surfaces and geometrically tame ends, see [37] or [22].

Let $inj_N(x)$ denote the injectivity radius at $x \in N$. For the purposes of this section, we shall assume that there exists $\epsilon_0 > 0$ such that $inj_N(x) > \epsilon_0$ for all $x \in N$. Further, $\pi_1(N)$ is assumed to be freely indecomposable. By [5], N is geometrically tame. In order to apply Theorem 3.10 we need some preliminary Lemmas.

Let E be a simply degenerate end of N. Then E is homeomorphic to $S \times [0, \infty)$ for some closed surface S of genus greater than one.

Lemma 4.1 [37]. *There exists $D_1 > 0$ such that for all $x \in N$, there exists a pleated surface $g : (S, \sigma) \to N$ with $g(S) \cap B_{D_1}(x) \neq \emptyset$.*

The following Lemma follows easily from the fact that $inj_N(x) > \epsilon_0$:

Lemma 4.2 [5],[37]. *There exists $D_2 > 0$ such that if*

$$g : (S, \sigma) \to N$$

is a pleated surface, then $dia(g(S)) < D_2$.

The following Lemma due to Minsky [22] follows from compactness of pleated surfaces. It says that pleated surfaces that are close in N have to be close in the Teichmüller metric.

Lemma 4.3 [22]. *Fix S and $\epsilon > 0$. For all $a > 0$ there exists $b > 0$ such that if $g : (S, \sigma) \to N$ and $h : (S, \rho) \to N$ are homotopic pleated surfaces which are isomorphisms on π_1 and $inj_N(x) > \epsilon$ for all $x \in N$, then*

$$d_N(g(S), h(S)) \leq a \Rightarrow d_{Teich}(\sigma, \rho) \leq b,$$

where d_{Teich} denotes Teichmüller distance.

Definition. The **universal curve** over $X \subset Teich(S)$ is a bundle whose fiber over $x \in X$ is x itself. [20]

Lemma 4.4. *There exist K, ϵ and a homeomorphism h from E to the universal curve over a Lipschitz path in Teichmüller space, such that h is a (K, ϵ)-quasi-isometry.*

Proof. We can assume that $S \times \{0\}$ is mapped to a pleated surface $S_0 \subset N$ under the homeomorphism from $S \times [0, \infty)$ to E. We shall construct inductively a sequence of 'equispaced' pleated surfaces $S_i \subset E$ exiting the end. Assume that $S_0, \cdots, S_n$ have been constructed such that:

1. If E_i is the non-compact component of $E \backslash S_i$, then $S_{i+1} \subset E_i$.
2. Hausdorff distance between S_i and S_{i+1} is bounded above by $3(D_1 + D_2)$.
3. $d_N(S_i, S_{i+1}) \geq D_1 + D_2$.
4. From Lemma 4.3 and condition (2) above there exists D_3 depending on D_1, D_2 and S such that $d_{Teich}(S_i, S_{i+1}) \leq D_3$.

Next choose $x \in E_n$, such that $d_N(x, S_n) = 2(D_1 + D_2)$. Then by Lemma 4.1, there exists a pleated surface $g : (S, \tau) \to N$ such that $d_N(x, g(S)) \leq D_1$. Let $S_{n+1} = g(S)$. Then by the triangle inequality and Lemma 4.2, if $p \in S_n$ and $q \in S_{n+1}$, then

$$D_1 + D_2 \leq d_N(p, q) \leq 3(D_1 + D_2).$$

This allows us to continue inductively. S_i corresponds to a point x_i of $Teich(S)$. Joining the x_i's in order, one gets a Lipschitz path in $Teich(S)$. Mapping fibers over x_i to embedded incompressible surfaces lying in uniform bounded neighborhoods of the pleated surface S_i and extending over intermediate product regions we get the desired homeomorphism h. The Lemma follows. q.e.d.

Note that in the above Lemma, pleated surfaces are not assumed to be embedded. This is because immersed pleated surfaces with a uniform lower bound on injectivity radius are uniformly quasi-isometric to the corresponding Riemann surfaces.

Observe that the universal cover $\widetilde{E}$ of E is quasi-isometric to a tree (in fact a ray) of hyperbolic metric spaces by setting $T = [0, \infty)$, with vertex set $\{n : n \in \mathbb{N} \cup \{0\}\}$, edge set $\{[n-1, n] : n \in \mathbb{N}\}$, $X_n = \widetilde{S_n} = X_{[n-1,n]}$. Further, by Lemma 4.3 this tree of hyperbolic metric spaces satisfies the quasi-isometrically embedded condition. We shall now describe $\widetilde{C(N)}$ as a tree of hyperbolic metric spaces. Assume $M \subset C(N)$ and $\partial M = \{F_1, \cdots, F_n\}$ where F_i are pleated surfaces in N cutting off ends E_i.

Lemma 4.5 [2]. *$\pi_1(N)$ is hyperbolic in the sense of Gromov. Also, if $i : E \to N$, denotes inclusion, then $i_*\pi_1(E)$ is a quasiconvex subgroup of $\pi_1(N)$.*

Remark. In fact there exists a geometrically finite hyperbolic manifold homeomorphic to N. This is part of Thurston's monster theorem. See [19] for a different proof of the fact. Also, the limit set of a geometrically finite manifold is locally connected [1]. This shall be of use later.

Recall that $M \subset N$ is the Scott-core of N and that $\pi_1(N)$ is freely indecomposable. Note that $\widetilde{M} \subset \widetilde{N}$ is quasi-isometric to the Cayley graph of $\pi_1(N)$. Hence, $\widetilde{M}$ is a hyperbolic metric space. Let $\widetilde{F_i} \subset \widetilde{N}$ represent a lift of F_i to $\widetilde{N}$. Then, by Lemma 4.5 above, $\widetilde{F_i}$ is a word-hyperbolic metric space. If $\widetilde{E_i}$ is a lift of E_i containing $\widetilde{F_i}$, then from our previous discussion, $\widetilde{E_i}$ is a ray of hyperbolic metric spaces. Since there are only finitely many ends E_i, we have thus shown:

Lemma 4.6. *The hyperbolic metric space $\widetilde{C(N)}$ is quasi-isometric to a tree (T) of hyperbolic metric spaces satisfying the qi-embedded condition. Further, we can choose a base vertex v_0 of T such that X_{v_0} is homeomorphic to $\widetilde{M}$.*

Applying Theorem 3.10, we get

Theorem 4.7. *Let Γ be a freely indecomposable Kleinian group, such that $\mathbb{H}^3/\Gamma = M$ has injectivity radius uniformly bounded below by some $\epsilon > 0$. Then there exists a continuous map from the Gromov boundary of Γ (regarded as an abstract group) to the limit set of Γ in $\mathbb{S}^2_\infty$.*

The above theorem has been independently proven by Klarreich [17] (using different techniques), where different examples of maps between boundaries of hyperbolic metric spaces are considered.

Lemma 4.8. *Let N be a geometrically tame 3-manifold with $inj_N(x) > \epsilon_0 > 0$ for all $x \in N$. Then the Gromov boundary of $\pi_1(N)$ is locally connected.*

Proof. This follows from the fact that there exists a geometrically finite manifold $M = \mathbb{H}^3/\Gamma$ homeomorphic to N [19] and that for such an M, the limit set of Γ is locally connected [1]. q.e.d.

Lemma 4.8 also follows from recent work of Bowditch and Swarup who show that the boundary of a one-ended hyperbolic group is locally connected.

Since a continuous image of a compact locally connected set is locally connected [16], Lemma 4.8 and Theorem 4.7 give:

Corollary 4.9. *Let $N = \mathbb{H}^3/\Gamma$ be a freely indecomposable 3-manifold with $inj_N(x) > \epsilon_0$ for all $x \in N$. Then the limit set of Γ is locally connected.*

The rest of this section is devoted to a somewhat different approach to a theorem of Minsky [21] .

It is well known that geodesics in hyperbolic metric spaces diverge exponentially. See [35, p. 36], for instance. The following proposition 'quasi-fies' this statement:

Proposition 4.10. *Given δ, $A_0 \geq 0$ there exist $\beta_1 > 1$, $A_1, B > 0$, such that if $[x,y], [y,z]$ and $[z,w]$ are geodesics in a δ-hyperbolic metric space (X,d) with $(x,z)_y \leq A_0$, $(y,w)_z \leq A_0$ and $d(y,z) \geq B$, then any path joining x to w and lying outside a D-neighborhood of $[y,z]$ has length greater than or equal to $A_1 {\beta_1}^D d(y,z)$, where*

$$D = min\{(d(x,[y,z]) - 1), (d(w,[y,z]) - 1)\}.$$

Lemma 4.4 shows that there exists a quasi-isometry from a lift $\widetilde{E}$ of an end to the universal cover of a universal curve over a Lipschitz path σ in $Teich(S)$. We want to show further that σ is a Teichmüller quasigeodesic.

In order to do this we need to construct a quasiconvex set B_λ as in the previous section.

Let $S_0 = \partial E$ be a pleated surface containing a closed geodesic l of N. This can always be arranged by taking a simple closed geodesic sufficiently far out in E and mapping in a pleated surface containing it [37], [5]. Construct a sequence of equispaced pleated surfaces as in Lemma 4.4. $\widetilde{E}$ is quasi-isometric to a ray of hyperbolic metric spaces (X, d), with vertex set $\{n : n \in \mathbb{N} \cup \{0\}\}$, edge set $\{[n-1, n] : n \in \mathbb{N}\}$, $X_n = \widetilde{S_n} = X_{[n-1,n]}$.

We need to go back and forth between X and $\widetilde{E}$. First let us deal with the geometry of X.

Let λ be a geodesic segment in X. Further assume that, in fact, $\lambda \subset X_0$. Let p, q be the end-points of λ. Recall that X is a ray of hyperbolic metric spaces. Since each edge of X has length one, there exist geodesic rays $r_p, r_q \subset X$ starting at p, q respectively such that $r_p(n), r_q(n)$ lie in X_n. Here r_p, r_q may be regarded as lifts of the ray to which X projects when regarded as a ray of spaces. Let λ_n denote a shortest path in X_n joining $r_p(n), r_q(n)$. Note that $\lambda = \lambda_0$. Then as in Theorem 3.8 $B_\lambda = \cup_i \lambda_i$ is uniformly quasiconvex and hence a δ-hyperbolic metric space for some (uniform) $\delta > 0$. Note that

$$d(r_p(n), p) = n = d(r_q(n), q),$$
$$d(r_p(n), X_0) = n = d(r_q(n), X_0).$$

Hence there exists a uniform $A_0 > 0$ (independent of p, q) such that $(r_p(n), q)_p \leq A_0$ and $(r_q(n), p)_q \leq A_0$.

Let $|\lambda_n|$ denote the length of λ_n. From Proposition 4.10 , there exist $\beta_1 > 1$ and $A_1, B > 0$ such that if $|\lambda_0| > B$ then

$$A_1\beta_1^n \leq |\lambda_n|.$$

Further, since the map between X_i and X_{i+1} is a uniform quasi-isometry, there exist $\beta_2 > 1$ and $A_2 > 0$ such that

$$A_1\beta_1^n \leq |\lambda_n| \leq A_2\beta_2^n.$$

Hence for all $C_1 > 1$ there exists $m \geq 1$ such that

$$|\lambda_{n+m}| \geq C_1|\lambda_n| \qquad \text{for all } n > 0.$$

Note that the above argument goes through for λ a quasigeodesic provided we change our constants appropriately. Summarizing the above discussion and adopting the notation used, we have the following Lemma:

Lemma 4.11. *Given $K, C_1 > 1$ and $\epsilon > 0$, there exist $m, B > 0$ such that if $\lambda \subset X_0$ is a (K, ϵ) - quasigeodesic in X with $|\lambda| > B$, then $|\lambda_{n+m}| > C_1|\lambda_n|$ for all $n \geq 0$.*

We want to translate the above inequality to $\widetilde{E}$ and prove that σ is a Teichmüller quasigeodesic. The idea is the following: λ_n's in X correspond to certain geodesics μ_n in $\widetilde{S_n}$. In going from $\widetilde{S_n}$ to $\widetilde{S_{n+km}}$, μ_n gets stretched by at least a factor close to C_1^k. Hence the Teichmüller distance between S_n and S_{n+km} is greater than or equal to k log (C_1). Since σ is already Lipschitz, this shows σ is a Teichmüller quasigeodesic. We formalize this below.

Lemma 4.12. *σ is a Teichmüller quasigeodesic.*

Proof. Fix x_0 in S_0. Inductively, define x_n to be the image of x_{n-1} under the Teichmüller map from S_{n-1} to S_n. Let r denote an embedding of $[0, \infty)$ into E sending $[n, n+1]$ to a (any) shortest geodesic from x_n to x_{n+1}. Since $dia(S_n)$ and $d_N(S_{n-1}, S_n)$ are uniformly bounded, r is a quasigeodesic in E. Let h denote a quasi-isometric homeomorphism between X and $\widetilde{E}$ sending X_n to $\widetilde{S_n}$. Note that in general the pleated surfaces constructed need not be embedded. But there is an embedded surface at a uniformly bounded distance from any such pleated surface. Call this new (not necessarily pleated) surface S_n in that case. This changes distances between x_n and x_{n+1} by a uniformly bounded amount. Also the quasi-isometry between the lift of a pleated surface and the lift of a nearby embedded surface can be taken to be a $(1, \epsilon_0)$-quasi-isometry for some uniformly bounded $\epsilon_0 > 0$. For ease of exposition therefore, we assume that our pleated surfaces are embedded.

Recall that l is a closed geodesic in S_0. Let $\tilde{l}$ be a bi-infinite geodesic in $\widetilde{E}$ covering l. Let $[a, b]$ be a segment in $\tilde{l}$ covering l. Thus $[a, b]$ is a geodesic segment whose projection covers l.

Let r_1, r_2 be the lifts of r through a, b. Assume, after reparametrization if necessary, $r_1(n)$, $r_2(n) \in \widetilde{S_n}$. Let μ_n be the shortest path in $\widetilde{S_n}$ joining $r_1(n), r_2(n)$. So $\mu_0 = [a, b]$. Let $\lambda \subset X$ be a shortest path in X_0 joining $h^{-1}(a) = p$ and $h^{-1}(b) = q$. Then $\lambda = \lambda_0$ is a (K, ϵ) quasigeodesic in X. Using the notation of Lemma 4.11 , for all $C_1 > 0$ there exist $m, B > 0$, such that if $|\lambda_0| > B$ then $|\lambda_{n+m}| > C_1|\lambda_n|$ for all $n \geq 0$.

Note that μ_n and $h(\lambda_n)$ have the same end-points. Let $|\mu_n|$ denote the length of μ_n. Then

$$\frac{1}{K_1}|\lambda_n| - \epsilon_1 \leq |\mu_n| \leq K_1|\lambda_n| + \epsilon_1$$

for some $K_1, \epsilon_1 > 0$ since h is a quasi-isometry.

Hence, for all $C_1 > 0$ there exist $m, B > 0$ such that if $|\mu_0| \geq B$, then $|\lambda_{n+m}| > C_1|\lambda_n|$ for all $n \geq 0$.

Fix $C_1 = e$. Then there exist $m, B > 0$ such that if $|\mu_0| \geq B$ then $|\mu_{n+km}| \geq e^k|\mu_n|$ for all $n \geq 0$.

Hence $d_{Teich}(S_{n+km}, S_n) \geq k$ for all $n \geq 0$.

Since σ was shown to be Lipschitz in Lemma 4.4, this proves that σ is a Teichmüller quasigeodesic. q.e.d.

Combining Lemma 4.4 and Lemma 4.12 above we get:

Lemma 4.13. *For each simply degenerate end E of a geometrically tame manifold N with indecomposable fundamental group there exist $K, \epsilon > 0$ and a homeomorphism h from E to the universal curve over a Teichmüller quasigeodesic such that h is a (K, ϵ)-quasi-isometry.*

So far arguments have been coarse. The argument above circumvents the construction of a model manifold in [21]. At this stage, we need to quote a part of the main theorem of [22], the common ingredient of both proofs.

Theorem 4.14 [22]. *If N is a geometrically tame hyperbolic 3-manifold with indecomposable fundamental group, such that there exists $\epsilon_0 > 0$ with $inj_N(x) > \epsilon_0$ for all $x \in N$, then for each simply degenerate end E of N we can choose a Teichmüller ray r, such that every pleated surface in E lies at a uniformly bounded distance from r. Further, any two such rays corresponding to the same ending lamination lie in a bounded neighborhood of each other.*

That last statement in Theorem 4.14 above was proven by Masur [18].

Combining Lemma 4.13 and Theorem 4.14 we have a proof of the main theorem of [21] : the ending lamination theorem for 3-manifolds with freely indecomposable fundamental group and a uniform lower bound on injectivity radius.

Theorem 4.15. *Let N_1 and N_2 be homeomorphic hyperbolic 3-manifolds with freely indecomposable fundamental group. Suppose there exists a uniform lower bound $\epsilon > 0$ on the injectivity radii of N_1 and*

N_2. If the end invariants of corresponding ends of N_1 and N_2 are equal, then N_1 and N_2 are isometric.

Proof. From Lemma 4.13, corresponding simply degenerate ends E_{i1}, E_{i2} of N_1 and N_2 are homeomorphic via quasi-isometries to universal curves over Teichmüller quasi-geodesics l_{i1} and l_{i2}. From Theorem 4.14 l_{i1} and l_{i2} lie in bounded neighbourhoods of Teichmüller geodesic rays which in turn lie in bounded neighborhoods of each other. Therefore l_{i1} and l_{i2} are quasigeodesics lying in bounded neighborhoods of each other, and the corresponding ends are homeomorphic via quasi-isometries to each other. Hence N_1, N_2 are homeomorphic by a quasi-isometry. Finally, by [36] N_1 and N_2 are isometric. q.e.d.

Note that to prove Theorem 4.15 above we do need the fact that l_{i1} and l_{i2} are quasigeodesics. This is to ensure that the paths over which the corresponding ends fiber, 'track' each other. In fact this is the point where the above argument offers an alternate approach to Minsky's technique of building a model manifold [21]. Building a model manifold is the new ingredient (over and above the main theorem of [22]) that Minsky needs in [21] to complete the proof of Theorem 4.15 above. In [21] the model manifold is then used to prove the existence of Cannon-Thurston maps when $\pi_1(N)$ is a surface group. Thus our approach in this section is, in some sense, opposite to that of Minsky's. We construct a Cannon-Thurston map first (in greater generality than Minsky) and use the techniques in conjunction with a part of the main theorem of [22] to prove Theorem 4.15 .

5. Examples

Let H be a hyperbolic subgroup of a hyperbolic group G.

Definition [15] [9]. If $i : \Gamma_H \to \Gamma_G$ is an embedding of the Cayley graph of H into that of G, then the *distortion* function is given by

$$disto(R) = R^{-1} Diam_{\Gamma_H}(\Gamma_H \cap B(R)),$$

where $B(R)$ is the ball of radius R around $1 \in \Gamma_G$.

All previously known examples of non-quasiconvex hyperbolic subgroups of hyperbolic groups exhibit exponential distortion. We construct in this section some examples exhibiting greater distortion. Some of these will be shown to have Cannon-Thurston maps. For the rest, existence of Cannon-Thurston maps is not yet known. Further, we shall describe certain examples of free subgroups of $PSL_2\mathbb{C}$ and show that they exhibit arbitrarily large distortion. The existence of Cannon-Thurston maps for some of these is not yet known.

Our starting point for constructing distorted subgroups of hyperbolic groups is the following Lemma of Bestvina, Feighn and Handel [3]:

Lemma 5.1 [3]. *There exists a hyperbolic group G such that $1 \to F \to G \to F \to 1$ is exact, where F is free of rank 3.*

Let $F_1 \subset G$ denote the normal subgroup, and $F_2 \subset G$ a section of the quotient group. Conjugation by generators of F_2 increases lengths of elements of F_1 by at most a multiplicative factor $\lambda > 1$. Hence the distortion of F_1 in G is at most exponential. But since the automorphisms induced by F_2 are hyperbolic [3], the distortion of F_1 in G is exponential.

Let $G_1, \cdots, G_n$ be n distinct copies of G. Let F_{i1} and F_{i2} denote copies of F_1 and F_2 respectively in G_i. Let

$$X_n = G_1 *_{H_1} G_2 * \cdots *_{H_{n-1}} G_n,$$

where each H_i is a free group of rank 3, the image of H_i in G_i is F_{i2}, and the image of H_i in G_{i+1} is $F_{(i+1)1}$. Then X_n is hyperbolic. This follows inductively from the main combination theorem of [2] and the fact that the image of H_i in G_i is quasiconvex in $G_1 *_{H_1} G_2 * \cdots *_{H_{i-1}} G_i$. Further, by the preceding paragraph, the distortion of X_m in X_{m+1} is exponential.

Let $H = F_{11} \subset X_n$. Then the distortion of H is superexponential for $n > 1$. In fact, the distortion function is an iterated exponential of height n. To see this one notes that since the distortion of X_m in X_{m+1} is exponential, the distortion of H in X_n is at most an iterated exponential of height n. To see that the distortion is in fact an iterated exponential of height n, we sketch an argument for $n = 2$. Let $G_1 *_{H_1} G_2$ be generated by a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 where a_i's are generators of F_{11}, b_i's are generators of H_1 and c_i's are images of generators of the quotient free group in G_2 under a section. Then $({c_1}^{-m} b_1 {c_1}^m)^{-1} a_1 ({c_1}^{-m} b_1 {c_1}^m)$ is in F_{11} and has length (in F_{11}) an iterated exponential (in m) of height 2. Hence the distortion of H in X_2 is at least an iterated exponential of height 2.

Note further that $G_1 *_{H_1} G_2$ can be regarded as a graph of groups with one vertex and three edges, where the vertex group is G_1 and edge groups are isomorphic to F. Then from Corollary 3.11, the pair

$(G_1, G_1 *_{H_1} G_2)$ has a Cannon-Thurston map. Proceeding inductively and observing that a composition of Cannon-Thurston maps is a Cannon-Thurston map, we see that (H, G) has a Cannon-Thurston map.

The next class of examples are not known to have Cannon-Thurston maps.

Our starting point is again Lemma 5.1. Let a_1, a_2, a_3 be generators of F_1 and b_1, b_2, b_3 be generators of F_2. Then

$$G = \{a_1, a_2, a_3, b_1, b_2, b_3 : b_i^{-1} a_j b_i = w_{ij}\},$$

where w_{ij} are words in a_i's. We add a letter c conjugating a_i's to 'sufficiently random' words in b_j's to get G_1. Thus,

$$G_1 = \{a_1, a_2, a_3, b_1, b_2, b_3, c : b_i^{-1} a_j b_i = w_{ij}, c^{-1} a_i c = v_i\},$$

where v_i's are words in b_j's satisfying a small-cancellation type condition to ensure that G_1 is hyperbolic. See [14, p. 151], for details on addition of 'random' relations.

Let H be the subgroup of G_1 generated by the a_i's. It can be checked that H has distortion function greater than any iterated exponential. To see this consider the sequence of words given by $w_1 = a_1$ and (inductively) $w_{i+1} = (c^{-1} w_i c)^{-1} w_i (c^{-1} w_i c)$. These are elements of H with length (in H) growing faster than any iterated expoential in i.

The above set of examples were motivated largely by examples of distorted cyclic subgroups in [15, p. 67].

So far, there is no satisfactory way of manufacturing examples of hyperbolic subgroups of hyperbolic groups exhibiting arbitrarily high distortion. It is easy to see that a subgroup of sub-exponential distortion is quasiconvex [15]. Not much else is known. For instance, one does not know if A^{n^2} can appear as a distortion function.

The situation is far more satisfactory in the case of Kleinian groups. We calculate below the distortion functions for a class of examples appearing in work of Minsky [23]:

Let S be a hyperbolic punctured torus so that the two shortest geodesics a and b are orthogonal and of equal length. Let S_0 denote S minus a neighborhood of the cusp. Let $N_\delta(a)$ and $N_\delta(b)$ be regular collar neighborhoods of a and b in S_0. For $n \in \mathbb{N}$, define $\gamma_n = a$ if n is even and equal to b if n is odd. Let T_n be the open solid torus neighborhood of $\gamma_n \times \{n + \frac{1}{2}\}$ in $S_0 \times [0, \infty)$ given by

$$T_n = N_\delta(\gamma_n) \times (n, n+1),$$

and let $M_0 = (S_0)\times[0,\infty)\backslash\bigcup_{n\in\mathbb{N}}T_n$.

Let $a(n)$ be a sequence of positive integers greater than one. Let $\hat{\gamma_n} = \gamma_n\times\{n\}$ and let μ_n be an oriented meridian for ∂T_n with a single positive intersection with $\hat{\gamma_n}$. Let M denote the result of gluing to each ∂T_n a solid torus $\hat{T_n}$, such that the curve $\hat{\gamma_n}^{a(n)}\mu_n$ is glued to a meridian. Let q_{nm} be the mapping class from S_0 to itself obtained by identifying S_0 to $S_0\times m$, pushing through M to $S_0\times n$ and back to S_0. Then $q_{n(n+1)}$ is given by $\Phi_n = D_{\gamma_n}^{a(n)}$, where D_c^k denotes Dehn twist along c, k times. Matrix representations of Φ_n are given by

$$\Phi_{2n} = \begin{pmatrix} 1 & a(2n) \\ 0 & 1 \end{pmatrix}$$

and

$$\Phi_{2n+1} = \begin{pmatrix} 1 & 0 \\ a(2n+1) & 1 \end{pmatrix}.$$

Recall that the metric on M_0 is the restriction of the product metric. $\hat{T_n}$'s are given hyperbolic metrics such that their boundaries are uniformly quasi-isometric to $\partial T_n \subset M_0$. Then from [23], M is quasi-isometric to the complement of a rank-one cusp in the convex core of a hyperbolic manifold $M_1 = \mathbb{H}^3/\Gamma$. Let σ_n denote the shortest path from $S_0\times 1$ to $S_0\times n$. Let $\overline{\sigma_n}$ denote σ_n with reversed orientation. Then $\tau_n = \sigma_n\gamma_n\overline{\sigma_n}$ is a closed path in M of length $2n+1$. Further τ_n is homotopic to a curve $\rho_n = \Phi_1\cdots\Phi_n(\gamma_n)$ on S_0. Then

$$\Pi_{i=1\cdots n}a(i) \leq l(\rho_n) \leq \Pi_{i=1\cdots n}(a(i)+2).$$

Hence

$$\Pi_{i=1\cdots n}a(i) \leq (2n+1)disto(2n+1) \leq \Pi_{i=1\cdots n}(a(i)+2).$$

Since M is quasi-isometric to the complement of the cusp of a hyperbolic manifold and γ_n's lie in a complement of the cusp, the distortion function of Γ is of the same order as the distortion function above. In particular, functions of arbitraily fast growth may be realised. This answers a question posed by Gromov [15, p. 66].

Manifolds with unbounded $a(n)$'s are not known to have Cannon-Thurston maps.

One should point out that in [7], Cannon and Thurston give an explicit description of the boundary maps in terms of ending laminations.

In [26] , a similar description is given for hyperbolic normal subgroups of hyperbolic groups. In analogy with [26] one might be able to develop a theory of ending laminations parametrized by the boundary of T and thereby give an explicit description of the boundary maps occurring in this paper.

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