Tohoku Math. J.
60 (2008), 287–301

CANONICAL FILTRATIONS AND STABILITY OF DIRECT IMAGES BY FROBENIUS MORPHISMS

YUKINORI KITADAI AND HIDEYASU SUMIHIRO*

(Received December 12, 2006, revised October 17, 2007)

Abstract. We study the stability of direct images by Frobenius morphisms. First, we compute the first Chern classes of direct images of vector bundles by Frobenius morphisms modulo rational equivalence up to torsions. Next, introducing the canonical filtrations, we prove that if X is a nonsingular projective minimal surface of general type with semistable Ω^1_X with respect to the canonical line bundle K_X, then the direct images of line bundles on X by Frobenius morphisms are semistable with respect to K_X.

1. Introduction. Let k be an algebraically closed field of characteristic $p > 0$, X a nonsingular projective variety over k of dimension n, $F = F_X$ the absolute Frobenius morphism of X, and H a numerically positive divisor on X. A divisor H on X is called *numerically positive* if it is numerically effective and $H^n > 0$. Then one can define the *slope* of a torsion free sheaf $\mathscr{E}$ on X with respect to H by

$$\mu(\mathscr{E}) = \frac{c_1(\mathscr{E})H^{n-1}}{\mathrm{rk}(\mathscr{E})},$$

where $c_1(\mathscr{E})$ is the first Chern class of $\mathscr{E}$ and $\mathrm{rk}(\mathscr{E})$ is the rank of $\mathscr{E}$. Then a torsion free sheaf $\mathscr{E}$ on X is called *semistable* (resp. *stable*) with respect to H if for all nonzero torsion free subsheaves $\mathscr{F}$ of $\mathscr{E}$, $\mu(\mathscr{F}) \le \mu(\mathscr{E})$ (resp. $\mu(\mathscr{F}) < \mu(\mathscr{E})$).

As for the semistability of Frobenius pull-backs of vector bundles, a lot of useful and important results have been obtained (see, for examples, [5], [8], [16]). On the other hand, Lange and Pauly proved recently the following theorem on the stability of Frobenius direct images of line bundles.

THEOREM (Lange-Pauly [11]). *Let X be a nonsingular projective curve over k of genus $g(X) \ge 2$ and $\mathscr{L}$ a line bundle on X. Then $F_*\mathscr{L}$ is stable.*

Hence it is quite natural to consider the following question for higher dimensional cases:

PROBLEM. Let X be a nonsingular projective variety of general type over k of dimension $n \ge 2$ and $\mathscr{E}$ a semistable vector bundle with respect to a numerically positive divisor H on X. Then is $F_*\mathscr{E}$ semistable with respect to H?

2000 *Mathematics Subject Classification.* Primary 14J60; Secondary 13A35, 14J29.
Key words and phrases. Vector bundles, stability, Frobenius morphisms, canonical filtrations, geography.
* The second author is partly supported by the Grant-in-Aid for Scientific Research (C), Japan Society for the Promotion of Science.

It is well-known that the de Rham complex $(F_*(\Omega_X^\bullet), d)$ of X plays an important role in the proof of Deligne and Illusie's theorem [2]. Hence it seems that our claim of the Problem might be useful in the studies, such as geography, Kodaira vanishing theorem or etc., of nonsingular projective varieties of general type in positive characteristic. In this paper, we shall prove the following theorems which give affirmative answers to the Problem when X is a surface and $\mathscr{E}$ is a line bundle on X.

THEOREM 5.1. *Let X be a nonsingular projective surface over k, and let H be a numerically positive divisor on X such that $|mH|$ is base point free and it contains a nonsingular member for sufficiently large integers m. Assume that $K_X H > 0$ and Ω_X^1 is semistable with respect to H. Then $F_*\mathscr{L}$ is semistable with respect to H for any line bundle $\mathscr{L}$ on X.*

and

THEOREM 5.3. *Let X be a nonsingular projective surface over k, and let H be a numerically positive divisor on X such that $|mH|$ is base point free and it contains a nonsingular member for sufficiently large integers m. Assume that $K_X \equiv 0$ (numerically equivalent to 0) and Ω_X^1 is semistable with respect to H. Then $F_*\mathscr{L}$ is semistable with respect to H for any line bundle $\mathscr{L}$ on X.*

As corollaries of Theorem 5.1, we obtain the following

COROLLARY 5.2. *Let X be a nonsingular projective minimal surface of general type over k. Assume that Ω_X^1 is semistable with respect to K_X. Then $F_*\mathscr{L}$ is semistable with respect to K_X for any line bundle $\mathscr{L}$ on X.*

Further, we obtain the following on geography of minimal projective surface of general type in positive characteristic.

COROLLARY 6.1. *Let X be a nonsingular projective minimal surface of general type over k. Assume that Ω_X^1 is semistable with respect to K_X.*

(1) *(Bogomolov's inequality) If Ω_X^1 is strongly semistable, i.e., $(F^k)^*(\Omega_X^1)$ is semistable for every $k \in \mathbf{N}$ with respect to K_X, then we have*

$$c_1^2(X) \le 4c_2(X)\,.$$

(2) *If $(F^{k-1})^*(\Omega_X^1)$ is semistable and $(F^k)^*(\Omega_X^1)$ is not semistable with respect to K_X for a positive integer k, then we have*

$$c_1^2(X) \le \frac{4p^{2k}}{p^{2k} - (p-1)^2} c_2(X)\,.$$

In particular, we obtain that $c_2(X) > 0$.

We introduced a natural filtration $W^\bullet = \{W^i\}$ $(0 \le i \le n(p-1)+1)$ of $F^*F_*(\mathscr{E})$ called a canonical filtration of $F^*F_*\mathscr{E}$ (cf. Definition 3.1)

$$F^*F_*\mathscr{E} = W^0 \supset W^1 \supset \cdots \supset W^{n(p-1)+1} = 0$$

as follows. Let $\varphi : F^*F_*\mathcal{O}_X \to \mathcal{O}_X$ be the canonical surjective homomorphism and $I = \mathrm{Ker}(\varphi)$ the kernel of φ. Since $F^*F_*\mathcal{E}$ is an $F^*F_*\mathcal{O}_X$ module, we define

$$W^i = F^*F_*\mathcal{E} \cdot I^i , \quad 0 \le i \le n(p-1)+1 .$$

This canonical filtration is also introduced by Joshi, Ramanan, Xia and Yu [8] and Sun [19] using the canonical connection $\nabla : F^*F_*\mathcal{E} \to F^*F_*\mathcal{E} \otimes \Omega^1_X$ of $F^*F_*\mathcal{E}$, which is the positive characteristic version of the Gauss-Manin connection [9]. It is easily seen that both filtrations coincide with each other by local calculations. The canonical filtration and canonical connection play an essential role in our proofs of Theorem 5.1 and Theorem 5.3.

The authors would like to express their sincere gratitude to Professor Kei-ichi Watanabe for pointing out Lange and Pauly's Theorem ([11]) to them and to the referee for the referee's comments.

2. A formula for first Chern classes of the direct images by Frobenius morphisms. Let X be a nonsingular projective variety of dimension n over k and $\mathcal{E}$ a vector bundle of rank r on X. The aim of this section is to compute the first Chern class $c_1(F_{X*}\mathcal{E})$ to determine its slope.

THEOREM 2.1. *Let $\mathcal{E}$ be a vector bundle on X of rank r. Then*

$$c_1(F_*\mathcal{E}) \equiv \frac{p^n - p^{n-1}}{2} r K_X + p^{n-1} c_1(\mathcal{E}) ,$$

where $\equiv$ denotes rational equivalence up to torsions and K_X is the canonical divisor of X.

PROOF. Applying Grothendieck-Riemann-Roch Theorem (cf. [4]) to $F : X \to X$, we obtain

$$\mathrm{ch}(F_*\mathcal{E}) \cdot \mathrm{td}(X) = F_*(\mathrm{ch}(\mathcal{E}) \cdot \mathrm{td}(X))$$

in the Chow ring $A(X)_{\boldsymbol{Q}}$. Hence it follows that

$$c_1(F_*\mathcal{E}) - \frac{1}{2} p^n r K_X = F_*(c_1(\mathcal{E})) - \frac{r}{2} F_* K_X .$$

Since $F_*D = p^{n-1}D$ for every divisor D of X, we obtain the desired formula. □

REMARK 2.2. Kurano proved a similar formula by using Singular Riemann-Roch Theorem.

THEOREM 2.3 (Kurano [10]). *Let k be a perfect field*, $\mathrm{char}(k) = p > 0$, *and X a normal algebraic variety of dimension n over k. Then*

$$c_1(F_*\mathcal{O}_X) = \frac{p^n - p^{n-1}}{2} K_X$$

in $A_{n-1}(X)_{\boldsymbol{Q}}$.

3. Canonical filtrations. In this section, we introduce a useful filtration on $F^*F_*\mathcal{E}$, where $\mathcal{E}$ is a vector bundle on X. Let I be the kernel of the natural surjection $F^*F_*\mathcal{O}_X \to \mathcal{O}_X$.

Since $F^*F_*\mathscr{O}_X$ is an $\mathscr{O}_X$-algebra, we obtain a descending filtration

$$I^0 := F^*F_*\mathscr{O}_X \supset I^1 := I \supset I^2 \supset I^3 \supset \cdots$$

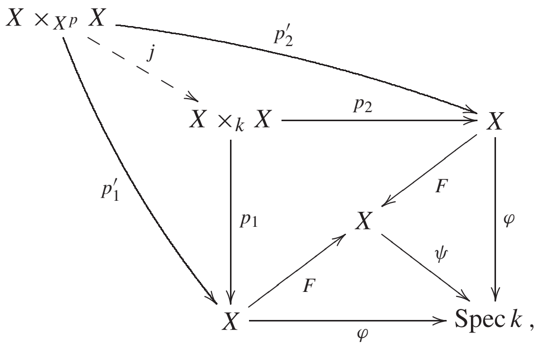
on $F^*F_*\mathscr{O}_X$. Here we consider $F^*F_*\mathscr{O}_X$ as an $\mathscr{O}_X$-module from right. Further, we have a descending filtration on $F^*F_*\mathscr{E}$:

$$W^0 = F^*F_*\mathscr{E} \supset W^1 = F^*F_*\mathscr{E} \cdot I \supset \cdots \supset W^i = F^*F_*\mathscr{E} \cdot I^i \supset \cdots \supset W^{n(p-1)+1} = (0)\,.$$

DEFINITION 3.1. We call this filtration $W^\bullet$ (resp. $I^\bullet$) the *canonical filtration* on $F^*F_*\mathscr{E}$ (resp. $F^*F_*\mathscr{O}_X$).

REMARK 3.2. It is observed that the canonical filtration on $F^*F_*\mathscr{E}$ introduced in [8] and [19] by using the canonical connection $\nabla : F^*F_*\mathscr{E} \to F^*F_*\mathscr{E} \otimes \Omega^1_X$ (cf. [9]) coincides with our canonical filtration $W^\bullet = \{W^i = F^*F_*\mathscr{E} \cdot I^i \mid 0 \leq i \leq n(p-1)+1\}$ on $F^*F_*\mathscr{E}$.

Consider the following commutative diagram:



where φ is the structure morphism of X, ψ is the morphism induced from the map taking the p-th root of elements of k, and p_i, p'_i are natural projections. Then there exists the morphism j in the diagram which is a closed immersion.

Let J (resp. I') be the kernel of the natural surjection $\mathscr{O}_{X\times_k X} \to \mathscr{O}_X$ (resp. $\mathscr{O}_{X\times_{X^p} X} \to \mathscr{O}_X$). Then there exists the following commutative diagram with exact rows of sheaves on $X \times_k X$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J & \longrightarrow & \mathscr{O}_{X\times_k X} & \longrightarrow & \mathscr{O}_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & j_*I' & \longrightarrow & j_*\mathscr{O}_{X\times_{X^p} X} & \longrightarrow & j_*\mathscr{O}_X & \longrightarrow & 0\,.
\end{array}$$

We have $I = p'_{2*}I'$ because $F^*F_*\mathscr{O}_X = p'_{2*}\mathscr{O}_{X\times_{X^p} X}$ and $p'_2 = p_2 \circ j$ is an affine morphism. Hence the morphism $J^i/J^{i+1} = S^i(\Omega^1_X) \twoheadrightarrow j_*(I'^i/I'^{i+1}) \cong I^i/I^{i+1}$ is surjective on X, where Ω^1_X is the vector bundle of regular differential forms of degree 1.

Let $U = \operatorname{Spec} A \subset X$ be a nonempty affine open subset. Then the exact sequence

$$0 \to I \to F^*F_*\mathscr{O}_X \to \mathscr{O}_X \to 0$$

is locally expressed in the following way:

$$0 \to I \to A \otimes_{A^p} A \to A \to 0$$

and $I = \langle a \otimes 1 - 1 \otimes a \mid a \in A \rangle A$. We consider $A \otimes_{A^p} A$ as an A-module from right. Let $\{x_1, \ldots, x_n\}$ be a regular system of parameters. For any element $a \in A$, write $a = \sum_{0 \le i_1, \ldots, i_n \le p-1} a^p_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n}$, where $a_{i_1,\ldots,i_n} \in A$ by shrinking U sufficiently small if necessary. Then we have

$$\begin{aligned} a \otimes 1 - 1 \otimes a &= \sum a^p_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n} \otimes 1 - 1 \otimes \sum a^p_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n} \\ &= \sum (x_1^{i_1} \cdots x_n^{i_n} \otimes a^p_{i_1,\ldots,i_n} - 1 \otimes a^p_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n}) \\ &= \sum (x_1^{i_1} \cdots x_n^{i_n} \otimes 1 - 1 \otimes x_1^{i_1} \cdots x_n^{i_n}) a^p_{i_1,\ldots,i_n} . \end{aligned}$$

Therefore, $I = \langle x_1^{i_1} \cdots x_n^{i_n} \otimes 1 - 1 \otimes x_1^{i_1} \cdots x_n^{i_n} \mid 0 \le i_1, \ldots, i_n \le p-1 \rangle A$ locally.

Let us put $\omega_i = x_i \otimes 1 - 1 \otimes x_i$ $(1 \le i \le n)$ and for $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$, $(0 \le \alpha_k, \beta_k \le p-1, 1 \le k \le n)$ denote by $\beta \le \alpha$ if $\beta_k \le \alpha_k$ for all k.

LEMMA 3.3. *$x_1^{\alpha_1} \cdots x_n^{\alpha_n} \otimes 1 - 1 \otimes x_1^{\alpha_1} \cdots x_n^{\alpha_n} = \sum_{\beta \le \alpha} \omega_1^{\beta_1} \cdots \omega_n^{\beta_n} a_{\beta_1,\cdots,\beta_n}$ for some $a_{\beta_1,\ldots,\beta_n} \in A$.*

PROOF. We shall prove by induction on $|\alpha| = \alpha_1 + \cdots + \alpha_n$. When $|\alpha| = 1$, it is obvious by definition. Assume that we have

$$x_1^{\alpha_1} \cdots x_n^{\alpha_n} \otimes 1 - 1 \otimes x_1^{\alpha_1} \cdots x_n^{\alpha_n} = \sum_{\beta \le \alpha} \omega_1^{\beta_1} \cdots \omega_n^{\beta_n} a_{\beta_1,\cdots,\beta_n}$$

for some $a_{\beta_1,\ldots,\beta_n} \in A$. Multiplying by $\omega_k = x_k \otimes 1 - 1 \otimes x_k$, we have

$$\begin{aligned} \sum_{\beta \le \alpha} \omega_1^{\beta_1} \cdots \omega_k^{\beta_k+1} \cdots \omega_n^{\beta_n} a_{\beta_1,\cdots,\beta_n} &= (x_k \otimes 1 - 1 \otimes x_k)(x_1^{\alpha_1} \cdots x_n^{\alpha_n} \otimes 1 - 1 \otimes x_1^{\alpha_1} \cdots x_n^{\alpha_n}) \\ &= x_1^{\alpha_1} \cdots x_k^{\alpha_k+1} \cdots x_n^{\alpha_n} \otimes 1 - 1 \otimes x_1^{\alpha_1} \cdots x_k^{\alpha_k+1} \cdots x_n^{\alpha_n} \\ &\quad + (1 \otimes x_k - x_k \otimes 1) x_1^{\alpha_1} \cdots x_n^{\alpha_n} \\ &\quad + (1 \otimes x_1^{\alpha_1} \cdots x_n^{\alpha_n} - x_1^{\alpha_1} \cdots x_n^{\alpha_n} \otimes 1) x_k . \end{aligned}$$

Hence it follows that

$$\begin{aligned} & x_1^{\alpha_1} \cdots x_k^{\alpha_k+1} \cdots x_n^{\alpha_n} \otimes 1 - 1 \otimes x_1^{\alpha_1} \cdots x_k^{\alpha_k+1} \cdots x_n^{\alpha_n} \\ &\quad = \sum_{\beta \le \alpha} \omega_1^{\beta_1} \cdots \omega_k^{\beta_k+1} \cdots \omega_n^{\beta_n} a_{\beta_1,\cdots,\beta_n} + \sum_{\beta \le \alpha} \omega_1^{\beta_1} \cdots \omega_n^{\beta_n} a_{\beta_1,\cdots,\beta_n} x_k + \omega_k x_1^{\alpha_1} \cdots x_n^{\alpha_n} . \end{aligned}$$

□

Thus, we see that $I = \langle \omega_1^{\alpha_1} \cdots \omega_n^{\alpha_n} \mid (\alpha_1, \ldots, \alpha_n) \ne (0) \rangle A$. In addition, we observe that

LEMMA 3.4. *If $\sum_\alpha \omega_1^{\alpha_1} \cdots \omega_n^{\alpha_n} a_{\alpha_1,\ldots,\alpha_n} = 0$, $a_{\alpha_1,\ldots,\alpha_n} \in A$, then $a_{\alpha_1,\ldots,\alpha_n} = 0$ for all $\alpha = (\alpha_1, \ldots, \alpha_n)$.*

PROOF. Multiplying by $\omega_1^{p-2}\omega_2^{p-1}\cdots\omega_n^{p-1}$, we have

$$\omega_1^{p-1}\omega_2^{p-1}\cdots\omega_n^{p-1}a_{10\cdots 0}=0\,,$$

since $\omega_i^p=0$ for $1\le i\le n$. Hence

$$\begin{aligned}0&=\omega_1^{p-1}\omega_2^{p-1}\cdots\omega_n^{p-1}a_{10\cdots 0}\\&=\sum_{\beta\le(p-1,\cdots,p-1),\beta\ne 0}(x_1^{\beta_1}\cdots x_n^{\beta_n}\otimes 1-1\otimes x_1^{\beta_1}\cdots x_n^{\beta_n})b_{\beta_1,\dots,\beta_n}a_{10\cdots 0}\end{aligned}$$

implies $a_{10\cdots 0}=0$ because $\{x_1^{\beta_1}\cdots x_n^{\beta_n}\otimes 1 \mid 0\le\beta_1,\dots,\beta_n\le p-1\}$ is a free basis of $A\otimes_{A^p}A$ as an A-module. Iterating this procedure, we observe that $a_{\alpha_1,\dots,\alpha_n}=0$ for all $\alpha=(\alpha_1,\dots,\alpha_n)$. □

Therefore summing up the above, we obtain the following.

COROLLARY 3.5. (1) *I is a free A-module with a basis $\{\omega_1^{\alpha_1}\cdots\omega_n^{\alpha_n} \mid (\alpha_1,\dots,\alpha_n)\ne 0, 0\le\alpha_k\le p-1, 1\le k\le n\}$.*

$$I=\bigoplus_{(\alpha_1,\dots,\alpha_n)}\omega_1^{\alpha_1}\cdots\omega_n^{\alpha_n}A\,.$$

(2) $I^i/I^{i+1}=\bigoplus_{\alpha_1+\cdots+\alpha_n=i}\omega_1^{\alpha_1}\cdots\omega_n^{\alpha_n}A$ *for* $0\le i\le n(p-1)$.

Next, we will calculate the filtration for the case of curves or surfaces.

3.1. Curve case. Assume that X is a curve. Let x be a regular parameter and put $\omega:=x\otimes 1-1\otimes x$. Then we have $I=\bigoplus_{1\le i\le p-1}\omega^i A$ and $I^i/I^{i+1}=\omega^i A$.

Since I^i/I^{i+1} is a line bundle on X, the surjection $J^i/J^{i+1}=K_X^{\otimes i}\to I^i/I^{i+1}$ is an isomorphism for $0\le i\le p-1$. Hence we obtain

PROPOSITION 3.6. *Let X be a nonsingular projective curve over k and $I^\bullet$ the canonical filtration on $F^*F_*\mathscr{O}_X$. Then it follows that*

$$F^*F_*\mathscr{O}_X\supset I\supset I^2\supset\cdots\supset I^{p-1}\supset I^p=(0)$$

and $I^i/I^{i+1}=K_X^{\otimes i}$ for $0\le i\le p-1$.

3.2. Surface case. Assume that X is a surface. Let $\{x,y\}$ be a regular system of parameters and let $\omega:=x\otimes 1-1\otimes x$ and $\eta:=y\otimes 1-1\otimes y$. Then we see that

$$F^*F_*\mathscr{O}_X\supset I\supset I^2\supset\cdots\supset I^{2p-2}\supset I^{2p-1}=(0)\,,$$

$$I^i=\bigoplus_{\substack{k+l\ge i,\\0\le k,l\le p-1}}\omega^k\eta^l A\,,\quad I^i/I^{i+1}=\bigoplus_{\substack{k+l=i,\\0\le k,l\le p-1}}\omega^k\eta^l A\,.$$

LEMMA 3.7. $\binom{p-1}{n}=(-1)^n$ *in positive characteristic p.*

PROOF. It is obvious. □

LEMMA 3.8. $I^{2p-2}\cong K_X^{\otimes(p-1)}$.

PROOF. I^{2p-2} is a line bundle on X because $I^{2p-2} = \omega^{p-1}\eta^{p-1}A$ locally. Let $\{x', y'\}$ be a regular system of parameters of another open subset V. Let $\omega' := x' \otimes 1 - 1 \otimes x'$ and $\eta' := y' \otimes 1 - 1 \otimes y'$. Put $\omega = \omega' a + \eta' b +$ (higher terms) and $\eta = \omega' c + \eta' d +$ (higher terms) for $a, b, c, d \in \Gamma(U \cap V, \mathscr{O}_X)$ on the intersection $U \cap V$. Then since $\omega'^p = \eta'^p = 0$, we have

$$\begin{aligned}\omega^{p-1}\eta^{p-1} &= \sum_{0 \le m,n \le p-1} \omega'^{m+n}\eta'^{2p-2-(m+n)} \cdot \binom{p-1}{m}\binom{p-1}{n} a^m b^{p-1-m} c^n d^{p-1-n} \\ &= \omega'^{p-1}\eta'^{p-1} \sum_{0 \le m \le p-1} \binom{p-1}{m}\binom{p-1}{p-1-m} (ad)^m (bc)^{p-1-m}.\end{aligned}$$

By Lemma 3.7, it turns out that

$$\begin{aligned}&\sum_{0 \le m \le p-1} \binom{p-1}{m}\binom{p-1}{p-1-m} (ad)^m (bc)^{p-1-m} \\ &= \sum \binom{p-1}{m} (ad)^m (-1)^{p-1-m} (bc)^{p-1-m} \\ &= (ad - bc)^{p-1}.\end{aligned}$$

This is the transition function of $K_X^{\otimes(p-1)}$. □

If $i \le p-1$, then J^i/J^{i+1} and I^i/I^{i+1} are vector bundles on X of the same rank, and so it follows that $I^i/I^{i+1} \cong J^i/J^{i+1} \cong S^i(\Omega_X^1)$. On the other hand, there exists the following perfect pairing:

$$\begin{array}{ccc} I^i/I^{i+1} \otimes_{\mathscr{O}_X} I^{2p-2-i}/I^{2p-1-i} & \longrightarrow & I^{2p-2}/I^{2p-1} = I^{2p-2} \cong \omega_X^{\otimes(p-1)}. \\ \cup & & \cup \\ \omega^k\eta^l \otimes \omega^{k'}\eta^{l'} & \longmapsto & \omega^{k+k'}\eta^{l+l'} \end{array}$$

Thus, combining the above, we obtain

PROPOSITION 3.9. *Let X be a nonsingular projective surface over k and $I^\bullet$ the canonical filtration on $F^*F_*\mathscr{O}_X$. Then it holds that*

$$F^*F_*\mathscr{O}_X \supset I \supset I^2 \supset \cdots \supset I^{2p-2} \supset I^{2p-1} = (0)$$

and

$$I^i/I^{i+1} = \begin{cases} S^i(\Omega_X^1), & 0 \le i \le p-1, \\ K_X^{\otimes(i-p+1)} \otimes S^{2p-2-i}(\Omega_X^1), & p \le i \le 2p-2. \end{cases}$$

4. Canonical connections. Let $\mathscr{E}$ be a quasi-coherent sheaf on a nonsingular projective variety X of dimension n. Then there exists a connection $\nabla : F^*\mathscr{E} \to F^*\mathscr{E} \otimes \Omega_X^1$, which is called the *canonical connection* (cf. [9]). This is locally written as

$$\begin{array}{ccc} M \otimes_A A & \longrightarrow & M \otimes_A A \otimes_A \Omega^1_{A/k} \cong M \otimes_A \Omega^1_{A/k}, \\ \cup & & \cup \\ m \otimes f & \longmapsto & m \otimes df \end{array}$$

where $A = \Gamma(U, \mathcal{O}_X)$ and $M = \Gamma(U, \mathcal{E})$ for an affine open subset U of X. Here A is considered as an A-module through Frobenius morphism. In particular, we get a connection on $F^*F_*\mathcal{O}_X$

$$\nabla : F^*F_*\mathcal{O}_X \to F^*F_*\mathcal{O}_X \otimes \Omega^1_X .$$

Let $\{x_1, \dots, x_n\}$ be a regular system of parameters on $U = \operatorname{Spec} A$ and $\omega_i = x_i \otimes 1 - 1 \otimes x_i$ for $1 \le i \le n$. Then we have by straightforward computation,

$$\nabla(\omega_1^{\alpha_1} \cdots \omega_n^{\alpha_n} f) = \sum_{k=1}^{n} \left(-\alpha_k \omega_1^{\alpha_1} \cdots \omega_k^{\alpha_k - 1} \cdots \omega_n^{\alpha_n} f + \omega_1^{\alpha_1} \cdots \omega_n^{\alpha_n} \frac{\partial f}{\partial x_k} \right) \otimes dx_k, \quad f \in A .$$

5. Main results. Using the canonical filtrations (Proposition 3.9), we can prove the following theorem, which is a generalization of Lange and Pauly's Theorem to the surface case.

THEOREM 5.1. *Let X be a nonsingular projective surface over k, and let H be a numerically positive divisor on X such that $|mH|$ is base point free and it contains a nonsingular member for sufficiently large integers m. Assume that $K_X H > 0$ and Ω^1_X is semistable with respect to H. Then $F_*\mathcal{L}$ is semistable with respect to H for any line bundle $\mathcal{L}$ on X.*

PROOF. Assuming that $F_*\mathcal{L}$ is not semistable with respect to H, we shall derive a contradiction. Let $S \subset F_*\mathcal{L}$ be a destabilizing subsheaf. Then $\mu(S) > \mu(F_*\mathcal{L}) = ((1 - 1/p)/2)K_X H + (1/p)c_1(\mathcal{L})H$.

Let $I^\bullet$ (resp. $W^\bullet$) be the canonical filtration of $F^*F_*\mathcal{O}_X$ (resp. $F^*F_*\mathcal{L}$).

By definition, we have $W^i := F^*F_*\mathcal{L} \cdot I^i$ for $0 \le i \le 2p - 1$. Let $U = \operatorname{Spec} A \subset X$ be an affine open subset of X and $M = \Gamma(U, \mathcal{L})$. Then $M \otimes_{A^p} A = M \otimes_A (A \otimes_{A^p} A)$ is an $(A \otimes_{A^p} A)$-module, and so $M \otimes_{A^p} A \cdot I^i = M \otimes_A (A \otimes_{A^p} A) \cdot I^i = M \otimes_A I^i$. Thus, considering I^i as an $\mathcal{O}_X$-module from both sides, we observe that $W^i = F^*F_*\mathcal{L} \cdot I^i \cong \mathcal{L} \otimes_{\mathcal{O}_X} I^i$. Hence $W^\bullet$ is a filtration of $F^*F_*\mathcal{L}$ such that

$$F^*F_*\mathcal{L} \supset W^1 \supset W^2 \supset \cdots \supset W^{2p-2} \supset W^{2p-1} = (0)$$

and

$$\operatorname{Gr}^i(W^\bullet) = \begin{cases} \mathcal{L} \otimes S^i(\Omega^1_X), & 0 \le i \le p - 1, \\ \mathcal{L} \otimes K_X^{\otimes(i-p+1)} \otimes S^{2p-2-i}(\Omega^1_X), & p \le i \le 2p - 2 \end{cases} \tag{1}$$

by Proposition 3.9.

Moreover, $F^*S \cap W^\bullet$ is a filtration of F^*S. Let m be the integer such that $F^*S \cap W^m \ne (0)$ and $F^*S \cap W^{m+1} = (0)$.

CLAIM. *$F^*S \cap W^i \supsetneq F^*S \cap W^{i+1}$ for $0 \le i \le m$.*

Indeed, it is trivial in the case $i = m$ by definition of m. Let $i < m$. Locally, we use the same notation of ω, η as in 3.2. Pick a nonzero element

$$g \otimes \left(\sum_{k+l=i+1} \omega^k \eta^l f_{kl} + \theta \right) \in F^*S \cap W^{i+1} ,$$

where $g \in \Gamma(U, \mathscr{L})$, $f_{kl} \in A$, and for some $f_{k_0 l_0} \neq 0$ and $\theta \in I^{i+2}$. Then we have

$$\begin{aligned} &\nabla\left(g \otimes \left(\sum_{k+l=i+1} \omega^k \eta^l f_{kl} + \theta \right)\right) \\ &\quad = g \otimes \sum_{k+l=i+1} (-k\omega^{k-1}\eta^l f_{kl} + \text{(higher)}) \otimes dx \\ &\qquad + g \otimes \sum_{k+l=i+1} (-l\omega^k \eta^{l-1} f_{kl} + \text{(higher)}) \otimes dy . \end{aligned}$$

Since the restriction of ∇ to F^*S is a connection of F^*S, we see that $g \otimes \sum(-k\omega^{k-1}\eta^l f_{kl} + \text{(higher)}) \in F^*S \cap W^i \setminus F^*S \cap W^{i+1}$ and $g \otimes \sum(-l\omega^k\eta^{l-1} f_{kl} + \text{(higher)}) \in F^*S \cap W^i \setminus F^*S \cap W^{i+1}$.

Therefore, $F^*S \cap W^i / F^*S \cap W^{i+1}$ is a nonzero subsheaf of $W^i/W^{i+1} \cong \mathscr{L} \otimes I^i/I^{i+1}$. Since Ω^1_X is semistable by assumption, $\mathrm{Gr}^i(W^\bullet)$ $(0 \leq i \leq m)$ is semistable by above (1) and Ilangovan-Mehta-Parameswaran's Theorem ([7, 15]) and the restriction theorem (cf. [12, Corollary 5.4]). Hence we obtain

$$\mu(F^*S \cap W^i / F^*S \cap W^{i+1}) \leq \mu(\mathscr{L} \otimes I^i/I^{i+1}) .$$

Let r_i be the rank of $F^*S \cap W^i / F^*S \cap W^{i+1}$. Then it follows that

$$\frac{(c_1(F^*S \cap W^i) - c_1(F^*S \cap W^{i+1}))H}{r_i} \leq c_1(\mathscr{L})H + \frac{i}{2} K_X H .$$

Summing up for all i, we have

$$c_1(F^*S)H \leq \sum_{i=0}^m r_i c_1(\mathscr{L})H + \frac{1}{2}\sum_{i=0}^m i r_i K_X H ,$$

$$\mu(F^*S) = \frac{c_1(F^*S)H}{\sum_{i=0}^m r_i} \leq c_1(\mathscr{L})H + \frac{\sum i r_i}{2\sum r_i} K_X H .$$

On the other hand,

$$\begin{aligned} \mu(F^*S) = p \cdot \mu(S) &> p \cdot \left(\frac{1 - 1/p}{2} K_X H + \frac{1}{p} c_1(\mathscr{L})H \right) \\ &= \frac{p-1}{2} K_X H + c_1(\mathscr{L})H . \end{aligned}$$

Thus, we obtain

$$(p-1)\sum_{i=0}^m r_i < \sum_{i=0}^m i r_i . \tag{2}$$

In particular, we may assume $m \geq p$ from the inequality (2).

Let $K = k(X)$ be the rational function field of X. Then we see that $W^i \otimes K = \langle \omega^\alpha \eta^\beta \mid \alpha+\beta \geq i \rangle K$ and $W^i/W^{i+1} \otimes K = \langle \omega^\alpha \eta^\beta \mid \alpha+\beta = i \rangle K$. Let $\delta_l = \sum_{\alpha+\beta=i} \omega^\alpha \eta^\beta f_{\alpha\beta}^{(l)}$ + (higher) $\in (F^*S \cap W^i) \otimes K$ $(f_{\alpha\beta}^{(l)} \in K)$ be a lifting of an element $\bar{\delta}_l \in (F^*S \cap W^i) \otimes K/(F^*S \cap W^{i+1}) \otimes K$, and put $\varepsilon_l = (f_{\alpha\beta}^{(l)}) \in K^{\oplus(i+1)} \cong W^i/W^{i+1} \otimes K$. Then it is easily seen that $\{\varepsilon_l \mid 1 \leq l \leq r_i\}$ is linearly independent over K if and only if $\{\bar{\delta}_l \mid 1 \leq l \leq r_i\}$ is linearly independent over K.

CLAIM. $r_i \geq r_{2p-2-i}$ *for* $1 \leq i \leq p-2$.

Indeed, let $\{\bar{\delta}_l \mid 1 \leq l \leq r_{2p-2-i}\}$ be a basis of $(F^*S \cap W^{2p-2-i}) \otimes K/(F^*S \cap W^{2p-1-i}) \otimes K$ and take $\{\delta_l \mid 1 \leq l \leq r_{2p-2-i}\}$ a set of liftings of $\{\bar{\delta}_l \mid 1 \leq l \leq r_{2p-2-i}\}$. Put $\delta_l = \sum_{\alpha+\beta=2p-2-i} \omega^\alpha \eta^\beta f_{\alpha\beta}^{(l)}$ + (higher) $\in (F^*S \cap W^{2p-2-i}) \otimes K$ and $\varepsilon_l = (f_{\alpha\beta}^{(l)}) \in K^{\oplus(i+1)}$. Then we see that $\{\varepsilon_l \mid 1 \leq l \leq r_{2p-2-i}\}$ is linearly independent over K. On the other hand, we have

$$\begin{aligned}\nabla(\delta_l) = &\sum_{\alpha+\beta=2p-2-i} (-\alpha\omega^{\alpha-1}\eta^\beta f_{\alpha\beta}^{(l)} + (\text{higher})_x) \otimes dx \\ &+ \sum_{\alpha+\beta=2p-2-i} (-\beta\omega^\alpha\eta^{\beta-1} f_{\alpha\beta}^{(l)} + (\text{higher})_y) \otimes dy\,.\end{aligned}$$

Hence, if we put

$$\nabla_x(\delta_l) = \sum_{\alpha+\beta=2p-2-i} (-\alpha\omega^{\alpha-1}\eta^\beta f_{\alpha\beta}^{(l)} + (\text{higher})_x)\,,$$

$$\nabla_y(\delta_l) = \sum_{\alpha+\beta=2p-2-i} (-\beta\omega^\alpha\eta^{\beta-1} f_{\alpha\beta}^{(l)} + (\text{higher})_y)\,,$$

then we see that $\nabla_x(\delta_l)$ and $\nabla_y(\delta_l)$ are contained in $(F^*S \cap W^{2p-3-i}) \otimes K$. Further, it follows that $\{\overline{\nabla_x(\delta_l)} \mid 1 \leq l \leq r_{2p-2-i}\}$ is linearly independent over K, where $\overline{\nabla_x(\delta_l)}$ is the residue class of $\nabla_x(\delta_l)$ in $(F^*S \cap W^{2p-3-i}) \otimes K/(F^*S \cap W^{2p-2-i}) \otimes K$, since $\{\varepsilon_{xl} \mid 1 \leq l \leq r_{2p-2-i}\}$ is linearly independent over K, where $\varepsilon_{xl} = (-\alpha f_{\alpha\beta}^{(l)}) \in K^{\oplus(i+2)}$. Similarly, it turns out that $\{\overline{\nabla_y(\delta_l)} \mid 1 \leq l \leq r_{2p-2-i}\}$ is linearly independent over K. Thus, iterating these operations $2(p-1-i)$-times, we observe that $\{\overline{\nabla_y^{p-1-i}\nabla_x^{p-1-i}(\delta_l)} \mid 1 \leq l \leq r_{2p-2-i}\}$ is linearly independent over K, where $\overline{\nabla_y^{p-1-i}\nabla_x^{p-1-i}(\delta_l)}$ is the residue class of $\nabla_y^{p-1-i}\nabla_x^{p-1-i}(\delta_l)$ in $(F^*S \cap W^i) \otimes K/(F^*S \cap W^{i+1}) \otimes K$. Therefore we get

$$r_i \geq r_{2p-2-i}\,, \quad 1 \leq i \leq p-2\,.$$

Hence we have

$$
\begin{aligned}
&(p-1)\sum_{i=0}^{m} r_i - \sum_{i=0}^{m} i r_i \\
&\quad = \left((p-1)\sum_{i=0}^{p-1} r_i + (p-1)\sum_{j=p}^{m} r_j\right) - \left(\sum_{i=0}^{p-1} i r_i + \sum_{j=p}^{m} j r_j\right) \\
&\quad = \sum_{i=0}^{p-1}(p-1-i)r_i + \sum_{j=2p-2-m}^{p-2} (p-1-(2p-2-j))r_{2p-2-j} \\
&\quad \geq \sum_{i=0}^{p-1}(p-1-i)r_i + \sum_{j=2p-2-m}^{p-2} (j-(p-1))r_j \\
&\quad = \sum_{i=0}^{2p-3-m} (p-1-i)r_i \geq 0\,.
\end{aligned}
$$

This contradicts the inequality (2). □

In particular, we obtain the following from Theorem 5.1.

COROLLARY 5.2. *Let X be a nonsingular projective minimal surface of general type over k. Assume that Ω_X^1 is semistable with respect to K_X. Then $F_*\mathscr{L}$ is semistable with respect to K_X for any line bundle $\mathscr{L}$ on X.*

PROOF. We see by [3] that $|mK_X|$ $(m \geq 5)$ is base point free and it contains a nonsingular member. □

Using the canonical filtrations, we can also prove a similar result in the case that K_X is numerically trivial.

THEOREM 5.3. *Let X be a nonsingular projective surface over k, and let H be a numerically positive divisor on X such that $|mH|$ is base point free and it contains a nonsingular member for sufficiently large integers m. Assume that $K_X \underset{\text{num}}{\equiv} 0$ and Ω_X^1 is semistable with respect to H. Then $F_*\mathscr{L}$ is semistable with respect to H for any line bundle $\mathscr{L}$ on X.*

PROOF. Consider the canonical filtration $W^\bullet$ as in the proof of Theorem 5.1. Since $K_X \underset{\text{num}}{\equiv} 0$, the graded components of $W^\bullet$ have the same slope $c_1(\mathscr{L})H$. Hence $F^*F_*\mathscr{L}$ is semistable, since it is an extension of semistable vector bundles with the same slope $c_1(\mathscr{L})H$ and so $F_*\mathscr{L}$ is semistable. □

EXAMPLE 5.4 (cf. Noma [14]). If $X \subset \boldsymbol{P}^3$ is a general surface of degree $d \geq 4$, we can prove Ω_X^1 is strongly stable, i.e., $(F^k)^*\Omega_X^1$ is stable for every $k \in \boldsymbol{N}$, with respect to any ample divisor H. So X satisfies the conditions of above theorems.

PROOF. Let $\mathscr{L}$ be a sub-line bundle of $(F^k)^*\Omega_X^1$. By Noether's Theorem (cf. [1]), $\mathrm{Pic}(\boldsymbol{P}^3) \cong \boldsymbol{Z} \to \mathrm{Pic}(X)$ is an isomorphism. So write $\mathscr{L} = \mathscr{O}_X(m)$ for some integer m.

Consider the following commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X(-p^k d) & \longrightarrow & Q & \longrightarrow & \mathcal{L} = \mathcal{O}_X(m) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X(-p^k d) & \longrightarrow & (F^k)^*\Omega^1_{\boldsymbol{P}^3}|_X & \longrightarrow & (F^k)^*\Omega^1_X & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
$$

Since $H^1(X, \mathcal{O}_X(-p^k d - m)) = 0$, the extension Q splits as $Q \cong \mathcal{O}_X(-p^k d) \oplus \mathcal{O}_X(m)$. There is the exact sequence:

$$0 \to (F^k)^*\Omega^1_{\boldsymbol{P}^3}|_X \to \bigoplus^4 \mathcal{O}_X(-p^k) \to \mathcal{O}_X \to 0\,.$$

So the composition

$$Q \cong \mathcal{O}_X(-p^k d) \oplus \mathcal{O}_X(m) \hookrightarrow (F^k)^*\Omega^1_{\boldsymbol{P}^3}|_X \hookrightarrow \bigoplus^4 \mathcal{O}_X(-p^k)$$

is an injection. If $m \geq 0$, then this map cannot be an injection. So $m < 0$. Therefore, $\mu(\mathcal{L}) = mH^2 < 0$. On the other hand, $\mu((F^k)^*\Omega^1_X) = p^k K_X H/2 \geq 0$, since X is of degree $d \geq 4$. Thus Ω^1_X is strongly stable with respect to H. □

REMARK 5.5. Using the canonical filtrations in the curve case, we can also prove the following.

(1) $F_*\mathcal{O}_X \otimes F_*\mathcal{O}_X$ is not semistable, whereas $F_*\mathcal{O}_X$ is stable for nonsingular curves X with genus ≥ 2 ([5]).

(2) The following generalization of Lange and Pauly's Theorem ([11]) by applying the arguments in Theorem 5.1 to the curve case.

THEOREM 5.6 (Mehta-Pauly [13], Sun [19]). *Let X be a nonsingular projective curve over k of genus $g(X) \geq 2$ and $\mathcal{E}$ a stable* (*resp. semistable*) *vector bundle on X. Then $F_*\mathcal{E}$ is stable* (*resp. semistable*).

REMARK 5.7. Under the assumption of Theorem 5.1, we see that the canonical filtration $W^\bullet$ on $F^*F_*\mathcal{L}$ is the Harder-Narasimhan filtration of $F^*F_*\mathcal{L}$. In fact, it follows from Proposition 3.9 that $W^i/W^{i+1} = \mathcal{L} \otimes I^i/I^{i+1}$ is semistable with respect to H and the slope $\mu(W^i/W^{i+1}) = c_1(\mathcal{L})H + (i/2)K_X H$ for $0 \leq i \leq 2(p-1)$.

6. An application. In this section, we show an application to the geography of nonsingular projective minimal surfaces of general type.

THEOREM 6.1. *Let X be a nonsingular projective minimal surface of general type over k. Assume that Ω_X^1 is semistable with respect to K_X.*

(1) (*Bogomolov's inequality*) *If Ω_X^1 is strongly semistable, i.e., $(F^k)^*\ \Omega_X^1$ is semistable for every $k \in \boldsymbol{N}$, with respect to K_X, then we have*

$$c_1^2(X) \leq 4c_2(X)\,.$$

(2) *If $(F^{k-1})^*\Omega_X^1$ is semistable with respect to K_X and $(F^k)^*\Omega_X^1$ is not semistable with respect to K_X for a positive integer k, then we have*

$$c_1^2(X) \leq \frac{4p^{2k}}{p^{2k}-(p-1)^2}c_2(X)\,.$$

In particular, we obtain $c_2(X) > 0$.

PROOF. (1) is well-known. We shall prove (2).

Assuming $\mathcal{O}_X(L)$ is a maximal destabilizing subsheaf of $(F^k)^*\Omega_X^1$, consider the following exact sequence

$$0 \to \mathcal{O}_X(L) \to (F^k)^*\Omega_X^1 \to I_Z \otimes \mathcal{O}_X(M) \to 0\,,$$

where I_Z is the ideal of a 0-dimensional closed subscheme Z of X. Then we have a nonzero map $(F^{k-1})^*\Omega_X^1 \to F_*\mathcal{O}_X(M)$ by the adjoint of $(F^k)^*\Omega_X^1 \to \mathcal{O}_X(M)$. Since $(F^{k-1})^*\Omega_X^1$ and $F_*\mathcal{O}_X(M)$ are semistable by Corollary 5.2, it follows that $\mu((F^{k-1})^*\Omega_X^1) \leq \mu(F_*\mathcal{O}_X(M))$. Thus, we obtain

$$\frac{p^k}{2}K_X^2 < LK_X \leq \frac{p^k+p-1}{2}K_X^2$$

by Theorem 2.1. Moreover, we have $0 \leq \deg(Z) = c_2(\mathcal{O}_X(-L) \otimes (F^k)^*\Omega_X^1) = p^{2k}c_2(X) - p^kK_XL + L^2 = p^{2k}c_2(X) - ML$.

Assume $L^2 \leq 0$. In this case, since we have $L^2 = p^kK_XL - ML \leq 0$ and $ML \leq p^{2k}c_2(X)$, we obtain

$$K_X^2 < 2c_2(X)\,.$$

Assume $M^2 \leq 0$. Then since $M^2 = p^kK_XM - LM \leq 0$, it turns out that $p^kK_XM \leq LM \leq p^{2k}c_2(X)$. On the other hand, we have $MK_X = p^kK_X^2 - LK_X \geq (p^k - p + 1/2)K_X^2$. Hence it follows that

$$K_X^2 \leq \frac{2p^{2k}}{p^k-p+1}c_2(X)\,.$$

Assume that $L^2 > 0$ and $M^2 > 0$. Then we have $L^2M^2 \leq (LM)^2$ by the Hodge index theorem. Since $L^2M^2 = (p^kK_XL - ML)(p^kK_XM - ML)$, it follows that $(K_XL)(K_XM) \leq K_X^2(LM) \leq p^{2k}K_X^2c_2(X)$. Since $(K_XL)(K_XM) = (p^kK_X^2 - K_XM)(K_XM) \geq (p^kK_X^2 - (p^k - p + 1/2)K_X^2)((p^k - p + 1/2)K_X^2)$, we obtain

$$K_X^2 \leq \frac{4p^{2k}}{p^{2k}-(p-1)^2}c_2(X)\,.$$

Combining the above, we get the claim. □

REMARK 6.2. In the case that Ω_X^1 is semistable with respect to K_X, Shepherd-Barron [16, 17] gave a better inequality than ours by different arguments.

REFERENCES

[1] P. DELIGNE AND N. KATZ, Groupes de monodromie en géométrie algébrique. II, Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 II), Lecture Notes in Mathematics, Vol. 340, Springer-Verlag, Berlin-New York, 1973.

[2] P. DELIGNE AND L. ILLUSIE, Relèvements modulo p^2 et décomposition du complexe de De Rham, Invent. Math. 89 (1987), 247–270.

[3] T. EKEDAHL, Canonical models of surfaces of general type in positive characteristic, Inst. Hautes Études Sci. Publ. Math. 67 (1988), 97–144.

[4] W. FULTON, Intersection theory Second edition, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 2., Springer-Verlag, Berlin, 1998.

[5] D. GIESEKER, Stable vector bundles and the Frobenius morphism, Ann. Sci. École Norm. Sup. (4) 6 (1973), 95–101.

[6] D. HUYBRECHTS AND M. LEHN, The geometry of moduli spaces of sheaves, Aspects Math. E31, Friedr. Vieweg & Sohn, Braunschweig, 1997.

[7] S. ILANGOVAN, V. B. MEHTA AND A. J. PARAMESWARAN, Semistability and semisimplicity in representations of low height in positive characteristic, A tribute to C. S. Seshadri (Chennai, 2002), 271–282, Trends Math., Birkhäuser, Basel, 2003.

[8] K. JOSHI, S. RAMANAN, E. XIA AND J. K. YU, On vector bundles destabilized by Frobenius pull-back, Compos. Math. 142 (2006), 616–630.

[9] N. M. KATZ, Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin, Inst. Hautes Études Sci. Publ. Math. 39 (1970), 175–232.

[10] K. KURANO, The singular Riemann-Roch theorem and Hilbert-Kunz functions, J. Algebra 304 (2006), 487–499.

[11] H. LANGE AND C. PAULY, On Frobenius-destabilized rank-2 vector bundles over curves, arXiv.math. AG/0309456 v2, (2005).

[12] A. LANGER, Semistable sheaves in positive characteristic, Ann. of Math. (2) 159 (2004), 251–276.

[13] V. B. MEHTA AND C. PAULY, Semistability of Frobenius direct images over curves, arXiv.math.AG/0607565 v1, (2006).

[14] A. NOMA, Stability of Frobenius pull-backs of tangent bundles of weighted complete intersections, Math. Nachr. 221 (2001), 87–93.

[15] J. P. SERRE, Sur la semi-simplicité des produits tensoriels de représentations de groupes, Invent. Math. 116 (1994), 513–530.

[16] N. I. SHEPHERD-BARRON, Unstable vector bundles and linear systems on surfaces in characteristic p, Invent. Math. 106 (1991), 243–262.

[17] N. I. SHEPHERD-BARRON, Geography for surfaces of general type in positive characteristic, Invent. Math. 106 (1991), 263–274.

[18] N. I. SHEPHERD-BARRON, Semi-stability and reduction mod p, Topology 37 (1998), 659–664.

[19] X. SUN, Stability of direct images under Frobenius morphism, arXiv.math.AG/0608043 v2, (2006).

DEPARTMENT OF MATHEMATICS
GRADUATE SCHOOL OF SCIENCE
HIROSHIMA UNIVERSITY
HIGASHI-HIROSHIMA 739–8526
JAPAN

E-mail address: Nyoho@hiroshima-u.ac.jp
: sumihiro@math.sci.hiroshima-u.ac.jp