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Canonical Coordinates for Coadjoint Orbits of Completely Solvable Groups

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Abstract. We show that when the methods of [2] are combined with the explicit stratification and orbital parameters of [9] and [10], the result is a construction of explicit analytic canonical coordinates for any coadjoint orbit $\mathcal{O}$ of a completely solvable Lie group. For each layer in the stratification, the canonical coordinates and the orbital cross-section together constitute an analytic parametrization for the layer.

Finally, we quantize the minimal open layer with the Moyal star product and prove that the coordinate functions are in a convenient completion of spaces of polynomial functions on $\mathfrak{g}^*$, for a metric topology naturally related to the star product.
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0. Introduction

Let G be an exponential Lie group with Lie algebra $\mathfrak{g}$, and let $\mathcal{O} \subset \mathfrak{g}^*$ be a coadjoint orbit of G. Then $\mathcal{O}$ carries a canonical symplectic structure, meaning that $\mathcal{O}$ is equipped with a distinguished, closed two-form $\omega_{\mathcal{O}}$ with the property that $\omega_{\mathcal{O}}$ is non-degenerate at each point of $\mathcal{O}$. If (U, c) is chart in $\mathcal{O}$ with $c = (p_1, p_2, \ldots, p_d, q_1, q_2, \ldots, q_d)$, then $c = (p_1, p_2, \ldots, p_d, q_1, q_2, \ldots, q_d)$ are called canonical coordinates on U if

$$\omega_{\mathcal{O}}|_U = \sum_{r=1}^{d} dp_r \wedge dq_r.$$

A standard geometric result says that there is a chart for a neighborhood of every point of $\mathcal{O}$ with canonical coordinates, and a natural question is to what extent can such coordinates be defined "globally".

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In the nilpotent case, the approaches to this question are based upon the fundamental descriptions of the coadjoint orbit space and algebra of rational G-invariant functions on $\mathfrak{g}^*$ [14]. M. Vergne shows that there is a G-invariant Zariski open subset of coadjoint orbits having maximal dimension, and rational functions $p_1, p_2, \ldots, p_d$, $q_1, q_2, \ldots, q_d$ on $\mathfrak{g}^*$ that are non-singular on this open set and whose restriction to each of these orbits defines canonical coordinates ([17], Sect. 4, Théorème). Also building on the results of [14], N. V. Pedersen shows in ([12], Theorem 5.1.1) that it is possible to classify orbits into algebraically defined families or "layers", so that within each layer of orbits, there are functions globally defined on the entire layer whose restriction to each coadjoint orbit within the layer are canonical coordinates. Moreover, in this case the procedure by which orbits are classified into layers, and the construction for each layer, are entirely explicit and algorithmic. As in the result of Vergne, the coordinate functions are rational.

In general (i.e., for G exponential), N. V. Pedersen shows in ([11], Theorem 2.2.1) that, for each coadjoint orbit $\mathcal{O}$, canonical coordinates can be defined whose domain is all of $\mathcal{O}$. In [2], there is a method for (1) selecting a partition of $\mathfrak{g}^*$ into G-invariant Borel subsets, and (2) on each Borel subset, defining functions $p_1, p_2, \ldots, p_d$, $q_1, q_2, \ldots, q_d$ which are canonical coordinates on each coadjoint orbit inside. In both of these papers the results are obtained without the benefit of a description of the orbit space which resembles that of [14] for the nilpotent case. Because of this, the results are not as explicit as in the nilpotent case.

Such a description of the coadjoint orbit space for exponential groups is given in [9] and [10]. As in the nilpotent case, the orbit classification is based upon "jump indices", but the procedure is necessarily more refined and the resulting picture of the orbit space more complex. Nevertheless, it is natural to ask whether this description, which is a precise generalization of the Pukanszky description in [14], can be used to construct explicit canonical coordinates. In particular, what is the relationship between the partition of [2], and the (refined) stratification of [9] and [10]? Can the methods of [2] be combined with those of [9] and [10] to give canonical coordinates in a more explicit form? We answer both of these questions in this paper for the case where $\mathfrak{g}$ is completely solvable.

In Section 1 we examine a "fine" stratification of $\mathfrak{g}^*$ as defined in [10] and describe its relationship to the corresponding stratification of a codimension one subgroup. In Section 2 we specialize to the case where G is completely solvable and recall the method of [2] by which one obtains globally defined canonical coordinates on G-invariant sets $\Omega_{(d)}$. We show that, in the case where G is completely solvable, each fine layer Ω is contained in some $\Omega_{(d)}$. In Section 3 we examine the case where $\mathfrak{g}$ is nilpotent, and show that it is possible to carry out the method of Vergne [17] so that the resulting Zariski-open set coincides with the Zariski-open fine layer, and so that the coordinate functions coincide with those of [2]. In Section 4, we recall the "ultra-fine" stratification of [9] and [10], and the resulting orbital description by means of cross-sections. We show that the cross-section mapping can be extended to an analytic function defined on an explicit dense and open subset of the complexification of $\mathfrak{g}^*$. When the method of [2] is applied within this context, the canonical coordinates are explicit and can be also extended to analytic functions defined on an open subset of the complexification of $\mathfrak{g}^*$. In Section 5, for the each layer, we describe the cross-section of [10] which is an algebraic set, but, in general, not a manifold. This cross-section is the graph of a rational function defined on a natural algebraic set. We state an explicit form of Théorème 1.6 in [2] and we give two examples. Finally, in Section 6 we recall the definition of the deformed Weyl algebra, of the $\star$ metric and the corresponding completion. We prove a global version of a result of [3], that for each generic coadjoint orbit, the coordinates p_r, q_r are in fact quantizable functions and belong to the completion of an algebra of polynomial functions.

1. The fine stratification

Let G be a connected, simply connected exponential solvable Lie group with Lie algebra $\mathfrak{g}$. Let $\{X_1, X_2, \ldots, X_n\}$ be a basis for $\mathfrak{g}$ and set

$$\mathfrak{g}_j = \text{span } \{X_1, X_2, \ldots, X_j\}, 1 \leq j \leq n.$$

We choose the basis $(X_1, X_2, \ldots, X_n)$ so that it satisfies
(i) for some p, $1 \leq p \leq n$, $\mathfrak{g}_p$ is the nilradical of $\mathfrak{g}$, and
(ii) if $\mathfrak{g}_j$ is not an ideal, then $\mathfrak{g}_{j+1}$ and $\mathfrak{g}_{j-1}$ are ideals, $1 \leq j \leq n$.
Let

$$I = \{j\colon\ \mathfrak{g}_j \text{ is an ideal }\},\ I' = \{j\colon\ j \in I \ \text{ and } \ j-1 \in I\} \quad \text{and} \quad I'' = I \setminus I'.$$

Denote the complexification of $\mathfrak{g}$ by $\mathfrak{g}_{\mathbb{C}}$, and regard $\mathfrak{g} \subset \mathfrak{g}_{\mathbb{C}}$ as a real subalgebra, extending elements of $\mathfrak{g}^*$ to $\mathfrak{g}_{\mathbb{C}}$ in the natural way. Define elements Z_j $(1 \leq j \leq n)$ of $\mathfrak{g}_{\mathbb{C}}$ as follows: fix j in $\{1, \ldots, n\}$, if $j \in I'$ set $Z_j = X_j$, and if $j \in I''$, set $Z_{j-1} = X_{j-1} + iX_j$ and $Z_j = X_{j-1} - iX_j$. We say that $(X_1, X_2, \ldots, X_n)$ is a "good basis" for $\mathfrak{g}$ if it satisfies conditions (i) and (ii), together with the condition
(iii) the set $\{Z_1, Z_2, \ldots, Z_n\}$ as defined above is a Jordan-Hölder basis for $\mathfrak{g}_{\mathbb{C}}$.
Note that a good basis exists [6]. We refer to $\{Z_1, Z_2, \ldots, Z_n\}$ as the Jordan-Hölder basis corresponding to the good basis $\{X_1, X_2, \ldots, X_n\}$ of $\mathfrak{g}$. Let $\mathfrak{s} = \mathfrak{g}_{\mathbb{C}}$ and

$$\mathfrak{s}_j = \text{span}\{Z_1, Z_2, \ldots, Z_j\}, \quad 1 \leq j \leq n;$$

then $\mathfrak{s}_j$ is an ideal in $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{s}_j = \bar{\mathfrak{s}}_j$ if and only if j is in I'.

With a good basis for $\mathfrak{g}$ in place, we set some more notation. Denote by $(X_1^*, \ldots, X_n^*)$ the dual basis of $\mathfrak{g}^*$. For ℓ in $\mathfrak{g}^*$, X in $\mathfrak{g}$ and x in G, let us put $[X, \ell] = ad^*X(\ell)$ and $x.\ell = Ad^*x\ell$. For each $\ell \in \mathfrak{g}^*$, and for any subset $\mathfrak{h}$ of $\mathfrak{s}$, let

$$\mathfrak{h}^\ell = \{X \in \mathfrak{s}\colon\ \ \langle \ell, [X, Y]\rangle = 0 \text{ for all } Y \in \mathfrak{h}\}$$

and

$$\mathbf{e}_\ell = \{j\colon\ \ \mathfrak{s}_j^\ell \neq \mathfrak{s}_{j-1}^\ell\}.$$

It is easily seen that $\mathbf{e}_\ell$ is the set $\{j\colon\ \ Z_j \notin \mathfrak{s}_{j-1} + \mathfrak{s}^\ell\}$. Let

$$\mathcal{E} = \{\mathbf{e}_\ell\colon\ \ell \in \mathfrak{g}^*\}.$$

The set $\mathcal{E}$ has a natural total ordering: let $\mathbf{e} = \{e_1 < \cdots < e_{2d}\}$ and $\mathbf{e}' = \{e'_1 < \cdots < e'_{2d'}\}$, we say that $\mathbf{e} < \mathbf{e}'$ if:

$$\begin{cases} d > d' \\ \text{or} \\ d = d' \text{ and } e_r < e'_r, \text{ where } r = \min\{s\colon\ \ e_s \neq e'_s\}. \end{cases}$$

For each $\mathbf{e} \in \mathcal{E}$, let $\Omega_\mathbf{e}$ be the set $\{\ell \in \mathfrak{g}^*\colon\ \ \mathbf{e}_\ell = \mathbf{e}\}$. Each $\Omega_\mathbf{e}$ is a G-invariant algebraic set, the collection $\{\Omega_\mathbf{e}\}$ constitutes a partition of $\mathfrak{g}^*$, and for each $\mathbf{e}$, the set $\bigcup\limits_{\mathbf{e}' \leq \mathbf{e}} \Omega_{\mathbf{e}'}$ is a Zariski-open subset of $\mathfrak{g}^*$. As it is shown in [12], there are semi-invariant polynomials $Q_\mathbf{e}$, $\mathbf{e} \in \mathcal{E}$, such that:

$$\Omega_\mathbf{e} = \{\ell \in \mathfrak{g}^*\colon\ \ Q_{\mathbf{e}'}(\ell) = 0, \text{ if } \mathbf{e}' < \mathbf{e} \text{ and } Q_\mathbf{e}(\ell) \neq 0\}.$$

Following [12], the partition $\{\Omega_\mathbf{e}\}$ is referred to herein as the "coarse stratification". As the name suggests, this partition is too coarse for some purposes, even in the case where $\mathfrak{g}$ is nilpotent, and various procedures have been given for its refinement. In [12], where $\mathfrak{g}$ is nilpotent, a refinement (the "fine stratification") is the first step in an explicit but rather complex procedure for construction of "quantizable" canonical coordinates for coadjoint orbits ([12], Theorem 5.1.1). (The definition of quantizable function will be given in Section 6). On the other hand in [1], a simple procedure for constructing quantizable canonical coordinates for coadjoint orbits over a Zariski-open subset is given, again in the nilpotent case. The procedure of [1] is then found to generalize to the exponential solvable case [2] but in a somewhat less explicit form. At about the same time a two-step refinement of the coarse layering was shown in the exponential case to yield an algorithm for the simultaneous parametrization of coadjoint orbits and the orbit space within each layer [9], [10]. We refer to this doubly-refined partition as the "ultra-fine layering". It is our aim to reconcile these procedures in the case where $\mathfrak{g}$ is completely solvable, producing an algorithm for explicit construction of quantizable canonical coordinates on coadjoint orbits across entire ultra-fine layers.

In the remainder of this section we examine the first step of layer refinement as defined in [9] and [10], and the relationship between fine layers in $\mathfrak{g}^*$ and $\mathfrak{g}^*_{n-1}$ is made explicit. It is then shown that in the case where $\mathfrak{g}$ is completely solvable, this first step of refinement yields the same partition as the "fine stratification" of [12].

Fix a non-empty $\mathbf{e}$ in $\mathcal{E}$. Let $2d$ be the number of elements in $\mathbf{e}$. We consider the set $J_\mathbf{e}$ of all pairs $(\mathbf{i},\mathbf{j})$ where $\mathbf{i} = \{i_1, i_2, \ldots, i_d\}$ and $\mathbf{j} = \{j_1, j_2, \ldots, j_d\}$ are index sequences whose values taken together constitute the index set $\mathbf{e}$, and which satisfy the conditions $i_r < j_r$ and $i_r < i_{r+1}$ $(1 \leq r \leq d)$. To each $\ell \in \Omega_\mathbf{e}$, we associate subalgebras $\mathfrak{h}_r(\ell)$ of $\mathfrak{s}$, $r = 0, 1, \ldots, d$, and a sequence pair $(\mathbf{i}(\ell),\mathbf{j}(\ell))$ by the following inductive scheme: set $\mathfrak{h}_0(\ell)=\mathfrak{s}$, and for $r=1,2,\ldots,d$, let

$$\begin{aligned} i_r(\ell) &= \min\{j:\ \ \mathfrak{s}_j\cap\mathfrak{h}_{r-1}(\ell)\not\subset\mathfrak{h}_{r-1}(\ell)^{\ell}\cap\mathfrak{h}_{r-1}(\ell)\},\\ \mathfrak{h}_r(\ell) &= \mathfrak{h}_{r-1}(\ell)\cap\left(s_{i_r}\cap\mathfrak{h}_{r-1}(\ell)\right)^{\ell},\\ j_r(\ell) &= \min\{j:\ \ \mathfrak{s}_j\cap\mathfrak{h}_{r-1}(\ell)\not\subset\mathfrak{h}_r(\ell)\}. \end{aligned}$$

The sequence $(\mathbf{i}(\ell),\mathbf{j}(\ell))$ belongs to $J_{\mathbf{e}}$, $\mathfrak{h}_r(\ell)$, $(0\le r\le d)$ is a subalgebra of $\mathfrak{s}$ of codimension r, and $\mathfrak{h}_d(\ell)$ is totally isotropic with respect to the skewsymmetric bilinear form on $\mathfrak{s}$ defined by ℓ.

Put $\mathcal{F}_{\mathbf{e}}=\{(\mathbf{i}(\ell),\mathbf{j}(\ell)):\ \ell\in\Omega_{\mathbf{e}}\}$ and $\mathcal{H}=\{(\mathbf{e},\mathbf{i},\mathbf{j}):\ \mathbf{e}\in\mathcal{E},\ (\mathbf{i},\mathbf{j})\in\mathcal{F}_{\mathbf{e}}\}$. Then $\mathcal{H}$ has a total ordering:
let $(\mathbf{e},\mathbf{i},\mathbf{j})$ and $(\mathbf{e}',\mathbf{i}',\mathbf{j}')$ be two elements in $\mathcal{H}$, we say $(\mathbf{e},\mathbf{i},\mathbf{j})<(\mathbf{e}',\mathbf{i}',\mathbf{j}')$ if:

$$\begin{cases} \mathbf{e}<\mathbf{e}' \\ \text{or} \\ \mathbf{e}=\mathbf{e}' \text{ and } j_r<j'_r,\ \text{where } r=\min\{s:\ \ j_s\ne j'_s\}. \end{cases}$$

From now on, we shall represent $(\mathbf{e},\mathbf{i},\mathbf{j})$ by $(\mathbf{e},\mathbf{j})$. Set

$$\Omega_{\mathbf{e},\mathbf{j}}=\{\ell\in\Omega_{\mathbf{e}}:\ \ \mathbf{j}(\ell)=\mathbf{j}\}.$$

For any subset $\mathbf{h}$ of $\mathbf{e}$ ($\mathbf{e}\in\mathcal{E}$, $\mathbf{e}\ne\emptyset$), let $M_{\mathbf{h}}(\ell)$ be the corresponding skew-symmetric submatrix of $M_{\mathbf{e}}(\ell)$:

$$M_{\mathbf{h}}(\ell)=(\langle\ell,[Z_{j_r},Z_{j_s}]\rangle)_{j_r,j_s\in\mathbf{h}}.$$

We denote the Pfaffian of $M_{\mathbf{h}}(\ell)$ by $P_{\mathbf{h}}(\ell)$.

Now fix $(\mathbf{i},\mathbf{j})$ in $\mathcal{F}_{\mathbf{e}}$,

$$\mathbf{i}=\{i_1<i_2<\cdots<i_d\},\ \mathbf{j}=\{j_1,j_2,\ldots,j_d\}.$$

For each r, $1\le r\le d$, let $\mathbf{h}_r=\{i_1,i_2,\ldots,i_r,\ j_1,j_2,\ldots,j_r\}$, set

$$P_{\mathbf{e},\mathbf{j},r}(\ell)=P_{\mathbf{h}_r}(\ell)\quad\text{and}\quad P_{\mathbf{e},\mathbf{j}}(\ell)=\prod_{r=1}^{d}P_{\mathbf{e},\mathbf{j},r}(\ell).$$

Set also

$$\Omega_{\mathbf{e},\mathbf{j},r}=\{\ell\in\Omega_{\mathbf{e}}:\ \ j_t(\ell)=j_t,\ 1\le t\le r\},\quad(1\le r\le d)\quad\text{and}\quad\Omega_{\mathbf{e},\mathbf{j},0}=\Omega_{\mathbf{e}}.$$

We know that (see [7])

$$\Omega_{\mathbf{e},\mathbf{j}}=\{\ell\in\Omega_{\mathbf{e}}:\ \ P_{\mathbf{e},\mathbf{j}'}(\ell)=0\quad\text{for all}\quad(\mathbf{e},\mathbf{j}')<(\mathbf{e},\mathbf{j})\quad\text{and}\quad P_{\mathbf{e},\mathbf{j}}(\ell)\ne0\}.$$

To begin with, our aim is to determine the relationship between the layers in $\mathfrak{g}^*$ and those of $\mathfrak{g}^*_{n-1}$. Denote $\mathfrak{g}_{n-1}$ by $\mathfrak{g}^0$, and $\mathfrak{g}^0_{\mathbb{C}}$ by $\mathfrak{s}^0$ and consider the restriction map $\pi:\mathfrak{g}^*\to\mathfrak{g}^{0*}$, $\ell\mapsto\ell^0=\pi(\ell)=\ell|_{\mathfrak{g}^0}$. Since n is in I', the preceding construction for $\mathfrak{g}^0$, using $(X_1,\ldots,X_{n-1})$ gives the same basis $(Z_1,\ldots,Z_{n-1})$ thus the same ideals $\mathfrak{s}_j$ for $j<n$. We define $\mathbf{e}^0$ to be the set of jump indices for ℓ^0:

$$\mathbf{e}^0=\{j:\ \ 1\le j\le n-1\ \text{ and }\ Z_j\notin\mathfrak{s}_{j-1}+(s^0)^{\ell^0}\}$$

and we put $(\mathbf{i}^0,\mathbf{j}^0)=(\mathbf{i}^0(\ell^0),\mathbf{j}^0(\ell^0))$.

Lemma 1.1. $\mathbf{e}^0$ *is a subset of* $\mathbf{e}$.

Proof. Put:

$$\mathbf{e}^j = \{r\colon\ 1 \leq r \leq j \ \text{ and } \ Z_r \notin \mathfrak{s}_{r-1} + \mathfrak{s}_j^{\ell^j}\},$$

$\mathbf{e}^j$ is the set of jump indices for ℓ^j. We clearly have

$$\mathbf{e} = \{j\colon\ 1 \leq j \leq n \ \text{ and } \ Z_j \notin \mathfrak{s}_{j-1} + \mathfrak{s}^\ell \cap \mathfrak{s}_j\}.$$

Now $\mathfrak{s}^\ell \cap \mathfrak{s}_j \subset \mathfrak{s}_j^{\ell^j}$, thus

$$Z_r \in \mathfrak{s}_{r-1} + \mathfrak{s}^\ell \cap \mathfrak{s}_j \Longrightarrow Z_r \in \mathfrak{s}_{r-1} + \mathfrak{s}_j^{\ell^j},$$

and therefore

$$Z_r \notin \mathfrak{s}_{r-1} + \mathfrak{s}_j^{\ell^j} \Longrightarrow Z_r \notin \mathfrak{s}_{r-1} + \mathfrak{s}^\ell \cap \mathfrak{s}_j.$$

This implies that $\mathbf{e}^j$ is a subset of $\mathbf{e}$. More precisely we have

$$\emptyset = \mathbf{e}^1 \subset \mathbf{e}^2 \subset \cdots \subset \mathbf{e}^{n-1} = \mathbf{e}^0 \subset \mathbf{e}^n = \mathbf{e},$$

and

$$|\mathbf{e}^j| = |\mathbf{e}^{j-1}| \ \text{ or } \ |\mathbf{e}^j| = |\mathbf{e}^{j-1}| + 2,$$

where $|\mathbf{e}|$ is the cardinal of the set $\mathbf{e}$. ∎

Lemma 1.2. *Put*

$$\tilde{\mathfrak{h}}_r(\ell) = \mathfrak{h}_r(\ell) \cap \mathfrak{s}^0, \quad r = 0, \dots, d$$

If n is a jump index, then there is an unique k in $\{1, \dots, d\}$ such that

$$\tilde{\mathfrak{h}}_k(\ell) = \tilde{\mathfrak{h}}_{k-1}(\ell).$$

Proof. Thus

$$\tilde{\mathfrak{h}}_d(\ell) = \left(\sum_{j=1}^{n} \mathfrak{s}_j^{\ell^j}\right) \cap \mathfrak{s}^0 = \left(\sum_{j=1}^{n-1} \mathfrak{s}_j^{\ell^j} + \mathfrak{s}_n^\ell\right) \cap \mathfrak{s}^0.$$

But n is a jump index, thus X_n is not in $\mathfrak{s}^\ell + \mathfrak{s}^0$ and $\mathfrak{s}^\ell \subset \mathfrak{s}^0$. Then

$$\tilde{\mathfrak{h}}_d(\ell) = \sum_{j=1}^{n-1} \mathfrak{s}_j^{\ell_j} = \mathfrak{h}_{d-1}^0(\ell^0).$$

Indeed, since:

$$\tilde{\mathfrak{h}}_r(\ell) = \operatorname{Ker}\left(Z_n^*|_{\mathfrak{h}_r(\ell)}\right),$$

$\tilde{\mathfrak{h}}_r(\ell)$ $(r = 0, \dots, d)$ is a subalgebra of $\mathfrak{h}_r(\ell)$ of codimension 0 or 1. On the other hand,

$$\mathfrak{s}^0 = \tilde{\mathfrak{h}}_0(\ell) \supset \tilde{\mathfrak{h}}_1(\ell) \supset \cdots \supset \tilde{\mathfrak{h}}_d(\ell) = \mathfrak{h}_{d-1}^0(\ell^0),$$

$\dim\ \mathfrak{s}^0 = n - 1$ and $\dim\ \tilde{\mathfrak{h}}_d(\ell) = n - d$. We conclude that there is an unique k in $\{1, \dots, d\}$ such that $\tilde{\mathfrak{h}}_k(\ell) = \tilde{\mathfrak{h}}_{k-1}(\ell)$, this proves our lemma. ∎

Remark 1.1. As a consequence of Lemma 1.2., we have

$$\tilde{\mathfrak{h}}_0(\ell) \neq \mathfrak{h}_0(\ell), \ldots, \tilde{\mathfrak{h}}_{k-1}(\ell) \neq \mathfrak{h}_{k-1}(\ell),$$

and

$$\tilde{\mathfrak{h}}_{k-1}(\ell) = \tilde{\mathfrak{h}}_k(\ell) = \mathfrak{h}_k(\ell),\ \tilde{\mathfrak{h}}_{k+1}(\ell) = \mathfrak{h}_{k+1}(\ell), \ldots, \tilde{\mathfrak{h}}_d(\ell) = \mathfrak{h}_d(\ell).$$

Let us choose $Y_1, Y_2, \ldots, Y_d$ in $\mathfrak{s}$ such that Y_j is in $\mathfrak{h}_{j-1}(\ell) \setminus \mathfrak{h}_j(\ell)$. Then $(Y_1, Y_2, \ldots, Y_r)$ is a basis of $\mathfrak{s}$ mod $\mathfrak{h}_r(\ell)$ $(r = 1, 2, \ldots, d)$, and

$$\mathfrak{h}_r(\ell) = \operatorname{span}\{Y_{r+1}, \ldots, Y_d\} \oplus \mathfrak{h}_d(\ell), \quad r = 0, 1, \ldots, d-1.$$

For $r \geq k$, by a dimension argument, we have $\tilde{\mathfrak{h}}_r(\ell) = \mathfrak{h}_r(\ell) \subset \mathfrak{s}^0$, thus $Y_{k+1}, \ldots, Y_d$ are in $\mathfrak{s}^0$. Since $\tilde{\mathfrak{h}}_k(\ell) = \tilde{\mathfrak{h}}_{k-1}(\ell)$ then Y_k does not belong to $\mathfrak{s}^0$. By adding, if necessary, a multiple of Y_k, to $Y_{k-1}, \ldots, Y_1$, we suppose that $Y_{k-1}, \ldots, Y_1$ are also in $\mathfrak{s}^0$. Now we have:

$$Y_j \in \mathfrak{s}^0, \quad \text{for } j \neq k \text{ and } Y_k \notin \mathfrak{s}^0$$

and

$$\mathfrak{h}_r(\ell) = \tilde{\mathfrak{h}}_r(\ell) \oplus \mathbb{C}Y_k, \quad \text{for } r < k \text{ and } \mathfrak{h}_r(\ell) = \tilde{\mathfrak{h}}_r(\ell), \quad \text{for } r \geq k.$$

Now the relationship between the jump indices of ℓ in $\mathfrak{g}^*$ and those of ℓ^0 in $\mathfrak{g}^{0*}$ is given by the following:

Proposition 1.3. *1. If n does not belong to $\mathbf{e}$, then for any r, $1 \leq r \leq d$,*

$$i_r(\ell) = i_r^0(\ell^0), \quad \tilde{\mathfrak{h}}_r(\ell) = \mathfrak{h}_r^0(\ell^0) \quad \text{and} \quad j_r(\ell) = j_r^0(\ell^0).$$

2. If n belongs to $\mathbf{e}$, then $n = j_k \in \mathbf{j}$ where k is defined in Lemma 1.2. and for $1 \leq r < k$,

$$i_r^0(\ell^0) = i_r(\ell), \quad \mathfrak{h}_r^0(\ell^0) = \tilde{\mathfrak{h}}_r(\ell) \quad \text{and} \quad j_r^0(\ell^0) = j_r(\ell),$$

while for $k < r \leq d$,

$$i_{r-1}^0(\ell^0) = i_r(\ell), \quad \mathfrak{h}_{r-1}^0(\ell^0) = \tilde{\mathfrak{h}}_r(\ell) \quad \text{and} \quad j_{r-1}^0(\ell^0) = j_r(\ell).$$

Proof. 1. In this case, $\mathbf{e} = \mathbf{e}^0$, $\mathfrak{s}^\ell = (\mathfrak{s}^0)^{\ell^0} \oplus \mathbb{C}Z$, with $Z \in \mathfrak{s} \setminus \mathfrak{s}^0$. We consider:

$$\mathfrak{s} = \mathfrak{h}_0(\ell) \supset \mathfrak{h}_1(\ell) \supset \cdots \supset \mathfrak{h}_d(\ell)$$

and

$$\mathfrak{s}^0 = \mathfrak{h}_0^0(\ell^0) \supset \mathfrak{h}_1^0(\ell^0) \supset \cdots \supset \mathfrak{h}_d^0(\ell^0).$$

But $\mathfrak{h}_d(\ell)$ being the Vergne polarization of ℓ relative to the basis $\{Z_j\}$, $\mathfrak{h}_d(\ell) = \mathfrak{h}_d^0(\ell^0) \oplus \mathbb{C}Z$ and thus

$$\tilde{\mathfrak{h}}_r(\ell) = \mathfrak{h}_r^0(\ell^0), \quad \text{for all } r = 1, \ldots, d.$$

Consider now

$$i_r(\ell) = \min\{j\colon\ \mathfrak{s}_j \cap \mathfrak{h}_{r-1}(\ell) \not\subset \mathfrak{h}_{r-1}(\ell)^{\ell} \cap \mathfrak{h}_{r-1}(\ell)\}.$$

But for $j < n$, $\mathfrak{s}_j \cap \mathfrak{h}_{r-1}(\ell) = \mathfrak{s}_j \cap \tilde{\mathfrak{h}}_{r-1}(\ell)$ and thus

$$i_r(\ell) = \min\{j\colon\ \mathfrak{s}_j \cap \mathfrak{h}^0_{r-1}(\ell^0) \not\subset \mathfrak{h}^0_{r-1}(\ell^0)^{\ell^0} \cap \mathfrak{h}^0_{r-1}(\ell^0)\} = i^0_r(\ell^0).$$

Finally, for the $j_r(\ell)$, since $j_r(\ell) < n$,

$$\begin{aligned} j_r(\ell) &= \min\{j\colon\ \mathfrak{s}_j \cap \mathfrak{h}_{r-1}(\ell) \not\subset \mathfrak{h}_r(\ell)\} \\ &= \min\{j\colon\ \mathfrak{s}_j \cap \tilde{\mathfrak{h}}_{r-1}(\ell) \not\subset \tilde{\mathfrak{h}}_r(\ell)\} \\ &= \min\{j\colon\ \mathfrak{s}_j \cap \mathfrak{h}^0_{r-1}(\ell^0) \not\subset \mathfrak{h}^0_r(\ell^0)\} = j^0_r(\ell^0). \end{aligned}$$

2. For the second assertion, we claim first that if $r \neq k$ then $j_r(\ell) < n$. Suppose $j_r(\ell) = n$, then we have

$$\tilde{\mathfrak{h}}_{r-1}(\ell) = \mathfrak{s}^0 \cap \mathfrak{h}_{r-1}(\ell) \subset \mathfrak{h}_r(\ell), \quad \dim \tilde{\mathfrak{h}}_{r-1}(\ell) \leq \dim \tilde{\mathfrak{h}}_r(\ell).$$

But

$$\dim \tilde{\mathfrak{h}}_{r-1}(\ell) = \dim \mathfrak{h}_{r-1}(\ell) \quad \text{or} \quad \dim \tilde{\mathfrak{h}}_{r-1}(\ell) = \dim \mathfrak{h}_{r-1}(\ell) - 1$$

and thus

$$\tilde{\mathfrak{h}}_{r-1}(\ell) = \mathfrak{h}_r(\ell),$$

which is not the case. That means that the jump index n is in fact $j_k(\ell)$. Now to complete the proof of 2, we have to consider two cases, case 1: $1 \leq r < k$ and case 2: $k < r \leq d$.

<u>Case 1:</u> Let us prove by induction that

$$i_r(\ell) = i^0_r(\ell^0), \quad j_r(\ell) = j^0_r(\ell^0) \quad \text{and} \quad \tilde{\mathfrak{h}}_r(\ell) = \mathfrak{h}^0_r(\ell^0).$$

For $r = 1$, we have

$$\begin{aligned} &i_1(\ell) = \min\{j,\ \ \mathfrak{s}_j \not\subset \mathfrak{s}^{\ell}\}, \quad \mathfrak{h}_1(\ell) = \mathfrak{s}^{\ell}_{i_1}, \\ &j_1(\ell) = \min\{j,\ \ \mathfrak{s}_j \not\subset \mathfrak{h}_1(\ell)\}. \end{aligned}$$

Since Z_{j_1} does not belong to $\mathfrak{h}_1(\ell)$, then $j_1(\ell) \neq n$, indeed, if $j_1(\ell)$ would be n, then $k = 1$, which is impossible in case 1. On the other hand, $\mathfrak{s}^{\ell}$ is a subspace of $(\mathfrak{s}^0)^{\ell^0}$ with codimension 1, thus $i_1(\ell) \leq i^0_1(\ell^0)$. But if $i_1(\ell) < i^0_1(\ell^0)$, then Z_{i_1} is in $(\mathfrak{s}^0)^{\ell^0}$ and $(\mathfrak{s}^0)^{\ell^0} = \mathfrak{s}^{\ell} \oplus \mathbb{C}Z_{i_1}$ and so for all $X \in \mathfrak{s}^0$,

$$\langle \ell, [X, \mathfrak{s}_{i_1}]\rangle = \langle \ell^0, [X, \mathfrak{s}_{i_1}]\rangle = 0,$$

since $\mathfrak{s}_{i_1} \subset \mathfrak{s}_{i^0_1} \subset (\mathfrak{s}^0)^{\ell^0}$. This implies that $\mathfrak{s}^0 \subset \mathfrak{h}_1(\ell)$, but they have same dimension, thus $\mathfrak{s}^0 = \mathfrak{h}_1(\ell)$ and so $j_1(\ell) = n$ which is not the case. We thus have

$$i_1(\ell) = i^0_1(\ell^0).$$

Now we get:

$$\begin{aligned}&\tilde{\mathfrak{h}}_1(\ell)=\{X\in\mathfrak{s}^0:\ \langle\ell,[X,\mathfrak{s}_{i_1}]\rangle=0\}=\{X\in\mathfrak{s}^0:\ \langle\ell^0,[X,\mathfrak{s}_{i_1^0}]\rangle=0\}=\mathfrak{h}_1^0(\ell^0),\\ &j_1(\ell)=\min\{j:\ \mathfrak{s}_j\not\subset\mathfrak{h}_1(\ell)\}.\end{aligned}$$

Since $j_1(\ell)<n$, thus for $j<n$, $\mathfrak{s}_j\subset\mathfrak{h}_1(\ell)$ if and only if $\mathfrak{s}_j\subset\mathfrak{h}_1^0(\ell^0)$, therefore,

$$j_1(\ell)=\min\{j:\ \ \mathfrak{s}_j\not\subset\mathfrak{h}_1^0(\ell^0)\}=j_1^0(\ell^0).$$

The conclusion holds for $r=1$.

Suppose now that, for some $r=2,\dots,k-1$, $i_{r-1}(\ell)=i^0_{r-1}(\ell^0)$, $\tilde{\mathfrak{h}}_{r-1}(\ell)=\mathfrak{h}^0_{r-1}(\ell^0)$ and $j_{r-1}(\ell)=j^0_{r-1}(\ell^0)$ and prove the result for r. We have

$$\begin{aligned}&i_r(\ell)=\min\{j:\ \ \mathfrak{s}_j\cap\mathfrak{h}_{r-1}(\ell)\not\subset\mathfrak{h}_{r-1}(\ell)^\ell\cap\mathfrak{h}_{r-1}(\ell)\},\\ &\mathfrak{h}_r(\ell)=\mathfrak{h}_{r-1}(\ell)\cap(\mathfrak{s}_{i_r}\cap\mathfrak{h}_{r-1}(\ell))^\ell,\\ &j_r(\ell)=\min\{j:\ \ \mathfrak{s}_j\cap\mathfrak{h}_{r-1}(\ell)\not\subset\mathfrak{h}_r(\ell)\}.\end{aligned}$$

Consider first $i_r(\ell)$, for all $j<i_r(\ell)$, we have

$$\mathfrak{s}_j\cap\mathfrak{h}_{r-1}(\ell)\subset\mathfrak{h}_{r-1}(\ell)^\ell\cap\mathfrak{h}_{r-1}(\ell)$$

so that

$$\mathfrak{s}_j\cap\tilde{\mathfrak{h}}_{r-1}(\ell)\subset\mathfrak{h}_{r-1}(\ell)^\ell\cap\tilde{\mathfrak{h}}_{r-1}(\ell)$$

that is to say:

$$\mathfrak{s}_j\cap\mathfrak{h}^0_{r-1}(\ell^0)\subset\mathfrak{h}^0_{r-1}(\ell^0)^{\ell^0}\cap\mathfrak{h}^0_{r-1}(\ell^0).$$

Thus $i_r(\ell)\le i_r^0(\ell^0)$. Now since Y_r belongs to $\mathfrak{h}_{r-1}(\ell)\setminus\mathfrak{h}_r(\ell)$, there exists Z in $\mathfrak{s}_{i_r}\cap\mathfrak{h}_{r-1}(\ell)$ such that $\langle\ell,[Y_r,Z]\rangle\neq0$. But we have

$$\mathfrak{s}_{i_r}\cap\mathfrak{h}_{r-1}(\ell)=\mathfrak{s}_{i_r}\cap\tilde{\mathfrak{h}}_{r-1}(\ell),$$

thus Z is in $\mathfrak{s}_{i_r}\cap\mathfrak{h}^0_{r-1}(\ell^0)$, so Z does not belong to $\mathfrak{h}^0_{r-1}(\ell^0)^{\ell^0}$ and we have $i_r^0(\ell^0)\le i_r(\ell)$, we conclude then $i_r^0(\ell^0)=i_r(\ell)$. As for $\tilde{\mathfrak{h}}_r(\ell)$,

$$\begin{aligned}\tilde{\mathfrak{h}}_r(\ell)&=\big(\mathfrak{s}_{i_r}\cap\mathfrak{h}_{r-1}(\ell)\big)^\ell\cap\tilde{\mathfrak{h}}_{r-1}(\ell)\\ &=\mathfrak{s}^0\cap\big(\mathfrak{s}_{i_r}\cap\mathfrak{h}_{r-1}(\ell)\big)^\ell\cap\mathfrak{h}^0_{r-1}(\ell^0)\\ &=\big(\mathfrak{s}_{i_r^0}\cap\mathfrak{h}^0_{r-1}(\ell^0)\big)^{\ell^0}\cap\tilde{\mathfrak{h}}^0_{r-1}(\ell^0)=\mathfrak{h}^0_r(\ell^0).\end{aligned}$$

Finally, because $j_r(\ell)<n$, then

$$j_r(\ell)=\min\{j:\ \mathfrak{s}_j\cap\tilde{\mathfrak{h}}_{r-1}(\ell)\not\subset\tilde{\mathfrak{h}}_r(\ell)\}=\min\{j:\ \mathfrak{s}_j\cap\mathfrak{h}^0_{r-1}(\ell^0)\not\subset\mathfrak{h}^0_r(\ell^0)\}=j_r^0(\ell^0).$$

Case 2: We prove by induction that for $r>k$,

$$i_r(\ell)=i^0_{r-1}(\ell^0),\ \ \tilde{\mathfrak{h}}_r(\ell)=\mathfrak{h}_r(\ell)=\mathfrak{h}^0_{r-1}(\ell^0)\ \text{ and }\ j_r(\ell)=j^0_{r-1}(\ell^0).$$

Let us suppose $r = k+1$. Since $\mathfrak{h}_k(\ell) = \tilde{\mathfrak{h}}_{k-1}(\ell) = \mathfrak{h}^0_{k-1}(\ell^0)$,

$$\mathfrak{h}_k(\ell)^\ell \cap \tilde{\mathfrak{h}}_k(\ell) = \mathfrak{h}_k(\ell)^\ell \cap \tilde{\mathfrak{h}}_{k-1}(\ell) = \mathfrak{h}^0_{k-1}(\ell^0)^{\ell^0} \cap \mathfrak{h}^0_{k-1}(\ell^0),$$

but for $j < n$,

$$\mathfrak{s}_j \cap \mathfrak{h}_k(\ell) = \mathfrak{s}_j \cap \tilde{\mathfrak{h}}_k(\ell) = \mathfrak{s}_j \cap \tilde{\mathfrak{h}}_{k-1}(\ell) = \mathfrak{s}_j \cap \mathfrak{h}^0_{k-1}(\ell^0),$$

thus $i_{k+1}(\ell) = i^0_k(\ell^0)$. Let us consider now $\mathfrak{h}_{k+1}(\ell)$:

$$\begin{aligned}\mathfrak{h}_{k+1}(\ell) = \tilde{\mathfrak{h}}_{k+1}(\ell) &= \Big(\mathfrak{s}_{i^0_k} \cap \mathfrak{h}_k(\ell)\Big)^\ell \cap \mathfrak{h}_k(\ell)\\ &= \Big(\mathfrak{s}_{i^0_k} \cap \mathfrak{h}^0_{k-1}(\ell^0)\Big)^{\ell^0} \cap \mathfrak{h}^0_{k-1}(\ell^0) = \mathfrak{h}^0_k(\ell^0).\end{aligned}$$

Finally,

$$j_{k+1}(\ell) = \min\{j\colon\ \ \mathfrak{s}_j \cap \mathfrak{h}_k(\ell) \not\subset \mathfrak{h}_{k+1}(\ell)\}.$$

But $j_{k+1}(\ell) \neq n$ and so

$$\begin{aligned}j_{k+1}(\ell) &= \min\{j\colon\ \ \mathfrak{s}_j \cap \tilde{\mathfrak{h}}_k(\ell) \not\subset \tilde{\mathfrak{h}}_{k+1}(\ell)\}\\ &= \min\{j\colon\ \ \mathfrak{s}_j \cap \mathfrak{h}^0_{k-1}(\ell^0) \not\subset \mathfrak{h}^0_k(\ell^0)\} = j^0_k(\ell^0).\end{aligned}$$

Now we suppose that for some $r > k$, we have:

$$i_r(\ell) = i^0_{r-1}(\ell^0), \quad \mathfrak{h}_r(\ell) = \tilde{\mathfrak{h}}_r(\ell) = \mathfrak{h}^0_{r-1}(\ell^0) \ \text{ and } \ j_r(\ell) = j^0_{r-1}(\ell^0).$$

Then, for $r+1$, we get

$$i_{r+1}(\ell) = \min\{j\colon\ \ \mathfrak{s}_j \cap \mathfrak{h}_r(\ell) \not\subset \mathfrak{h}_r(\ell)^\ell \cap \mathfrak{h}_r(\ell)\}.$$

But

$$\mathfrak{s}_j \cap \mathfrak{h}_r(\ell) = \mathfrak{s}_j \cap \tilde{\mathfrak{h}}_r(\ell) = \mathfrak{s}_j \cap \mathfrak{h}^0_{r-1}(\ell^0)$$

and

$$\mathfrak{h}_r(\ell)^\ell \cap \mathfrak{h}_r(\ell) = \mathfrak{h}^0_{r-1}(\ell^0)^{\ell^0} \cap \mathfrak{h}^0_{r-1}(\ell^0).$$

Therefore $i_{r+1}(\ell) = i^0_r(\ell^0)$. Consider now $\mathfrak{h}_{r+1}(\ell)$:

$$\begin{aligned}\mathfrak{h}_{r+1}(\ell) = \tilde{\mathfrak{h}}_{r+1}(\ell) &= \mathfrak{h}_r(\ell) \cap \big(\mathfrak{s}_{i_{r+1}} \cap \mathfrak{h}_r(\ell)\big)^\ell\\ &= \mathfrak{h}^0_{r-1}(\ell^0) \cap \big(\mathfrak{s}_{i^0_r} \cap \mathfrak{h}^0_{r-1}(\ell^0)\big)^{\ell^0} = \mathfrak{h}^0_r(\ell^0).\end{aligned}$$

Finally, because $j_{r+1}(\ell) < n$, thus

$$\begin{aligned}j_{r+1}(\ell) &= \min\{j\colon\ \ \mathfrak{s}_j \cap \mathfrak{h}_r(\ell) \not\subset \mathfrak{h}_{r+1}(\ell)\}\\ &= \min\{j\colon\ \ \mathfrak{s}_j \cap \mathfrak{h}^0_{r-1}(\ell^0) \not\subset \mathfrak{h}^0_r(\ell^0)\} = j^0_r(\ell^0).\end{aligned}$$ ■

Now let $\Omega_{\mathbf{e},\mathbf{j}}$ be any layer belonging to the fine stratification, and define $\mathbf{e^0}$ and $\mathbf{j^0}$ exactly as in Proposition 1.3, according as $n \notin \mathbf{e}$ or $n \in \mathbf{e}$. We are near to the completion of our goal for this section, which is to understand clearly the relationship between $\Omega_{\mathbf{e},\mathbf{j}}$ and the layer $\Omega^0_{\mathbf{e}^0,\mathbf{j}^0}$ in $(\mathfrak{g}^0)^*$. Recall that they are defined by means of Pfaffian polynomials:

$$\Omega_{\mathbf{e},\mathbf{j}} = \{\ell \in \Omega_{\mathbf{e}}\colon \ P_{\mathbf{e},\mathbf{j}'}(\ell) = 0 \ \text{ for all } \ (\mathbf{e},\mathbf{j}') < (\mathbf{e},\mathbf{j}) \ \text{ and } \ P_{\mathbf{e},\mathbf{j}}(\ell) \neq 0\},$$

where each polynomial $P_{\mathbf{e},\mathbf{j}}(\ell)$ is the product of the Pfaffians $P_{\mathbf{e},\mathbf{j},r}$, $1 \leq r \leq d$. Similarly

$$\Omega^0_{\mathbf{e}^0,\mathbf{j}^0} = \{\ell^0 \in \Omega^0_{\mathbf{e}^0}:\ P^0_{\mathbf{e}^0,\mathbf{j}'^0}(\ell^0) = 0 \text{ for all } (\mathbf{e}^0,\mathbf{j}'^0) < (\mathbf{e}^0,\mathbf{j}^0) \text{ and } P^0_{\mathbf{e}^0,\mathbf{j}^0}(\ell^0) \neq 0\}.$$

If n is not a jump index for ℓ, then the corresponding Pfaffians $P_{\mathbf{e},\mathbf{j},r}$ and $P^0_{\mathbf{e}^0,\mathbf{j}^0,r}$ coincide and so we get

$$\Omega_{\mathbf{e},\mathbf{j}} = \pi^{-1}(\Omega^0_{\mathbf{e}^0,\mathbf{j}^0}) \cap \Omega_{\mathbf{e}}.$$

Suppose then that n is a jump index for ℓ. Here the Pfaffians do not coincide but, with the help of Proposition 1.3, we can find a precise relationship between them. In order to compute $P_{\mathbf{e},\mathbf{j},r}$, one considers the space $W_r = \operatorname{span}\{Z_{i_1}, Z_{j_1}, \ldots, Z_{i_r}, Z_{j_r}\}$, equipped with the skew-symmetric bilinear form β:

$$\beta(X,Y) = \langle \ell, [X,Y]\rangle.$$

The matrix of β is put in the canonical form:

$$M' = \begin{pmatrix} \begin{pmatrix} 0 & -Q_1(\ell) \\ Q_1(\ell) & 0 \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{pmatrix} 0 & -Q_r(\ell) \\ Q_r(\ell) & 0 \end{pmatrix} \end{pmatrix},$$

by using a modified basis $\{\rho_{t-1}(Z_{i_t},\ell), \rho_{t-1}(Z_{j_t},\ell)\colon \ 1 \leq t \leq r\}$ for W_r. The functions $\rho_t(\cdot,\ell) = \rho_t(\cdot)$ and Q_t are recursively defined for $t = 1,\ldots,d$ by:

$$\begin{aligned} \rho_0(Z) &= Z, \\ \rho_t(Z) &= \rho_{t-1}(Z) - \frac{\langle \ell, [\rho_{t-1}(Z), \rho_{t-1}(Z_{j_t})]\rangle}{\langle \ell, [\rho_{t-1}(Z_{i_t}), \rho_{t-1}(Z_{j_t})]\rangle}\rho_{t-1}(Z_{i_t}) \\ &\qquad - \frac{\langle \ell, [\rho_{t-1}(Z), \rho_{t-1}(Z_{i_t})]\rangle}{\langle \ell, [\rho_{t-1}(Z_{j_t}), \rho_{t-1}(Z_{i_t})]\rangle}\rho_{t-1}(Z_{j_t}) \\ Q_t(\ell) &= \langle \ell, [\rho_{t-1}(Z_{i_t}), \rho_{t-1}(Z_{j_t})]\rangle, \quad 1 \leq t \leq d. \end{aligned}$$

Then for each $1 \leq r \leq d$, one has ([10], Lemma 1.5):

$$P_{\mathbf{e},\mathbf{j},r}(\ell) = P_{\mathbf{e},\mathbf{j},r-1}(\ell)Q_r(\ell) = Q_1(\ell)Q_2(\ell)\cdots Q_r(\ell).$$

Note that for each t, Q_t depends only upon the restriction of ℓ to $\mathfrak{s}^0$, so we could just as well write $Q_t(\ell^0)$. At the same time, for $\mathfrak{s}^0$ we build the same mappings $\rho^0_r(\cdot,\ell^0)$ and $Q^0_t(\ell^0)$ getting:

$$P^0_{\mathbf{e}^0,\mathbf{j}^0,r}(\ell^0) = P^0_{\mathbf{e}^0,\mathbf{j}^0,r-1}(\ell^0)Q^0_r(\ell^0) = Q^0_1(\ell^0)Q^0_2(\ell^0)\cdots Q^0_r(\ell^0).$$

So we must describe the relationship between the functions Q_t and Q_t^0, or what is essentially the same, the relationship between the functions ρ_t and ρ_t^0 on $\mathfrak{s}^0$.

First of all, it is easy to check that for each t, the function ρ_t has the property that

$$\langle \ell, [\rho_t(X), Y] \rangle = \langle \ell, [\rho_t(X), \rho_t(Y)] \rangle = \langle \ell, [X, \rho_t(Y)] \rangle$$

holds for any X, $Y \in \mathfrak{s}$. Next, suppose that $n = j_k$ for some $1 \leq k \leq d$, and set $Y = \rho_{k-1}(Z_{i_k})$. Then $Y \in \mathfrak{s}_{i_k} \cap \mathfrak{h}_{k-1}(\ell)$ and for any $Z \in \mathfrak{s}^0$, $\rho_{k-1}(Z) \in \mathfrak{s}^0 \cap \mathfrak{h}_{k-1}(\ell)$, therefore by definition of j_k and the property above we have

$$\langle \ell, [Y, Z] \rangle = \langle \ell, [Y, \rho_{k-1}(Z)] \rangle = 0.$$

Thus $Y \in (\mathfrak{s}^0)^\ell$. We use this observation in the following lemma.

Lemma 1.4. *Suppose that n is a jump index for ℓ so that $n = j_k$ for some $1 \leq k \leq d$. Then for each index t, $0 \leq t \leq d$, one has the following:*
(a) if $t < k$, then $\rho_t(Z) = \rho_t^0(Z)$ holds for every $Z \in \mathfrak{s}^0$,
(b) if $t \geq k$, then $\rho_t(Z) = \rho_{t-1}^0(Z) \bmod \mathbb{C}Y$ holds for every $Z \in \mathfrak{s}^0$.

Proof. We proceed by induction on t. If $t = 0$, then $t < k$ and $\rho_0(Z) = Z = \rho_0^0(Z)$ holds for each $Z \in \mathfrak{s}^0$. Suppose then that $0 < t \leq d$ and assume that the lemma is true for $t-1$. If $t < k$, then $t - 1 < k$ also, and so by our assumption and Proposition 1.3, we have

$$\begin{aligned}
\rho_t(Z) &= \rho_{t-1}(Z) - \frac{\langle \ell, [\rho_{t-1}(Z), \rho_{t-1}(Z_{j_t})] \rangle}{\langle \ell, [\rho_{t-1}(Z_{i_t}), \rho_{t-1}(Z_{j_t})] \rangle} \rho_{t-1}(Z_{i_t}) \\
&\qquad\qquad\qquad - \frac{\langle \ell, [\rho_{t-1}(Z), \rho_{t-1}(Z_{i_t})] \rangle}{\langle \ell, [\rho_{t-1}(Z_{j_t}), \rho_{t-1}(Z_{i_t})] \rangle} \rho_{t-1}(Z_{j_t}) \\
&= \rho_{t-1}^0(Z) - \frac{\langle \ell^0, [\rho_{t-1}^0(Z), \rho_{t-1}^0(Z_{j_t^0})] \rangle}{\langle \ell^0, [\rho_{t-1}^0(Z_{i_t^0}), \rho_{t-1}^0(Z_{j_t^0})] \rangle} \rho_{t-1}^0(Z_{i_t^0}) \\
&\qquad\qquad\qquad - \frac{\langle \ell^0, [\rho_{t-1}^0(Z), \rho_{t-1}^0(Z_{i_t^0})] \rangle}{\langle \ell^0, [\rho_{t-1}^0(Z_{j_t^0}), \rho_{t-1}^0(Z_{i_t^0})] \rangle} \rho_{t-1}^0(Z_{j_t^0}) \\
&= \rho_t^0(Z).
\end{aligned}$$

Next suppose that $t = k$. Then by the observation preceding the lemma we have

$$\langle \ell, [\rho_{t-1}(Z), \rho_{t-1}(Z_{i_t})] \rangle = 0,$$

and by our assumption the statement *(a)* holds for $t - 1$. Hence

$$\rho_t(Z) = \rho_{t-1}^0(Z) - \frac{\langle \ell^0, [\rho_{t-1}(Z), \rho_{t-1}(Z_{j_t})] \rangle}{\langle \ell^0, [\rho_{t-1}(Z_{i_t}), \rho_{t-1}(Z_{j_t})] \rangle} \rho_{t-1}(Z_{i_t}) = \rho_{t-1}^0(Z) \mod \mathbb{C}Y.$$

Finally, suppose that $t > k$. Then our assumption entails that the statement *(b)* holds for $t - 1$. Combining this with our observation, we have that

$$\langle \ell, [\rho_{t-1}(Z), \rho_{t-1}(W)] \rangle = \langle \ell^0, [\rho_{t-2}^0(Z), \rho_{t-2}^0(W)] \rangle$$

holds for any Z, $W \in \mathfrak{s}^0$. Now apply Proposition 1.3 to see that for any $Z \in \mathfrak{s}^0$,

$$\begin{aligned}
\rho_t(Z) &= \rho_{t-1}(Z) - \frac{\langle \ell, [\rho_{t-1}(Z), \rho_{t-1}(Z_{j_t})]\rangle}{\langle \ell, [\rho_{t-1}(Z_{i_t}), \rho_{t-1}(Z_{j_t})]\rangle}\rho_{t-1}(Z_{i_t}) \\
&\qquad\qquad - \frac{\langle \ell, [\rho_{t-1}(Z), \rho_{t-1}(Z_{i_t})]\rangle}{\langle \ell, [\rho_{t-1}(Z_{j_t}), \rho_{t-1}(Z_{i_t})]\rangle}\rho_{t-1}(Z_{j_t}) \\
&= \rho^0_{t-2}(Z) - \frac{\langle \ell^0, [\rho^0_{t-2}(Z), \rho^0_{t-2}(Z_{j^0_{t-1}})]\rangle}{\langle \ell^0, [\rho^0_{t-2}(Z_{i^0_{t-1}}), \rho^0_{t-2}(Z_{j^0_{t-1}})]\rangle}\rho^0_{t-2}(Z_{i^0_{t-1}}) \\
&\qquad - \frac{\langle \ell^0, [\rho^0_{t-2}(Z), \rho^0_{t-2}(Z_{i^0_{t-1}})]\rangle}{\langle \ell^0, [\rho^0_{t-2}(Z_{j^0_{t-1}}), \rho^0_{t-2}(Z_{i^0_{t-1}})]\rangle}\rho^0_{t-2}(Z_{j^0_{t-1}}) \mod \mathbb{C}Y \\
&= \rho^0_{t-1}(Z) \mod \mathbb{C}Y.
\end{aligned}$$

This completes the proof. ∎

Now, it follows from Lemma 1.4 that for $1 \le t < k$, $Q^0_t(\ell^0) = Q_t(\ell^0)$, while for $k \le t < d$, $Q^0_t(\ell^0) = Q_{t+1}(\ell^0)$. Hence for each $1 \le r < k$, we have

$$P^0_{\mathbf{e}^0,\mathbf{j}^0,r}(\ell^0) = P_{\mathbf{e},\mathbf{j},r}(\ell),$$

while for $k \le r < d$,

$$P^0_{\mathbf{e}^0,\mathbf{j}^0,r}(\ell^0)Q_k(\ell) = P_{\mathbf{e},\mathbf{j},r+1}(\ell).$$

Thus the formula

$$P_{\mathbf{e},\mathbf{j}}(\ell) = P_{\mathbf{e}^0,\mathbf{j}^0}(\ell^0)Q_k(\ell^0)^{d-k+1}$$

holds. We sum up the preceding discussion as follows.

Theorem 1.5. *Let $\mathbf{e} = \{i_1 < \cdots < i_d,\ j_1, \dots, j_d\} \in \mathcal{E}$:*
1- if n is not in $\mathbf{e}$, then:

$$\Omega_{\mathbf{e},\mathbf{j}} = \pi^{-1}(\Omega^0_{\mathbf{e}^0,\mathbf{j}^0}) \cap \Omega_{\mathbf{e}}.$$

2- If n is in $\mathbf{e}$, so that $n = j_k$ for some $1 \le k \le d$, then:

$$\pi(\Omega_{\mathbf{e},\mathbf{j}}) = \Omega^0_{\mathbf{e}^0,\mathbf{j}^0} \cap \{\ell^0 \in \mathfrak{g}^{0*}:\ \ Q_k(\ell^0) \neq 0\}$$

where $Q_k(\ell^0) = \langle \ell^0, [\rho_{k-1}(Z_{i_k}), \rho_{k-1}(Z_{j_k})]\rangle$ and

$$\Omega_{\mathbf{e},\mathbf{j}} = \pi^{-1}(\pi(\Omega_{\mathbf{e},\mathbf{j}})).$$ ∎

2. Construction of canonical coordinates and the fine stratification

We begin with an excerpt from the proof of Théorème 1.6 in [2]. Let $\mathfrak{g}$ be an exponential solvable Lie algebra (over $\mathbb{R}$). Let $\mathfrak{g}^0$ be a codimension one ideal in $\mathfrak{g}$, and let $\pi : \mathfrak{g}^* \to \mathfrak{g}^{0*}$ be the restriction map. For $\ell \in \mathfrak{g}^*$, denote by $\mathcal{O}_\ell$ the G-orbit of ℓ, by $\mathcal{O}^0_{\ell^0}$ the G^0-orbit of $\ell^0 = \pi(\ell)$ and by $\mathfrak{g}(\ell)$ the Lie algebra of the stabilizer $G(\ell)$ of ℓ.

Lemma 2.1. *Let Ω be a G-invariant Borel subset of $\mathfrak{g}^*$ such that $\mathfrak{g}(\ell) \subset \mathfrak{g}^0$ holds for all $\ell \in \Omega$. Choose $X \in \mathfrak{g} \setminus \mathfrak{g}^0$ and choose a Borel cross-section $\Sigma \subset \Omega$ for the G-orbits in Ω, with $\sigma : \Omega \to \Sigma$ the G-invariant Borel cross-section map. Then there is a unique (Borel) function $q : \Omega \to \mathbb{R}$ such that*

$$\pi(\exp q(\ell)X\ell) \in \mathcal{O}^0_{\sigma(\ell)^0}$$

holds for each $\ell \in \Omega$.

Proof. For each $\ell \in \Omega$, there exists g in G such that:

$$\ell = g\sigma(\ell).$$

But g can be written in an unique way as $g = \exp(-qX)g^0$ with g^0 in G^0 and q in $\mathbb{R}$. Thus:

$$\pi(\exp(qX)\ell) = \pi(g^0\sigma(\ell)) = g^0\sigma(\ell)^0 \in \mathcal{O}^0_{\sigma(\ell)^0}.$$

Now, if q and q' satisfy this relation, then there are g^0 and g'^0 in G^0 such that:

$$\exp(qX)\ell^0 = g^0\sigma(\ell)^0, \quad \exp(q'X)\ell^0 = g'^0\sigma(\ell)^0.$$

Then:

$$\exp(-q'X)g'^0 {g^0}^{-1}\exp(qX) \in G(\ell^0).$$

But, in our case, $\mathfrak{g}(\ell)$ is of codimension 1 in $\mathfrak{g}^0(\ell^0) = \mathfrak{g}(\ell^0)$, thus $G(\ell^0) \subset G^0$ and the above relation can be written as:

$$\exp(q-q')Xg''^0 \in G(\ell^0) \subset G^0,$$

which implies $q' = q$.

Now q is a well defined function on Ω, it is proved in [2] that q is a Borel function. We shall not use this fact here. ∎

Note that the definition of q depends upon the choice of the cross-section Σ with cross-section map σ, as well as the choice of $X \in \mathfrak{g} \setminus \mathfrak{g}^0$.

From now on, we assume that $\mathfrak{g}$ is completely solvable. We choose the basis $\{Z_j\}$ real, so that $\mathfrak{g}_j$ is an ideal for all j. Let $\pi^j : \mathfrak{g}^* \to \mathfrak{g}_j^*$ be the restriction map and for $\ell \in \mathfrak{g}^*$, denote $\pi^j(\ell)$ by ℓ^j. We recall the definition of the partition $\{\mathfrak{g}^*_{(d)}\}$ of $\mathfrak{g}^*$ defined in [2]. For each $\ell \in \mathfrak{g}^*$, and $1 \le j \le n$, set

$$d_j(\ell) = \frac{1}{2}\dim G_j\ell^j,$$

and set $(d(\ell)) = (d_1(\ell), d_2(\ell), \ldots, d_n(\ell))$. For each non-decreasing n-tuple (d) of non-negative integers set

$$\Omega_{(d)} = \{\ell \in \mathfrak{g}^* : \ (d(\ell)) = (d)\}.$$

The non-empty $\Omega_{(d)}$ constitute the "dimension-based" partition of $\mathfrak{g}^*$; we shall call this partition the d-partition. Note that $\pi^j(\Omega_{(d)}) = \Omega_{(d_1,d_2,\ldots,d_j)} \subset \mathfrak{g}_j^*$ holds for each j. There is a natural ordering on the layers $\Omega_{(d)}$, given by reversing the lexicographic ordering on the n-tuples $(d_1,\ldots,d_n)$ ($(d_1,\ldots,d_n)<(d'_1,\ldots,d'_n)$ if $d_j>d'_j$ where j is the smallest index i for which $d_i\neq d'_i$). On the other hand, if $M_t(\ell)=(\langle\ell,[Z_i,Z_j]\rangle)_{1\le i,j\le t}$, then $2d_t$ is the rank of $M_t(\ell)$ thus for any (d),

$$\bigcup_{(d')\le(d)}\Omega_{(d')}$$

is Zariski-open in $\mathfrak{g}^*$.

Now fix $\Omega_{(d)}$. Because each $\mathfrak{g}_j$ is an ideal, it follows that $\Omega_{(d)}$ is G-invariant and hence coincides with the set $\mathfrak{g}^*_{(d)}$ defined in ([2], Définition 1.4).

The methods of ([2], Théorème 1.6) show that, for $\mathfrak{g}$ completely solvable, there is a global, but non explicit, construction of canonical coordinates for all orbits in $\Omega_{(d)}$. We shall give in section 4 an explicit version of this construction *i.e.* an explicit choice of canonical coordinates. Before that, we first recall briefly the methods used in [2]. In order to execute the method of [2], we must assume that for each $1\le j\le n$, we have some cross-section $\Sigma^j\subset\pi^j(\Omega_{(d)})$ for G_j-orbits in $\pi^j(\Omega_{(d)})$. Because the cross-sections are not precisely specified in this assumption, we shall call this method the "non parametric" construction.

Then ([2], Théorème 1.6) provides a method for construction of a (Borel) bijection:

$$\psi:(\Omega_{(d)}/G)\times\mathbb{R}^{2d}\to\Omega_{(d)}$$

with the following properties: for each $\mathcal{O}\in\Omega_{(d)}/G$, $\psi(\mathcal{O},\cdot)$ maps $\mathbb{R}^{2d}$ diffeomorphically onto $\mathcal{O}$, and if we write $\psi^{-1}(\ell)=\big(\mathcal{O}_\ell,p(\ell),q(\ell)\big)$, then the canonical 2-form ω on $\mathcal{O}$ is given by

$$\omega=\sum_{r=1}^{d}dp_r\wedge dq_r. \qquad (*)$$

Observe that the existence of such a map ψ is equivalent to the existence of a map $c:\Omega_{(d)}\to\mathbb{R}^{2d}$ with the property that for each $\mathcal{O}\in\Omega_{(d)}/G$, $c|_{\mathcal{O}}$ provides canonical coordinates in the sense of $(*)$. With this in mind we describe the method of [2] inductively. Suppose that

$$c^{n-1}=(p_1^{n-1},\ldots,p_{d_{n-1}}^{n-1},q_1^{n-1},\ldots,q_{d_{n-1}}^{n-1})$$

is given.

Case 1: $d_n=d_{n-1}$.

The restriction map π defines a diffeomorphism from each G-orbit $\mathcal{O}$ onto a G_{n-1}-orbit which preserves their canonical symplectic forms. Thus we put:

$$p_r=p_r^{n-1}\circ\pi,\quad q_r=q_r^{n-1}\circ\pi,\quad 1\le r\le d_n.$$

Case 2: $d_n=d_{n-1}+1$.

We keep our usual notations $\mathfrak{g}^0=\mathfrak{g}_{n-1}$, $\ell^0=\pi(\ell)$. We have the cross-section Σ^0 for $\pi(\Omega_{(d)})$. We denote by σ^0 the corresponding cross-section mapping. In this case, for each $\ell \in \Omega_{(d)}$, $\mathfrak{g}(\ell) \subset \mathfrak{g}^0$, the coadjoint orbit $\mathcal{O}_\ell$ of ℓ is:

$$\mathcal{O}_\ell = \{\exp(-qX_n)\ell' + pX_n^*\colon \ q \in \mathbb{R},\ p \in \mathbb{R},\ \ell' \in \mathcal{O}^0_{\sigma^0(\ell^0)}\},$$

and the hypothesis of Lemma 2.1 is satisfied in the obvious way. Set $d = d_n$ and set $q_d(\ell) = q(\ell)$, where $q(\ell)$ is the function obtained by Lemma 2.1. Then we put $p_d^n(\ell) = \langle \ell, X_n\rangle = \ell_n$, and for $1 \le r \le d_{n-1}$,

$$p_r(\ell) = p_r^{n-1}\big(\exp(-q(\ell)X_n)\ell\big), \quad q_r(\ell) = q_r^{n-1}\big(\exp(-q(\ell)X_n)\ell\big).$$

Given all the cross sections Σ^j for $\pi^j(\Omega_{(d)})$, the above becomes a construction of c: beginning with c^1 defined on $\pi^1(\Omega_{(d)})$ (necessarily as a trivial map since $d_1 = 0$), we then use the above to construct c^2 and then construct c^3 from c^2, and so on.

We will show that the methods of [9] and [10] can be combined with the above construction to get explicit analytic coordinate functions. First we establish the relationship between the d-partition and the fine layers $\Omega_{\mathbf{e},\mathbf{j}}$ of Section 1.

Lemma 2.2. *Fix any fine layer $\Omega_{\mathbf{e},\mathbf{j}}$ and choose any $1 \le t \le n$. Set*

$$\begin{aligned} R^t &= \{1 \le r \le d\colon\ j_r \le t\} = \{r_1^t < r_2^t < \cdots < r_a^t\},\\ \mathbf{i}^t &= \{i_{r_1^t}, i_{r_2^t}, \ldots, i_{r_a^t}\},\\ \mathbf{j}^t &= \{j_{r_1^t}, j_{r_2^t}, \ldots, j_{r_a^t}\},\\ \mathbf{e}^t &= \{i_r,\ j_r :\ r \in R^t\} \end{aligned}$$

and let $\Omega^t_{\mathbf{e}^t,\mathbf{j}^t}$ be the fine layer in $\mathfrak{g}_t^*$ corresponding to the data $\mathbf{e}^t$, $\mathbf{i}^t$ and $\mathbf{j}^t$. Then $\pi^t(\Omega_{\mathbf{e},\mathbf{j}}) \subset \Omega^t_{\mathbf{e}^t,\mathbf{j}^t}$.

Proof. Use induction on the dimension of $\mathfrak{g}$ and Proposition 1.3. ∎

In ([12], Section 4.2), N. V. Pedersen considers the sets

$$\Omega_{(\mathbf{e}^1,\mathbf{e}^2,\ldots,\mathbf{e}^n)} = \{\ell \in \mathfrak{g}^*\colon\ \ \mathbf{e}^t(\ell) = \mathbf{e}^t,\ \ 1 \le t \le n\}$$

where $(\mathbf{e}^1, \mathbf{e}^2, \ldots, \mathbf{e}^n)$ is a fixed n-tuple of jump indices. The non-empty such sets constitute what he calls the "fine stratification" of $\mathfrak{g}^*$. With this stratification, he gives, where $\mathfrak{g}$ is nilpotent a method to construct canonical coordinates for the orbits of $\Omega_{(\mathbf{e}^1,\mathbf{e}^2,\ldots,\mathbf{e}^n)}$. This method is different of the non-parametric construction of [2]. We shall see below that this stratification coincides with the fine stratification defined by the $\{\Omega_{\mathbf{e},\mathbf{j}}\}$.

Lemma 2.3. *Let $\ell, \ell' \in \mathfrak{g}^*$ such that for each $1 \le t \le n$, $\mathbf{e}^t(\ell) = \mathbf{e}^t(\ell')$. Then $\mathbf{j}(\ell) = \mathbf{j}(\ell')$.*

Proof. Suppose that $\mathbf{j}(\ell) \ne \mathbf{j}(\ell')$. We claim that $\mathbf{i}(\ell) \ne \mathbf{i}(\ell')$. Let $r_0 = \min\{1 \le r \le d\colon j_r(\ell) \ne j_r(\ell')\}$, say that $j_{r_0}(\ell) < j_{r_0}(\ell')$. Set $m = j_{r_0}(\ell)$, and set

$$R^m(\ell) = \{1 \le r \le d\colon j_r(\ell) \le m\}, \quad R^m(\ell') = \{1 \le r \le d\colon j_r(\ell') \le m\}.$$

Observe that, by virtue of our assumptions above, $r_0 \notin R^m(\ell')$. Also, by repeated application of Proposition 1.3, we find that $\mathbf{e}^m(\ell) = \{i_r(\ell), j_r(\ell)\colon r \in R^m(\ell)\}$, and similarly for ℓ'. Now since $\mathbf{e}^m(\ell) = \mathbf{e}^m(\ell')$, then $m \in \mathbf{e}^m(\ell')$ so that $m = j_{s_0}(\ell')$, for some s_0. By definition of r_0 we have $s_0 > r_0$. Now set $i = i_{r_0}(\ell)$; since $\mathbf{e}^m(\ell) = \mathbf{e}^m(\ell')$, i belongs to $\mathbf{e}^m(\ell')$, so $i \in \{i_u(\ell'), j_u(\ell')\}$ for some $u \in R^m(\ell')$. If $i = i_u(\ell')$, then our claim is evident. On the other hand if $i = j_u(\ell')$, then definition of r_0 implies $u > r_0$, hence $i_{r_0}(\ell') < i_u(\ell') < i$, so again our claim follows.

Having established that $\mathbf{i}(\ell) \neq \mathbf{i}(\ell')$, set

$$k_0 = \min\{1 \leq k \leq d\colon i_k(\ell) \neq i_k(\ell')\},$$

and let us assume that $i_{k_0}(\ell) < i_{k_0}(\ell')$. Set $m = i_{k_0}(\ell)$, and note that $k_0 \notin R^m(\ell)$ since $j_{k_0}(\ell) > i_{k_0}(\ell) = m$. Again, $\mathbf{e}^m(\ell) = \mathbf{e}^m(\ell')$ and now $m \notin \mathbf{e}^m(\ell)$, hence $m \notin \mathbf{e}^m(\ell')$. But $m \in \mathbf{e}(\ell')$, so $m = i_u(\ell')$. Definition of k_0 implies that $u > k_0$ hence $i_{k_0}(\ell) = i_u(\ell') > i_{k_0}(\ell')$, contradicting our assumption above. This completes the proof. ∎

Proposition 2.4. *Let $\mathfrak{g}$ be a completely solvable Lie algebra with a fixed Jordan-Hölder sequence $\{\mathfrak{g}_j\}$, let $\Omega_{\mathbf{e},\mathbf{j}}$ be a fine layer, and for each $1 \leq t \leq n$, let $\mathbf{e}^t$ be defined as in Lemma 2.2 and $d_t = \frac{1}{2}|e^t|$. Then*

$$\Omega_{\mathbf{e},\mathbf{j}} = \Omega_{(\mathbf{e}^1,\mathbf{e}^2,\ldots,\mathbf{e}^n)} \subset \Omega_{(d)} \cap \Omega_e.$$

Proof. Let $\ell \in \Omega_{\mathbf{e},\mathbf{j}}$; by Lemma 2.2, $\ell^t \in \Omega^t_{\mathbf{e}^t,\mathbf{j}^t}$, so $\mathbf{e}^t(\ell^t) = \mathbf{e}^t, 1 \leq t \leq n-1$, hence $\ell \in \Omega_{(\mathbf{e}^1,\mathbf{e}^2,\ldots,\mathbf{e}^n)}$. On the other hand, if $\ell \in \Omega_{(\mathbf{e}^1,\mathbf{e}^2,\ldots,\mathbf{e}^n)}$, then Lemma 2.3 shows that $\ell \in \Omega_{\mathbf{e},\mathbf{j}}$, and for each $1 \leq t \leq n$, $d_t(\ell^t) = \frac{1}{2}|e^t|$. ∎

Remark 2.1. The dimension sequence $(d_1, d_2, \ldots, d_n)$ can be obtained directly from the data of $\Omega_{\mathbf{e},\mathbf{j}}$ as follows. Let $\mathbf{r} = \mathbf{r}(\mathbf{j})$ be the increasing sequence whose values are those of $\mathbf{j}$ (that is, $\mathbf{r}$ is the rearrangement of $\mathbf{j}$ into increasing order). Define $(d) = (d_1 \leq d_2 \leq \cdots \leq d_n)$ by:

$$d_t = 0, \quad 1 \leq t < r_1, \qquad d_t = 1, \quad r_1 \leq t < r_2,$$
$$\cdots \qquad d_t = d, \quad r_d \leq t \leq n.$$

3. The Vergne construction

In [17], M. Vergne constructs canonical coordinates for generic coadjoint orbits in the dual of a nilpotent Lie algebra. In this section, we shall compare this construction with the fine stratification and the non-parametric construction.

We assume that $\mathfrak{g}$ is a nilpotent Lie algebra over $\mathbb{R}$. As usual we assume that we have chosen a Jordan-Hölder sequence $\{\mathfrak{g}_j\}$ of ideals in $\mathfrak{g}$. Let $\Omega_{\mathbf{e}}$ be the minimal coarse Zariski-open. Let $T_{\mathbf{e}} = \{\ell \in \mathfrak{g}^*\colon \ \ell(Z_j) = 0, \ \ \forall j \in \mathbf{e}\}$. As it is well-known, the set

$$\Omega_{\mathbf{e}} \cap T_{\mathbf{e}}$$

is a cross-section for coadjoint orbits in $\Omega_{\mathbf{e}}$, and the projection onto the cross-section is given by rational functions λ_i, $i \notin \mathbf{e}$, which are regular on $\Omega_{\mathbf{e}}$. Finally, each λ_i is of the form

$$\lambda_i(\ell) = \ell_i + f_i(\ell_1, \ell_2, \dots, \ell_{i-1}), \qquad \big(\ell_j = \ell(Z_j)\big)$$

and all G-invariant rational functions are rational combinations of the λ_i, $i \notin \mathbf{e}$. The same comments hold for the coarse partition of $\mathfrak{g}_t^*$, $1 \leq t \leq n-1$ corresponding to the sequence $\{\mathfrak{g}_j\colon \ j = 1, \dots, t\}$, and we let

$$\lambda_i^t, \quad i \notin \mathbf{e}^t$$

denote the corresponding rational functions. In [1] and [17], a construction of an invariant Zariski-open set Ω' and *rational* functions $p_1, p_2, \dots, p_d$, $q_1, q_2, \dots, q_d$ on Ω' is given so that for each coadjoint orbit $\mathcal{O}$ in Ω', $p_1|_{\mathcal{O}}, p_2|_{\mathcal{O}}, \dots, p_d|_{\mathcal{O}}$, $q_1|_{\mathcal{O}}, q_2|_{\mathcal{O}}, \dots, q_d|_{\mathcal{O}}$ are canonical coordinates for $\mathcal{O}$.

We now show that this procedure can be carried out so that Ω' coincides with the Zariski-open fine layer

$$\Omega_{\mathbf{e},\mathbf{j}} \subset \Omega_{\mathbf{e}} \cap \Omega_{(d)_0},$$

((d_0) being defined from $\mathbf{e}$, $\mathbf{j}$ as in the above remark) and so that the functions $p_1, p_2, \dots, p_d$, $q_1, q_2, \dots, q_d$ are precisely those obtained by the non parametric construction, restricted to $\Omega_{\mathbf{e},\mathbf{j}}$, and with the cross-sections indicated above.

Assume that, for $\mathfrak{g}^0 = \mathfrak{g}_{n-1}$, we have

$$\Omega'^0 = \Omega^0_{\mathbf{e}^0,\mathbf{j}^0} = \Omega_{(\mathbf{e}^1,\dots,\mathbf{e}^{n-1})}$$

and that

$$p_1^0, p_2^0, \dots, p_{d^0}^0, \quad q_1^0, q_2^0, \dots, q_{d^0}^0$$

are defined on Ω'^0 as indicated. Put $\pi = \pi^{n-1}$.

Case 1: there is a G-invariant polynomial function of the form $z(\ell) = \alpha(\ell)\ell_n + \beta(\ell)$ where $\alpha(\ell)$ is also G-invariant.

Let $\Omega' = \pi^{-1}(\Omega'^0) \cap \{\ell \in \mathfrak{g}^*\colon \ \alpha(\ell) \neq 0\}$. Then it is easily seen that $\mathfrak{g}(\ell) \not\subset \mathfrak{g}^0$ holds for all $\ell \in \Omega'$, and hence $\mathbf{e}^n(\ell) = \mathbf{e}^{n-1}(\ell)$ holds for all $\ell \in \Omega'$, hence $\Omega' \subset \Omega_{(\mathbf{e}^1,\mathbf{e}^2,\dots,\mathbf{e}^{n-1},\mathbf{e}^n)}$. This case coincides with the first case of the non-parametric construction. For each orbit $\mathcal{O}$ in Ω', the projection π restricted to $\mathcal{O}$ is injective, and the functions p_r, q_r are obtained by composing p_r^0, q_r^0 with π.

Now we *choose* the function $z(\ell)$ to be

$$z(\ell) = P^N_{\mathbf{e},\mathbf{j},d}(\ell)\lambda_n(\ell)$$

so that $\alpha(\ell) = P_{\mathbf{e},\mathbf{j},d}(\ell)^N$, where N is chosen large enough that z is polynomial. Then

$$\Omega' = \pi^{-1}(\Omega_{(\mathbf{e}^1,\mathbf{e}^2,\dots,\mathbf{e}^{n-1})}) \cap \{\ell \in \mathfrak{g}^*\colon \ P_{\mathbf{e},\mathbf{j},d}(\ell) \neq 0\} = \Omega_{\mathbf{e},\mathbf{j}}.$$

Case 2: case 1 does not hold.

Here there is a G_{n-1}-invariant polynomial function y on $\mathfrak{g}^{0*}$ which is not G-invariant, and which has the property that

$$\ell \mapsto [Z_n, y](\ell) := \frac{d}{dt}\bigg|_{t=0} y\big(\exp(-tZ_n)\ell\big)$$

is a non-zero G-invariant polynomial function, say z. It is easily seen that if $z(\ell) \neq 0$, then $\mathfrak{g}(\ell) \subset \mathfrak{g}^0$. One sets

$$q(\ell) = \frac{y(\ell)}{z(\ell)}, \qquad \varphi(\ell) = \pi\big(\exp(q(\ell)Z_n)\ell\big),$$

and

$$\Omega' = \varphi^{-1}({\Omega'}^0) \cap \{\ell \in \mathfrak{g}^* : \ z(\ell) \neq 0\}.$$

By virtue of our assumption that ${\Omega'}^0 = \Omega_{(\mathbf{e}^1, \mathbf{e}^2, \dots, \mathbf{e}^{n-1})}$, we have $\pi(\ell) \in {\Omega'}^0$ if and only if $\varphi(\ell) \in {\Omega'}^0$, so $\Omega' = \pi^{-1}({\Omega'}^0) \cap \{\ell \in \mathfrak{g}^* : \ z(\ell) \neq 0\}$.

Now since y is a rational combination of the G_{n-1} invariants λ_j^{n-1}, $j \notin \mathbf{e}^{n-1}$, there is j such that $[Z_n, \lambda_j^{n-1}] \neq 0$. Let i be the smallest such j. Then $[Z_n, \lambda_i^{n-1}]$ is G_{n-1} invariant and depends only upon $\pi^{i-1}(\ell)$, it must be a rational combination of the λ_j^{n-1}, $j < i$. By minimality of i, $[Z_n, \lambda_i^{n-1}] = z$ must be G-invariant. The point is that we can take $y = \lambda_i^{n-1}$, and we make this natural choice so that

$$z(\ell) = [Z_n, \lambda_i^{n-1}](\ell) \quad \text{ and } \quad q(\ell) = \frac{\lambda_i^{n-1}(\ell)}{z(\ell)}.$$

Let $\mathbf{e} = \mathbf{e}^{n-1} \cup \{i, n\}$; we claim that, with this choice of $y(\ell)$, $z(\ell)$, and $q(\ell)$ above, one has $\Omega' = \Omega_{(\mathbf{e}^1, \mathbf{e}^2, \dots, \mathbf{e}^{n-1}, \mathbf{e}^n)}$. Write $\mathbf{i}^{n-1} = \{i_1 < i_2 < \cdots < i_{d_{n-1}}\}$ and set $k - 1 = \max\{r : \ i_r < i\}$. Then for any $\ell \in \pi^{-1}(\Omega_{(\mathbf{e}^1, \mathbf{e}^2, \dots, \mathbf{e}^{n-1})})$, we have $\mathbf{e}(\ell) = \mathbf{e}^{n-1} \cup \{i, n\}$ if and only if $\langle \ell, [\rho_{k-1}(Z_n, \ell), Z_i] \rangle \neq 0$. Recall that $\rho_{k-1}(Z_n, \ell)$ is of the form $Z_n + W(\ell)$ where $W(\ell)$ belongs to $\mathfrak{g}_{n-1}$, and that $\rho_{k-1}(Z_n, \ell)$ belongs to $\mathfrak{g}_{i-1}^{\ell}$. Write

$$y(\ell) = \ell_i + f(\ell_1, \ell_2, \dots, \ell_{i-1}) = Z_i(\ell) + f(\ell).$$

Now since y is $\mathfrak{g}_{n-1}$-invariant, then $[W(\ell), y] = 0$ holds for each ℓ belonging to $\pi^{-1}(\Omega_{(\mathbf{e}^1, \mathbf{e}^2, \dots, \mathbf{e}^{n-1})})$. On the other hand, since $\rho_{k-1}(Z_n, \ell) \in \mathfrak{g}_{i-1}^{\ell}$ we have $[\rho_{k-1}(Z_n, \ell), f](\ell) = 0$ also. Hence

$$z(\ell) = [Z_n, y](\ell) = [\rho_{k-1}(Z_n, \ell), y](\ell) = [\rho_{k-1}(Z_n, \ell), Z_i](\ell).$$

Thus $z(\ell) \neq 0$ if and only if $\langle \ell, [\rho_{k-1}(Z_n, \ell), Z_i] \rangle \neq 0$. Our claim follows.

We sum up the preceding as follows:

Proposition 3.1. *Let $\mathfrak{g}$ be a nilpotent Lie algebra over $\mathbb{R}$, let $\{\mathfrak{g}_j : \ 1 \leq j \leq n\}$ be a Jordan Hölder sequence of ideals in $\mathfrak{g}$, and let $\Omega_{(\mathbf{e}^1, \mathbf{e}^2, \dots, \mathbf{e}^{n-1}, \mathbf{e}^n)}$ be the Zariski-open fine layer in $\mathfrak{g}^*$. Then there is a determination of the construction in ([17], Section 4, proof of Théorème) whereby the Zariski-open set obtained precisely coincides with the fine layer $\Omega_{(\mathbf{e}^1, \mathbf{e}^2, \dots, \mathbf{e}^{n-1}, \mathbf{e}^n)}$, and such that the functions $p_1, p_2, \dots, p_d$, $q_1, q_2, \dots, q_d$ obtained are precisely those of the non-parametric construction.* ∎

4. Explicit canonical coordinates

We return to the setup of Section 2 now: $\mathfrak{g}$ is completely solvable over $\mathbb{R}$, with the basis $\{Z_j\}$ chosen so that $\mathfrak{g}_j$ is an ideal for all j. For each j, $1 \le j \le n$, let γ_j be the real-valued homomorphism on $\mathfrak{g}$ defined by

$$[X, Z_j^*] = \gamma_j(X) Z_j^* \mod \operatorname{span}\{Z_{j+1}^*, \dots, Z_n^*\},$$

and let μ_j be the corresponding positive character of G: $\mu_j(\exp X) = \exp \gamma_j(X)$.

Our goal is to combine the methods of [2] with the explicit constructions of [10] (see also [8] and [9]) in order to obtain explicit formulas for globally-defined orbital canonical coordinates. We have seen that in the nilpotent case, the fine layering is sufficient for this purpose, but, as it is shown in [10], the completely solvable case requires a stratification that is further refined. We shall now describe this so-called "ultra-fine" stratification.

Let $\Omega_{\mathbf{e},\mathbf{j}}$ be a fine layer, with d the dimension of the orbits contained in $\Omega_{\mathbf{e},\mathbf{j}}$. For each $1 \le r \le d$, we have the mapping

$$\rho_r : \mathfrak{s} \times \Omega_{\mathbf{e},\mathbf{j}} \to \mathfrak{s}$$

as defined in Section 1, where, for each $\ell \in \Omega_{\mathbf{e},\mathbf{j}}$, the functions $\rho_r(\cdot, \ell)$ are used to compute Pfaffians corresponding to the alternating form β_ℓ. Define, for $1 \le r \le d$,

$$b_{i_r}(\ell) = \frac{\gamma_{i_r}(\rho_{r-1}(Z_{j_r}, \ell))}{\langle \ell, [Z_{i_r}, \rho_{r-1}(Z_{j_r}, \ell)]\rangle}.$$

Then b_{i_r} is a real semi-invariant rational function on Ω with multiplier $\mu_{i_r}^{-1}$, and the function $\theta_{i_r}(\ell) = \ell_{i_r} - b_{i_r}(\ell)^{-1}$ depends only upon $\ell_1, \ell_2, \dots, \ell_{i_r - 1}$ ([10], Lemma 4.1). For $\ell \in \Omega_{\mathbf{e},\mathbf{j}}$, set $\varphi(\ell) = \{i \in \mathbf{i} : b_i(\ell) \neq 0\}$ and for each subset φ of $\mathbf{i}$, set

$$\Omega_{\mathbf{e},\mathbf{j},\varphi} = \{\ell \in \Omega_{\mathbf{e},\mathbf{j}} : \varphi(\ell) = \varphi\}.$$

The non-empty layers $\Omega_{\mathbf{e},\mathbf{j},\varphi}$ constitute a G-invariant partition of $\mathfrak{g}^*$ that we call the ultra-fine stratification corresponding to the basis $\{Z_j\}$ chosen.

Now fix an ultra-fine layer $\Omega = \Omega_{\mathbf{e},\mathbf{j},\varphi}$. We employ the results of [10], as described in, and with the notation of [8], with simplifications appropriate for the completely solvable case. Write $\mathbf{e} = \{e_1 < e_2 < \dots < e_{2d}\}$, Then functions $r_b : \Omega \to \mathfrak{g}, 1 \le b \le 2d$, are defined so that for each $\ell \in \Omega$,

$$t \mapsto \exp(t_1 r_1(\ell)) \exp(t_2 r_2(\ell)) \cdots \exp(t_{2d} r_{2d}(\ell))\ell \qquad (*)$$

defines an analytic diffeomorphism of $\mathbb{R}^{2d}$ with the coadjoint orbit of ℓ. If e_b is a value of the sequence $\mathbf{i}$, say $e_b = i_r$, then $r_b(\ell) \in \mathfrak{g}$ has the form

$$r_b(\ell) = \frac{\rho_{r-1}(Z_{j_r}, \ell)}{\langle \ell, [Z_{i_r}, \rho_{r-1}(Z_{j_r}, \ell)]\rangle}.$$

If $e_b = j_r$, the formula for r_b is the same but with the letters "i" and "j" reversed. A substitution procedure (described in the proof of Lemma 4.1 below) transforms $(*)$ into a G-invariant map $P^* : \Omega \to \Omega$ whose image Σ is a cross-section for the coadjoint orbits in Ω. Write $P^*(\ell) = \sum_j P_j^*(\ell) Z_j^*$. Then for each $1 \le j \le n$, explicit, real-valued functions $\mu_j(\ell)$ and $y_j(\ell)$ are constructed on Ω, so that

$$P_j^*(\ell) = \begin{cases} \mu_j(\ell)\ell_j + y_j^\circ(\ell), & j \notin \mathbf{e} \\ 0, & j \in \mathbf{e}, j \notin \varphi \\ \operatorname{sign}(b_j(\ell)) \; + y_j(\ell), & j \in \varphi. \end{cases}$$

It is shown ([8], Corollary 1.3.14) that for each $\ell \in \Omega$, $\mu_j(\ell)$ and $y_j(\ell)$ depend only upon $\ell_1, \ell_2, \ldots, \ell_{j-1}$. Let $V = \operatorname{span}\{Z_j^* : j \notin \mathbf{e} \text{ or } j \in \varphi\}$. The orbital cross-section $\Sigma = P^*(\Omega) \subset \Omega$ is precisely the set

$$\Sigma = \{\ell \in \Omega \cap V : |b_i(\ell)| = 1, \text{ if } i \in \varphi\}.$$

If Ω is not the minimal layer in the ultra-fine stratification, then it is not an open subset of $\mathfrak{g}^*$, and may not even be a submanifold. With the following lemma, we extend the construction of P^* to an open set containing Ω.

Lemma 4.1. *Associated to the index data* $\mathbf{e}, \mathbf{j}, \varphi$*, there is an explicit, dense open subset* $\mathcal{U}$ *of* $\mathfrak{s}^* = (\mathfrak{g}^*)_{\mathbb{C}}$*, and an explicit analytic function* $\sigma : \mathcal{U} \to \mathfrak{s}^*$ *whose restriction to* Ω *is* P^*.

Proof. For each $\ell \in \mathfrak{s}^*$ let $\rho_0(Z, \ell) = Z$, set

$$\mathcal{U}_1 = \{\ell \in \mathfrak{s}^* : \langle \ell, [Z_{i_1}, Z_{j_1}] \rangle \neq 0\}$$

and $\mathfrak{m}_1 = \operatorname{span}\{Z_{i_1}, Z_{j_1}\}$. For each $\ell \in \mathcal{U}_1$, we have $\mathfrak{m}_1 \cap \mathfrak{m}_1^\ell = (0)$, and we let $\rho_1(Z, \ell)$ be the projection of Z into $\mathfrak{m}_1^\ell$ parallel to $\mathfrak{m}_1$. Set $\mathcal{U}_2 = \{\ell \in \mathcal{U}_1 : \langle \ell, [Z_{i_2}, \rho_1(Z_{j_2}, \ell)] \rangle \neq 0\}$ and $\mathfrak{m}_2(\ell) = \mathfrak{m}_1 +$ span $\{\rho_1(Z_{i_2}, \ell), \rho_1(Z_{j_2}, \ell)\}$. For each $\ell \in \mathcal{U}_2$, let $\rho_2(Z, \ell)$ be the projection of Z into $\mathfrak{m}_2(\ell)^\ell$ parallel to $\mathfrak{m}_2(\ell)$. Continuing in this way, we obtain a Zariski open set $\mathcal{U}_d$ of $\mathfrak{s}^*$ and for $0 \le r \le d-1$, a rational projection function $\rho_r : \mathfrak{s} \times \mathcal{U}_d \to \mathfrak{s}$. (We abuse notation slightly here, as the restriction of ρ_r to $\mathfrak{s} \times \Omega$ is also called ρ_r above.) For each $i = i_r \in \mathbf{i}$ and $\ell \in \mathcal{U}_d$, we set

$$b_i(\ell) = \frac{\gamma_i(\rho_{r-1}(Z_{j_r}, \ell))}{\langle \ell, [Z_i, \rho_{r-1}(Z_{j_r}, \ell)] \rangle}.$$

For each $i \in \varphi$, $b_i(\ell)$ is a rational function that is non-singular, hence analytic, on $\mathcal{U}_d$. Set

$$\mathcal{U} = \{\ell \in \mathcal{U}_d : Re(b_i(\ell)) \neq 0, i \in \varphi\}.$$

It is clear that $\Omega \subset \mathcal{U}$. The first step in the construction of σ is to extend certain functions that are used in the construction of P^* to the set $\mathcal{U}$. (We have already done this for the functions b_i.) For each $1 \le b \le 2d$, we define $r_b(\ell)$, $\ell \in \mathcal{U}$ by exactly the same formula as above; it is clear from this formula that $\ell \mapsto r_b(\ell)$ is analytic at each point of $\mathcal{U}$. Also for each $i = i_r \in \mathbf{i}$ we have that $\theta_i(\ell) = \ell_i - b_i(\ell)^{-1}$ is analytic on $\mathcal{U}$.

Now fix $j, 1 \le j \le n$, and let a be the smallest subindex b such that $e_b \ge j$ (if $j > e_{2d}$ set $a = 2d+1$). For each multi-index $q \in \{0,1,2,\ldots\}^{a-1}$, set

$$u_j^\circ(q,\ell) = \langle \ell, p_{j-1}\big(ad(r_{a-1}(\ell))^{q_{a-1}} \cdots ad(r_2(\ell))^{q_2} ad(r_1(\ell))^{q_1}(Z_j)\big)\rangle$$

where $p_{j-1} : \mathfrak{s} \to \mathfrak{s}_{j-1}$ is projection parallel to span$\{Z_j, \ldots, Z_n\}$. It is clear that for each multi-index q, $u_j^\circ(q,\ell)$ is analytic at each ℓ in $\mathcal{U}$. For each $t \in \mathbb{R}^{a-1}$, set

$$y_j^\circ(t,\ell) = \sum_{q \ne (0,0,\ldots,0)} \frac{t^q}{q!}\, u_j^\circ(q,\ell).$$

To prove convergence of this series, we let $\mathfrak{s}$ and $\mathfrak{s}^*$ have the Euclidean metrics obtain by their identifications with $\mathbb{C}^n$ via their respective bases $\{Z_j\}$ and $\{Z_j^*\}$. Let B be any compact subset of $\mathcal{U}$. For each $1 \le b \le a-1$, we have $M_b > 0$ such that $\|ad\ r_b(\ell)\| < M_b$ holds for all $\ell \in B$. Thus

$$\|p_{j-1}\Big((ad(r_{a-1}(\ell))^{q_{a-1}} \cdots ad(r_2(\ell))^{q_2} ad(r_1(\ell))^{q_1}(Z_j)\Big)\| < M_1^{q_1} M_2^{q_2} \cdots M_{a-1}^{q_{a-1}}$$

holds for all $\ell \in B$. Now let $N > 0$ such that $\|\ell\| < N$ holds in B; then for each multi-index q and $\ell \in B$ we have $|u_j^\circ(q,\ell)| < N\ M_1^{q_1} M_2^{q_2} \cdots M_{a-1}^{q_{a-1}}$. Now for all $t \in \mathbb{C}^{a-1}$ and $\ell \in B$ we have

$$\begin{aligned}\sum_{q \ne (0,\ldots,0)} \Big|\frac{t^q}{q!} u_j^\circ(q,\ell)\Big| &\le N \sum_{q \ne (0,\ldots,0)} \Big|\frac{t^q}{q!}\Big|\ M_1^{q_1} M_2^{q_2} \cdots M_{a-1}^{q_{a-1}} \\ &= N\big(\ e^{|t_1|M_1} e^{|t_2|M_2} \cdots e^{|t_{a-1}|M_{a-1}} - 1\big).\end{aligned}$$

Hence the power series for $y_j^\circ(t,\ell)$ converges absolutely on each set $\mathbb{C}^{a-1} \times B$. In particular, $y_j^\circ(t,\ell)$ is analytic at each point of $\mathbb{C}^{a-1} \times \mathcal{U}$.

Finally, set

$$\mu_j(t,\ell) = \exp\left(\sum_{b=1}^{a-1} t_b\ \gamma_j\big(r_b(\ell)\big)\right),$$

It is clear that $\mu_j(t,\ell)$ is also analytic at each point of $\mathbb{C}^{a-1} \times \mathcal{U}$ (and that it is defined by a convergent power series of a similar form as $y_j^\circ(t,\ell)$). This completes the first step of the construction. We pause to remark that the restrictions of the functions above to Ω have a number of special properties (see [8], [10]) that do not necessarily continue to hold for the extended functions.

The second step is to define a function $Q_j : \mathbb{C}^{2d} \times \mathcal{U} \to \mathfrak{s}^*$ whose restriction to $\mathbb{R}^{2d} \times \Omega$ parametrizes each orbit: $\mathcal{O}_\ell = \{Q(t,\ell) : t \in \mathbb{R}^{2d}\}$. Write $Q(t,\ell) = \sum_{j=1}^n Q_j(t,\ell) Z_j^*$, and fix $1 \le j \le n$. Then Q_j is defined as follows. If $j \notin \mathbf{e}$ set

$$Q_j(t,\ell) = \mu_j(t,\ell)\ell_j + y_j^\circ(t,\ell).$$

If $j \in \mathbf{e}$, then set

$$Q_j(t,\ell) = \mu_j(t,\ell)\big(\ell_j + t_a\big) + y_j^\circ(t,\ell),$$

if $j \notin \varphi$ while if $j \in \varphi$ set

$$Q_j(t,\ell) = \mu_j(t,\ell)\big(e^{t_a\gamma_j(r_a(\ell))}b_j(\ell)^{-1} + \theta_j(\ell)\big) + y_j^\circ(t,\ell).$$

It is clear from the preceding that for each j, Q_j is analytic on $\mathbb{C}^{2d} \times \mathcal{U}$. Now consider the restriction of Q_j to $\mathbb{R}^{2d} \times \Omega$. An examination of ([8], the proof of Lemma 1.3.12, and Proposition 1.3.13), shows that for each $\ell \in \Omega$,

$$Q(t,\ell) = \exp(t_1r_1(\ell))\exp(t_2r_2(\ell))\cdots\exp(t_{2d}r_{2d}(\ell))\ell,$$

and Q parametrizes each orbit as claimed.

The next step is to define substitution functions $\Phi_a(\ell), 1 \le a \le 2d$. For each $\epsilon \in \{-1,1\}^\varphi$, set $\mathcal{U}_\epsilon = \{\ell \in \mathcal{U} : \mathrm{sign}(Re(b_i(\ell))) = \epsilon_i, i \in \varphi\}$ so that $\mathcal{U} = \cup\, \mathcal{U}_\epsilon$. Choose a branch of the logarithm on $\mathbb{C}$ that is analytic on $Re(z) > 0$. For $a = 1$, if $j = e_1 \notin \varphi$, then set $\Phi_1(\ell) = -\ell_j$, while if $j = e_a \in \varphi$, then for $\ell \in \mathcal{U}_\epsilon$, set $\Phi_1(\ell) = (\gamma_j(r_1(\ell))^{-1}\log(\epsilon_j b_j(\ell))$. It is clear that these functions are analytic at each point of $\mathcal{U}$. Assume that for $1 \le b \le a-1$, $\Phi_b(\ell)$ is defined and anaytic on $\mathcal{U}$. Set $y_j^\circ(\ell) = y_j^\circ(\Phi(\ell),\ell)$. Then y_j°, being the composition of the analytic functions $\ell \to (\Phi_1(\ell),\Phi_2(\ell),\ldots,\Phi_{a-1}(\ell),\ell)$ and $y_j^\circ(t,\ell)$, is analytic on $\mathcal{U}$. Similarly we set $\mu_j(\ell) = \mu_j(\Phi_1(\ell),\Phi_2(\ell),\ldots,\Phi_{a-1}(\ell),\ell)$ and we see that μ_j is analytic on $\mathcal{U}$. Now if $j = e_a \notin \varphi$, set

$$\Phi_a(\ell) = -y_j^\circ(\ell)\mu_j(\ell)^{-1} - \ell_j$$

while if $j = e_a \in \varphi$, then

$$\Phi_a(\ell) = \gamma_j(r_a(\ell))^{-1}\log\left(\mu_j(\ell)^{-1}\epsilon_j b_j(\ell)\right).$$

In light of the preceding we have that $\Phi_a(\ell)$ is analytic at each point ℓ in $\mathcal{U}$.

We are now in a position to construct the mapping σ: set

$$\sigma_j(\ell) = Q_j(\Phi(\ell),\ell),\ \ell \in \mathcal{U}.$$

It is clear from our work thus far in this proof that σ is analytic at each point ℓ in $\mathcal{U}$. To see that the restriction of σ to Ω is P^*, we rely upon the description of the construction of P^* in [10] (or [8] for a description with notation more closely matching the present notation). By a substitution procedure, explicit functions $\Phi_a(z_1,z_2,\ldots,z_a,\ell), 1 \le a \le 2d$ are obtained so that $Q(t,\ell)$ is transformed into a function $P(z_1,z_2,\ldots,z_{2d},\ell)$:

$$P(z,\ell) = Q(\Phi_1(z,\ell),\ldots,\Phi_{2d}(z,\ell),\ell).$$

The function P^* is then defined as $P^*(\ell) = P(z^*,\ell)$ where $z_a^* = 0$ if $e_a \notin \varphi$ and $z_a^* = \epsilon_j$ if $j = e_a \in \varphi$, $\ell \in \Omega_\epsilon$. For the functions $\Phi_a(\ell)$ defined above, an examination of the functions $\Phi_a(z^*,\ell)$ from [10] shows that for each $\ell \in \Omega$, $\Phi_a(z^*,\ell) = \Phi_a(\ell)$. Thus for each $\ell \in \Omega$,

$$P^*(\ell) = P(z^*,\ell) = Q(\Phi(z^*,\ell),\ell) = \sigma(\ell). \qquad \blacksquare$$

Remark 4.1. For every $1 \le j \le n$, the complex function σ_j has the same form as its real-valued restriction P_j^* to Ω: for $\ell \in \mathcal{U}$,

$$\sigma_j(\ell) = \begin{cases} \mu_j(\ell)\ell_j + y_j^\circ(\ell), & j \notin \mathbf{e} \\ 0, & j \in \mathbf{e}, j \notin \varphi \\ \operatorname{sign}(b_j(\ell)) \ + y_j(\ell), & j \in \varphi. \end{cases}$$

where, in the case $j \in \varphi$, $y_j(\ell) = \mu_j(\ell)\theta_j(\ell) + y_j^\circ(\ell)$.

We observe that for each $1 \le t < n$, this procedure can be carried out for the Lie algebra $\mathfrak{g}_t$, with respect to the basis $Z_j, 1 \le j \le t$, yielding the ultra-fine layers $\Omega^t_{\mathbf{e}^t,\mathbf{j}^t,\varphi^t}$, and for each layer, the associated Zariski open set $\mathcal{U}^t$, the cross-section map σ^t, and the cross-section Σ^t. In order to apply the non parametric construction of [2], we must verify that the projection π^t maps each ultra-fine layer Ω into the corresponding layer Ω^t.

Fix t, $1 \le t \le n-1$, and recall the index set $R^t = \{r_1 < r_2 < \cdots < r_{d_t}\}$ from Lemma 2.2, so that, with $\mathbf{i}^t$ defined by $i^t_a = i_{r_a}$, $j^t_a = j_{r_a}$, $1 \le a \le d_t$, and $\mathbf{e}^t$ the union of the values of $\mathbf{i}^t$ and $\mathbf{j}^t$, we have $\pi^t(\Omega_{\mathbf{e},\mathbf{j}}) \subset \Omega^t_{\mathbf{e}^t,\mathbf{j}^t}$. We also have the mappings $\rho^t_a : \mathfrak{g}_t \times \Omega^t_{\mathbf{e}^t,\mathbf{j}^t} \to \mathfrak{g}_t$, and in light of Lemma 2.2, we shall describe the relationship between the maps ρ_r and ρ^t_a. First we set some additional notation for convenience. Having fixed $\ell \in \Omega_{\mathbf{e},\mathbf{j}}$, we write $\rho_r(Z) = \rho_r(Z,\ell)$ and $\rho^t_a(Z) = \rho^t_a(Z,\ell^t)(Z \in \mathfrak{s}_t)$. For $1 \le r \le d$ put

$$Y_r = \rho_{r-1}(Z_{i_r}),\ X_r = \rho_{r-1}(Z_{j_r}),$$

and for $1 \le a \le d_t$, put

$$Y^t_a = \rho^t_{a-1}(Z_{i_{r_a}}),\ X^t_a = \rho^t_{a-1}(Z_{j_{r_a}}).$$

Then for each $1 \le r \le d$ one has

$$\rho_r(Z) = Z - \sum_{s=1}^{r} \frac{\langle \ell, [Z, X_s]\rangle}{\langle \ell, [Y_s, X_s]\rangle} Y_s - \sum_{s=1}^{r} \frac{\langle \ell, [Z, Y_s]\rangle}{\langle \ell, [X_s, Y_s]\rangle} X_s$$

and similarly for $\rho^t_a(Z)$.

Lemma 4.2. *Fix $\ell \in \Omega_{\mathbf{e},\mathbf{j}}$ with $\ell^t = \pi^t(\ell)$. For $1 \le a \le d_t$, set*

$$\mathcal{Y}^t_a = \operatorname{span}\{Y_r : r \notin R^t, r < r_a\}.$$

Then one has the following.
(a) $\mathcal{Y}^t_a \subset \mathfrak{s}^\ell_t$.
(b) If $r_{a-1} < r < r_a$, then for $Z \in \mathfrak{s}_t$, $\rho_r(Z) = \rho^t_{a-1}(Z) \mathrm{mod} \mathcal{Y}^t_a$.
(c) For $Z \in \mathfrak{s}_t$, $1 \le a \le d_t$,

$$\langle \ell, [Z, Y_{r_a}]\rangle = \langle \ell^t, [Z, Y^t_a]\rangle, \quad \langle \ell, [Z, X_{r_a}]\rangle = \langle \ell^t, [Z, X^t_a]\rangle,$$
$$\rho_{r_a}(Z) = \rho^t_a(Z) \mod \mathcal{Y}^t_a.$$

Proof. We proceed by induction on $1 \le a \le d_t$; first suppose that $a = 1$. Let Z be any element of $\mathfrak{s}_t$. For each $r < r_1$, we have $t < j_r$ so by definition of j_r,

$$\mathfrak{s}_t \cap \mathfrak{h}_{r-1}(\ell) \subset \mathfrak{h}_r(\ell).$$

Now $\rho_{r-1}(Z) \in \mathfrak{s}_t \cap \mathfrak{h}_{r-1}(\ell)$ so we have

$$\langle \ell, [Z, Y_r] \rangle = \langle \ell, [\rho_{r-1}(Z), Y_r] \rangle = 0.$$

Hence $\mathfrak{s}_t \subset \{Y_r\}^\ell$ so $Y_r \in (\mathfrak{s}_t)^\ell$. We conclude from the preceding that $\mathcal{Y}_1^t \subset (\mathfrak{s}_t)^\ell$. It follows that for $r < r_a$,

$$\rho_r(Z) = Z - \sum_{s=1}^{r} \frac{\langle \ell, [Z, X_s] \rangle}{\langle \ell, [Y_s, X_s] \rangle} Y_s = \rho_0^t(Z) \mod \mathcal{Y}_1^t.$$

Now we apply this formula to $Z_{i_{r_1}}$ and $Z_{j_{r_1}}$ to get

$$\langle \ell, [Z, Y_{r_1}] \rangle = \langle \ell, [Z, \rho_{r_1-1}(Z_{i_{r_1}})] \rangle = \langle \ell, [Z, \rho_0^t(Z_{i_{r_1}})] \rangle = \langle \ell, [Z, Y_1^t] \rangle,$$

as well as $\langle \ell, [Z, X_{r_1}] \rangle = \langle \ell, [Z, X_1^t] \rangle$ (in a similar way), and then

$$\begin{aligned}
\rho_{r_1}(Z) &= \rho_{r_1-1}(Z) - \frac{\langle \ell, [Z, X_{r_1}] \rangle}{\langle \ell, [Y_{r_1}, X_{r_1}] \rangle} Y_{r_1} - \frac{\langle \ell, [Z, Y_{r_1}] \rangle}{\langle \ell, [X_{r_1}, Y_{r_1}] \rangle} X_{r_1} \\
&= \rho_0^t(Z) - \frac{\langle \ell, [Z, X_1^t] \rangle}{\langle \ell, [Y_1^t, X_1^t] \rangle} Y_1^t - \frac{\langle \ell, [Z, Y_1^t] \rangle}{\langle \ell, [X_1^t, Y_1^t] \rangle} X_1^t \mod \mathcal{Y}_1^t \\
&= \rho_1^t(Z) \mod \mathcal{Y}_1^t.
\end{aligned}$$

The lemma is now verified for $a = 1$.

Suppose then that $a > 1$ and that the lemma holds for $a - 1$. Let $Z \in \mathfrak{s}_t$ and $r_{a-1} < r < r_a$. Again by definition of j_r it follows that $Z \in \{Y_r\}^\ell$ so $Y_r \in (\mathfrak{s}_t)^\ell$. Having $\mathcal{Y}_{a-1}^t \subset (\mathfrak{s}_t)^\ell$ by induction, we conclude that $\mathcal{Y}_a^t \subset (\mathfrak{s}_t)^\ell$. Hence

$$\begin{aligned}
\rho_r(Z) &= \rho_{r_{a-1}}(Z) - \sum_{s=r_{a-1}+1}^{r} \frac{\langle \ell, [Z, X_s] \rangle}{\langle \ell, [Y_s, X_s] \rangle} Y_s - \sum_{s=r_{a-1}+1}^{r} \frac{\langle \ell, [Z, Y_s] \rangle}{\langle \ell, [X_s, Y_s] \rangle} X_s \\
&= \rho_{a-1}^t(Z) - \sum_{s=r_{a-1}+1}^{r} \frac{\langle \ell, [Z, X_s] \rangle}{\langle \ell, [Y_s, X_s] \rangle} Y_s \mod \mathcal{Y}_{a-1}^t \\
&= \rho_{a-1}^t(Z) \mod \mathcal{Y}_a^t.
\end{aligned}$$

As in the case $a = 1$, we apply this formula with $r = r_a - 1$ to $Z_{i_{r_a}}$ and $Z_{j_{r_a}}$ to get

$$\langle \ell, [Z, Y_{r_a}] \rangle = \langle \ell, [Z, \rho_{r_a-1}(Z_{i_{r_a}})] \rangle = \langle \ell, [Z, \rho_{a-1}^t(Z_{i_{r_a}})] \rangle = \langle \ell, [Z, Y_a^t] \rangle,$$

similarly $\langle \ell, [Z, X_{r_a}] \rangle = \langle \ell, [Z, X_a^t] \rangle$, and finally

$$\begin{aligned}
\rho_{r_a}(Z) &= \rho_{r_a-1}(Z) - \frac{\langle \ell, [Z, X_{r_a}] \rangle}{\langle \ell, [Y_{r_a}, X_{r_a}] \rangle} Y_{r_a} - \frac{\langle \ell, [Z, Y_{r_a}] \rangle}{\langle \ell, [X_{r_a}, Y_{r_a}] \rangle} X_{r_a} \\
&= \rho_{a-1}^t(Z) - \frac{\langle \ell, [Z, X_a^t] \rangle}{\langle \ell, [Y_a^t, X_a^t] \rangle} Y_a^t - \frac{\langle \ell, [Z, Y_a^t] \rangle}{\langle \ell, [X_a^t, Y_a^t] \rangle} X_a^t \mod \mathcal{Y}_a^t \\
&= \rho_a^t(Z) \mod \mathcal{Y}_a^t.
\end{aligned}$$

This completes the proof. ∎

We observe that for each $1 \leq a \leq d_t$, $\mathcal{Y}_a^t \subset \text{Ker } \gamma_{i_{r_a}}$. With this in mind, an application of Lemma 4.2 (b) and (c), when $Z = Z_{i_{r_a}}$ and with $r_a = r_a^t$, shows that for $\ell \in \Omega_{\mathbf{e},\mathbf{j}}$,

$$b_i^t(\ell^t) = \frac{\gamma_{i_{r_a}}(\rho_{a-1}^t(Z_{j_{r_a}}, \ell^t))}{\langle \ell^t, [Z_{i_{r_a}}, \rho_{a-1}^t(Z_{j_{r_a}}, \ell^t)]\rangle} = \frac{\gamma_{i_{r_a}}(\rho_{r_a-1}(Z_{j_{r_a}}, \ell))}{\langle \ell, [Z_{i_{r_a}}, \rho_{r_a-1}(Z_{j_{r_a}}, \ell)]\rangle} = b_i(\ell)$$

The following is now immediate.

Proposition 4.3. *Let $\mathfrak{g}$ be a completely solvable Lie algebra over $\mathbb{R}$, choose a real Jordan-Hölder basis for $\mathfrak{g}$, and let $\Omega = \Omega_{\mathbf{e},\mathbf{j},\varphi}$ be an ultra-fine layer in $\mathfrak{g}^*$. Fix $t, 1 \leq t < n$ and set*

$$\varphi^t = \mathbf{i}^t \cap \varphi.$$

Then $\pi^t(\Omega) \subset \Omega_{\mathbf{e}^t,\mathbf{j}^t,\varphi^t}$ and $\pi^t(\Sigma) \subset \Sigma^t$. Hence the associated cross-section map σ^t is defined and analytic on the neighborhood $\pi^t(\mathcal{U}) \cap \mathcal{U}^t$ of $\pi^t(\Omega)$. ■

Lemma 4.4. *For the data $\mathbf{e}$, $\mathbf{j}$ and φ associated to the ultra-fine layer Ω, let $\mathcal{U}$ and $\mathcal{U}^{n-1}$ be the open sets associated with $\mathbf{e}$, $\mathbf{j}, \varphi$ and $\mathbf{e}^{n-1}$, $\mathbf{j}^{n-1}, \varphi^{n-1}$ (resp.) via Lemma 4.1. Suppose that $n \in \mathbf{e}$, so that we are in the situation of Lemma 2.1. Set $n = j_k$, $i = i_k$, and set $\pi = \pi^{n-1}$. Define $q(\ell)$ on $\mathcal{U} \cap \pi^{-1}(\mathcal{U}^{n-1})$ as follows:*
(1) If $i \notin \varphi$,

$$\begin{aligned} q(\ell) &= \frac{\sigma_i^{n-1}(\pi(\ell))}{\mu_i(\ell)\langle \ell, [\rho_{k-1}(Z_n, \ell), Z_i]\rangle} \\ &= \frac{\ell_i}{\langle \ell, [\rho_{k-1}(Z_n, \ell), Z_i]\rangle} + \frac{y_i^{n-1}(\pi(\ell))}{\mu_i^{n-1}(\pi(\ell))\langle \ell, [\rho_{k-1}(Z_n, \ell), Z_i]\rangle}. \end{aligned}$$

(2) If $i \in \varphi$,

$$q(\ell) = \frac{\log(\epsilon_i b_i(\ell))}{\gamma_i\big(\rho_{k-1}(Z_n, \ell)\big)} - \frac{\log\big(\mu_i^{n-1}(\pi(\ell))\big)}{\gamma_i\big(\rho_{k-1}(Z_n, \ell)\big)}.$$

Then $q(\ell)$ is analytic on $\mathcal{U} \cap \pi^{-1}(\mathcal{U}^{n-1})$, and its restriction to Ω is precisely the function of Lemma 2.1, associated to the cross-section Σ.

Proof. It is clear from the proof of Lemma 4.1 that $q(\ell)$ is analytic on $\mathcal{U} \cap \pi^{-1}(\mathcal{U}^{n-1})$, so we need only prove that its restriction to Ω satisfies the conditions for the function described in Lemma 2.1.

Let $\ell \in \Omega$; then $\sigma(\ell) = P^*(\ell)$ and we use the simpler notation $\sigma(\ell)$ here. Recall that $\sigma(\ell)$ belongs to the cross-section Σ, while $\sigma^{n-1}(\ell)$ belongs to the cross-section Σ^{n-1} for G_{n-1}-orbits in Ω^{n-1}. We begin by observing that the function $q(\ell)$ of Lemma 2.1 must satisfy

$$\sigma^{n-1}\Big(\pi\big(\exp(q(\ell))Z_n\ell\big)\Big) = \sigma^{n-1}\Big(\pi\big(\sigma^n(\ell)\big)\Big).$$

By Proposition 4.3, $\sigma^{n-1}\big(\pi(\sigma^n(\ell))\big) = \pi\big(\sigma^n(\ell)\big)$, and hence $q(\ell)$ satisfies

$$\sigma^{n-1}\Big(\pi\big(\exp(q(\ell))Z_n\ell\big)\Big) = \pi\Big(\sigma^n(\ell)\Big).$$

Since $\rho_{k-1}(Z_n,\ell) = Z_n \mod \mathfrak{g}_{n-1}$, it follows that

$$\sigma^{n-1}\Big(\pi\big(\exp(q(\ell)Z_n)\ell\big)\Big) = \sigma^{n-1}\Big(\pi\big(\exp(q(\ell)\rho_{k-1}(Z_n,\ell))\ell\big)\Big),$$

so

$$\sigma^{n-1}\Big(\pi\big(\exp(q(\ell)\rho_{k-1}(Z_n,\ell))\ \ell\big)\Big) = \pi\Big(\sigma^n(\ell)\Big).$$

Put $\ell' = \exp\Big(q(\ell)Z_n\Big)\ell$, we compute the above in the i-th direction. Observe first that the left hand side is

$$\mu_i^{n-1}(\pi(\ell'))\ \ell'_i + y_i^{n-1}(\pi(\ell')),$$

and since $\rho_{k-1}(Z_n,\ell) \in \mathfrak{g}_{i-1}^{\ell}$, then $\ell'_j = \ell_j, 1 \le j < i$, and hence

$$\mu_i^{n-1}(\pi(\ell')) = \mu_i^{n-1}(\pi(\ell)), \quad y_i^{n-1}(\pi(\ell')) = y_i^{n-1}(\pi(\ell)).$$

Now suppose that we are in case *(1)*. Then

$$\ell'_i = \ell_i - q(\ell)\langle \ell, [\rho_{k-1}(Z_n,\ell), Z_i]\rangle$$

so we have

$$\begin{aligned}\mu_i^{n-1}(\pi(\ell))\big(\ell_i - q(\ell) < \ell, [\rho_{k-1}(Z_n,\ell), Z_i]\rangle\big) + y_i^{n-1}(\pi(\ell)) =\\ = \big(\sigma^{n-1}(\pi(\exp(q(\ell)\rho_{k-1}(Z_n,\ell))\ \ell)\big)_i = 0.\end{aligned}$$

Solving for $q(\ell)$ in the above gives the result for case *(1)*.

In case *(2)*, one has

$$\ell'_i = e^{q(\ell)\gamma_i(\rho_{k-1}(Z_n,\ell))} b_i(\ell)^{-1} + \theta_i(\ell)$$

so

$$\begin{aligned}\mu_i^{n-1}(\pi(\ell))\big(e^{q(\ell)\gamma_i(\rho_{k-1}(Z_n,\ell))} b_i(\ell)^{-1} + \theta_i(\ell)\big) + y_i^{n-1}(\pi(\ell)) =\\ = |b_i(\ell)| b_i(\ell)^{-1} + y_i^n(\pi(\ell)).\end{aligned}$$

Now by ([8] Proposition 1.3.13), we have

$$y_i^n(\ell) = \mu_i^{n-1}(\pi(\ell))\theta_i(\ell) + y_i^{n-1}(\pi(\ell)). \tag{$*$}$$

On the other hand, using the formula for σ_i^n and the formula above one computes that

$$\begin{aligned}\sigma_i^n(\ell) &= |b_i(\ell)| b_i(\ell)^{-1} + y_i^n(\ell) = \sigma_i^{n-1}(\pi(\ell))\\ &= \mu_i^{n-1}(\pi(\ell))(e^{q(\ell)\gamma_i(\rho_{k-1}(Z_n,\ell))} b_i(\ell)^{-1} + \theta_i(\ell)) + y_i^{n-1}(\pi(\ell)).\end{aligned} \tag{$**$}$$

Combining the two equations $(*)$ and $(**)$ gives

$$\mu_i^{n-1}(\pi(\ell)) e^{q(\ell)\gamma_i(\rho_{k-1}(Z_n,\ell))} b_i(\ell)^{-1} = |b_i(\ell)| b_i(\ell)^{-1}.$$

Now again solving for $q(\ell)$ gives the result for case *(2)*. ■

The following theorem summarizes our results thus far.

Theorem 4.5. *Let G be a completely solvable Lie group with Lie algebra $\mathfrak{g}$, choose a real Jordan-Hölder basis $\{Z_j\}$. Let $\mathcal{P}$ be the corresponding ultra-fine stratification of $\mathfrak{g}^*$, and let Ω be a layer belonging to $\mathcal{P}$. Then there is an explicit construction of an open set $\mathcal{V}$ in $\mathfrak{g}^*_{\mathbb{C}}$, and complex-valued functions $p_1, p_2, \ldots, p_d,\ \ q_1, q_2, \ldots, q_d$ on $\mathcal{V}$, such that $\mathcal{V}$ contains Ω, and such that for each coadjoint orbit $\mathcal{O}$ in Ω,*

$$p_1|_{\mathcal{O}}, p_2|_{\mathcal{O}}, \ldots, p_d|_{\mathcal{O}},\ q_1|_{\mathcal{O}}, q_2|_{\mathcal{O}}, \ldots, q_d|_{\mathcal{O}}$$

are real-valued, global canonical coordinates for $\mathcal{O}$. Moreover, for each $1 \leq j \leq n$, $0 \leq r \leq d$, there is an entire function $\alpha_{j,r}(\sigma(\ell), \cdot)$ such that for each $1 \leq j \leq n$ and $\ell \in \Omega$ one has

$$\ell_j = \sum_{r:\ j_r \leq j} \alpha_{j,r}(\sigma(\ell), q(\ell)) p_r(\ell) + \alpha_{j,0}(\sigma(\ell), q(\ell)).$$

Proof. Let $\mathbf{e}$, $\mathbf{j}, \varphi$ be the index data associated to the ultra-fine layer Ω, and let $\mathcal{U}$ be the open set associated with this data via Lemma 4.1. Suppose that $p_1^{n-1}, \ldots, p_{d_{n-1}}^{n-1}, q_1^{n-1}, \ldots, q_{d_{n-1}}^{n-1}$ have been constructed explicitly so that for some open set $\mathcal{V}^{n-1} \subset \mathcal{U}$, each is analytic on $\mathcal{V}^{n-1}$. If $n \notin \mathbf{e}$, then we are done, so suppose that $n \in \mathbf{e}$, so that $d_{n-1} = d-1$. Set $p_d(\ell) = \ell_n$ and define $q_d(\ell)$ as in Lemma 4.4, according as $i_k \notin \varphi$ or $i_k \in \varphi$. Recall that $\ell \to q_d(\ell)$ is analytic on $\mathcal{U} \cap \pi^{-1}(\mathcal{U}^{n-1})$. Define

$$\mathcal{V} = \{\ell \in \mathcal{U} \cap \pi^{-1}(\mathcal{U}^{n-1}) : \exp\big(q_d(\ell) Z_n\big)\ell \in \mathcal{V}^{n-1}\}$$

and for $\ell \in \mathcal{V}$ and $1 \leq r \leq d-1$, set

$$p_r(\ell) = p_r^{n-1}\big(\exp(q_d(\ell)Z_n)\ell\big) \quad \text{and} \quad q_r(\ell) = q_r^{n-1}\big(\exp(q_d(\ell)Z_n)\ell\big).$$

Then $\mathcal{V}$ is open, and p_r (resp. q_r) is a composition of the function $\ell \to \exp\big(q_d(\ell)Z_n\big)\ell$, which is analytic on $\mathcal{V}$ with values in $\mathcal{V}^{n-1}$, followed by the function p_r^{n-1} (resp. q_r^{n-1}), which is analytic on $\mathcal{V}^{n-1}$.

The last formula is obtained by induction from the definition of the functions q_r and $p_r, 1 \leq r \leq d$ (see [2]). ∎

5. Global Parametrization of a Layer

Let $\Omega = \Omega_{\mathbf{e},\mathbf{j},\varphi}$ be an ultra-fine layer with $\Sigma = P^*(\Omega)$. Recall that, if $\varphi \neq \emptyset$, then for each $\epsilon \in \{-1,1\}^{\varphi}$, $\Omega_\epsilon = \{\ell \in \Omega :\ \operatorname{sign}(b_i(\ell)) = \epsilon_i, i \in \varphi\}$, so $\Sigma = \bigcup_\epsilon \Sigma_\epsilon$ where $\Sigma_\epsilon = \Sigma \cap \Omega_\epsilon = \{\ell \in \Sigma :\ b_i(\ell) = \epsilon_i, i \in \varphi\}$. Set $V_0 = \operatorname{span}\{Z_i^*\colon\ i \notin \mathbf{e}\}$ and $V_\varphi = \operatorname{span}\{Z_i^*\colon\ i \in \varphi\}$. Let π_0 be the projection onto V_0 parallel to the vectors $Z_j^*, j \in \mathbf{e}$. Similarly let π_φ be the orthogonal projection onto V_φ. We claim that Σ_ϵ is in fact the graph of a rational function $f_\epsilon : \pi_0(\Sigma_\epsilon) \to V_\varphi$.

Before proving this claim, we outline some important features of the constructions of [10] (again using notation from [8]).

(i) For $\ell \in \Omega$ and $j \in \varphi$, $y_j(\ell) = \mu_j(\ell)\theta_j(\ell) + y_j^\circ(\ell)$.
(ii) For $\ell \in \Sigma$, $\Phi_a(\ell) = 0, 1 \le a \le 2d$ (since $\sigma(\ell) = \ell$ in this case).
(iii) For any $\ell \in \Sigma$, $y_j^\circ(\ell) = 0$ and $\mu_j(\ell) = 1$, hence $y_j(\ell) = \theta_j(\ell)$ if $j \in \varphi$. Recall that $\theta_j(\ell)$ is a rational function depending only upon the restriction $\pi^{j-1}(\ell)$ of ℓ to $\mathfrak{g}_{j-1}$.

Write $f_\epsilon(\ell) = \sum_{i\in\varphi} f_{\epsilon,i}(\ell)Z_i^*$ where $f_{\epsilon,i}$ is real-valued. For each $1 \le j \le n$ let $\pi_0^j = \pi^j \circ \pi_0$, $\pi_\varphi^j = \pi^j \circ \pi_\varphi$, $V_0^j = \pi^j(V_0)$ and $V_\varphi^j = \pi^j(V_\varphi)$. We construct $f_\epsilon^j : \pi_0^j(\Sigma_\epsilon) \to V_\varphi^j, 1 \le j \le n$ inductively, so that its graph coincides with $\pi^j(\Sigma_\epsilon)$.

Suppose that $j = 1$. If $j \notin \mathbf{e}$, $V_0^1 = \mathbb{R}Z_1^*$, and we define $f_\epsilon^1 = 0$ on $\pi_0^1(\Sigma_\epsilon)$. It is obvious that the graph of f_ϵ^1 coincides with $\pi^1(\Sigma_\epsilon)$. If $1 \in \mathbf{e}$ then $1 \in \varphi$, so $V_\varphi^1 = \mathbb{R}Z_1^*$ and $\pi_0^1(\Sigma) = (0)$. So we set $f_\epsilon^1(0) = \epsilon_1$ in this case, and it is clear that $\pi^1(\Sigma_\epsilon) = \epsilon_1 Z_1^* =$ graph f_ϵ^1.

Now suppose that we have a rational function $f_\epsilon^{j-1} : \pi_0^{j-1}(\Sigma_\epsilon) \to V_\varphi^{j-1}$ such that graph$(f_\epsilon^{j-1}) = \pi^{j-1}(\Sigma_\epsilon)$, that is

$$\pi^{j-1}(\Sigma_\epsilon) = \{\pi_0^{j-1}(\ell) + f_\epsilon^{j-1}(\pi_0^{j-1}(\ell)) : \ \pi_0^{j-1}(\ell) \in \pi_0^{j-1}(\Sigma_\epsilon)\}.$$

Case 1: $j \notin \mathbf{e}$.

Here $V_\varphi^j = V_\varphi^{j-1}$ and $\pi_\varphi^j = \pi_\varphi^{j-1}$. Define $f_\epsilon^j : \pi_0^j(\Sigma_\epsilon) \to V_\varphi^j$ by

$$f_\epsilon^j(\pi_0^j(\ell)) = f_\epsilon^{j-1}(\pi_0^{j-1}(\ell)).$$

Let $\ell^j = \pi^j(\ell) \in \pi^j(\Sigma_\epsilon)$. Then $\pi_0^j(\ell) \in \pi_0^j(\Sigma_\epsilon)$ and by induction

$$\pi_\varphi^j(\ell) = \pi_\varphi^{j-1}(\ell) = f_\epsilon^{j-1}(\pi_0^{j-1}(\ell)) = f_\epsilon^j(\pi_0^j(\ell)).$$

Hence $\ell^j = \pi_0^j(\ell) + f_\epsilon^j(\pi_0^j(\ell)) \in$ graph(f_ϵ^j). On the other hand, if $g \in$ graph(f_ϵ^j), then for some $\ell \in \Sigma_\epsilon$, $g = \pi_0^j(\ell) + f_\epsilon^j(\pi_0^j(\ell))$. Now with the induction assumption we have

$$\pi^{j-1}(g) = \pi^{j-1}\big(\pi_0^j(\ell) + f_\epsilon^j(\pi_0^j(\ell))\big) = \pi_0^{j-1}(\ell) + f_\epsilon^{j-1}(\pi_0^{j-1}(\ell)) = \pi^{j-1}(\ell)$$

and $g_j = \ell_j$, hence

$$g = \pi^j(g) = \pi^{j-1}(g) + g_j Z_j^* = \pi^j(\ell).$$

Case 2: $j \in \mathbf{e} \setminus \varphi$.

Here by construction of Σ_ϵ, we have that $\pi^j(\ell) = \pi^{j-1}(\ell)$ holds for every $\ell \in \Sigma_\epsilon$. Also, $V_\varphi^j = V_\varphi^{j-1}$ and $V_0^j = V_0^{j-1}$, so we define $f_\epsilon^j : \pi_0^j(\Sigma_\epsilon) \to V_\varphi^j$ by

$$f_\epsilon^j = f_\epsilon^{j-1}.$$

By induction, graph$(f_\epsilon^j) =$ graph$(f_\epsilon^{j-1}) = \pi^{j-1}(\Sigma_\epsilon) = \pi^j(\Sigma_\epsilon)$.

Case 3: $j \in \varphi$.

Here we have $\pi_0^j(\ell) = \pi_0^{j-1}(\ell)$ for every $\ell \in Sigma_\epsilon$. For $i \in \varphi$, $i < j$, set $f_{\epsilon,i}^j = f_{\epsilon,i}^{j-1}$ and define

$$f_{\epsilon,j}^j(\pi_0^j(\ell)) = \epsilon_j + \theta_j\big(\pi_0^{j-1}(\ell) + f_\epsilon^{j-1}(\pi_0^{j-1}(\ell))\big).$$

Let $\pi^j(\ell) \in \pi^j(\Sigma_\epsilon)$. By induction we have $\pi^{j-1}(\ell) = \pi_0^{j-1}(\ell) + f_\epsilon^{j-1}(\pi_0^{j-1}(\ell))$. Since $\ell \in \Sigma_\epsilon$, then by (iii) above and the definition of $f_{\epsilon,j}^j$ we have

$$\ell_j = P_j^*(\ell) = \operatorname{sign}(b_j(\ell)) + y_j(\ell) = \epsilon_j + \theta_j(\pi^{j-1}(\ell)) = f_{\epsilon,j}^j(\pi_0^j(\ell)).$$

Hence

$$\begin{aligned}\pi^j(\ell) &= \pi^{j-1}(\ell) + \ell_j Z_j^* \\ &= \pi_0^{j-1}(\ell) + f_\epsilon^{j-1}(\pi_0^{j-1}(\ell)) + f_{\epsilon,j}^j(\pi_0^j(\ell))Z_j^* = \pi_0^j(\ell) + f_\epsilon^j(\pi_0^j(\ell))\end{aligned}$$

On the other hand, if $g \in \operatorname{graph}(f_\epsilon^j)$, so that for some $\ell \in \Sigma_\epsilon$, $g = \pi_0^j(\ell) + f_\epsilon^j(\pi_0^j(\ell))$, then

$$\pi^{j-1}(g) = \pi^{j-1}\big(\pi_0^j(\ell) + f_\epsilon^j(\pi_0^j(\ell))\big) = \pi_0^{j-1}(\ell) + f_\epsilon^{j-1}(\pi_0^{j-1}(\ell)) = \pi^{j-1}(\ell).$$

But since $\ell \in \Sigma_\epsilon$,

$$g_j = f_{\epsilon,j}^j(\pi_0^j(\ell)) = \epsilon_j + \theta_j(\pi^{j-1}(\ell)) = P_j^*(\ell) = \ell_j.$$

Thus $g = \pi^j(g) = \pi^j(\ell)$, and the claim is proved.

Observe that the restriction of π_0 to Σ_ϵ is injective. Set $\mathcal{W}_\epsilon = \pi_0(\Sigma_\epsilon)$. It is now clear that $\mathcal{W}_\epsilon$ is precisely the set of all $\ell_0 \in V_0$ such that f_ϵ is defined at ℓ_0 and such that $\ell_0 + f_\epsilon(\ell_0)$ satisfies the algebraic conditions that define Ω_ϵ. We sum up the preceding in the following

Proposition 5.1. *Let $\Omega = \Omega_{\mathbf{e},\mathbf{j},\varphi}$ be an ultra-fine layer, with $\Sigma = \bigcup_\epsilon \Sigma_\epsilon$ the cross-section defined in [10]. Set $V_0 = \operatorname{span}\{Z_j^* : j \notin \mathbf{e}\}$ and $V_\varphi = \operatorname{span}\{Z_j^* : j \in \varphi\}$. Then for each $\epsilon \in \{-1,1\}^\varphi$, there is an algebraic subset $\mathcal{W}_\epsilon$ of V_0 and a rational function $f_\epsilon : \mathcal{W}_\epsilon \to V_\varphi$ such that Σ_ϵ is the graph of f_ϵ. The set $\mathcal{W}_\epsilon$ is explicitly described as follows. Set $D(f_\epsilon) = \{\ell_0 \in V_0 : f_\epsilon \text{ is defined at } \ell_0\}$. Then*

$$\mathcal{W}_\epsilon = \{\ell_0 \in D(f_\epsilon) : \ \ell_0 + f_\epsilon(\ell_0) \in \Omega_\epsilon\}. \qquad \blacksquare$$

Next we state an explicit version of ([2], Théorème 1.6) for completely solvable Lie algebras.

Proposition 5.2. *Let $\mathfrak{g}$ be a completely solvable Lie algebra over $\mathbb{R}$, choose a real Jordan-Hölder basis $\{Z_j\}$ for $\mathfrak{g}$ and let $\Omega = \Omega_{\mathbf{e},\mathbf{j},\varphi}$ be an ultra-fine layer with cross-section Σ as defined in [10]. Then Ω is parametrized as follows: for any ℓ in Ω, let $\mathcal{O}_\ell$ be the orbit of ℓ and $P^*(\ell) = \mathcal{O} \cap \Sigma$. Choose ϵ so that $P^*(\ell) \in \Sigma_\epsilon$ and write $P^*(\ell) = \lambda(\ell) + f_\epsilon(\lambda(\ell))$ where $\lambda(\ell) \in \mathcal{W}_\epsilon$. Then the map*

$$\begin{aligned}\psi : \Omega_\epsilon &\longrightarrow \mathcal{W}_\epsilon \times \mathbb{R}^{2d} \\ \ell &\longmapsto \big(\lambda(\ell), p(\ell), q(\ell)\big),\end{aligned}$$

is a bijection and a global parametrization of Ω_ϵ in the sense of ([2] Théorème 1.6). $\blacksquare$

The following example shows that the cross-sections Σ need not be a submanifold if $\mathfrak{g}^*$. Let $\mathfrak{g}$ be the nilpotent Lie algebra with basis

$$(Z_{11}, Z_{22}, Z_{13}, Z_{31}, Y_1, Y_2, Y_3, X_1, X_2, X_3)$$

and the non-vanishing brackets:

$$[X_1, Y_1] = Z_{11}, \quad [X_2, Y_2] = Z_{22}, \quad [X_1, Y_3] = Z_{13}, \quad [X_3, Y_1] = Z_{31}.$$

We look for the (ultra) fine layer $\Omega_{\mathbf{e},\mathbf{j}}$ with $\mathbf{e} = \{5, 6, 8, 9\}$ and $\mathbf{j} = \{8, 9\}$. Then $\Omega_{\mathbf{e},\mathbf{j}}$ is defined by the following relations:

$$\lambda_{11} = \langle \ell, Z_{11} \rangle \neq 0, \ \lambda_{22} = \langle \ell, Z_{22} \rangle \neq 0, \ \lambda_{31}\lambda_{13} = \langle \ell, Z_{31} \rangle \langle \ell, Z_{13} \rangle = 0$$

and Σ is

$$\Sigma = \{\ell \in \Omega_{\mathbf{e},\mathbf{j}} : \ \ \langle \ell, X_1 \rangle = \langle \ell, X_2 \rangle = \langle \ell, Y_1 \rangle = \langle \ell, Y_2 \rangle = 0\}.$$

The points ℓ in Σ such that $\lambda_{31} = \lambda_{13} = 0$ are singular.

Remark 5.1. In [15] M. Saint Germain built a formal Weinstein local chart for any ℓ in the dual $\mathfrak{g}^*$ of a nilpotent Lie algebra $\mathfrak{g}$. That means, he gave formal series:

$$z_1, \dots, z_{n-2d}, \quad q_1, \dots, q_d, \quad p_1, \dots, p_d,$$

such that formally:

$$\{z_i, z_j\} = \mu_{ij}(z), \ \{z_i, p_j\} = \{z_i, q_j\} = \{p_i, p_j\} = \{q_i, q_j\} = 0, \text{ and } \{p_i, q_j\} = \delta_{ij}.$$

Any ℓ' in an open set containing ℓ can be expressed as a formal series of the p_j, q_j and z_j. All these formal series belong to the projective limit $\hat{S}(\mathfrak{g})$ of the space of polynomial functions on $\mathfrak{g}^*$ with respect to the successive powers of the ideal $I_{\mathcal{O}}$ of functions vanishing on the orbit $\mathcal{O}$ of ℓ.

The construction of Proposition 5.2 is explicitly computable rather than formal, but it does not give a Weinstein chart. Indeed, although the functions p_i, q_i are defined on an open set $\mathcal{V}$, λ is only defined on the ultra-fine layer Ω. Even if $\lambda(\ell)$ is a regular point in $\Omega \cap V_0$, and (λ, p, q) gives a local chart in Ω, there is no natural extension of this chart to a local chart in $\mathfrak{g}^*$. For instance, we consider the nilpotent Lie algebra $\mathfrak{g}$ with basis (Z, Y, X, W) and non vanshing brakets:

$$[W, X] = Y, \quad [X, Y] = Z.$$

If $\ell = \zeta Z^* + \gamma Y^* + \beta X^* + \mu W^*$ belongs to the layer

$$\Omega = \{\ell \colon \zeta = 0, \ \gamma \neq 0\}$$

then $q(\ell) = \frac{\beta}{\gamma}$, $p(\ell) = \mu$, $V_0 = \text{span}\{Z^*, Y^*\}$, but $\Omega \cap V_0 = \{(0, \gamma) \colon \ \gamma \neq 0\}$ and $\lambda(\ell) = \gamma$. The open $\mathcal{V}$ in Theorem 4.5 is $\mathcal{V} = \{\ell \colon \ \gamma \neq 0\}$, the extension $\lambda(\ell) = \gamma$ to $\mathcal{V}$ does not give a Weinstein chart. Instead, we can set

$$\lambda(\ell) = \text{sign}(\gamma)\sqrt{\gamma^2 + 2\mu\zeta}.$$

This gives a local Weinstein chart extending (λ, p, q).

Remark 5.2. The situation is much better for the minimal layer Ω because $V_0 \cap \Omega$ is now an open set in V_0 which provides coordinates for the manifold Σ and Proposition 5.2 gives us a global G-invariant Weinstein chart for Ω.

6. Star algebras

In this section we restrict ourselves to the minimal Zariski-open layer $\Omega_{\mathbf{e},\mathbf{j},\varphi}$. Let us denote it by Ω, and let Σ be the cross-section in Ω as constructed in [10] and described above. Recalling the spaces V_0 and V_φ, the set $\mathcal{W}_\epsilon$ is now an open set in V_0 and Σ_ϵ is the graph of a rational mapping f_ϵ from $\mathcal{W}_\epsilon$ to V_φ.

Now, it is easy to define a G-invariant local chart around each point ℓ_0 in Ω_ϵ. We choose on the vector space V_0 the Euclidean norm associated to the basis $(Z_s^*)_{s\notin\mathbf{e}}$, there is $\eta > 0$ such that the ball $B(\pi_0\left(P^*(\ell_0)\right), \eta)$ centered in $\pi_0\left(P^*(\ell_0)\right)$,with radius η is included in $\mathcal{W}_\epsilon$. Let us put $h(\ell) = \pi_0\left(P^*(\ell) - P^*(\ell_0)\right)$. The domain of our chart is
$\mathcal{U} = \{\ell \colon \pi_0(P^*(\ell)) \in B\left(\pi_0(P^*(\ell_0)), \eta\right)\}$, it is diffeomorphic to
$B\left(\pi_0(P^*(\ell_0)), \eta\right) \times \mathbb{R}^{2d}$ by $\Psi(\ell) = (h_s(\ell),\ q_i(\ell),\ p_i(\ell))$ with
$h(\ell) = \sum_{s\notin\mathbf{e}} h_s(\ell) Z_s^*$, and $1 \le i \le d$.

As an application of Theorem 4.5 and Section 5, we shall now extend results of [3] to the global parametrization of $\mathcal{U}$. We consider thus deformed structure related to the problem of quantization not only for a particular orbit in $\mathfrak{g}^*$ but globally, for the minimal ultra-fine layer Ω.

Let us recall briefly the general setup of geometric quantization for a Poisson manifold like $\mathcal{U}$ (see [18]). First, by construction, the Vergne polarization at $P^*(\ell)$ ($\ell \in \mathcal{U}$) is (see [2]):

$$\mathfrak{h}^d(P^*(\ell)) = \left\{ X \in \mathfrak{s}\colon \quad \frac{\partial(X \circ \Psi^{-1})}{\partial p_i}\bigg|_{\Psi(P^*(\ell))} = 0, \quad 1 \le i \le d \right\}.$$

This (algebraic) polarization allows us to define the space of “polarized” functions on $\mathcal{U}$:

$$\begin{aligned} C^\infty(\mathcal{U})_0 &= \left\{ u \in C^\infty(\mathcal{U}) :\ X^- u(\ell) = \partial_t u(\exp(-tX)\ell)|_{t=0} \equiv 0,\ X \in \mathfrak{h}^d(P^*(\ell)) \right\} \\ &= \left\{ u \in C^\infty(\mathcal{U}) :\ \frac{\partial(u \circ \Psi^{-1})}{\partial p_i} \equiv 0, \quad 1 \le i \le d \right\}. \end{aligned}$$

We identify the function v on $\mathcal{U}$ with the function $v \circ \Psi^{-1}$. Let us define:
-the Liouville form $\theta = \sum_i p_i dq_i$,
-the Hamiltonian vector field $X_v = \sum_i \partial_{p_i} v \partial_{q_i} - \partial_{q_i} v \partial_{p_i}$ of v,
- the Poisson bracket $\{v, u\} = X_v u$ of v and u.
The geometric quantization of v is an operator Q_v defined on $C^\infty(\mathcal{U})_0$ by:

$$Q_v(u) = vu + \frac{\hbar}{i}\{v, u\} - \theta(X_v)u,$$

where $\hbar$ is the Planck constant. The operator Q_v is well defined if and only if v is "quantizable" *i. e.* $v \in C^\infty(\mathcal{U})_1$, where:

$$C^\infty(\mathcal{U})_1 = \{v \in C^\infty(\mathcal{U}): \ \ \{v, C^\infty(\mathcal{U})_0\} \subset C^\infty(\mathcal{U})_0\}$$
$$= \left\{v \in C^\infty(\mathcal{U}): \ \ \frac{\partial^2 v}{\partial p_i \partial p_j} \equiv 0, \ 1 \le i,j \le d\right\}.$$

Q is thus a linear map from the space $C^\infty(\mathcal{U})_1$ to the space $\mathcal{L}(C^\infty(\mathcal{U})_0)$ of linear endomorphisms of $C^\infty(\mathcal{U})_0$ such that:

$$Q_1 u = u, \quad Q_{q_j} u = q_j u \quad \text{and} \quad Q_{p_j} u = \frac{\hbar}{i}\frac{\partial u}{\partial q_j}.$$

Moreover Q satisfies, for quantizable functions v and v',

$$Q_{\frac{\hbar}{i}\{v,v'\}} = Q_v \circ Q_{v'} - Q_{v'} \circ Q_v.$$

Such a quantization can also be described, without operator, as a formal associative deformation of the ususal pointwise product in $C^\infty(\mathcal{U})$ whose the first non trivial term is given by the Poisson bracket (see [4]). More precisely, we define the Moyal star product by the formal series:

$$u \star v = \sum_{s=0}^{\infty} \left(\frac{\hbar}{2i}\right)^s \frac{1}{s!} P^s(u,v), \tag{$*$}$$

where P^s is the s^{th}-power of the Poisson bracket. On $\mathbb{R}^n \simeq V_0 \oplus \mathbb{R}^{2d}$ with coordinates $(x_1, \dots, x_n) = (h_s, \ q_i, \ p_i)$, we consider the bilinear skew-symmetric form:

$$\omega(x,x') = \sum_{i=1}^{d} p_i q_i' - q_i p_i'.$$

Let $\Lambda = (\Lambda^{ij})_{1 \le i,j \le n}$ be the matrix of ω, then

$$P^0(u,v) = uv, \quad P^r(u,v) = \sum_{\substack{i_1,\cdots,i_r=1 \\ j_1,\cdots,j_r=1}}^{n} \Lambda^{i_1 j_1} \cdots \Lambda^{i_r j_r} \frac{\partial^r u}{\partial x_{i_1} \cdots \partial x_{i_r}} \frac{\partial^r v}{\partial x_{j_1} \cdots \partial x_{j_r}}.$$

The series $(*)$ is considered as a formal series in $\hbar$ and $\star$ is an associative product on $C^\infty(\mathcal{U})[[\hbar]]$.

Let us restrict ourselves for a moment to the functions $v(h,q,p)$ which are restriction to $\mathbb{R}^n$ of entire functions on $\mathbb{C}^n$. We shall just say v is an entire function. If $u(h,q)$ is a polarized entire function, we define Tu as the function:

$$Tu(h,q,p) = \exp\Big(\frac{2}{i\hbar}\sum_j p_j q_j\Big) u(h, 2q).$$

Then a direct computation shows that T is one-to-one with range the space of entire functions w such that $w \star q_j = 0$ for all j and if

$$v(h,q,p) = \sum_{j=1}^{d} v_j(h,q)p_j + v_0(h,q)$$

is quantizable and u polarized, both entire functions, then we claim $v \star Tu$ is defined by a series $(*)$ converging for any value of $\hbar \neq 0$ and:

$$v \star Tu = T\left(Q_v u\right).$$

Indeed, if $v(h,q)$ is polarized,

$$\begin{aligned}
(v \star Tu)(h,q,p) &= \sum_{s=0}^{\infty} \left(\frac{\hbar}{2i}\right)^s \frac{1}{s!} P^s(v,Tu)(h,q,p) \\
&= \sum_{s=0}^{\infty} \left(\frac{\hbar}{2i}\right)^s \frac{1}{s!} \sum_{i_1,\dots,i_s=1}^{d} (-1)^s \frac{\partial^s v}{\partial q_{i_1}\dots\partial q_{i_s}}(h,q) \frac{\partial^s e^{\frac{2}{i\hbar}p\cdot q}}{\partial p_{i_1}\dots\partial p_{i_s}} u(h,2q) \\
&= e^{\frac{2}{i\hbar}p\cdot q}\left(\sum_{s=0}^{\infty} \frac{1}{s!} \sum_{i_1,\dots,i_s=1}^{d} q_{i_1}\dots q_{i_s} \frac{\partial^s v}{\partial q_{i_1}\dots\partial q_{i_s}}(h,q)\right) u(h,2q) \\
&= e^{\frac{2}{i\hbar}p\cdot q} v(h,2q)u(h,2q) = T(vu)(h,q,p).
\end{aligned}$$

Similarly:

$$\begin{aligned}
\left(p_j \star Tu\right)(h,q,p) &= p_j(Tu)(h,q,p) + \frac{\hbar}{2i}\frac{\partial}{\partial q_j}(Tu)(h,q,p) \\
&= p_j(Tu)(h,q,p) - p_j e^{\frac{2}{i\hbar}p\cdot q}u(h,2q) + \frac{\hbar}{2i} e^{\frac{2}{i\hbar}p\cdot q} 2\frac{\partial u}{\partial q_j}(h,2q) \\
&= T\left(\frac{\hbar}{i}\frac{\partial u}{\partial q_j}\right)(h,q,p).
\end{aligned}$$

This proves our claim since $\star$ is associative and any quantizable function v can be written as:

$$v = \sum_{j=1}^{d} v_j p_j + v_0 = \sum_{j=1}^{d} v_j \star p_j + v_0 + \sum_{j=1}^{d} \frac{\hbar}{2i}\frac{\partial v_j}{\partial q_j}.$$

Thus the Moyal star product allows us to extend the definition of the quantization $v \mapsto Q_v$ to a larger class of functions by putting $Q_w u = T^{-1}\left(w \star Tu\right)$ and we get, by construction:

$$Q_{w_1 \star w_2} = Q_{w_1} \circ Q_{w_2}.$$

We consider now the space $\mathbb{C}[[h,q]][p]$ of polynomial functions in the variables p_i with coefficients formal power series in the variables h_s and q_i. The

Moyal product is in fact a finite sum if u and v are in $\mathbb{C}[[h,q]][p]$ and, in this case, $u \star v$ is an element of $\mathbb{C}[[h,q]][p]$, for any value of $\hbar$.

Let Φ be the mapping:

$$\Phi : S(\mathfrak{g}) \longrightarrow \mathbb{C}[[h,q]][p]$$

which associates to each polynomial function $\ell \longmapsto U(\ell)$, on $\mathfrak{g}^*$, the Taylor expansion in 0 of the analytic function $U \circ \Psi^{-1}(h,q,p)$. Since $(\mathcal{U},\Psi)$ is a chart, Φ is injective. Let us put $\mathcal{A} = \Phi(S(\mathfrak{g}))$. We consider among the elements of $\mathcal{A}$ the polarized and the quantizable ones:

$$\mathcal{A}_0 = \left\{ u \in \mathcal{A}\colon \ \frac{\partial u}{\partial p_i} \equiv 0 \right\},$$

$$\mathcal{A}_1 = \left\{ u \in \mathcal{A}\colon \ \frac{\partial^2 u}{\partial p_i \partial p_j} \equiv 0 \right\}.$$

In order to globalize the study beginning in [3], we shall prove that our coordinates functions h, q, p are in a convenient completion of the spaces $\mathcal{A}_0$ and $\mathcal{A}_1$.

Definition 6.1. Let $u = \sum_{a,b,c} \alpha_{a,b,c} h^a q^b p^c$ be in $\mathcal{A}$, here we use the usual notations for $a = (a_s)$ in $\mathbb{N}^{\{1,\dots,n\}\setminus\mathbf{e}}$, $|a| = \sum_{s\notin\mathbf{e}} a_s$, $h^a = \prod_{s\notin\mathbf{e}} h_s^{a_s}$, for $b = (b_j)$ in $\mathbb{N}^d$, $|b| = \sum_{j=1}^d b_j$, $q^b = \prod_{j=1}^d q_j^{b_j}$ and similarly for the variables p_j. We define
1. the (h,q)-valuation of u by $val_{h,q}(0) = +\infty$ and:

$$val_{h,q}(u) = \min\{|a| + |b|\colon \ \exists\ c\colon \ \alpha_{a,b,c} \neq 0\},$$

2. the $\star$-valuation by $val_\star(0) = +\infty$ and:

$$val_\star(u) = val_\star\left(\sum_c u_c(h,q)p^c\right) = \min\left\{(val_{h,q}(u_c) - |c|)\colon \ c \in \mathbb{N}^d\right\},$$

3. the metric $d_\star$ on $\mathcal{A}$ by:

$$d_\star(u,v) = d_\star(u-v,0) = e^{-val_\star(u-v)}.$$

In fact, $\star$ is continuous on the metric space $(\mathcal{A},\ d_\star)$ and can be extended to the completion $\overline{\mathcal{A}}$ of $\mathcal{A}$ for $d_\star$. We define $\overline{\mathcal{A}}_0$, $\overline{\mathcal{A}}_1$ as the completions of $\mathcal{A}_0$, $\mathcal{A}_1$ with respect to $d_\star$. Due to the continuity of $\star$, it is easy to prove that $\overline{\mathcal{A}}_0$ is an algebra for the usual pointwise product and $\overline{\mathcal{A}}_1$ is an $\overline{\mathcal{A}}_0$-module. See [5] for a detailed study of these properties.

Theorem 6.2. *Let (h_s, q_i, p_i) be the coordinate functions defined above. Then*
1. for all s not in $\mathbf{e}$, h_s belongs to $\overline{\mathcal{A}}_0$,
2. for all i, $1 \leq i \leq d$, q_i belongs to $\overline{\mathcal{A}}_0$,
3. for all i, $1 \leq i \leq d$, p_i belongs to $\overline{\mathcal{A}}_1$.

Proof. Let $\mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_n = \mathfrak{g}$ be the increasing sequence of ideals used to define recursively the variables (h,q,p). Denote by Ω^t, $\mathcal{W}^t$, h_s^t, q_i^t, p_i^t, $\mathcal{A}^t$,

$\mathcal{A}_0^t$ and so on the objects constructed for $\mathfrak{g}_t^*$. We shall prove the result for all intermediate variables (h^t, q^t, p^t). For $t = 1$, this is trivial and to prove Theorem 6.2 by induction on t, it is enough to prove it at the last step. As usual, π is the restriction map from $\mathfrak{g}^*$ onto $\mathfrak{g}_{n-1}^*$. Let us suppose the result holds for $\mathfrak{g}_{n-1}$. There is two cases.

Case 1: $n \notin \mathbf{e}$.
By definition the set $\mathcal{W}_\epsilon$ is

$$\mathcal{W}_\epsilon = \left\{\ell \in V_0^{n-1} \oplus \mathbb{R}Z_n^* : \ P_{\mathbf{e},\mathbf{j}}(\ell + f_\epsilon(\ell)) \neq 0, \ \mathrm{sign}(b_j(\ell + f_\epsilon(\ell)) = \epsilon_j, \ j \in \varphi\right\}.$$

We saw that in this case:

$$P_{\mathbf{e},\mathbf{j}}(\ell + f_\epsilon(\ell)) = P^{n-1}_{\mathbf{e}^{n-1},\mathbf{j}^{n-1}}(\ell^{n-1} + f_\epsilon(\ell^{n-1}))$$

and

$$b_j(\ell + f_\epsilon(\ell)) = b_j(\ell^{n-1} + f_\epsilon(\ell^{n-1})).$$

Thus

$$\mathcal{W}_\epsilon = \pi^{-1}\left(\mathcal{W}_\epsilon^{n-1}\right) = \mathcal{W}_\epsilon^{n-1} \times \mathbb{R}.$$

Fix ℓ_0 in Ω_ϵ, choose $\eta > 0$ such that $B\left(\pi_0(P^*(\ell_0)), \eta\right)$ is included in $\mathcal{W}_\epsilon$ then

$$B^{n-1}\left(\pi^{n-1}(P^{*n-1}(\ell_0^{n-1})), \eta\right) \subset \mathcal{W}_\epsilon^{n-1}.$$

There is a new invariant function, $h_n(\ell) = P_n^*(\ell) - P_n^*(\ell_0)$, the other coordinate functions $\ell_j : \ell \mapsto \langle \ell, Z_j \rangle$ do not change:

$$\Psi(\ell) = \left(h_s^{n-1}(\pi(\ell)), \ h_n(\ell), \ q_i^{n-1}(\pi(\ell)), \ p_i^{n-1}(\pi(\ell))\right).$$

Thus, by definition, $\mathcal{A}_0^{n-1}$ can be viewed as a subspace of $\mathcal{A}_0$ and $\mathcal{A}_1^{n-1}$ as a subspace of $\mathcal{A}_1$, moreover h_s^n, for $s < n$, q_i^n and p_i^n are in $\overline{\mathcal{A}}_0$, respectively $\overline{\mathcal{A}}_1$. The only function to consider is $h_n = h_n^n$.

Thanks to Theorem 4.5, we have:

$$\ell_n \circ \Psi^{-1}(h, q, p) = \sum_{i=1}^{d} \alpha_i(h, q) p_i + \alpha_0(h, q).$$

But the functions $\alpha_i(h, q)$ are analytic on $B\left(\pi_0(P^*(\ell_0)), \eta\right) \times \mathbb{R}^{2d}$. We can write:

$$\alpha_i(h, q) = \sum_{a,b} \alpha_{i,a,b} h^a q^b.$$

Since each h_s, q_j is in $\overline{\mathcal{A}}_0$ and $val_\star(h_s) = val_\star(q_j) = 1$, each of these series converges to an element in $\overline{\mathcal{A}}_0$. Since p_i is in $\overline{\mathcal{A}}_1$,

$$\alpha_0(h, q) = \ell_n \circ \Psi^{-1}(h, q, p) - \sum_{i=1}^{d} \alpha_i(h, q) p_i$$

is in $\overline{\mathcal{A}}_1$. But α_0 does not depend upon p_i, thus it is in $\overline{\mathcal{A}}_0$. Now by definition,

$$\alpha_0(h,0) = \ell_n \circ \Psi^{-1}(h,0,0) = P_n^*(\ell) = P_n^*(\ell_0) + h_n.$$

We can write:

$$\alpha_0(h,q) = P_n^*(\ell_0) + h_n + \sum_{|b|\geq 1} \sum_a \alpha_{0,a,b} h^a q^b.$$

The same argument as above prove that h_n is in $\overline{\mathcal{A}}_0$.
<u>**Case 2**</u>: $n = j_k \in \mathbf{e}$.
In this case, we define the function q_d as in Section 2. The cross-section Σ is the subset of the cross-section Σ_{n-1} defined by $h_{i_k}^{n-1}(\ell) = 0$. Thus $V_0^{n-1} = V_0 \oplus \mathbb{R} Z_{i_k}^*$ and the chart centered in $\ell_0 \in \Omega$ is defined as the function Ψ:

$$\Psi(\ell) =$$
$$= \Big(h_s^{n-1}\left(\exp(q_d(\ell)Z_n)\ell\right)_{(s\notin\mathbf{e})},\ q_j^{n-1}\left(\exp(q_d(\ell)Z_n)\ell\right),\ p_j^{n-1}\left(\exp(q_d(\ell)Z_n)\ell\right) \Big).$$

Let u be in $\mathcal{A}_0^{n-1}$. That means there is a polynomial function U in the variables $\ell_1, \dots, \ell_{n-1}$ such that at the step $n-1$, $u = U \circ (\Psi^{n-1})^{-1}$. We shall write $u^{n-1} = U \circ (\Psi^{n-1})^{-1}$. But we can consider the function U as an element in $S(\mathfrak{g})$ and u as the function $u^n = U \circ \Psi^{-1}$. Let us define $Z_n^- U$ as the function:

$$Z_n^- U(\ell) = \frac{d}{dt} U(\exp(-tZ_n)\ell)|_{t=0}.$$

Since by construction,

$$\Psi_\star(Z_n^-)u^n = Z_n^- U \circ \Psi^{-1} = \frac{\partial}{\partial q_d} u^n.$$

Thus:

$$u^n(h^n, q^n, p^n) = \sum_{m\geq 0} \frac{(-q_d)^m}{m!} \Psi_\star^{n-1}(Z_n^-)^m u^{n-1}(h^{n-1}, q^{n-1}, p^{n-1})$$

where

$$h_{i_k}^{n-1} = 0,\ h_s^{n-1} = h_s^n \ \ (s \neq i_k),\ q_i^{n-1} = q_i^n,\ \text{and } p_i^{n-1} = p_i^n.$$

Let us define x_a^{n-1} as q_s^{n-1} if $a = i_s$, p_s^{n-1} if $a = j_s$ and h_a^{n-1} otherwise. Then, thanks to Theorem 4.5, we get:

$$\ell_a^{n-1} = F(x_1^{n-1}, \dots, x_a^{n-1}, q_1^{n-1}, \dots, q_{d-1}^{n-1}).$$

or

$$x_a^{n-1} = H(\ell_1^{n-1}, \dots, \ell_a^{n-1}, q_1^{n-1}, \dots, q_{d-1}^{n-1}).$$

Especially, if $r = j_s > a$, $\partial_{p_s^{n-1}} \ell_a^{n-1} = 0$ and if $C_{a,b}^c$ are the structure constants of $\mathfrak{g}$:

$$\partial_{p_s^{n-1}}(Z_n^- \ell_a) = \partial_{p_s^{n-1}} \left(\sum_{b \le a} C_{n,a}^b \ell_b^{n-1} \right) = 0.$$

Thus $\partial_{p_s^{n-1}}(Z_n^- q_t^{n-1}) = 0$ if $t \le s$. But the vector field Z_n^- preserves the Poisson bracket on $\mathcal{U}^{n-1}$, thus:

$$\begin{aligned} 0 = Z_n^- \{q_s^{n-1}, q_t^{n-1}\} &= \{(Z_n^- q_s^{(n-1)}), q_t^{n-1}\} + \{q_s^{n-1}, (Z_n^- q_t^{n-1})\} \\ &= \partial_{p_t^{n-1}}(Z_n^- q_s^{n-1}) - \partial_{p_s^{n-1}}(Z_n^- q_t^{n-1}), \end{aligned}$$

hence $\partial_{p_s^{n-1}}(Z_n^- q_t^{n-1}) = 0$ for all t. Now:

$$\begin{aligned} 0 &= \exp(-q_d Z_n^-)\{q_i^{n-1}, u^{n-1}\} = \left\{\exp(-q_d Z_n^-) q_i^{n-1}, \exp(-q_d Z_n^-) u^{n-1}\right\} \\ &= \{q_i^n, u^n\} \end{aligned}$$

then u^n is still in $\mathcal{A}_0^n$.

Let us compare the valuations of u^{n-1} and u^n. Let X be an element of $\mathfrak{g}_{n-1}$. We have:

$$X \circ \Psi^{-1}(h^n, q^n, p^n) = \sum_{m=0}^{\infty} \frac{(-q_d^n)^m}{m!} (ad\ Z_n)^m (X) \circ (\Psi^{n-1})^{-1}(h^{n-1}, q^{n-1}, p^{n-1})$$

where

$$h_{i_k}^{n-1} = 0,\ h_s^{n-1} = h_s^n\ (s \ne i_k),\ q_i^{n-1} = q_i^n,\ p_i^{n-1} = p_i^n.$$

If the (h,q)-valuation of $X \circ (\Psi^{n-1})^{-1}$ is 0, then the (h,q)-valuation of $X \circ \Psi^{-1}$ is still 0 and if the (h,q)-valuation of $X \circ (\Psi^{n-1})^{-1}$ is strictly positive, then the (h,q)-valuation of $X \circ \Psi^{-1}$ satisfies:

$$val_{h,q}(X \circ \Psi^{-1}) \ge val_{h,q}(X \circ (\Psi^{n-1})^{-1}).$$

Indeed,

$$0 = \exp(-q_d Z_n^-) h_{i_k}^{n-1} = h_{i_k}^{n-1} - q_d Z_n^-(h_{i_k}^{n-1}) + \sum_{m \ge 2} \frac{(-q_d)^m}{m!} (\Psi_* Z_n^-)^m (h_{i_k}^{n-1}).$$

But $Z_n^-(h_{i_k}^{n-1}) \ne 0$, thus the (h,q)-valuation of $h_{i_k}^{n-1}$ is at least 1.

Let now u be in $\overline{\mathcal{A}}_0^{n-1}$, then u is the sum of a series:

$$u^{n-1} = \sum_{m=0}^{\infty} u_m^{n-1}$$

where $u_m^{n-1} = U_m \circ (\Psi^{n-1})^{-1}$, U_m is a polynomial function in the variables ℓ_t and $\lim_{m\to\infty} val_{h,q} u_m^{n-1}$ is $+\infty$. We consider each u_m as a function in $\mathcal{A}_0^n$, then:

$$val_{h,q} u_m^n \ge val_{h,q} u_m^{n-1}, \quad \lim_{m\to\infty} val_{h,q} u_m^n = +\infty$$

and u belongs to $\overline{\mathcal{A}}_0^n$. Moreover $val_{h,q}u^n \geq val_{h,q}u^{n-1}$.

Consider now the new coordinate function q_d^n. We saw it is an analytic G_{n-1}-invariant function on $\mathcal{U}^{n-1}$, thus it depends analytically on the variables h_j^{n-1} only. But these variables are in $\overline{\mathcal{A}}_0^{n-1}$, we can see them as functions λ_j^n in $\overline{\mathcal{A}}_0^n$ and:

$$val_{h,q}(\lambda_j^n) \geq val_{h,q}^{n-1} h_j^{n-1} = 1.$$

The power series defining q_d^n thus converges in $\overline{\mathcal{A}}_0$, this coordinate function belongs to $\overline{\mathcal{A}}_0$.

With the same argument as above, if u^{n-1} is in $\overline{\mathcal{A}}_0^{n-1}$ (respectively in $\overline{\mathcal{A}}_1^{n-1}$) then, if u^n is the function u^{n-1} viewed as an element of $\overline{\mathcal{A}}_0^{n-1}$ (respectively in $\overline{\mathcal{A}}_1^{n-1}$), $\Psi_\star(Z_n^-)(u^n)$ is in $\overline{\mathcal{A}}_0^n$ (respectively in $\overline{\mathcal{A}}_1^n$). Now, if u^{n-1} is either h_j^{n-1} or q_i^{n-1} or p_i^{n-1}, then the new coordinates h_j^n, q_i^n and p_i^n have the following form:

$$f^n = \sum_{m=0}^{\infty} \frac{(-q_d^n)^m}{m!} \left(\Psi_\star Z_n^-\right)^m (u^n).$$

All these functions are in the spaces $\overline{\mathcal{A}}_0$ (respectively $\overline{\mathcal{A}}_1$). Finally, by construction the coordinate p_d^n is in $\mathcal{A}_1$. ■

Remark 6.1. As it is proved in [3], if $\mathfrak{g}$ is an exponential, non completely solvable Lie algebra, then for some orbit it can be impossible to put the q coordinates in $\overline{\mathcal{A}}_0$. In fact, examples indicate that a natural parametrization of Ω needs to use complex coordinates.

Remark 6.2. If $\mathfrak{g}$ is nilpotent, the analytic functions used become rational, with invariant denominators. Thus we can introduce a new metric d_h, defined only from the h-valuation, in this case the h and q functions are in the closure of $\mathcal{A}_0$ for d_h and the p functions in the closure for d_h of $\mathcal{A}_1$. Since the topology defined by d_h is stronger than the topology defined by $d_\star$ on $\mathcal{A}_0$, this gives a better localization of these coordinate functions. We recover the result of [15] for the minimal layer.

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