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CANONICAL MODELS OF SURFACES OF GENERAL TYPE

by E. BOMBIERI (Pisa)

CONTENTS

1. Introduction and results.

Let S be an algebraic surface, complete and non-singular, over an algebraically closed field k of characteristic $\mathrm{char}(k) = 0$. Let K denote the canonical bundle of S (determinant of the cotangent bundle) and let mK be its m-th tensor power. One defines with Mumford the *pluricanonical ring* of the surface S as the graded ring

$$R = \sum_{m=0}^{\infty} H^0(S, \mathcal{O}(mK)) .$$

The graded ring R is a birational invariant of S and

$$\kappa(S) = -1 + \mathrm{tr}\deg_k R$$

is an important birational invariant of S. Surfaces with $\kappa(S) = -1, 0, 1$ are called of *special type* and one of the main results of the theory of classification of surfaces determines their structure: rational, ruled, abelian, K3, Enriques' surfaces, hyperelliptic, elliptic. Surfaces with $\kappa(S) = 2$ are called of *general type*.

It is known (Zariski [17], Mumford [10]) that if S is of general type then R is a finitely generated graded noetherian ring. The scheme

$$X = \mathrm{Proj}\ (R)$$

will be called the *abstract canonical model* of S; note that X depends only on the birational equivalence class of S.

Let $R = \sum_{m=0}^{\infty} R_m$, where $R_m = H^0(S, \mathcal{O}(mK))$, be the canonical ring of S and let for $n \geqq 1$

$$R^{(n)} = \sum_{m=0}^{\infty} R_{mn},$$

$$R^{[n]} = \sum_{m=0}^{\infty} R_n^m,$$

where $R_n^m \subseteq R_{mn}$ is the subspace of $H^0(S, \mathcal{O}(mnK))$ generated by products of m sections in $H^0(S, \mathcal{O}(nK))$. Since R is finitely generated we have that $R^{(n)}$ and $R^{[n]}$ are again graded noetherian rings and

$$R^{(n)} = R^{[n]} \quad \text{for} \quad n \gg 0.$$

We define

$$X^{(n)} = \mathrm{Proj}(R^{(n)}),$$

$$X^{[n]} = \mathrm{Proj}(R^{[n]})$$

and call $X^{[n]}$ the *n-canonical image* of S. It is well-known that $X^{(n)}$ is isomorphic to X for $n \geqq 1$, the isomorphism $X \rightarrow X^{(n)}$ being induced by the inclusion $R^{(n)} \subseteq R$. The inclusion $R^{[n]} \subseteq R$ induces a rational map

$$\varphi_n : X \rightarrow X^{[n]}$$

of the abstract canonical model into the n-canonical image of S.

The object of this paper is the detailed study of the rational map φ_n. We may ask for instance whether φ_n is a morphism, or is birational, or is a homeomorphism or an isomorphism.

In order to explain the significance of these questions we recall the answer to the analogous problem for curves. Curves of " general type " are exactly those for which the genus is $p \geqq 2$; then X is isomorphic to C and

$$X \rightarrow X^{[2]} \quad \text{is an isomorphism if} \quad p \geqq 3,$$

$$X \rightarrow X^{[n]} \quad \text{is an isomorphism if} \quad n \geqq 3,$$

while if $p = 2$ then $X^{[2]}$ is a projective line embedded in $\mathbf{P}^2(k)$ as a conic. Moreover $X \rightarrow X^{[1]}$ is an isomorphism if and only if the curve C is not hyperelliptic.

In the case of surfaces, it is known (Mumford [10]) that X is a birational model of S and is a normal surface with a finite number of rational double points. A minimal desingularization of X exists and is an absolutely minimal model of S.

We have the following characterization of surfaces of general type. Let

$$P_m = P_m(S) = \dim_k H^0(S, \mathcal{O}(mK))$$

be the *m-th* plurigenus of S; if $m = 1$, we write $p_g(S)$ instead of $P_1(S)$.

Theorem 1. — *A minimal surface* S *is of general type if and only if*

$$K^2 \geqq 1, \quad P^2 \geqq 2.$$

Theorem 1 is due to Kodaira [7].

We shall prove:

Main Theorem. — *Let* S *be a surface of general type and let* K^2 *be the self-intersection of the canonical bundle of a minimal model of* S.

We have

(i) $X \rightarrow X^{[n]}$ *is an isomorphism for all surfaces* S *and* $n \geqq 5$;

(ii) $X \rightarrow X^{[4]}$ *is an isomorphism if* $K^2 \geqq 2$;

(iii) $X \rightarrow X^{[n]}$, $n \geqq 3$ *is a birational map, except in the following cases:*

a) $K^2 = 1$, $p_g = 2$, $n = 3$ *and* 4, *where* $X^{[3]}$ *is a rational ruled surface* F_2 *of degree* 4 *embedded in* $\mathbf{P}^5(k)$ *and where* $X^{[4]}$ *is a quadric cone embedded in* $\mathbf{P}^8(k)$;

b) $K^2 = 2$, $p_g = 3$, $n = 3$, *where* $X^{[3]}$ *is a projective plane* $\mathbf{P}^2(k)$ *embedded in* $\mathbf{P}^9(k)$ *by means of the linear system of plane cubics*;

c) *some surfaces with* $K^2 = 1$, $p_g = 0$, $n = 3$ *and* 4 *and with* $K^2 = 2$, $p_g = 0$, $n = 3$;

(iv) $X \rightarrow X^{[2]}$ *is a birational map if*

$$K^2 \geqq 10, \quad p_g \geqq 6$$

except if S *has the structure of a fiber space*

$$f: S \rightarrow B$$

over a non-singular curve B, *with generic fiber a non-singular curve of genus* 2.

Conversely, let S *be a surface with the above structure* $f: S \rightarrow B$ *of fiber space and with* $K^2 \geqq 10$, $p_g \geqq 6$; *then* $X \rightarrow X^{[2]}$ *is generically a double covering and* $X^{[2]}$ *is a surface birationally equivalent to a rational or ruled surface.*

Remark 1. — It is doubtful whether the exceptions in (iii) c) really occur. I can prove that they do not occur if $n = 4$, while if $n = 3$ these surfaces should satisfy rather strong conditions.

Remark 2. — The fact that if S has a pencil of curves of genus 2 then $X \rightarrow X^{[2]}$ is not birational has been remarked by Kodaira [8]. Part (iv) of our Main Theorem shows that, except for a finite number of algebraic families of surfaces, $X \rightarrow X^{[2]}$ is not birational if and only if S has a pencil of curves of genus 2. We shall also prove that the condition $K^2 \geqq 10$ in (iv) is best possible.

Results in this range of ideas are not new. The first systematic study of surfaces of general type, mainly in case of irregularity $q = 0$, is due to Enriques [5] and surfaces with $K^2 = 1$, $p_g = 2$, $K^2 = 2$, $p_g = 3$ were already encountered by him as pathological examples. The structure of the abstract canonical model was determined by Mumford [10] and the n-canonical images $X^{[n]}$ have been studied from the birational point of view by Moisezon and Šafarevič [15]; from their results, it follows that there are only finitely many families of surfaces for which $X \rightarrow X^{[3]}$ is not a birational map. The more important biregular point of view was considered by Kodaira [7], [8], obtaining in some cases the best possible results; he proved in particular that $X \rightarrow X^{[n]}$ is birational if $n \geqq 5$ and $n=4$, $K^2 \geqq 2$. Refinements of Kodaira's results have been obtained in Bombieri [3]. Kodaira also points out very clearly how the problem can be attacked using the connectedness properties of pluricanonical divisors and the vanishing theorems.

Our techniques differ somewhat from those used previously. Using a strong vanishing theorem (Theorem A) due to Ramanujam [14], we are able to use directly the connectedness properties of pluricanonical divisors, avoiding the cumbersome use of composition series of [7] and [3]. A few cases with small K^2 are dealt with directly.

I want to express here by indebtedness to several mathematicians who helped me during the preparation of this work; in particular, Artin and Mumford for discussions on isolated rational singularities of surfaces, Van de Ven for conversations on irregular surfaces of general type, Ramanujam for the vanishing theorems, and Deligne for several useful suggestions. I also wish to thank the Mathematics Institute of the University of Warwick and the Institut des Hautes Études Scientifiques for providing financial support and a stimulating atmosphere during the preparation of part of this paper.

2. The geometry of the map $X \rightarrow X^{[n]}$.

We shall review here some fundamental properties of surfaces of general type and see how properties of the map $X \rightarrow X^{[n]}$ correspond to properties of the n-canonical bundle nK.

Let S be a complete non-singular algebraic surface over an algebraically closed field k and let L be a line bundle on it, $\mathscr{L}$ being the associated invertible sheaf. A point x of S (not necessarily a closed point) such that $s(x)=0$ for every global section s of L is called a base point of the linear system $|L|$ associated to $H^0(S, \mathscr{L})$. If

$$V = \operatorname{Proj}\Big(\sum_{m=0}^{\infty} H^0(S, \mathscr{L})^m\Big)$$

then it is well-known that the choice of a basis in $H^0(S, \mathscr{L})$ defines a rational map

$$\Phi_L : S \rightarrow V$$

which is a regular map outside the base point set of $|L|$. If $|L|$ has no base points then Φ_L is a morphism and

$$\mathscr{L} = \Phi_L^*(\mathcal{O}_V(1)).$$

Now let S be a minimal surface of general type. We have (Mumford [10], Kodaira [7]):

Proposition 1. — If C *is an irreducible curve on* S *then* $KC \geqslant 0$ *and if* $KC=0$, *then* $C^2=-2$ *and* C *is a rational non-singular curve.*

Moreover, the irreducible curves E *with* $KE=0$ *form a finite set and are numerically independent on* S.

Proof. — If an irreducible curve C on a surface S satisfies $C^2\geqq 0$, it is clear that $CD\geqq 0$ for every effective divisor D of S. Since mK is not empty for large m, we see that if $KC<0$ we must have also $C^2<0$. Hence

$$2p(C)-2=KC+C^2\leqq -2$$

and since C is irreducible we would have

$$p(C)=0, \qquad C^2=-1,$$

which contradicts the minimality of S.

Next, assume $KC=0$. Write

$$|mK|=|D|+F,$$

where F is a fixed part. Since S is of general type, we have $D^2>0$ for large m. Now

$$CD=mKC-FC=-FC$$

and if $C^2\geqq 0$ we would have $FC\geqq 0$, therefore $CD\geqq 0$ and thus $CD=0$. Since $D^2>0$ and $CD=0$, $C^2\geqq 0$, the Algebraic Index Theorem gives $C=0$. Thus we have proved that $C^2<0$. It follows that

$$2p(C)-2=KC+C^2<0$$

and thus

$$p(C)=0, \qquad C^2=-2,$$

as asserted. Note that this implies that $K^2>0$.

Finally, let $E_1, \ldots, E_n$ be distinct curves with $KE_i=0$. Assume a numerical equivalence relation

$$\sum_{i=1}^{r} m_i E_i \sim \sum_{j=r+1}^{n} m_j E_j$$

where $m_i\geqq 0$, $m_j\geqq 0$. Then

$$\Big(\sum_{i=1}^{r} m_i E_i\Big)^2=\Big(\sum_{i=1}^{r} m_i E_i\Big)\Big(\sum_{j=r+1}^{n} m_j E_j\Big)\geqq 0,$$

while the Algebraic Index Theorem gives

$$\Big(\sum_{i=1}^{r} m_i E_i\Big)^2<0$$

unless all $m_i=0$. Hence the curves E_i are numerically independent, and they form a finite set because, since $E_i^2<0$, they must be isolated in their numerical equivalence class, while

$$\operatorname{rank}_{\mathbf{Q}} \operatorname{Num}^1(S)<+\infty. \qquad \text{Q.E.D.}$$

Let $\mathscr{E}$ denote the set of such curves E and let $\mathscr{E}_\lambda$ be a maximal connected component of $\mathscr{E}$, $\mathscr{E}_\lambda = E_1 + \ldots + E_r$. Following Artin [1], we define the *fundamental cycle* Z of $\mathscr{E}_\lambda$ as the smallest divisor $m_1 E_1 + \ldots + m_r E_r = Z$, $m_i \geqq 1$, such that

$$ZE_i \leqq 0 \quad \text{for} \quad i = 1, \ldots, r.$$

We shall say for brevity that a divisor Z on S is a fundamental cycle if it is the fundamental cycle associated to a maximal connected component of $\mathscr{E}$.

The following facts are known:

Proposition 2 (Artin [1], [2]).

(i) *An effective divisor* Z *on* S *is a fundamental cycle if and only if it is a maximal cycle with*

$$KZ = 0, \quad Z^2 = -2.$$

(ii) *If* $\pi: S \to X$ *is a minimal resolution of singularities of the abstract canonical model* X, *the fibers of* π *over the singular points of* X *are the fundamental cycles of* S.

(iii) *We have*:

$$K_S = \pi^* K_X.$$

(iv) *If* Z *is a fundamental cycle on* S, $p = \pi(Z)$ *and* $\mathfrak{m}'$ *is the maximal ideal of* $\mathcal{O}_{X,p}$ *we have a canonical isomorphism*

$$H^0(S, \mathscr{I}_Z / \mathscr{I}_Z^2) \xrightarrow{\sim} \mathfrak{m}' / (\mathfrak{m}')^2.$$

If S is a minimal model, the morphism $\Phi_{nK}: S \to X^{[n]}$ factors as $\Phi_{nK} = \varphi_n \circ \pi$, where $\pi: S \to X$ is a minimal resolution of singularities of X. Let $\mathscr{E}_\lambda$, $\lambda = 1, \ldots, N$ be the maximal connected components of $\mathscr{E}$ and let Z_λ be the associated fundamental cycles. Following Kodaira, we say that Φ_{nK} is one-to-one mod $\mathscr{E}$ if, for x, y closed points of S, $x \neq y$, and not belonging to a same connected component $\mathscr{E}_\lambda$ of $\mathscr{E}$, we have that $\Phi_{nK}(x)$ and $\Phi_{nK}(y)$ are distinct closed points of $X^{[n]}$. In this case φ_n is a homeomorphism. Finally we note that $X \to X^{[n]}$ is an isomorphism if Φ_{nK} is one-to-one mod $\mathscr{E}$ and an isomorphism outside the set $\mathscr{E}$, and if the fibers of Φ_{nK} over the singular points of $X^{[n]}$ are the fundamental cycles of S. In terms of the pluricanonical ring R of S, this means that there exists an integer $a \geqq 0$ such that

$$R_n^{am} R_{nm} = R_n^{(a+1)m} \quad \text{for} \quad m \geqq 1.$$

If this holds with $a = 0$, then $X^{[n]}$ is a projectively normal surface, $X^{(n)}$ is projective and $X^{(n)} = X^n$ as polarized surfaces. This will certainly happen if n is large enough, and Kodaira [8] has proved that it is sufficient to take $n \geqq 8$.

We end this section with the following useful result (see Kodaira [7]).

Proposition 3. — *Let* S *be a minimal surface of general type. Then there are only finitely many irreducible curves* C *on* S *such that* KC *is bounded and* $C^2 < 0$.

Proof. — Let ξ be the (integral) numerical equivalence class of $(K^2)C - (KC)K$ in $\mathrm{Num}^1(S)$. Since C is irreducible we have

$$KC + C^2 = 2p(C) - 2 \geqq -2,$$

therefore

$$-\xi^2 = (\mathrm{KC})^2 (\mathrm{K}^2) - (\mathrm{K}^2)^2 \mathrm{C}^2$$

is bounded from above by a quantity depending only on KC and K^2. Now ξ belongs to the orthogonal of K in $\mathrm{Num}^1(\mathrm{S})$ and the quadratic form in the orthogonal of K given by the self-intersection is negative definite, by the Algebraic Index Theorem and $\mathrm{K}^2 > 0$. Hence ξ belongs to only finitely many classes, because $\mathrm{rank}_{\mathbf{Q}} \mathrm{Num}^1(\mathrm{S}) < +\infty$. Finally if $\mathrm{C}^2 < 0$ the curve C is isolated in its equivalence class. Q.E.D.

3. A vanishing theorem.

In this section we prove a vanishing theorem for a cohomology group $\mathrm{H}^1(\mathrm{S}, \mathscr{I}_{\mathrm{D}})$, where S is a complete non-singular algebraic surface over an algebraically closed field of characteristic 0, D is an effective divisor on S and $\mathscr{I}_{\mathrm{D}}$ is its sheaf of ideals. I am indebted to Ramanujam [14] for this proof.

If D is a divisor on S we denote by [D] the associated line bundle and by $\mathscr{O}([\mathrm{D}])$ the sheaf of germs of sections of D. If D is effective we have a canonical isomorphism

$$\mathscr{O}(-[\mathrm{D}]) \simeq \mathscr{I}_{\mathrm{D}}.$$

By an algebraic system $\{\mathrm{C}\}$ of divisors on S parametrized by V we always mean a flat family (see Mumford [11]). We also say that $\{\mathrm{C}\}$ is composed of a pencil if there is a morphism

$$f : \mathrm{S} \to \mathrm{B}$$

of S onto a non-singular curve B, such that every $\mathrm{C} \in \{\mathrm{C}\}$ is

$$\mathrm{C} \leqq f^{-1}(\mathrm{E})$$

for some $\mathrm{E} \in \mathrm{Div}(\mathrm{B})$. If B has genus $p \geqq 1$ we shall say that $\{\mathrm{C}\}$ is composed of an irrational pencil of genus p.

We have

Theorem A (Ramanujam [14]). — *Let* S *be as before and let* D *be an effective divisor on* S. *Assume that for some* $n > 0$ *the linear system* $|n[\mathrm{D}]|$ *has dimension*

$$\dim |n[\mathrm{D}]| \geqq 1$$

and is not composed of an irrational pencil.

Then we have

$$\dim_k \mathrm{H}^1(\mathrm{S}, \mathscr{I}_{\mathrm{D}}) = \dim_k \mathrm{H}^0(\mathrm{D}, \mathscr{O}_{\mathrm{D}}) - 1.$$

Let D be an effective divisor on S. We say that D is (numerically) *m-connected* if for every decomposition

$$\mathrm{D} = \mathrm{D}_1 + \mathrm{D}_2, \quad \mathrm{D}_i > 0$$

we have

$$\mathrm{D}_1 \mathrm{D}_2 \geqq m.$$

Now Ramanujam [14, Lemma 3] has proved that if D is connected (i.e., 1-connected) then

$$\dim_k H^0(D, \mathcal{O}_D) = 1.$$

We thus obtain:

Corollary. — *If* D *is connected and if* $D^2 > 0$ *we have*

$$H^1(S, \mathcal{I}_D) = 0.$$

Proof of Corollary. — If $D^2 > 0$ the Riemann-Roch theorem gives

$$\dim |n[D]| \geqq \frac{1}{2} n^2 D^2 + O(n)$$

which grows with n like n^2. Hence the linear system $|n[D]|$ cannot be composed of a pencil. Q.E.D.

Proof of Theorem A. — For any effective divisor C let

$$\alpha(C) = \dim_k \ker\{H^1(S, \mathcal{O}) \to H^1(C, \mathcal{O}_C)\}.$$

The conclusion of Theorem A is equivalent, in view of the exact sequence

$$0 \to \mathcal{I}_D \to \mathcal{O} \to \mathcal{O}_D \to 0,$$

to the statement that $\alpha(D) = 0$.

The basic result is Ramanujam's:

Lemma. — *For every* C *we have*

$$\alpha(C) = \alpha(C_{\mathrm{red}}).$$

Proof (Ramanujam [14], Lemma 6).

It is sufficient to prove that

$$\alpha(C_1) = \alpha(C_2)$$

for $C_1 \leqq C_2 \leqq 2C_1$. Since the ideal sheaf $\mathcal{I}_{C_1}\mathcal{O}_{C_2}$ of C_1 in C_2 is of square 0, we have the exact sequence

$$0 \longrightarrow \mathcal{I}_{C_1}\mathcal{O}_{C_2} \xrightarrow{1+x} \mathcal{O}^*_{C_2} \longrightarrow \mathcal{O}^*_{C_1} \longrightarrow 1$$

by means of the truncated exponential. This gives the cohomology sequence

$$H^0(\mathcal{O}^*_{C_1}) \to H^1(\mathcal{I}_{C_1}\mathcal{O}_{C_2}) \to \mathrm{Pic}(C_2) \to \mathrm{Pic}(C_1).$$

Since we are assuming $\mathrm{char}(k) = 0$, $H^0(\mathcal{O}^*_{C_1})$ is a divisible group, so that the kernel of $\mathrm{Pic}(C_2) \to \mathrm{Pic}(C_1)$ is torsion-free. Now let A_i denote the connected component at 0 of

$$\ker\{\mathrm{Pic}^0(S) \to \mathrm{Pic}(C_i)\}.$$

By Cartier's theorem, the A_i are abelian varieties and from what we have just proved we see that A_2/A_1 has no torsion. Hence $A_1 = A_2$ and the result follows because $\ker\{H^1(S, \mathcal{O}) \to H^1(C_i, \mathcal{O}_{C_i})\}$ is the dual of the Zariski tangent space at 0 of

$$\ker\{\mathrm{Pic}^0(S) \to \mathrm{Pic}(C_i)\}. \qquad \text{Q.E.D.}$$

Now we can complete the proof of Theorem A. Let n be so large so that

$$\dim |n[D]| = N - 1 \geqq 1.$$

By the previous lemma we have

$$\alpha(D) = \alpha(nD).$$

Now let

$$V \hookrightarrow S \times \mathbf{P}^{N-1}$$

be a relative effective Cartier divisor over $\mathbf{P}^{N-1}$ representing $|n[D]|$. Let also $f: V \to \mathbf{P}^{N-1}$ be the structure morphism of V, so that the elements of $|n[D]|$ can be viewed as the fibers $f^{-1}(x)$, x a closed point of $\mathbf{P}^{N-1}$. Since f is flat we have that

$$\dim_k H^0(f^{-1}(x), \mathcal{O}_{f^{-1}(x)}) = d(x)$$

is an *upper semicontinuous* function of x and it follows that

$$d(x) \leqq \dim_k H^0(nD, \mathcal{O}_{nD})$$

if x is *the* generic point of $\mathbf{P}^{N-1}$. The generic fiber $f^{-1}(x)$ is a divisor on $S' = S \otimes_k k(x)$ and the exact sequence

$$0 \to H^0(S', \mathcal{O}) \to H^0(f^{-1}(x), \mathcal{O}_{f^{-1}(x)}) \to H^1(S', \mathcal{J}_{f^{-1}(x)}) \to \ker\{H^1(S', \mathcal{O}) \to H^1(f^{-1}(x), \mathcal{O}_{f^{-1}(x)})\} \to 0$$

shows that

$$\begin{aligned} \alpha(f^{-1}(x)) + d(x) &= 1 + \dim_{k(x)} H^1(S', \mathcal{J}_{f^{-1}(x)}) \\ &= 1 + \dim_k H^1(S, \mathcal{O}_{nD}) \\ &= \alpha(nD) + \dim_k H^0(nD, \mathcal{O}_{nD}) \end{aligned}$$

therefore

$$\alpha(nD) \leqq \alpha(f^{-1}(x)).$$

Writing C_x for $f^{-1}(x)$, we thus see that there is a Zariski open set $U \subset \mathbf{P}^{N-1}$ such that for $x \in U$ we have

$$\alpha(nD) \leqq \alpha(C_x).$$

We have to prove that

$$\mathrm{Pic}^0(S) \to \mathrm{Pic}(C_x)$$

has finite kernel. By duality, this means that if $j: C_x \to S$ is the inclusion map then

$$j_* : \mathrm{Alb}(C_x) \to \mathrm{Alb}(S)$$

is an epimorphism, or in other words that

$$\psi : S \to \mathrm{Alb}(S)/j_* \mathrm{Alb}(C_x)$$

is a constant map. To be more explicit, if $\varphi : S \rightarrow \mathrm{Alb}(S)$ is the Albanese map, $j_* \mathrm{Alb}(C_x)$ is the abelian subvariety of $\mathrm{Alb}(S)$ generated by $\varphi(y)-\varphi(z)$, where y, z are points of $(C_x)_{\mathrm{red}}$. Note that this already implies that ψ is a constant map on every connected component of $(C_x)_{\mathrm{red}}$. Now by Chow's theorem (see Lang [9]) there is a Zariski open set $U' \subset \mathbf{P}^{N-1}$ such that $j_* \mathrm{Alb}(C_x)$, for $x \in U'$, is a fixed abelian subvariety, say $j_* \mathrm{Alb}(C)$, of $\mathrm{Alb}(S)$. It follows easily from this that

$$S \rightarrow \mathrm{Alb}(S)/j_* \mathrm{Alb}(C)$$

is a constant map on every connected component of every $C \in |n[D]|$.

Now if $|n[D]|$ is not composed of a pencil, any two closed points of S can be joined by a connected union of connected components of elements of $|n[D]|$, whence $S \rightarrow \mathrm{Alb}(S)/j_* \mathrm{Alb}(C)$ is a constant map. If instead $|n[D]|$ is composed of a pencil we see that there is a factorization

$$\begin{tikzcd} S \arrow[d, "f"'] \arrow[dr] & \\ B \arrow[r] & \mathrm{Alb}(S)/j_* \mathrm{Alb}(C) \end{tikzcd}$$

through the parametrizing curve B of the pencil. If $B = \mathbf{P}^1$ we get again a constant map. Q.E.D.

Remark. — If S is a regular surface, that is

$$\dim_k H^1(S, \mathcal{O}) = 0$$

we have trivially $\alpha(D) = 0$ and Theorem A holds with no conditions about D.

Theorem A requires D to be an effective divisor. The following result of Mumford takes care of the case in which D is not effective.

Theorem B. — Let S *be as before and let* $\mathcal{L}$ *be an invertible sheaf such that, for large n,* $\mathcal{L}^n$ *is spanned by its sections and has three algebraically independent sections. Then we have*

$$H^1(S, \mathcal{L}^{-1}) = 0.$$

For the proof, we refer to Mumford [12].

4. Connectedness of pluricanonical divisors.

Let S be a complete non-singular surface defined over an algebraically closed field k and let K be the canonical bundle of S. The two basic facts used in this section are:

(i) For any divisor D on S, we have:

$$D^2 + KD \equiv 0 \pmod 2;$$

(ii) (Algebraic Index Theorem) the quadratic form on $\mathrm{Num}^1(S)$ given by the self-intersection is non-degenerate, indefinite, with only one positive eigenvalue.

In what follows, S will also be a minimal surface of general type. By Proposition 1, we have:

(iii) For any effective divisor D on S we have:

$$KD \geqq 0.$$

Lemma 1. — If D *is effective and* $D \sim mK$, $m \geqq 1$, *then* D *is numerically 2-connected except in case* $K^2=1$, $m=2$, $D=D_1+D_2$, $D_1 \sim D_2 \sim K$.

Corollary. — If D *is as before, then* D *is numerically 1-connected.*

Proof (see also [7]). — If $D=D_1+D_2$ we can write

$$D_1 \sim rK+\xi, \qquad D_2 \sim (m-r)K-\xi,$$

where $r=(KD_1)/K^2$ and where ξ is a numerical equivalence class in $\mathrm{Num}^1(S)\otimes\mathbf{Q}$ with $K\xi=0$. Hence

$$D_1D_2=r(m-r)K^2-\xi^2;$$

since $r \geqq 0$, $m-r \geqq 0$ by (iii) and $\xi^2<0$ by (ii) unless $\xi=0$, and since we cannot have at the same time $r=0$ and $\xi=0$ (otherwise D_1 would be zero) we deduce that

$$D_1D_2 \geqq 1$$

and in particular D is numerically 1-connected.

We have also

$$D_1D_2=(m+1)KD_1-(D_1^2+KD_1)$$

hence by (i) we get that D_1D_2 is even unless m is even and KD_i both odd. However in this case we have $1/K^2 \leqq r \leqq m-(1/K^2)$ and we get

$$D_1D_2=r(m-r)K^2-\xi^2 \geqq m-(1/K^2)-\xi^2$$

which gives for $m \geqq 2$ the inequality $D_1D_2>1$, unless $m=2$, $r=1$, $K^2=1$, $\xi=0$.

Q.E.D.

The same method gives:

Lemma 2. — If D *is effective,* $D \sim mK$, $m \geqq 2$ *and if* $D=D_1+D_2$ *where* D_i *is effective with* $KD_i \geqq 1$, *we have*

$$D_1D_2 \geqq 3$$

except in the following cases:

$$K^2=1 \text{ or } 2,\ m=2,\ D_1 \sim D_2 \sim K;$$

$$K^2=1, \qquad m=3,\ D_1 \text{ or } D_2 \sim K.$$

Proof. — The method of Lemma 1 gives the result unless either $KD_1 \leqq 2$ or $KD_2 \leqq 2$ and $m \leqq 3$. Suppose $KD_1 = 1$. By (i) D_1^2 is odd; also the Algebraic Index Theorem shows that

$$D_1^2 \leqq \frac{(KD_1)^2}{K^2}$$

with equality only if $D_1 \sim \frac{(KD_1)}{K^2} K$. Hence $D_1^2 \leqq -1$ unless $K^2 = 1$ and $D_1 \sim K$. Since now

$$D_1 D_2 = mKD_1 - D_1^2 \geqq m + 1$$

unless $K^2 = 1$ and $D_1 \sim K$, we get what we want. Essentially the same argument applies if $KD_1 = 2$. Q.E.D.

Lemma 3. — Let Z *be a fundamental cycle of* S. *If* $D \sim mK - Z$, $m \geqq 1$, *then* D *is numerically 1-connected. Moreover if* $m \geqq 2$ *then* D *is numerically 2-connected, except in case* $K^2 = 1$, $m = 2$.

Proof. — Let $D = D_1 + D_2$ and suppose we are not in case $K^2 = 1$, $m = 2$. By Lemma 1 we have

$$D_1(D_2 + Z) \geqq 2, \quad D_2(D_1 + Z) \geqq 2;$$

summing these two inequalities and using $DZ = -Z^2 = 2$ we get what we want. Now if $K^2 = 1$, $m = 2$ the previous argument still applies unless either $D_1 \sim K$ or $D_2 \sim K$. If $D_1 \sim K$ we get

$$D_1 D_2 = KD_2 = K(K - Z) = 1,$$

as we wanted. The second part of Lemma 3 is proved in the same way using Lemma 2. Q.E.D.

The same argument gives:

Lemma 4. — Let Z_λ, Z_μ *be distinct fundamental cycles of* S. *If* $D \sim mK - Z_\lambda - Z_\mu$, $m \geqq 2$, *then* D *is numerically 1-connected.*

Lemma 5. — Let Z *be a fundamental cycle of* S. *If* $D \sim mK - 2Z$, $m \geqq 2$, *then* D *is 1-connected except in the following cases*:

$$K^2 = 1 \text{ or } 2, m = 2;$$
$$K^2 = 1, \qquad m = 3.$$

Now we shall study the connectedness of pluricanonical divisors after blowing up one or two points on S.

Let $\pi : \tilde{S} \to S$ be the blowing up of S at a closed point x of S, $x \notin \mathscr{E}$, and let $L = \pi^{-1}(x)$ be the exceptional curve of the first kind on $\tilde{S}$.

Lemma 6. — Let D *be an effective divisor on* $\tilde{S}$ *and let* $D \sim m\pi^*K - 2L$, $m \geqslant 1$. *Then* D *is numerically 1-connected except if* $K^2 = 1$, $m = 2$, $D = D_1 + D_2$, $D_1 \sim D_2 \sim \pi^*K - L$.

Proof. — Let $D = D_1 + D_2$ and let

$$\Delta_i = D_i + (D_i L) L.$$

Clearly the Δ_i are still effective divisors (even if $D_i L < 0$) possibly zero, and $\Delta_i L = 0$; hence there are effective divisors C_i on S such that $\pi^*[C_i] = [\Delta_i]$. Since $\Delta_1 + \Delta_2 \sim m\pi^* K$ we have $C_1 + C_2 \sim mK$, and if C_1 and C_2 are not zero the result follows from Lemma 1, from $D_1 L + D_2 L = 2$ and

$$D_1 D_2 = \Delta_1 \Delta_2 - (D_1 L)(D_2 L) = C_1 C_2 - (D_1 L)(D_2 L).$$

If $C_1 = 0$ then $D_1 = -(D_1 L) L$, hence

$$D_1 D_2 = -(D_1 L)(D_2 L) \geqq 3$$

because $D_1 L < 0$ and $D_1 L + D_2 L = 2$. Q.E.D.

Lemma 7. — *Let* D *be an effective divisor on* $\tilde{S}$ *and let* $D \sim m\pi^* K - 3L$, $m \geqq 2$. *Then* D *is 1-connected except in the following cases*:

$$K^2 = 1 \text{ or } 2, m = 2;$$
$$K^2 = 1, \quad m = 3.$$

Proof. — We define Δ_i, C_i as in the proof of Lemma 6 and use Lemma 2 instead of Lemma 1. We get the required result except possibly if $KC_1 = 0$ or $KC_2 = 0$. Assume $KC_1 = 0$; then C_1 has support in $\mathscr{E}$ and since $x \notin \mathscr{E}$ by hypothesis we see that the supports of Δ_1 and L are disjoint. This implies that $D_1 L < 0$, and Lemma 7 follows easily from this. Q.E.D.

Using Lemma 3 the same technique gives:

Lemma 8. — *Let* Z *be a fundamental cycle of* $\tilde{S}$ *and let* D *be an effective divisor on* $\tilde{S}$, $D \sim m\pi^* K - 2L - Z$, $m \geqq 2$. *Then* D *is 1-connected except if* $K^2 = 1$, $m = 2$.

Now let $\pi : \tilde{S} \to S$ be the blowing up of S at two closed points x, y, $x \neq y$, and x, $y \notin \mathscr{E}$ and let $L = \pi^{-1}(x)$, $M = \pi^{-1}(y)$ be the corresponding exceptional curves of the first kind on $\tilde{S}$. Using Lemma 2 we easily obtain:

Lemma 9. — *Let* D *be an effective divisor on* $\tilde{S}$ *and let* $D \sim m\pi^* K - 2L - 2M$, $m \geqq 2$. *Then* D *is 1-connected except in the following cases*:

$$K^2 = 1 \text{ or } 2, m = 2;$$
$$K^2 = 1, \quad m = 3.$$

Moreover if $K^2 = 2$, $m = 2$ *and* $D = D_1 + D_2$, *with* $D_1 D_2 \leqq 0$, *then*

$$D_1 \sim D_2 \sim \pi^* K - L - M.$$

Lemma 9A. — *Let* D *be an effective divisor on* $\tilde{S}$ *and let* $D \sim m\pi^* K - 2L - M$, $m \geqq 2$. *Then* D *is 1-connected except if* $K^2 = 1$, $m = 2$.

We can summarize the results so far obtained in the following table:

Numerical equivalence class of D		Exceptions to D being connected
mK	$m \geqq 1$	none
$mK-Z$	$m \geqq 1$	none
$m\pi^*K-2L$	$m \geqq 1$	$K^2=1, \quad m=2$
$mK-Z_\lambda-Z_\mu$	$m \geqq 2$	none
$m\pi^*K-2L-Z$	$m \geqq 2$	$K^2=1, \quad m=2$
$m\pi^*K-2L-M$	$m \geqq 2$	$K^2=1, \quad m=2$
$m\pi^*K-2L-2M$	$m \geqq 2$	$K^2=1$ or 2, $m=2$; $K^2=1$, $m=3$
$m\pi^*K-2Z$	$m \geqq 2$	$K^2=1$ or 2, $m=2$; $K^2=1$, $m=3$
$m\pi^*K-3L$	$m \geqq 2$	$K^2=1$ or 2, $m=2$; $K^2=1$, $m=3$

Moreover, in the exceptional cases, if $D=D_1+D_2$ and $D_1D_2 \leqq 0$, then one has the numerical equivalence classes of D_1 and D_2; this additional information will be used later in dealing with surfaces with $K^2=1$ or 2.

5. Normal canonical models.

Theorem 2. — Let S *be a minimal surface of general type. Then* $(m+1)K$ *is spanned by its global sections if*:

(i) $m \geqq 3$;
(ii) $m=2$, $K^2 \geqq 3$ *or* $K^2 \geqq 2$, $p_g \geqq 1$;
(iii) $m=1$, $K^2 \geqq 5$, $p_g \geqq 3$ *or* $p_g \geqq 3$, $q=0$.

Remark 1. — Statements (i) and (ii), second half, are already proved in Kodaira [7]; (ii), also in case $K^2=2$, $p_g=0$, is proved in Bombieri [3]. Result (iii) seems to be new.

Corollary (see [7]). — *If* $m \geqq 2$ *the m-th plurigenus* P_m *of* S *is given by*

$$P_m = \frac{1}{2} m(m-1) K^2 + \chi(\mathcal{O}),$$

and $\chi(\mathcal{O}) \geqq 1$.

Proof of the Corollary. — Since S is of general type, mK has three algebraically independent sections if $m \gg 0$. Now Theorem B applies and we get

$$H^1(S, \mathcal{O}(-mK)) = 0 \quad \text{for} \quad m \geqq 1.$$

The result follows by duality and the Riemann-Roch theorem. Finally the inequality $\chi(\mathcal{O}) \geqq 1$ follows from classification of surfaces; see [7] for details. Another proof will be found in Section X of this work.

Proof of Theorem 2. — Let x be a closed point of S, $x \notin \mathcal{E}$ and let $\mathcal{I}_x$ denote its sheaf of ideals. The sheaf $\mathcal{O}((m+1)K) \otimes \mathcal{I}_x$ is the sheaf of germs of sections of $(m+1)K$ vanishing at x. We have the exact sequence

$$0 \to \mathcal{O}((m+1)K) \otimes \mathcal{I}_x \to \mathcal{O}((m+1)K) \to \mathcal{F} \to 0$$

where $\mathcal{F}$ is a sheaf with support x and stalk $k(x)$ at x, and it is obvious that x will not be a base point of $|(m+1)K|$ if and only if

$$\dim_k H^0(S, \mathcal{O}((m+1)K) \otimes \mathcal{I}_x) = P_m - 1.$$

Let $\pi : \tilde{S} \to S$ be the blowing up of S at x, $x \notin \mathcal{E}$, and let $L = \pi^{-1}(x)$ be the exceptional curve of the first kind on $\tilde{S}$. Clearly

$$H^0(S, \mathcal{O}((m+1)K) \otimes \mathcal{I}_x) \quad \text{and} \quad H^0(\tilde{S}, \mathcal{O}((m+1)\pi^* K) \otimes \mathcal{I}_L)$$

are isomorphic, also $(\pi^* K)_L$ is a trivial bundle. Hence the cohomology sequence of

$$0 \to \mathcal{O}((m+1)\pi^* K) \otimes \mathcal{I}_L \to \mathcal{O}((m+1)\pi^* K) \to \mathcal{O}_L \to 0$$

shows that x cannot be a base point of $|(m+1)K|$ if

$$H^1(\tilde{S}, \mathcal{O}((m+1)\pi^* K) \otimes \mathcal{I}_L) = 0.$$

If instead $x \in \mathcal{E}$, let Z be the fundamental cycle of the connected component of $\mathcal{E}$ containing x. Since

$$\mathcal{O}_Z \otimes \mathcal{O}((m+1)K) = \mathcal{O}_Z$$

and

$$H^0(Z, \mathcal{O}_Z) = k$$

(use $KZ = 0$, the fact that invertible sheaves on Z are classified by the degree, by Artin [2], to get the first result, and use [2], Lemma 3 and Z numerically 1-connected to get the second equation), and since sections of $(m+1)K$ are constant on Z, we conclude as before that x cannot be a base point of $|(m+1)K|$ if

$$H^1(S, \mathcal{O}((m+1)K) \otimes \mathcal{I}_Z) = 0.$$

Assume that there exists a divisor $D \in |m\pi^* K - 2[L]|$. By Lemma 6, D is numerically 1-connected except in case $K^2 = 1$, $m = 2$, and we have also $D^2 > 0$ provided

$$m^2 K^2 \geqq 5.$$

Hence if $m^2 K^2 \geqq 5$ Theorem A applies and we obtain

$$H^1(\tilde{S}, \mathscr{I}_D) = 0;$$

note that by Theorem A, Remark 2, the above condition is not needed if $q = 0$. The canonical bundle of $\tilde{S}$ is $K = \pi^* K + [L]$, therefore by duality we find

$$H^1(\tilde{S}, \mathcal{O}((m+1)\pi^* K) \otimes \mathscr{I}_L) = 0$$

as we wanted.

A similar argument shows that if there exists a divisor $D \in |mK - [Z]|$, then

$$H^1(S, \mathcal{O}((m+1)K) \otimes \mathscr{I}_Z) = 0$$

provided $m^2 K^2 \geqq 3$.

In order to see which conditions give the existence of the divisor D, we may assume, by *reductio ad absurdum*, that x is a base point of $|(m+1)K|$, otherwise there is nothing to prove. Consider the case $x \notin \mathscr{E}$. Since

$$\mathcal{O}(m\pi^* K) \otimes \mathcal{O}_L = \mathcal{O}_L$$

we have the exact sequence

$$0 \to H^0(\tilde{S}, \mathcal{O}(m\pi^* K) \otimes \mathscr{I}_L^2) \to H^0(\tilde{S}, \mathcal{O}(m\pi^* K)) \to H^0(2L, \mathcal{O}_{2L})$$

and since $\dim_k H^0(2L, \mathcal{O}_{2L}) = 3$ we obtain $\dim_k H^0(\tilde{S}, \mathcal{O}(m\pi^* K) \otimes \mathscr{I}_L^2) \geqq P_m - 3$ and there is $D \in |(m\pi^* K) - [2L]|$ if $P_m \geqq 4$.

Now suppose $p_g \geqq 1$. If s is a non-trivial section of K then s^{m+1} is a non-trivial section of $(m+1)K$, hence vanishing on x, and it follows that if $m \geqq 2$ then

$$s^m \in H^0(S, \mathcal{O}(mK) \otimes \mathscr{I}_x^2)$$

and $(\pi^* s)^m$ is a section of $m\pi^* K$ vanishing on L of order at least 2. Hence D exists if $m \geqq 2$, $p_g \geqq 1$.

If $m = 1$, noting that every base point of $|2K|$ is also a base point of $|K|$ we get that there exists a section of K vanishing of order at least 2 at x, provided $p_g \geqq 3$, hence D exists if $m = 1$, $p_g \geqq 3$.

Finally if $x \in \mathscr{E}$ we see easily that D exists if $P_m \geqq 2$, and if $m = 1$, $p_g \geqq 1$.

Now the Riemann-Roch theorem and $\chi(\mathcal{O}) \geqq 1$ show that

$$P_m \geqq \frac{1}{2} m(m-1) K^2 + 1 \qquad \text{if} \qquad m \geqq 2,$$

and Theorem 2 follows from the conditions we have found for the existence of D.

Q.E.D.

Remark 2. — We have also proved that $|3K|$ has no fixed part in $\mathscr{E}$, and that $|2K|$ has no fixed part in $\mathscr{E}$ if $p_g \geqslant 1$.

Lemma 10. — *The map* $X \to X^{[m+1]}$ *is a homeomorphism if*

$$H^1(\tilde{S}, \mathcal{O}((m+1)\pi^*K) \otimes \mathscr{I}_L \otimes \mathscr{I}_M) = 0$$

for $x, y \notin \mathscr{E}$, $x \neq y$, $\tilde{S}$ *the blowing up of* S *at* x, y, $L = \pi^{-1}(x)$, $M = \pi^{-1}(y)$;

$$H^1(\tilde{S}, \mathcal{O}((m+1)\pi^*K) \otimes \mathscr{I}_L \otimes \mathscr{I}_Z) = 0$$

for $x \notin \mathscr{E}$ *and every fundamental cycle* Z *of* $\tilde{S}$, $\tilde{S}$ *the blowing up of* S *at* x, $L = \pi^{-1}(x)$;

$$H^1(S, \mathcal{O}((m+1)K) \otimes \mathscr{I}_{Z_\lambda} \otimes \mathscr{I}_{Z_\mu}) = 0$$

for Z_λ, Z_μ *distinct fundamental cycles of* S.

Moreover the map $X \to X^{[m+1]}$ *is an isomorphism if it is a homeomorphism and*

$$H^1(\tilde{S}, \mathcal{O}((m+1)\pi^*K) \otimes \mathscr{I}_L^2) = 0$$

for $x \notin \mathscr{E}$, $\tilde{S}$ *the blowing up of* S *at* x, $L = \pi^{-1}(x)$, *and if*

$$H^1(S, \mathcal{O}((m+1)K) \otimes \mathscr{I}_Z^2) = 0$$

for every fundamental cycle Z *of* S.

Proof. — The argument in the course of the proof of Theorem 2 giving the analogous criterion for $(m+1)K$ being spanned by its sections shows easily that $X \to X^{[m+1]}$ is a homeomorphism if the first three conditions of Lemma 10 are verified. In order to verify the second part of Lemma 10 we proceed as follows.

We have a commutative diagram

$$\begin{tikzcd} S \arrow[rr, "\pi"] \arrow[dr, "\Phi_{(m+1)K}"'] & & X \arrow[dl, "\varphi_{m+1}"] \\ & X^{[m+1]} & \end{tikzcd}$$

Now let Z denote a fundamental cycle, let $p = \Phi_{(m+1)K}(Z)$, $p' = \pi(Z)$ and let $(A, \mathfrak{m})$, $(A', \mathfrak{m}')$ be the local rings of p, p'. Since (X, φ_{m+1}) is a normalization of $X^{[m+1]}$ and since φ_{m+1} is a homeomorphism we may identify A with a subring of A', A' is a finite A-module and

$$A' = A + \mathfrak{m}'.$$

We have the exact sequence

$$0 \to \mathcal{O}((m+1)K) \otimes \mathscr{I}_Z^2 \to \mathcal{O}((m+1)K) \otimes \mathscr{I}_Z \to \mathscr{I}_Z / \mathscr{I}_Z^2 \to 0$$

(we use $KZ = 0$ and the fact that invertible sheaves on Z are classified by the degree, Artin [2]) hence we get the cohomology sequence

$$H^0(S, \mathcal{O}((m+1)K) \otimes \mathscr{I}_Z) \to H^0(S, \mathscr{I}_Z / \mathscr{I}_Z^2) \to 0.$$

On the other hand, by Proposition 2 we have an isomorphism

$$\mathrm{H}^0(\mathrm{S}, \mathscr{I}_\mathrm{Z}/\mathscr{I}_\mathrm{Z}^2) \cong \mathfrak{m}'/(\mathfrak{m}')^2$$

and since $\mathrm{H}^0(\mathrm{S}, \mathcal{O}((m+1)\mathrm{K})\otimes\mathscr{I}_\mathrm{Z}) \to \mathfrak{m}'/(\mathfrak{m}')^2$ factors through the maximal ideal of A, we deduce that the natural homomorphism

$$\mathfrak{m} \to \mathfrak{m}'/(\mathfrak{m}')^2$$

is surjective.

This clearly implies that $\mathfrak{m}' = \mathfrak{m}\mathrm{A}' + (\mathfrak{m}')^n$ for every n and since $\mathfrak{m}'/\mathfrak{m}\mathrm{A}'$ is an Artinian ring we must also have that $(\mathfrak{m}')^n$ is 0 in $\mathfrak{m}'/\mathfrak{m}\mathrm{A}'$ for large n. Hence $\mathfrak{m}' = \mathfrak{m}\mathrm{A}'$ and

$$\mathrm{A}' = \mathrm{A} + \mathfrak{m}\mathrm{A}'.$$

If we define $\mathrm{M} = \mathrm{A}'/\mathrm{A}$ then M is a finite A-module and by the previous equation we have:

$$\mathrm{M} = \mathfrak{m}\mathrm{A}'/\mathrm{A} = \mathfrak{m}\mathrm{M}.$$

By Nakayama's lemma, $\mathrm{M} = 0$ and $\mathrm{A} = \mathrm{A}'$.

Finally, in order to show that φ_{m+1} is an isomorphism at $\pi(z)$, $z \notin \mathscr{E}$, we use the same argument together with the isomorphism $\mathrm{H}^0(\tilde{\mathrm{S}}, \mathscr{I}_\mathrm{L}/\mathscr{I}_\mathrm{L}^2) \cong \mathfrak{m}'/(\mathfrak{m}')^2$. This is in fact clear, because π is an isomorphism at z, and $\tilde{\mathrm{S}} \to \mathrm{S}$ is the blowing up of S at z.

Q.E.D.

Theorem 3. — *Let* S *be a minimal surface of general type. Then* $\mathrm{X} \to \mathrm{X}^{[m+1]}$ *is an isomorphism if*:

(i) $m \geqq 4$;
(ii) $m = 3$, $\mathrm{K}^2 \geqq 2$;
(iii) $m = 2$, $\mathrm{K}^2 \geqq 6$ *or* $\mathrm{K}^2 \geqq 3$, $p_g \geqq 4$.

Proof. — The method of proof of Theorem 2 shows that the five conditions of Lemma 10 are satisfied if we can find D numerically 1-connected with $\mathrm{D}^2 > 0$ such that respectively

$$\mathrm{D} \in |m\pi^*\mathrm{K} - 2[\mathrm{L}] - 2[\mathrm{M}]|$$
$$\mathrm{D} \in |m\pi^*\mathrm{K} - 2[\mathrm{L}] - [\mathrm{Z}]|$$
$$\mathrm{D} \in |m\mathrm{K} - [\mathrm{Z}_\lambda] - [\mathrm{Z}_\mu]|$$
$$\mathrm{D} \in |m\pi^*\mathrm{K} - 3[\mathrm{L}]|$$
$$\mathrm{D} \in |m\mathrm{K} - 2[\mathrm{Z}]|.$$

The condition $\mathrm{D}^2 > 0$ is satisfied if

$$m^2\mathrm{K}^2 \geqq 10;$$

by Lemmas 9, 8, 4, 7 and 5, D is numerically 1-connected if

$$m + \mathrm{K}^2 \geqq 5;$$

and the existence of D is assured by

$$P_m \geqq 7 \quad \text{or} \quad m=2, \quad p_g \geqq 4;$$

the result now follows from Theorem 2, Corollary. Q.E.D.

Remark. — Since $X \rightarrow X^{[4]}$ is not an isomorphism if $K^2=1$, $p_g=2$, and also for some surfaces with $K^2=1$, $p_g=0$, we see that (ii) cannot be much improved. Result (iii) instead still leaves room for improvements and we can show that it remains true if $K^2 \geqq 3$, $p_g=3$. This is easily obtained by showing first that in this case it is sufficient to have $P_2 \geqq 6$, again using a *reductio ad absurdum*. Then one has to prove that $P_2 \geqq 6$ follows indeed from $K^2 \geqq 3$, $p_g=3$. In case $K^2=3$, $p_g=3$ the results of Section X show that $q=0$ and thus $P_2=7$. If instead $K^2=4$, $p_g=3$ the same results give $q=0$ or 2, and thus $P_2=8$ or 6. Since $P_2 \geqq K^2+1$ in any case, we have what we wanted.

Now we turn to the question of projectively normal models. We have:

Theorem 3A. — *If* $K^2 \geqq 5$, $p_g \geqq 3$, *then* $X^{[n]}$ *is a projectively normal model of* X *for* $n \geqq 6$.

Proof. — Let $\ell \geqq 1$ be such that ℓK is spanned by its sections, and let C be an irreducible non-singular curve in $|\ell K|$. We denote by s_0 a non-trivial section of ℓK vanishing on C. The exact sequence

$$0 \longrightarrow \mathcal{O}((m-\ell)K) \xrightarrow{s_0} \mathcal{O}(mK) \xrightarrow{r_C} \mathcal{O}_C(mK_C) \longrightarrow 0$$

shows that

$$R_m = s_0 R_{m-\ell} + A$$

where $R_m = H^0(S, \mathcal{O}(mK))$ and where A is any subspace of R_m such that $r_C R_m = r_C A$.

Now there is a section

$$\sigma_1 \in r_C R_\ell \subseteq H^0(C, \mathcal{O}_C(\ell K_C))$$

with only simple zeros; let X be the divisor of zeros of σ_1. We have the exact sequence

$$0 \longrightarrow \mathcal{O}_C((m-\ell)K_C) \xrightarrow{\sigma_1} \mathcal{O}_C(mK_C) \xrightarrow{r_X} \mathcal{O}_X \longrightarrow 0$$

and we deduce that

$$r_C R_m \subseteq \sigma_1 H^0(C, \mathcal{O}_C(m-\ell)K_C)) + B$$

where B is any subspace of $H^0(C, \mathcal{O}_C(mK_C))$ such that $r_X H^0(C, \mathcal{O}_C(mK_C)) = r_X B$.

Now consider the exact sequence:

$$0 \longrightarrow \mathcal{O}_C((m-2\ell)K_C) \xrightarrow{\sigma_1} \mathcal{O}_C((m-\ell)K_C) \xrightarrow{r_X} \mathcal{O}_X \longrightarrow 0.$$

If $\deg(m-2\ell)K_C > 2p(C)-2$ we have $H^1(C, \mathcal{O}_C((m-2\ell)K_C))=0$ and thus

$$r_X H^0(C, \mathcal{O}_C((m-\ell)K_C)) = H^0(X, \mathcal{O}_X).$$

Now since we are assuming that ℓK is spanned by its sections, we can find another section

$$\sigma_2 \in r_C R_\ell$$

such that σ_2 *never vanishes on* X. Hence

$$r_X \sigma_2 H^0(C, \mathcal{O}_C((m-\ell) K_C)) = H^0(X, \mathcal{O}_X)$$

and we may take $B = \sigma_2 H^0(C, \mathcal{O}_C((m-\ell) K_C))$.
Thus we get

$$r_C R_m \subseteq \sigma_1 H^0(C, \mathcal{O}_C((m-\ell) K_C)) + \sigma_2 H^0(C, \mathcal{O}_C((m-\ell) K_C)).$$

The exact sequence

$$0 \longrightarrow \mathcal{O}((m-2\ell) K) \xrightarrow{s_0} \mathcal{O}((m-\ell) K) \xrightarrow{r_C} \mathcal{O}_C((m-\ell) K_C) \longrightarrow 0$$

gives $r_C R_{m-\ell} = H^0(C, \mathcal{O}_C((m-\ell) K_C))$ provided $H^1(S, \mathcal{O}((m-2\ell) K)) = 0$, which is the case if $m-2\ell \neq 0, 1$. Hence $r_C R_m \subseteq \sigma_1 r_C R_{m-\ell} + \sigma_2 r_C R_{m-\ell}$; therefore if s_1, s_2 are sections of ℓK whose restriction to C is σ_1, σ_2, we see that

$$r_C R_m = r_C(s_1 R_{m-\ell} + s_2 R_{m-\ell}).$$

The conditions we have made are:

$$\deg(m-2\ell) K_C > 2p(C) - 2,$$
$$m - 2\ell \neq 0, \ 1$$

which are satisfied if

$$m \geqq 3\ell + 2.$$

We conclude with the following statement:

Lemma. — If ℓK *is spanned by its sections and if* $m \geqq 3\ell + 2$ *we have*:

$$R_m = R_\ell R_{m-\ell}.$$

The proof of Theorem 3A is now immediate, for if $\ell = 1$ we get

$$R_{m+4} = R_4 R_1^m, \ m = 1, 2, \ldots$$

which implies $R_{nm} = R_n^m$ for $n \geqq 4$ and all m.

If $K^2 \geqq 5$, $p_g \geqq 3$, by Theorem 2 we may take for ℓ any integer $\geqq 2$. We get

$$R_m = R_2 R_{m-2} \quad \text{if} \quad m \geqq 8$$
$$R_m = R_3 R_{m-3} \quad \text{if} \quad m \geqq 11$$

and it follows form this that

$$R_{nm} = R_n^m$$

for $n \geqq 6$ and all m. Q.E.D.

Remark. — It is possible to show that surfaces with $K^2 = 2$, $p_g = 3$ have $R_2 = R_1^2$, $R_3 = R_1^3$, $R_m = R_4 R_1^{m-4}$ for $m \geqq 4$.

6. Birational maps.

Theorem 4. — Let S *be a minimal surface of general type. Then* $X \to X^{[3]}$ *is a homeomorphism if* $K^2 \geqq 3$, $p_g \geqq 2$.

Proof. — We have to verify the first three conditions of Lemma 10 and, as in the proof of Theorem 3, this reduces to show the existence of a numerically 1-connected divisor D with $D^2>0$, in the appropriate linear system.

Reasoning by *reductio ad absurdum* as in the proof of Theorem 2, we readily see that D exists if $K^2 \geqq 3$, $p_g \geqq 2$. Q.E.D.

Theorem 5. — *Let* S *be a minimal surface of general type. Then* $X \to X^{[2]}$ *is a birational map if* $K^2 \geqq 10$, $p_g \geqq 6$, *except if* S *has the structure of a fiber space over a curve, with generic fiber a non-singular curve of genus 2. Conversely, if* S *has a pencil of curves of genus 2, the map* $X \to X^{[2]}$ *is generically a double covering and* $X^{[2]}$ *is birationally equivalent to a ruled surface.*

Proof. — Assume that φ_2 is not a birational map. Then there is a Zariski open set U of S such that for every closed point x of U there is y in U with

$$\Phi_{2K}(x)=\Phi_{2K}(y) \quad \text{and} \quad x \neq y.$$

Choosing a smaller U if necessary, by Proposition 3 we may assume that $U \subseteq S - \bigcup C$, where C runs over all irreducible curves C on S with $KC \leqq K^2$ and $C^2<0$. If $\mathscr{C}$ is the set of such curves, we have that $\mathscr{C}$ contains all irreducible curves with $KC=0$ and with $KC=1$, since if $K^2 \geqq 2$ and $KC=1$ the Algebraic Index Theorem gives $C^2 \leqq 0$, and clearly C^2 is odd.

Since $p_g \geqq 6$, we see that there is a non-zero section s of the sheaf $\mathcal{O}(K) \otimes \mathscr{J}_x^2 \otimes \mathscr{J}_y^2$; hence if $\pi : \tilde{S} \to S$ is the blowing up of S at x and y we have that there is a non-trivial section $\pi^* s$ of $\mathcal{O}(\pi^* K) \otimes \mathscr{J}_L^2 \otimes \mathscr{J}_M^2$, where as before L and M denote the exceptional curves of the first kind on $\tilde{S}$. It follows that there exists a divisor $D \in |\pi^* K - 2[L] - 2[M]|$ and since $K^2 \geqq 9$ we have $D^2>0$. If D were numerically connected, as in the proof of Theorem 2, we would obtain

$$H^1(\tilde{S}, \mathcal{O}(2\pi^* K) \otimes \mathscr{J}_L \otimes \mathscr{J}_M)=0,$$

which contradicts Lemma 10. Hence D *cannot be numerically connected.*

Let C be the divisor of zeros of the section s, so that $C \sim K$, x and y are multiple points of C and $\pi^{-1}(C)=D+2L+2M$. Let $D=D_1+D_2$, let

$$\Delta_i = D_i + (D_i L)\, L + (D_i M)\, M$$

and note that Δ_1, Δ_2 are effective divisors and that there are effective (possibly 0) divisors C_i on S such that $C=C_1+C_2$ and

$$\Delta_i = \pi^{-1}(C_i).$$

We have $D_1 D_2 = C_1 C_2 - (D_1 L)(D_2 L) - (D_1 M)(D_2 M)$ and clearly

$$D_1 L + D_2 L = 2, \qquad D_1 M + D_2 M = 2.$$

Hence $D_1 D_2 \geqq C_1 C_2 - 2$ and we have equality only if $D_i L = D_i M = 1$.

Suppose first that $C_1=0$. Then $D_1L\leqq 0$, $D_1M\leqq 0$ and since either $D_1L<0$ or $D_1M<0$ (otherwise $D_1=0$) we obtain

$$D_1D_2\geqq 3.$$

Now suppose that C_1 and C_2 are effective and not 0. By Lemma 1 and its proof C is numerically 2-connected and C_1C_2 is even, therefore if D is not numerically connected we must have:

$$C_1C_2=2, \qquad D_iL=D_iM=1.$$

Clearly this implies that x and y are simple points of C_1 and C_2. By our choice of U we may assume that $KC_i\geqq 2$.

If $KC_i\geqq 3$ and if $K^2\geqq 10$, the method of proof of Lemma 1 gives easily $C_1C_2\geqq 3$, hence $D_1D_2\geqq 1$. It follows that, if D is not numerically connected, then we may assume $KC_1=2$. Since $C_1C_2=C_1(K-C_1)$ we deduce that $C_1^2=0$. Now we can write

$$C_1=C+Z$$

where $x, y\in C$, $KC=2$, $KZ=0$ and C is irreducible. Since by our choice of U we cannot have $C^2<0$ and since C^2 is even, the Algebraic Index Theorem shows again that

$$C^2=0.$$

By the theorem of Bertini we may assume, by further restricting U, that C is non-singular of genus 2. Thus we have found an algebraic pencil $\{C\}$ on the surface S.

Conversely, let $\{C\}$ be a pencil of curves of genus 2 on S and let B denote the parametrizing curve of the pencil. The pencil $\{C\}$ can be viewed as a relative effective Cartier divisor $V\hookrightarrow S\times B$ over B, and the fibers of the morphism $f: S\rightarrow B$ over the closed points of B are the curves of the pencil. Let $C_x=f^{-1}(x)$ be the fiber of f over the point x of B, and, for x a closed point of B, let

$$\mathcal{N}_{C_x}=\mathcal{O}_{C_x}\otimes\mathcal{O}([C_x])$$

be the normal sheaf of C_x in S (if C_x is non-singular, then $\mathcal{N}_{C_x}$ is the sheaf of germs of sections of the normal bundle of C_x in S).

Since $\dim_k H^0(C_x,\mathcal{N}_{C_x})\geqq 1$ for every closed point x of B (this follows from Mumford [11], lecture 22), we obtain that if C_x is non-singular then its normal bundle in S is trivial, therefore the canonical bundle k_{C_x} of C_x is given by

$$k_{C_x}=K_{C_x}.$$

Now we readily see that, if $D\sim K-C_x$, then D is 1-connected, because $D+C_x$ is 2-connected by Lemma 1 (if $D=D_1+D_2$ then $D_1(D_2+C_x)\geqq 2$, $D_2(D_1+C_x)\geqq 2$, and $(D_1+D_2)C_x=2$), hence by Theorem A and $D^2>0$ we get

$$H^1(S,\mathcal{O}(2K-[C_x]))=0.$$

This shows that the restriction map

$$r_{C_x} : H^0(S, \mathcal{O}(2K)) \to H^0(C_x, \mathcal{O}_{C_x} \otimes \mathcal{O}(2K))$$

is surjective. Since $2K_{C_x} = 2k_{C_x}$, we deduce that the restriction of Φ_{2K} to C_x is the 2-canonical map of C_x. Hence $\Phi_{2K}(C_x)$ is a conic and $C_x \to \Phi_{2K}(C_x)$ is a covering of degree 2.

It remains to show that Φ_{2K} is a double covering, or in other words that, if C, C′ are distinct fibers of $f: S \to B$, then $\Phi_{2K}(C) \neq \Phi_{2K}(C')$.

Clearly we have $\Phi_{2K}(C) = \Phi_{2K}(C')$ if and only if every section of 2K vanishing on C also vanishes on C′. Since we have shown that

$$H^1(S, \mathcal{O}(2K - [C])) = 0$$

we have the exact sequence

$$0 \to H^0(S, \mathcal{O}(2K) \otimes \mathcal{I}_C) \to H^0(S, \mathcal{O}(2K)) \to H^0(C, \mathcal{O}_C \otimes \mathcal{O}(2K)) \to 0;$$

therefore assuming $\Phi_{2K}(C) = \Phi_{2K}(C')$, we find

$$\dim_k H^0(S, \mathcal{O}(2K) \otimes \mathcal{I}_C \otimes \mathcal{I}_{C'}) = \dim_k H^0(S, \mathcal{O}(2K) \otimes \mathcal{I}_C) = P_2 - 3.$$

It follows easily from this

$$\dim_k H^1(S, \mathcal{O}(2K) \otimes \mathcal{I}_C \otimes \mathcal{I}_{C'}) > 0$$

whence using Theorem A, Corollary we conclude that, if D is an effective divisor $D \in |K - [C] - [C']|$, then D *is not numerically connected* (note that $D^2 = K^2 - 8 > 0$). Let $D = D_1 + D_2$ where the divisors D_i are effective and non-zero. By Lemma 1 we have

$$D_1(D_2 + C + C') \geq 2, \qquad D_2(D_1 + C + C') \geq 2$$

and since $K^2 \geq 10$ by hypothesis, the argument given in the proof of Lemma 1 shows that there is equality only if $KD_i \leq 2$.

On the other hand, summing the two inequalities and using

$$(D_1 + D_2)\, C = (D_1 + D_2)\, C' = 2$$

we see that if D is not 1-connected then equality must hold. We deduce that $KD_i \leq 2$ and

$$K^2 = K(D_1 + D_2 + C + C') \leq 8,$$

a contradiction. Q.E.D.

Remark. — The conditions given in Theorem 5 cannot be weakened too much. For if B, B′ are non-singular curves of genus 2 then $S = B \times B'$ is a minimal surface of general type with $p_g = 4$, $K^2 = 8$ and the 2-canonical map Φ_{2K} is a covering of degree 4 of a quadric, rather than being of degree 2.

We end this section with the following

Example. — There is a regular minimal surface S with $K^2 = 9$, $p_g = 6$, $q = 0$ with the following properties:

(i) $|K|$ has one (closed) base point;
(ii) $X^{[1]}$ is a rational normal ruled surface F_2 of degree 4 in $\mathbf{P}^5(k)$;
(iii) $X^{[2]}$ is isomorphic with a quadric cone embedded in $\mathbf{P}^{15}(k)$;
(iv) if $\tilde{S}$ is the blowing up of S at the base point of $|K|$, then $\tilde{S}$ has a pencil of curves of genus 3 but no irreducible curve C with $p(C)=2$.

Proof. — The surface F_2 is a $\mathbf{P}^1$-bundle over a rational curve and it has a cross-section B with $B^2=-2$. If L is the fiber of F_2 then L, B form an integral basis for rational equivalence on F_2 and

$$L^2=0, \quad LB=1, \quad B^2=-2.$$

The canonical bundle of F_2 is

$$K_{F_2}=-4[L]-2[B]$$

while $3[L]+[B]$ is the hyperplane bundle of F_2. The surface F_2 is a quadric cone Q blown up at the vertex and $2[L]+[B]$ corresponds to the plane section of the cone Q. It follows that for every $m\geqq 1$ the linear system $|m(2[L]+[B])|$ has no base points.

We take $m=7$ and choose a curve

$$\Delta_0\in|7(2[L]+[B])|$$

which is irreducible and non-singular. Clearly Δ_0 is disjoint from B, therefore

$$\Delta=\Delta_0+B$$

is a non-singular effective divisor on F_2. Since the rational equivalence class of Δ is $14L+8B$, which is divisible by 2, there is a double covering

$$\pi:\tilde{S}\rightarrow F_2$$

which is branched on Δ. We claim that $\tilde{S}$ is the surface S of the example, blown up at the base point of $|K_S|$, and that $F_2=X^{[1]}$ is the 1-canonical image of $\tilde{S}$.

The canonical bundle of $\tilde{S}$ is

$$K_{\tilde{S}}=\pi^* K_{F_2}+\frac{1}{2}\pi^*[\Delta].$$

Now since π is a double covering and $B^2=-2$ we have, if $\Lambda=\pi^{-1}(B)$, that

$$\pi^*[B]=2[\Lambda]$$

and $\Lambda^2=-1$. Clearly Λ is isomorphic to B, therefore $p(\Lambda)=0$ and Λ is an exceptional curve of the first kind on $\tilde{S}$. If now $C=\pi^{-1}(L)$ we get

$$K_{\tilde{S}}=3[C]+4[\Lambda]$$

and thus $K_{\tilde{S}}^2=8$.

In order to compute $\chi(\mathcal{O}_{\tilde{S}})$ we use the signature formula for a cyclic covering which gives

$$\begin{aligned} K_{\tilde{S}}^2 - 8\chi(\mathcal{O}_{\tilde{S}}) &= \tau(\tilde{S}) \\ &= 2\tau(F_2) - \frac{1}{2}(\Delta^2) \\ &= -\frac{1}{2}(\Delta^2) = -48, \end{aligned}$$

from which it follows that

$$\chi(\mathcal{O}_{\tilde{S}}) = 7.$$

It is clear that Λ is the only exceptional curve of $\tilde{S}$, therefore a minimal model S of $\tilde{S}$ has $K_S^2 = 9$, $\chi(\mathcal{O}_S) = 7$. By a result which will be proved later (Theorem 9 of Section X) we have $p_g(S) \leqq 6$, therefore $\chi(\mathcal{O}_S) = 7$ implies

$$p_g(S) = p_g(\tilde{S}) = 6$$

and S is a regular surface. Now

$$|3[C] + 2[\Lambda]| + 2\Lambda \subseteq |K_{\tilde{S}}|$$

and since $3[C] + 2[\Lambda] = \pi^*(3[L] + [B])$ and

$$\dim |3[L] + [B]| = 5$$

we obtain that $|K_{\tilde{S}}| = |3[C] + 2[\Lambda]| + 2\Lambda$. This clearly implies that the linear system $|K_S|$ has one base point b and that $\tilde{S}$ is the blowing up of S at b; also it implies that $X^{[1]} = F_2$.

Let $\rho : \tilde{S} \rightarrow S$ be the contraction of Λ. Since $C\Lambda = 1$ and $(C + \Lambda)\Lambda = 0$ we see that if $\Gamma = \rho(C)$ then

$$\rho^*[\Gamma] = [C] + [\Lambda]$$

and $K_S = 3[\Gamma]$.

The linear system $|[\Gamma]|$ has dimension 1 and b as a base point. Moreover, since $\pi|_C : C \rightarrow L$ is a double covering, we have that the curves Γ of $|[\Gamma]|$ are hyperelliptic of genus 3.

Now we show that $X^{[2]}$ is a quadric cone Q embedded in $\mathbf{P}^{15}(k)$ by means of $\mathcal{O}_Q(3)$. It is sufficient to show that $X^{[2]}$ is not a birational model of S, since then $X^{[2]}$ will be the image of F_2 by means of $3(2[L] + [B])$ (note that 2Λ is a fixed part of $|2K_{\tilde{S}}|$). We shall prove in fact that Φ_{2K} identifies pairs of points of Γ belonging to the hyperelliptic pencil of Γ.

Let Γ be a non-singular element of $|[\Gamma]|$ and let

$$x + y \in |h_\Gamma|, \qquad x \neq y$$

where h_Γ is the hyperelliptic bundle of Γ.

We have the exact sequence

$$0 \rightarrow \mathcal{O}(2K) \otimes \mathcal{J}_\Gamma \rightarrow \mathcal{O}(2K) \otimes \mathcal{J}_x \otimes \mathcal{J}_y \rightarrow \mathcal{O}_\Gamma(2K_\Gamma - [x + y]) \rightarrow 0$$

and by Theorem A and $K = 3[\Gamma]$ we have

$$\dim_k H^1(S, \mathcal{O}(2K) \otimes \mathcal{J}_\Gamma) = \dim_k H^1(S, \mathcal{O}([\Gamma] - K)) = 0.$$

The Riemann-Roch theorem now gives

$$\dim_k H^0(S, \mathcal{O}(2K)\otimes\mathcal{I}_\Gamma) = 12$$

and we get

$$\dim_k H^0(S, \mathcal{O}(2K)\otimes\mathcal{I}_x\otimes\mathcal{I}_y) = 12 + \dim_k H^0(\Gamma, \mathcal{O}_\Gamma(2K_\Gamma - [x+y])).$$

Since $P_2 = 16$, we have to show that

$$\dim_k H^0(\Gamma, \mathcal{O}_\Gamma(2K_\Gamma - [x+y])) = 3$$

and since Γ has genus 3 and $\deg(2K_\Gamma - [x+y]) = 4$ this is equivalent to showing that $2K_\Gamma - [x+y]$ is the canonical bundle of Γ. The canonical bundle $k_\Gamma = 2h_\Gamma$ of Γ is $[K]_\Gamma + [\Gamma]_\Gamma = 4[\Gamma]_\Gamma$. Now $[\Gamma]_\Gamma$ has one non-trivial section s with only one simple zero, at the base point b of $|K|$. Clearly s^4 is a section of $2h_\Gamma$ and since $|2h_\Gamma|$ is composed of $|h_\Gamma|$ we get

$$s^4 = h_1 h_2$$

with $h_i \in H^0(\Gamma, \mathcal{O}_\Gamma(h_\Gamma))$. It follows that s^2 is a section of h_Γ and

$$h_\Gamma = 2[\Gamma]_\Gamma;$$

this proves what we want.

Finally we show that $\tilde{S}$ has no irreducible curve D with $p(D) = 2$.

If $p(D) = 2$ we have $D^2 \leqq 0$ by the Algebraic Index Theorem. Let $C = \pi(D)$. If $\pi: D \to C$ is a double covering we must have $C^2 = \frac{1}{2}D^2 \leqq 0$, and since the only irreducible curves on F_2 with non-positive self-intersection are L and B, we get a contradiction. If instead $\pi: D \to C$ is an isomorphism we get $p(C) = 2$, which implies easily

$$[C] = 5[L] + 2[B].$$

Now $\pi^{-1}(C) = D + D'$ and D' is isomorphic with D. Since

$$KD = KD' = (3L + B)(5L + 2B) = 7$$

and $p(D) = p(D') = 2$ we find $D^2 = D'^2 = -5$. Also $(D + D')^2 = 2C^2 = 24$, therefore $DD' = 17$. On the other hand we must have :

$$DD' \geqq C\Delta = 34. \qquad \text{Q.E.D.}$$

7. Birational maps, continued.

The arguments of the previous section fail to show that $X \to X^{[3]}$ is birational in all cases with $K^2 \geqq 3$, because we need the existence of a divisor D in the linear system $|2\pi^* K - 2[L] - 2[M]|$. Using additional arguments, we shall prove

Theorem 6. — Let S *be a minimal surface of general type and let* $q = \dim_k H^1(S, \mathcal{O})$ *be its irregularity. Then* $X \to X^{[3]}$ *is a birational map if:*

$$K^2 \geqq 4, \quad p_g = 0 \text{ or } 1,$$
$$K^2 = 3, \quad p_g = 1, \quad q = 0,$$
$$K^2 = 2, \quad p_g = 1 \text{ or } 2, \quad q = 0$$

and S *has no torsion.*

Remark. — We shall prove in Sections X and XI of this paper that if $K^2 = 2$, $p_g = 2$, then $q = 0$ and S has no torsion.

Proof. — Let $x, y \notin \mathscr{E}$ be two distinct closed points of S and let $\pi : \tilde{S} \to S$ denote the blowing up of S at x, y. Let also L, M be the exceptional lines of the first kind on $\tilde{S}$. Using Lemma 10 we have that $\Phi_{3K}(x) \neq \Phi_{3K}(y)$ will follow from

$$H^1(\tilde{S}, \mathcal{O}(2[L] + 2[M] - 2\pi^* K)) = 0.$$

Let $D \in |2\pi^* K - 2[L] - [M]|$, assuming for the time being that D exists. By Lemma 9A D is 1-connected if $K^2 \geqq 2$, therefore we get

$$H^1(\tilde{S}, \mathscr{J}_D) = 0$$

if $K^2 \geqq 2$, by Theorem A, Corollary. Our aim is to show that $H^1(\tilde{S}, \mathscr{J}_D \otimes \mathcal{O}([M])) = 0$.

Now suppose:

Assumption A. — M is not a component of D.

Then since $DM = 1$, D and M intersect transversally at a *simple point* z of D. We have a commutative exact diagram:

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r, dash] & \mathcal{O}_D \arrow[r] \arrow[u] & \mathcal{O}_D \otimes \mathcal{O}([M]) \arrow[r] \arrow[u] & \mathscr{F} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r, dash] & \mathcal{O} \arrow[r] \arrow[u] & \mathcal{O}([M]) \arrow[r] \arrow[u] & \mathscr{N}_M \arrow[r] \arrow[u] & 0 \\
0 \arrow[r, dash] & \mathscr{J}_D \arrow[r] \arrow[u] & \mathscr{J}_D \otimes \mathcal{O}([M]) \arrow[r] \arrow[u] & \mathscr{J}_D \otimes \mathscr{N}_M \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

where $\mathscr{N}_M$ is the normal sheaf of M in $\tilde{S}$ and where $\mathscr{F}$ is a sheaf with support z and stalk $k(z)$ at z.

Assumption B. — We have:

$$\dim_k H^0(D, \mathcal{O}_D \otimes \mathcal{O}([M])) = 1.$$

Using B and

$$\dim_k H^0(D, \mathcal{O}_D) = 1, \qquad H^i(S, \mathcal{N}_M) = 0, \qquad H^1(S, \mathscr{I}_D) = 0$$

we deduce the commutative exact diagram

$$\begin{tikzcd}
0 \arrow[r] & H^0(D, \mathscr{J}) \arrow[r] & H^1(D, \mathcal{O}_D) \arrow[r] & H^1(D, \mathcal{O}_D \otimes \mathcal{O}([M])) \arrow[r] & 0 \\
 & 0 \arrow[u] \arrow[r] & H^1(\tilde{S}, \mathcal{O}) \arrow[u] \arrow[r] & H^1(\tilde{S}, \mathcal{O}([M])) \arrow[u] \arrow[r] & 0 \\
 & & 0 \arrow[u] & H^1(\tilde{S}, \mathscr{I}_D \otimes \mathcal{O}([M])) \arrow[u] & \\
 & & & 0 \arrow[u] &
\end{tikzcd}$$

from which we obtain

$$\dim_k H^1(\tilde{S}, \mathscr{I}_D \otimes \mathcal{O}([M])) = \dim_k \{\operatorname{Im} H^0(D, \mathscr{F}) \cap \operatorname{Im} H^1(\tilde{S}, \mathcal{O})\}$$

the intersection being taken in $H^1(D, \mathcal{O}_D)$.

Clearly this implies that $H^1(\tilde{S}, \mathscr{I}_D \otimes \mathcal{O}([M])) = 0$ if S has irregularity $q = 0$; we shall prove that if $q = 1$ the intersection is 0 provided a certain condition on D is satisfied.

Since z is a simple point of D, there is a unique irreducible component Γ of D with $z \in \Gamma$, and Γ has multiplicity one in D. We shall denote by $\mu : C \to \Gamma$ a normalization of Γ and the point $\mu^{-1}(z)$ on C will be again indicated with z, since no confusion should arise in the next argument.

Since $q = 1$ by hypothesis, there is a surjective morphism $\tilde{S} \to E$ of $\tilde{S}$ onto an elliptic curve $E = \operatorname{Alb}(\tilde{S})$; composing this map with the inclusion of Γ in S we get a morphism

$$\varphi : C \to E$$

which is surjective unless $\varphi^* H^1(E, \mathcal{O}_E) = 0$. Now it is clear that

$$\begin{aligned} &\dim_k \{\operatorname{Im} H^0(D, \mathscr{F}) \cap \operatorname{Im} H^1(\tilde{S}, \mathcal{O})\} \\ &= \dim_k \{\delta H^0(C, \mathcal{O}_C([z])/\mathcal{O}_C) \cap \varphi^* H^1(E, \mathcal{O}_E)\} \end{aligned}$$

where δ is the coboundary map induced from

$$0 \to \mathcal{O}_C \to \mathcal{O}_C([z]) \to \mathcal{O}_C([z])/\mathcal{O}_C \to 0.$$

If we identify the space $H^1(C, \mathcal{O}_C)$ with the dual of $H^0(C, \Omega^1_C)$, the space of regular differentials on C, the elements of $H^1(C, \mathcal{O}_C)$ are linear functionals

$$< r, \omega > = \sum_{x \in C} \operatorname{res}_x(r_x \omega)$$

where r is a repartition on C. The subspace $\delta H^0(C, \mathcal{O}_C([z])/\mathcal{O}_C)$ of $H^1(C, \mathcal{O}_C)$ consists of the functionals

$$\mathrm{res}_z(r_z\,\omega)$$

where r_z is a rational function which at z has at most a simple pole. The image of $H^1(E, \mathcal{O}_E)$ in $H^1(C, \mathcal{O}_C)$ consists of the functionals

$$\sum_{y \in E} \sum_{\varphi(x)=y} \mathrm{res}_x((\varphi^* s_y)\,\omega)$$

where s is a repartition on E. If η is a regular differential on E, using the formula of traces one checks easily that the value of this functional on $\varphi^*\eta$ is $d\langle s, \eta\rangle$, where d is the degree of the morphism φ. It follows that if $H^1(S, \mathscr{I}_D \otimes \mathcal{O}([M]))$ is not 0 then

$$d\langle s, \eta\rangle = \mathrm{res}_z(r_z\,\varphi^*\eta)$$

therefore $\varphi^*\eta$ cannot vanish at z. Obviously this implies that the restriction to D o a non-zero regular differential $\tilde{\omega}$ of $\tilde{S}$ cannot vanish at z.

On the other hand, since M is an exceptional curve of the first kind, there is a closed point t of M such that $\tilde{\omega}$ itself vanishes at t. Hence $z \neq t$. We conclude that

$$\dim_k H^1(S, \mathscr{I}_D \otimes \mathcal{O}([M])) = 0$$

if the following conditions are satisfied:

A) M is not a component of D;
B) $\dim_k H^0(\tilde{S}, \mathcal{O}_D \otimes \mathcal{O}([M])) = 1$;
C) in case $q = 1$ we have $t \in D$.

Lemma 11. — Let D *be an effective divisor on* S *with* $DM = 1$. *Assume that* (A) *holds and that* D *is 1-connected. Let also* Γ *be the irreducible component of* D *with* $\Gamma M = 1$. *Then*

$$\dim_k H^0(\tilde{S}, \mathcal{O}_D \otimes \mathcal{O}([M])) = 1$$

except possibly if Γ *is a rational curve.*

Proof. — We follow Ramanujam's proof ([14], Lemma 3) of a similar result. Since D is 1-connected and Γ is not a rational curve we have that

$$H^0(\tilde{S}, \mathcal{O}_{D_{\mathrm{red}}} \otimes \mathcal{O}([M]))$$

consists only of constants. Now let σ be a non-zero section in

$$\ker\{H^0(\tilde{S}, \mathcal{O}_D \otimes \mathcal{O}([M])) \to H^0(\tilde{S}, \mathcal{O}_{D_{\mathrm{red}}} \otimes \mathcal{O}([M]))\}.$$

There is a maximal divisor D_1 such that σ goes to zero in $H^0(\tilde{S}, \mathcal{O}_{D_1} \otimes \mathcal{O}([M]))$ and $0 < D_1 < D$. Writing $D = D_1 + D_2$ one checks two exact sequences of sheaves

$$0 \to \mathcal{O}_{D_2} \xrightarrow{\sigma} \mathcal{O}_D \otimes \mathcal{O}([M]) \to \mathcal{O}_D \otimes \mathcal{O}([M])/\sigma\mathcal{O}_D \to 0$$

$$0 \to \mathscr{F} \to \mathcal{O}_D \otimes \mathcal{O}([M])/\sigma\mathcal{O}_D \to \mathcal{O}_{D_1} \otimes \mathcal{O}([M]) \to 0$$

where $\mathcal{F}$ is a sheaf with 0-dimensional support; the exactness of these sequences depends on the fact that D_1 is maximal. Taking Chern classes we get

$$c(\mathcal{O}_D \otimes \mathcal{O}([M])) = c(\mathcal{O}_{D_1} \otimes \mathcal{O}([M]))\, c(\mathcal{O}_{D_2})\, c(\mathcal{F})$$

whence

$$\frac{1+M}{1+M-D} = \frac{1+M}{1+M-D_1} \cdot \frac{1}{1-D_2} \cdot c(\mathcal{F}).$$

If $\mathcal{F}$ has support at the points p_i, then $c(\mathcal{F}) = 1 - \sum_i n_i p_i$ where $n_i \geqq 0$. Taking degrees in the previous equation we deduce

$$D_1 D_2 = D_2 M - \sum_i n_i \leqq D_2 M.$$

Now since Γ has *multiplicity one* in D, we have that Γ is a component of D_1, hence Γ is not a component of D_2. This implies that $D_2 M = 0$, hence $D_1 D_2 \leqq 0$ and D is not 1-connected, a contradiction. Q.E.D.

We can now prove Theorem 6. Suppose it is false for a surface S. Then there is a Zariski open set U of S such that for every closed point x of U there is $y \in U$, $y \neq x$, such that

$$\Phi_{3K}(x) = \Phi_{3K}(y),$$

that is, every section of 3K vanishing at x vanishes also at y. Since our hypothesis imply $P_2 \geqq 5$ except if $K^2 = 2$, $p_g = 1$ in which case $P_2 = 4$, we obtain easily

$$\dim_k H^0(\tilde{S}, \mathcal{O}(2\pi^* K - 2[L] - [M])) \geqq \begin{cases} 1 & \text{if } K^2 = 4,\ p_g = 0 \text{ or } K^2 = 2,\ p_g = 1 \\ 2 & \text{in the other cases} \end{cases}$$

because if $p_g \geqq 1$ our hypothesis $\Phi_{3K}(x) = \Phi_{3K}(y)$ implies that every section of 2K vanishing on x vanishes also on y (taking a smaller open set U, if needed). It follows that there is D with $D \in |2\pi^* K - 2[L] - [M]|$ and that, if t is a given closed point of $\tilde{S}$, we may take D such that $t \in D$, except possibly if $K^2 = 4$, $p_g = 0$ or $K^2 = 2$, $p_g = 1$. Hence we may choose D so that condition (C) will be satisfied.

Now we show that we may also suppose that condition (A) is satisfied too. For if (A) does not hold, we have that

$$\Delta = D - M \in |2\pi^* K - 2[L] - 2[M]|.$$

By Lemma 10 and the results of section V we have that the hypothesis $\Phi_{3K}(x) = \Phi_{3K}(y)$ implies $H^1(\tilde{S}, \mathcal{J}_\Delta) \neq 0$; if $K^2 \geqq 3$, since then $\Delta^2 > 0$, Theorem A, Corollary, shows that Δ cannot be 1-connected, which contradicts Lemma 9, while if $K^2 = 2$, since then $q = 0$ by hypothesis, the exact sequence

$$0 \to H^0(\tilde{S}, \mathcal{O}) \to H^0(\Delta, \mathcal{O}_\Delta) \to H^1(\tilde{S}, \mathcal{J}_\Delta) \to 0$$

shows again that Δ cannot be 1-connected. By Lemma 9, we must have $\Delta = \Delta_1 + \Delta_2$ with $\Delta_1 \sim \Delta_2 \sim \pi^* K - L - M$. Writing $C_i = \pi(\Delta_i)$ we have

$$C_1 \sim C_2 \sim K, \quad x, y \in C_i.$$

Since $q = 0$ and S has no torsion, numerical equivalence coincides with linear equivalence, therefore

$$C_i \in |K|.$$

Now if $p_g = 1$ there is only one effective canonical divisor and, restricting U if necessary, we cannot have $x, y \in C_i$. If instead $p_g = 2$, there is only one canonical divisor Γ with x, $y \in \Gamma$ (restricting U, we may assume that x, y are not base points of $|K|$) and since now $\dim |2\pi^* K - 2[L] - [M]| \geqq 1$ we see that $|2\pi^* K - 2[L] - [M]|$ cannot have M as a fixed curve. This proves that we may choose D satisfying (A) and (C).

Finally we may restrict U so that (B) is satisfied. Otherwise, if Γ is the component of D which meets M, Lemma 11 shows that $C = \pi(\Gamma)$ is a rational curve on S with $y \in C$. This clearly would imply that S would have an algebraic system of dimension at least 1 of rational curves, which is absurd because S is of general type. It follows that

$$H^1(\widetilde{S}, \mathscr{I}_D \otimes \mathcal{O}([M])) = 0 \quad \text{and} \quad \Phi_{3K}(x) \neq \Phi_{3K}(y),$$

a contradiction. Q.E.D.

8. Birational maps : $\mathbf{K^2 = 1}$ and $\mathbf{p_g = 1}$.

Theorem 7. — Let S *be a minimal surface of general type such that:*

$$K^2 = 1, \quad p_g = 1, \quad q = 0$$

and S *has no torsion.*

Then $X \to X^{[3]}$ *and* $X \to X^{[4]}$ *are birational maps.*

Remark. — We shall prove in Sections X and XI of this paper that if $K^2 = 1$ and $p_g = 1$ then $q = 0$ and S has no torsion.

Proof. — Since $p_g > 0$, it is sufficient to prove that $X \to X^{[3]}$ is birational.

Lemma 12. — Let S *be as in Theorem 7. Then a general element of* $|2K|$ *is irreducible and non-singular.*

Proof. — $|2K|$ has no fixed part. For if

$$|2K| = |C| + Z$$

we have $KC + KZ = 2$, hence $KZ = 1$ or 0. We cannot have $KZ = 1$ since then $KC = 1$ and the Algebraic Index Theorem would give $C \sim K$, hence $C = K$ because $q = 0$ and S has no torsion, which would contradict $p_g = 1$. Also we cannot have $KZ = 0$, by Theorem 2, Remark 2.

Now a general element $C \in |2K|$ cannot be singular. Otherwise by the Bertini theorem, since $C^2=4$, the curves C would not have variable intersections outside the base points, which contradicts

$$\dim|2K|=P_2-1=2. \qquad \text{Q.E.D.}$$

We have to prove that there is a Zariski open set U on S such that one may separate points of U using sections of 3K. Since $p_g=1$, there is *one* canonical curve Γ, therefore

$$|2K|+\Gamma\subseteq|3K|.$$

We take $U\subseteq S-\mathscr{E}$ and $x, y\notin\mathscr{E}$, $x\neq y$ such that $\Phi_{3K}(x)=\Phi_{3K}(y)$. Then every section of 2K vanishing at x vanishes also at y and we get

$$\dim H^0(S, \mathcal{O}(2K)\otimes\mathscr{I}_x\otimes\mathscr{I}_y)=P_2-1=2.$$

It follows from this that there exists a non-singular curve $C\in|2K|$ with

$$x+y\in C.$$

Let $\widetilde{S}$ be the blowing up of S at x and y and let L, M be the exceptional lines of the first kind on S. Let $D\in|\pi^*2K-[L]-[M]|$ with $\pi(D)=C$.

Since $\Phi_{3K}(x)=\Phi_{3K}(y)$, Lemma 10 gives

$$\dim_k H^1(S, \mathcal{O}(3\pi^*K)\otimes\mathscr{I}_L\otimes\mathscr{I}_M)>0$$

therefore by duality

$$\dim_k H^1(S, \mathscr{I}_D\otimes\mathcal{O}([L]+[M]))>0.$$

We have the exact sequence of sheaves

$$0\to\mathscr{I}_D\otimes\mathcal{O}([L]+[M])\to\mathcal{O}([L]+[M])\to\mathcal{O}_D([x]+[y])\to 0$$

where $x=D\cap L$, $y=D\cap M$, because D is non-singular, and we get the cohomology sequence

$$0\to H^0(\widetilde{S}, \mathcal{O}([L]+[M]))\to H^0(\widetilde{S}, \mathcal{O}_D([x]+[y]))\to$$
$$\to H^1(\widetilde{S}, \mathscr{I}_D\otimes\mathcal{O}([L]+[M]))\to H^1(\widetilde{S}, \mathcal{O}([L]+[M])).$$

The Riemann-Roch theorem implies

$$\dim_k H^1(\widetilde{S}, \mathcal{O}([L]+[M]))=0$$

because $p_g=1$ and $q=0$, hence $\chi(\mathcal{O})=2$. It follows from this

$$\dim_k H^0(\widetilde{S}, \mathcal{O}_D([x]+[y]))=2$$

and D is hyperelliptic and $[x]+[y]$ is its hyperelliptic bundle.

Hence the curve C is also hyperelliptic and its hyperelliptic bundle h_C is given by

$$h_C=[x]+[y].$$

Since C has genus 4 the canonical bundle k_C of C is

$$k_C=3h_C=3K_C.$$

On the other hand, K_C has a non-trivial section s and thus

$$s^3 \in H^0(C, \mathcal{O}_C(3h_C)).$$

Since $|3h_C|$ is composed of $|h_C|$ we get

$$s^3 = h_1 h_2 h_3$$

with $h_i \in H^0(C, \mathcal{O}_C(h_C))$. It follows from this that s is a section of h_C, whence

$$K_C = h_C.$$

Thus we have an exact sequence

$$0 \to \mathcal{O}(-K) \to \mathcal{O}(K) \to \mathcal{O}_C(h_C) \to 0$$

and a cohomology sequence

$$0 \to H^0(S, \mathcal{O}(K)) \to H^0(C, \mathcal{O}_C(h_C)) \to 0;$$

since $\dim_k H^0(C, \mathcal{O}_C(h_C)) = 2$ and $p_g = 1$, we have a contradiction. Q.E.D.

9. Birational maps : $K^2 = 2$ and $p_g = 1$.

Theorem 8. — *Let* S *be a minimal surface of general type such that*

$$K^2 = 2, \quad p_g = 1$$

and either $q = 1$ *or* $q = 0$ *and the torsion group of* S *is* $\mathbf{Z}_2$.

Then $X \to X^{[3]}$ *and* $X \to X^{[4]}$ *are birational maps.*

Remark. — We shall prove in Section XI that if $q = 0$ the torsion group of S is either (0) or $\mathbf{Z}_2$.

Proof.

a) *The torsion case.* — The proof of Theorem 6 shows that the result follows unless if for x, y such that $\Phi_{3K}(x) = \Phi_{3K}(y)$, letting $\tilde{S}$ be the blowing up of S at x and y and L, M the corresponding exceptional lines, we have:

$$|2\pi^*K - 2[L] - 2[M]| \text{ is not empty,}$$

and if $D \in |2\pi^*K - 2[L] - 2[M]|$, then D is not 1-connected and

$$D = D_1 + D_2$$

with $D_1 \sim D_2 \sim \pi^*K - L - M$.

Since the torsion group is $\mathbf{Z}_2$, we deduce that

$$D_1 = D_2$$

if x, y belong to a sufficiently small Zariski open set of $\tilde{S}$. Moreover, the cohomology sequence of

$$0 \to \mathcal{I}_{D_1} \otimes \mathcal{O}([L] + [M]) \to \mathcal{O}([L] + [M]) \to \mathcal{O}_{D_1} \otimes \mathcal{O}([L] + [M]) \to 0$$

shows that $\dim_k H^0(\tilde{S}, \mathcal{O}_{D_1}\otimes\mathcal{O}([L]+[M]))=2$, because $\dim_k H^1(\tilde{S}, \mathcal{O}([L]+[M]))=0$ by the Riemann-Roch theorem and

$$\dim_k H^1(\tilde{S}, \mathcal{J}_{D_1}\otimes\mathcal{O}([L]+[M]))\geqq 1$$

(by duality, this means that the sections of $K+[C]$, where $C=\pi(D_1)$, do not separate x and y on S). Hence, restricting the open set U if necessary, we arrive at the following statement: let [C] be the line bundle on S with

$$[C]\sim K, \qquad [C]\neq K.$$

Then $\dim_k |[C]|=1$, and if $|[C]|=|[F]|+X$ where X is a fixed part, we have that the generic element F of $|[F]|$ is irreducible, non-singular and hyperelliptic. The mapping Φ_{3K} identifies pairs of points of F which are zeros of sections of the hyperelliptic bundle h_F of F.

It is easily seen, using the Algebraic Index Theorem, that either $X=0$ or X is a fundamental cycle of S.

Suppose first that $X=0$, so that $[F]=[C]$.

The cohomology sequence of

$$0\rightarrow\mathcal{O}(3K-[C])\rightarrow\mathcal{O}(3K)\rightarrow\mathcal{O}_C(3K_C)\rightarrow 0$$

shows that the restriction map

$$r_C : H^0(S, \mathcal{O}(3K))\rightarrow H^0(C, \mathcal{O}_C(3K_C))$$

is surjective. Hence $\Phi_{3K|C}$ coincides with Φ_{3K_C} and is not birational. Therefore, since C has genus 3, the linear system $|3K_C|$ is composed of the hyperelliptic pencil $|h_C|$. Hence

$$3K_C=3h_C.$$

Since $p_g=1$, K_C has a non-zero section s and we obtain

$$s^3=h_1h_2h_3$$

with $h_i\in H^0(C, \mathcal{O}_C(h_C))$, from which it follows easily that $s\in H^0(C, \mathcal{O}_C(h_C))$ and thus

$$K_C=h_C.$$

This gives an exact sequence

$$0\rightarrow\mathcal{O}(K-[C])\rightarrow\mathcal{O}(K)\rightarrow\mathcal{O}_C(h_C)\rightarrow 0$$

and since $H^i(S, \mathcal{O}(K-[C]))=0$ for $i=1, 0$, the cohomology sequence shows that the restriction map

$$r_C : H^0(S, \mathcal{O}(K))\rightarrow H^0(C, \mathcal{O}_C(h_C))$$

is an *isomorphism*. This contradicts $p_g=1$.

Now suppose that $|[C]|=|[F]|+X$ has a fixed part. Reasoning as before, we see that the restriction map

$$r_F : H^0(S, \mathcal{O}(3K-[X]))\rightarrow H^0(F, \mathcal{O}_F(3K_F-[X]_F))$$

is *surjective* and since $\Phi_{3K|F}$ is not birational, we have *a fortiori* that $\Phi_{3K-[X]|F}$, and hence $\Phi_{3K_F-[X]_F}$, are not birational maps. Since F has genus 2 and $3K_F-[X]_F$ has degree 4, we get

$$3K_F-[X]_F=2h_F.$$

Clearly $h_F=K_F$ ($[F]_F$ is a trivial bundle) therefore

$$[X]_F=K_F.$$

We deduce the exact sequence of sheaves

$$0\to\mathcal{O}([X]-K)\to\mathcal{O}([X]+[F]-K)\to\mathcal{O}_F\to 0$$

and a cohomology sequence

$$0\to H^0(F,\mathcal{O}_F)\to H^1(S,\mathcal{O}([X]-K)).$$

By duality, this implies that

$$\dim_k H^1(S,\mathcal{O}(2K-[X]))\geqq 1$$

and since $H^1(S,\mathcal{O}(2K))=0$, the fundamental cycle X must be a fixed part of $|2K|$. This contradicts Theorem 2, Remark 2, and proves Theorem 8 in case the torsion group is $\mathbf{Z}_2$.

b) *The irregular case.* — Now we assume that $q=1$, hence

$$\chi(\mathcal{O})=1 \quad\text{and}\quad P_2=3,\ P_3=7.$$

Let $E=\mathrm{Pic}^0(S)$ and let L_u, $u\in E$ be a Poincaré family of bundles such that L_0 is a trivial bundle. We have:

$$L_u+L_v=L_{u+v}$$

for u, $v\in E$, where $u+v$ is the sum on the elliptic curve E. We define $K_u=K+L_u$ and the Riemann-Roch theorem gives

$$\dim_k H^0(S,\mathcal{O}(K_u))=1+\dim_k H^1(S,\mathcal{O}(K_u))$$

for $u\neq 0$. We cannot have $\dim_k H^0(S,\mathcal{O}(K_u))=2$, otherwise if s_1, s_2 were linearly independent sections of K_u, s_1^2, s_1s_2, s_2^2 would be linearly independent sections of $2K_u$. Now the Riemann-Roch theorem and Theorem B give easily

$$\dim_k H^0(S,\mathcal{O}(2K_u))=3$$

and we would get that sections of $2K_u$ would be products of sections of K_u. This is clearly impossible, because $2K_u=K_v+K_{2u-v}$ for every $v\in E$ and since K_v has a section for every v we readily get a contradiction. Thus we have proved

$$\dim_k H^0(S,\mathcal{O}(K_u))=1 \quad\text{for}\quad u\in E$$

$$\dim_k H^1(S,\mathcal{O}(K_u))=\begin{cases}1 & \text{for}\quad u=0\\ 0 & \text{for}\quad u\neq 0.\end{cases}$$

We denote by C_u the divisor of zeros of the (unique up to a scalar factor) non-trivial section of K_u.

The general curve C_u is irreducible. If not, we must have

$$C_u = F_u + X$$

where X is a fundamental cycle and where F_u is a curve with

$$KF_u = 2, \quad F_u^2 = 0, \quad F_u X = 2.$$

There is a surjective morphism

$$\rho : S \to E$$

with fiber F_u. This is impossible, because since X is made up of *rational* curves, we have that $\rho(X)$ is a point, therefore $F_u X = 0$, a contradiction.

Now assume that Φ_{3K} is not a birational map. Then the sections of 3K do not separate points on S; since

$$C_u + C_v + C_w \in |3K|$$

if $u+v+w=0$, we deduce that if $\Phi_{3K}(x) = \Phi_{3K}(y)$ and $x \in C_u$, then $y \in C_u$ and the restriction $\Phi_{3K|C_u}$ cannot be a rational map.

If C_u is non-singular, as in the torsion case we obtain that the restriction homomorphism

$$r_{C_u} : H^0(S, \mathcal{O}(3K)) \to H^0(C_u, \mathcal{O}_{C_u}(3K_{C_u}))$$

is surjective. Since C_u has genus 3 and $\Phi_{3K|C_u} = \Phi_{3K_{C_u}}$ is not birational, we find that C_u is hyperelliptic and, exactly as in the torsion case, that

$$K_{C_u} = h_{C_u}$$

where C_u is the hyperelliptic bundle of C_u.

Now we obtain the exact sequence of sheaves

$$0 \to \mathcal{O}(K - K_u) \to \mathcal{O}(K) \to \mathcal{O}_{C_u}(h_{C_u}) \to 0;$$

since by duality

$$\dim_k H^1(S, \mathcal{O}(K - K_u)) = \dim_k H^1(S, \mathcal{O}(K_u)) = 0$$

the cohomology sequence gives an isomorphism

$$H^0(S, \mathcal{O}(K)) \xrightarrow{\sim} H^0(C_u, \mathcal{O}_{C_u}(h_{C_u})),$$

contradicting that $p_g = 1$.

Hence the general curve C_u must the singular.

Proposition 4 (Van de Ven). — *Let* S *be a surface of general type. Then* S *does not contain an algebraic system of dimension* ≥ 1 *of elliptic curves.*

Proof. — Let Γ be the parameter curve of an algebraic system of dimension 1 of elliptic curves $\{E_z\}$, $z \in \Gamma$, on S. Let Σ be the graph in $\Gamma \times S$ of the correspondence $z \mapsto E_z$. If $\tilde{\Sigma}$ is a desingularization of Σ, it is easily seen that the general curve $z \times E_z$

on Σ becomes non-singular on $\widetilde{\Sigma}$, even if the general curve E_z is singular. Let $\widetilde{E}_z$ be the corresponding elliptic curve on $\widetilde{\Sigma}$. By the construction of Σ, we have $\widetilde{E}_z^2 = 0$, therefore $\widetilde{\Sigma} \rightarrow \Gamma$ defines a pencil of elliptic curves on $\widetilde{\Sigma}$ and $\widetilde{\Sigma}$ is of *special type*, by classification of surfaces.

On the other hand, the projection

$$p : \Sigma \rightarrow S$$

is surjective. Hence we obtain a proper surjective morphism

$$f : \widetilde{\Sigma} \rightarrow S$$

of a surface of special type onto a surface of general type. This is impossible, because one would get

$$P_m(S) \leqq P_m(\widetilde{\Sigma})$$

by lifting tensor 2-forms of S to $\widetilde{\Sigma}$, contradicting Theorem 2, Corollary and the fact that $P_m(\widetilde{\Sigma}) = O(m)$. Q.E.D.

By Proposition 4, C_u has genus 2 and only one double point p, which is either an ordinary double point or a cusp.

We blow up S at p and obtain a surface $\widetilde{S}$, with an exceptional line L. We denote by $\widetilde{C}$ the proper transform of C_u; clearly $\widetilde{C}$ is non-singular of genus 2.

If $\pi : \widetilde{S} \rightarrow S$ is the contraction of L we have

$$[\widetilde{C}] + 2[L] = \pi^* K_u,$$
$$K_{\widetilde{S}} = \pi^* K + [L],$$
$$L^2 = -1, \qquad \widetilde{C}^2 = -2, \qquad \widetilde{C}L = 2.$$

We also know that $\Phi_{3K|C}$ is not a birational map and *a fortiori* we obtain that $\Phi_{\pi^*(3K-K_v)|\widetilde{C}}$ is not birational for every $v \in E$. We have the exact sequence of sheaves on $\widetilde{S}$:

$$0 \rightarrow \mathcal{O}(\pi^*(3K - K_v) - [\widetilde{C}]) \rightarrow \mathcal{O}(\pi^*(3K - K_v)) \rightarrow \mathcal{O}_{\widetilde{C}}(\pi^*(3K - K_v)_{\widetilde{C}}) \rightarrow 0.$$

Now $\pi^*(3K - K_v) - [\widetilde{C}] = \pi^* K_{-u-v} - 2[L]$ and if v is a general point of E we have

$$H^0(\widetilde{S}, \mathcal{O}(\pi^* K_{-u-v} - 2[L])) = 0$$

otherwise p would be a multiple point of C_{-u-v} and $C_{-u-v} . C_u \geqq 4$, which is impossible. Also

$$\dim_k H^0(\widetilde{S}, \mathcal{O}(\pi^*(3K - K_v))) = \dim_k H^0(S, \mathcal{O}(3K - K_v)) = 3;$$

since $\widetilde{C}$ has genus 2 and $\pi^*(3K - K_v) . \widetilde{C} = 4$ we conclude that the restriction map

$$r_C : H^0(\widetilde{S}, \mathcal{O}(\pi^*(3K - K_v))) \rightarrow H^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(\pi^*(3K - K_v)_{\widetilde{C}}))$$

is an isomorphism. Hence $\Phi_{\pi^*(3K-K_v)\widetilde{C}}$ is not birational, and we conclude that

$$\pi^*(3K - K_v)_{\widetilde{C}} = 2h_{\widetilde{C}},$$

where $h_{\tilde{C}}$ is the hyperelliptic (canonical) bundle of $\tilde{C}$. Hence if v, w are general points of E we conclude that

$$\pi^*(K_v - K_w)_{\tilde{C}}$$

is a trivial bundle. Now we obtain the exact sequence of sheaves on $\tilde{S}$:

$$0 \to \mathcal{O}(\pi^*(K_v - K_w) - [\tilde{C}]) \to \mathcal{O}(\pi^*(K_v - K_w)) \to \mathcal{O}_{\tilde{C}} \to 0.$$

The cohomology sequence gives

$$\dim_k H^1(\tilde{S}, \mathcal{O}(\pi^*(K_v - K_w) - [\tilde{C}])) \geqq 1$$

and since
$$\pi^*(K_v - K_w) - [\tilde{C}] = -\pi^* K_{u+w-v} + 2[L]$$

we get
$$\dim_k H^1(\tilde{S}, \mathcal{O}(-\pi^* K_t + 2[L])) \geqq 1$$

if t is a general point of E. By duality, this implies

$$\dim_k H^1(\tilde{S}, \mathcal{O}(\pi^*(K + K_t) - [L])) \geqq 1,$$

therefore we conclude that p is a base point of the linear system $|K + K_t|$ on S.

This is clearly impossible, because for every z we have

$$C_z + C_{t-z} \in |K + K_t|$$

and this would imply that every curve C_z would pass through p. Since p was the double point of a general curve C_u, all curves C_z would have a double point at p, contradicting $C_z^2 = 2$. Q.E.D.

10. Irregular surfaces of general type.

In this section we study irregular surfaces of general type and prove that their genus is small.

We begin with an upper bound for the geometric genus

$$p_g = \dim_k H^2(S, \mathcal{O}),$$

first obtained by Noether (see for example [5]).

Theorem 9. — Let S *be a minimal surface of general type. Then we have:*

$$p_g \leqq \frac{1}{2} K^2 + 2, \quad K^2 \textit{ even}$$

$$p_g \leqq \frac{1}{2} K^2 + \frac{3}{2}, \quad K^2 \textit{ odd.}$$

Proof. — Let

$$|K| = |[C]| + X,$$

where X is a fixed part, possibly $X = 0$. We consider two cases depending on whether or not $|[C]|$ is composed of a pencil.

Case 1. — $|[C]|$ is composed of a pencil, possibly with base points.

Now we have $C \sim a[F]$ for some integer $a \geqq 1$ and some line bundle $[F]$, F irreducible.

We have $p_g \leqq a+1$ and if equality holds then

$$\dim_k H^0(S, \mathcal{O}([F])) = 2.$$

We have $KX \geqq 0$ therefore $K^2 \geqq aKF$ and since $F^2 \geqq 0$ we have $KF \geqq 2$, because $K^2 \geqq 2$. Hence $K^2 \geqq 2a$ and

$$p_g \leqq \frac{1}{2}K^2 + 1.$$

Case 2. — $|[C]|$ is not composed of a pencil.

We follow here an argument of Deligne.

The general element C of $|[C]|$ is reduced and irreducible. We have the exact sequence

$$0 \to \mathcal{O}(K) \to \mathcal{O}(K+[C]) \xrightarrow{r_C} \omega_C \to 0$$

where ω_C is the dualizing sheaf of C. Since C is connected and $C^2 > 0$, we have by Theorem A, Corollary

$$\dim_k H^1(S, \mathcal{O}(K+[C])) = 0,$$

therefore the cohomology sequence implies

$$\dim_k H^0(C, \omega_C) = \dim_k r_C H^0(S, \mathcal{O}(K+[C])) + q.$$

By the Riemann-Roch theorem, we have

$$\dim_k H^0(C, \omega_C) = \frac{1}{2}(C^2 + KC) + 1.$$

Also it is clear that

$$\dim_k r_C H^0(S, \mathcal{O}(K+[C])) \geqq 2\dim_k r_C H^0(S, \mathcal{O}([C])) - 1 = 2\dim_k H^0(S, \mathcal{O}([C])) - 3 = 2p_g - 3,$$

the last equality because $|[C]|$ is the non-fixed part of $|K|$. Hence

$$2p_g - 3 + q \leqq \frac{1}{2}(C^2 + KC) + 1.$$

We have $C^2 + KC = 2K^2 - 2KX - CX$ and since $KX \geqq 0$ and $CX \geqq 0$ we obtain

$$C^2 + KC \leqq 2K^2$$

and finally $p_g \leqq \frac{1}{2}K^2 + 2 - \frac{1}{2}q$. Q.E.D.

Remark. — We have in fact proved that if $|K|$ is not composed of a pencil plus a fixed part then if $|K| = |[C]| + X$ we have

$$p_g \leqq \frac{1}{2}K^2 + 2 - \frac{1}{2}q - \frac{1}{2}KX - \frac{1}{4}CX;$$

note that if $X > 0$ we have $CX \geqq 2$ by Lemma 1.

Lemma 13. — If $q \geqq 2$ *and if* $|K|$ *is composed of a pencil plus a fixed part we have*

$$p_g \leqq \frac{1}{2} K^2.$$

Proof. — A simple analysis shows that in Case 1 of the proof of Theorem 9 we have

$$p_g \leqq \frac{1}{2} K^2$$

unless $|K| = |a[F]| + X$, $a = p_g - 1$, $KF = 2$, $F^2 = 0$, $FX = 2$ and $\dim_k H^0(S, \mathcal{O}([F])) = 2$, in which case $p_g = \frac{1}{2}K^2 + 1$, K^2 even, $p_g = \frac{1}{2}K^2 + \frac{1}{2}$, K^2 odd. Since $|[F]|$ is a *rational* pencil, we have

$$\dim_k H^1(S, \mathcal{O}(-[F])) = 0$$

by Theorem A, therefore the natural map

$$H^1(S, \mathcal{O}) \to H^1(S, \mathcal{O}_F)$$

is injective. Since the general fiber F has genus 2 we have that

$$q = \dim_k H^1(S, \mathcal{O}) = 2.$$

Now for *every* fiber F the homomorphism

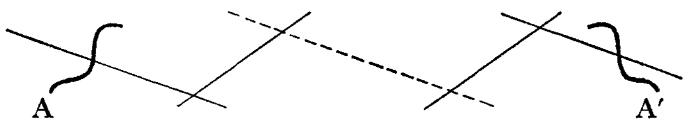
$$\mathrm{Pic}^0(S) \to \mathrm{Pic}(F)$$

has finite kernel, therefore

either F is irreducible and non-singular of genus 2;

or F has two distinct components A, A′ which are *elliptic* curves.

It follows that the special fibers of the pencil F are of the type



where A, A′ are non-singular elliptic

$$A^2 = A'^2 = -1,$$

and where the other components are non-singular rational curves $E \in \mathcal{E}$, with

$$E^2 = -2;$$

all components have multiplicity 1 in the fiber (see Ogg [13], who has classified the possible special fibers in a pencil of curves of genus 2).

Let us denote by $F^{(\nu)}$ a special fiber of the pencil with ν rational components and let m_ν be the number of such fibers. Computing the Euler-Poincaré characteristics (coefficients in $\mathbf{Q}$ or $\mathbf{Q}_l$) we obtain (see for instance [15], Chapter IV, Theorem 6)

$$\chi(S) = c_2[S] = \chi(\mathbf{P}^1)\,\chi(F) + \sum_{z \in \mathbf{P}^1} [\chi(F_z) - \chi(F)]$$

$$= -4 + \sum_{\nu=0}^{\infty} m_\nu(\nu + 1)$$

because (see [15], Chapter IV, Lemma 4 and its proof):

$$\chi(F^{(\nu)}) = \nu - 1.$$

We have

$$\begin{aligned} \frac{1}{12}(c_1^2 + c_2)[S] &= \chi(\mathcal{O}) \\ &= 1 - q + p_g \\ &= -1 + \frac{1}{2}K^2 + \begin{cases} 1 & K^2 \text{ even} \\ \frac{1}{2} & K^2 \text{ odd} \end{cases} \end{aligned}$$

therefore

$$c_2[S] = 5K^2 + \begin{cases} 0 & K^2 \text{ even} \\ -6 & K^2 \text{ odd} \end{cases}$$

and thus

$$\sum_{\nu=0}^{\infty} m_\nu(\nu + 1) = 5K^2 + \begin{cases} 4 & K^2 \text{ even} \\ -2 & K^2 \text{ odd.} \end{cases}$$

Let $F^{(\nu)}$ be a special fiber and write

$$F^{(\nu)} = A + E_1 + \ldots + E_\nu + A',$$

where A, A' are the elliptic components and $E_i \in \mathcal{E}$.

The curves A, $E_1, \ldots, E_\nu$ are linearly independent in $\mathrm{Num}^1(S)$, for if

$$nA + \sum_i n_i E_i \sim 0,$$

intersecting with A and $E_1, \ldots, E_\nu$ we get $n = n_1 = \ldots = n_\nu = 0$. Clearly if we take the union of these curves for all special fibers F, they are still linearly independent in $\mathrm{Num}^1(S)$. It follows that

$$\mathrm{rank}_{\mathbf{Q}} \mathrm{Num}^1(S) \geqq \sum_{\nu=0}^{\infty} m_\nu(\nu + 1).$$

On the other hand, since $\mathrm{char}(k) = 0$, we have

$$\mathrm{rank}_{\mathbf{Q}} \mathrm{Num}^1(S) \leqq \dim_k H^1(S, \Omega^1),$$

and

$$\begin{aligned} \dim_k H^1(S, \Omega^1) &= c_2[S] + 2q - 2\chi(\mathcal{O}_S) \\ &= 4K^2 + \begin{cases} 4 & K^2 \text{ even} \\ -1 & K^2 \text{ odd.} \end{cases} \end{aligned}$$

Hence

$$\begin{aligned} 4K^2 + \begin{cases} 4 & K^2 \text{ even} \\ -1 & K^2 \text{ odd} \end{cases} &= \dim_k H^1(S, \Omega^1) \geqq \mathrm{rank}_{\mathbf{Q}} \mathrm{Num}^1(S) \\ &\geqq \sum_{\nu=0}^{\infty} m_\nu(\nu + 1) \\ &= 5K^2 + \begin{cases} 4 & K^2 \text{ even} \\ -2 & K^2 \text{ odd.} \end{cases} \end{aligned}$$

It follows that $K^2 \leqq 1$, a contradiction. Q.E.D.

Lemma 14. — *Let* S *be a minimal surface of general type with irregularity*

$$q = \dim_k H^1(S, \mathcal{O}) \geqq 1.$$

Then we have $\chi(\mathcal{O}) \leqq \frac{1}{2}K^2$.

In particular, if $q = 1$ *we have* $p_g \leqq \frac{1}{2}K^2$.

Proof. — Since S is irregular, for every integer $m \geqq 2$ there is a cyclic unramified covering

$$\pi : \hat{S} \to S$$

with group $\mathbf{Z}_m$ acting *freely* on $\hat{S}$. We have

$$K_{\hat{S}} = \pi^* K_S, \quad K_{\hat{S}}^2 = mK_S^2, \quad \chi(\mathcal{O}_{\hat{S}}) = m\chi(\mathcal{O}_S)$$

and $\hat{S}$ is again a minimal surface of general type, with irregularity

$$q(\hat{S}) \geqq q(S).$$

We have $p_g(\hat{S}) = m\chi(\mathcal{O}_S) + q(\hat{S}) - 1$ therefore by Theorem 9

$$m\chi(\mathcal{O}_S) + q(\hat{S}) - 1 \leqq \frac{1}{2}mK_S^2 + 2;$$

the result follows by letting $m \to \infty$. Q.E.D.

Theorem 10. — *If* $p_g = \frac{1}{2}K^2 + 2$ *or if* $p_g = \frac{1}{2}K^2 + \frac{3}{2}$, *then* $q = 0$.

Proof. — This follows from Lemmas 13 and 14 and the Remark to Theorem 9.

In what follows, we use repeatedly the fact that $\chi(\mathcal{O}) \geqq 1$, which implies

$$q \leqq p_g.$$

Theorem 11 (Kodaira). — *If* $K^2 = 1$, *then* $q = 0$.

Proof. — By Lemma 14, if $q > 0$, we would have

$$1 \leqq \chi(\mathcal{O}) \leqq \frac{1}{2}K^2 = \frac{1}{2},$$

a contradiction. Q.E.D.

Theorem 12. — *If* $K^2 = 2$ *and if* $p_g = 2$ *then* $q = 0$.

Proof. — Lemma 14 shows that if $q > 0$ then we must have

$$q(S) = 2,$$

hence

$$\chi(\mathcal{O}_S) = 1.$$

We shall assume $\chi(\mathcal{O}_S) = 1$ and derive eventually a contradiction. Let $\hat{S}$ be an unramified covering surface of S with covering group $\mathbf{Z}_4$. We have

$$K_{\hat{S}}^2 = 8, \quad \chi(\mathcal{O}_{\hat{S}}) = 4$$

hence $p_g(\hat{S}) = 3 + q(\hat{S})$; moreover $q(\hat{S}) \geqq 2$.

By Lemma 13 $K_{\hat{S}}$ is not composed of a pencil and by Theorem 9, Remark it cannot have a fixed part. Since the base point set of $K_{\hat{S}}$ is invariant by the group $\mathbf{Z}_4$ acting freely on $\hat{S}$, we have either no base points or 4 base points.

By Theorem 10, $q(\hat{S})=2$. The canonical image $X^{[1]}$ of $\hat{S}$ is a surface of degree 8 or 4 in $\mathbf{P}^4(k)$, not contained in any hyperplane of $\mathbf{P}^4(k)$. Hence there are at most 2 linearly independent hyperquadrics of $\mathbf{P}^4(k)$ containing $X^{[1]}$. This means that if $s_1,\ldots,s_5$ are linearly independent sections of $K_{\hat{S}}$ then there are at most 2 linearly independent relations between the 15 sections s_is_j, $1\leqq i\leqq j\leqq 5$, of $2K_{\hat{S}}$. It follows that

$$P_2\geqq 15-2=13.$$

On the other hand,

$$P_2=K_{\hat{S}}^2+\chi(\mathcal{O}_{\hat{S}})=12,$$

a contradiction. Q.E.D.

We end this section by proving that, if S is a surface of general type, then $\chi(\mathcal{O})\geqq 1$. Since

$$\chi(\mathcal{O})=\frac{1}{12}(c_1^2+c_2)[S]$$

and since $c_1^2[S]\geqq 1$ if S is minimal, the result will follow if we show that $c_2[S]\geqq 0$. This is well-known, but we give the following theorem for completeness.

Theorem 13. — Let S *be a surface with* $c_2[S]<0$. *Then* S *is birationally equivalent to a ruled surface.*

Proof (Castelnuovo). — Since $c_2[S]<0$, S has irregularity

$$q=\dim_k H^1(S,\mathcal{O})\geqq 1.$$

Let $\pi:\hat{S}\to S$ be an unramified abelian covering with covering group $\mathbf{Z}_m$; we have

$$c_2[\hat{S}]=mc_2[S]\leqq -m.$$

Since

$$\begin{aligned} c_2[\hat{S}]&=2-4q(\hat{S})+b_2(\hat{S})\\ &\geqq 2-4q(\hat{S})+1+2p_g(\hat{S}) \end{aligned}$$

we deduce that $p_g(\hat{S})\leqq 2q(\hat{S})-4$ if m is sufficiently large, say $m\geqq 5$. Now let V be the vector space of regular 1-forms of $\hat{S}$ and let W be the vector space of regular 2-forms. The exterior product now defines a homomorphism

$$\alpha:\Lambda^2V\to W$$

and since $\dim_k W=p_g(\hat{S})$ we see that $\ker(\alpha)$ has codimension $\leqq p_g(\hat{S})$. The simple 2-vectors in Λ^2V form a cone of dimension $2\dim_k V-3=2q(\hat{S})-3$, therefore if

$$p_g(\hat{S})\leqq 2q(\hat{S})-4$$

we see that the kernel of α contains a non-zero simple 2-vector. Now this means that there are two regular differentials ω, ω' such that

$$\omega, \omega' \text{ are linearly independent,}$$
$$\omega \wedge \omega' = 0,$$

therefore there is a non-constant rational function φ on S such that

$$\omega = \varphi\omega'.$$

Now consider the linear system $|[C]|$ of level curves $\varphi = \text{const.}$ of φ. Since the non-zero 1-form

$$\omega - \varphi(C)\,\omega'$$

vanishes identically on C, we deduce that

$$\dim_k \ker\{H^1(\hat{S}, \mathcal{O}) \to H^1(C, \mathcal{O}_C)\} \geqq 1$$

and thus, by Theorem A, $|[C]|$ is composed of an irrational pencil. Let

$$f : \hat{S} \to B$$

be the corresponding morphism onto a curve B of genus $p \geqq 1$. Taking Euler-Poincaré characteristics we get

$$\begin{aligned} c_2[\hat{S}] = \chi(\hat{S}) &= \chi(F)\,\chi(B) + \sum_{z\in B} [\chi(F_z) - \chi(F)] \\ &\geqq \chi(F)\,\chi(B) \end{aligned}$$

where F is a general fiber and $F_z = f^{-1}(z)$. Since B has genus $\geqq 1$ we have $\chi(B) \leqq 0$ and since $c_2[S] < 0$ we must have

$$\chi(F) > 0.$$

Hence F is a rational curve and $\hat{S}$ is birationally equivalent to a ruled surface.

Finally, since the group $\mathbf{Z}_m$ acts freely on $\hat{S}$ it is immediate that $S = \hat{S}/\mathbf{Z}_m$ has again an irrational pencil

$$f : S \to B/\mathbf{Z}_m$$

with rational fibers. Q.E.D.

Remark. — A closer analysis of this proof shows that a minimal surface S has $c_2[S] < 0$ if and only if S is a $\mathbf{P}^1$-bundle over a non-singular curve of genus $p \geqq 2$.

11. The torsion group.

Theorem 14. — Let S *be a regular minimal surface of general type, with torsion group of order m. Then we have*

$$p_g \leqq \frac{1}{2}K^2 + \frac{3}{m} - 1.$$

Proof (Deligne). — If S has torsion group of order m, there is an unramified covering surface $\hat{S}$ of S with

$$\pi : \hat{S} \rightarrow S$$

of order m. We have

$$K_{\hat{S}}^2 = mK_S^2, \qquad \chi(\mathcal{O}_{\hat{S}}) = m\chi(\mathcal{O}_S).$$

If $\hat{S}$ has irregularity $q(\hat{S}) \geqq 1$, Lemma 14 gives $\chi(\mathcal{O}_S) \leqq \frac{1}{2}K_{\hat{S}}^2$, therefore

$$1 + p_g(S) \leqq \frac{1}{2}K^2.$$

If instead $\hat{S}$ is a regular surface, we obtain

$$m(1 + p_g(S)) = 1 + p_g(\hat{S}) \leqq \frac{m}{2}K_S^2 + 3$$

by Theorem 9.

Q.E.D.

Corollary. — If $K^2 = 1$, $p_g = 1$ *the torsion group is either* (0) *or* $\mathbf{Z}_2$. *If* $K^2 = 2$, $p_g = 1$ *and* $q = 0$ *the torsion group has order* $\leqq 3$. *If* $K^2 = 2$, $p_g = 2$ *the surface* S *has no torsion.*

Theorem 15. — If $K^2 = 1$, $p_g = 1$ *then* S *has no torsion. If* $K^2 = 2$, $p_g = 1$ *and* $q = 0$ *the torsion group of* S *is either* (0) *or* $\mathbf{Z}_2$.

Proof. — Assume $K^2 = 1$, $p_g = 1$. By Theorem 11, we have $q = 0$. Now suppose the torsion group is $\mathbf{Z}_2$. There is a line bundle [C] with

$$[C] \sim K, \qquad C \neq K, \qquad 2[C] = 2K$$

and the Riemann-Roch theorem shows easily that

$$\dim |[C]| = 1.$$

Since $P_2 = 3$ and $2[C] = 2K$, we deduce that 2K is composed of the linear system $|[C]|$. On the other hand, since $p_g = 1$, there is a canonical curve Γ, therefore

$$2\Gamma = C_1 + C_2$$

where $C_i \in |[C]|$. Since $K\Gamma = 1$, we must have

$$\Gamma = \Delta + Z$$

where Δ is irreducible with $K\Delta = 1$ and where $KZ = 0$, therefore $C_i = \Delta + Z_i$, where the Z_i are effective and $KZ_i = 0$. Since $C_1 \sim C_2 \sim \Gamma$, we get also

$$Z_1 \sim Z_2 \sim Z$$

therefore

$$Z_1 = Z_2 = Z$$

by Proposition 1. This gives $C_1 = C_2 = \Gamma$ and $[C] = K$, a contradiction, and the result follows from Theorem 14, Corollary.

Next assume $K^2 = 2$, $p_g = 1$, $q = 0$ and let [C] be a line bundle on S with

$$[C] \sim K, \qquad [C] \neq K.$$

We have

$$\dim_k H^0(S, \mathcal{O}([C])) = 2, \qquad \dim_k H^1(S, \mathcal{O}([C])) = 0.$$

Assume $2[C] \neq 2K$. Since $P_2 = 4$ and

$$\dim_k H^0(S, \mathcal{O}(2K - [C])) = 2,$$

we see that if $s_1, s_2, \sigma_1, \sigma_2$ are linearly independent sections of $[C]$ and $2K - [C]$ then $s_i \sigma_j$, $1 \leqq i, j \leqq 2$, are linearly independent sections of $2K$ and a basis of $H^0(S, \mathcal{O}(2K))$. By Theorem 2, Remark 2, $|2K|$ has no fixed part in $\mathscr{E}$, therefore $|[C]|$ has no fixed part, and by the argument of Lemma 12 the general element of $|[C]|$ is irreducible and non-singular, of genus 3.

The cohomology sequence of the exact sequence

$$0 \to \mathcal{O}(2K - 2[C]) \to \mathcal{O}(2K - [C]) \to \mathcal{O}_C((2K - [C])_C) \to 0$$

now shows that the restriction map

$$r_C : H^0(S, \mathcal{O}(2K - [C])) \to H^0(C, \mathcal{O}_C((2K - [C])_C))$$

is an isomorphism. It follows that C is hyperelliptic and

$$(2K - [C])_C = h_C,$$

where h_C is the hyperelliptic bundle of C. Since $2h_C$ is the canonical bundle of C, we deduce that

$$2(2K - [C])_C = (K + [C])_C$$

whence $3K_C = 3[C]_C$, and $3K_C = 3(2K - [C])_C = 3h_C$.

Since the restriction map

$$r_C : H^0(S, \mathcal{O}(K)) \to H^0(C, \mathcal{O}_C(K_C))$$

is an isomorphism, we have that

$$\dim_k H^0(C, \mathcal{O}_C(K_C)) = 1$$

and K_C has a non-trivial section s. Now since C is hyperelliptic of genus 3 we have that $|3h_C|$ is composed of the pencil $|h_C|$, therefore

$$s^3 = h_1 h_2 h_3$$

where $h_i \in H^0(C, \mathcal{O}_C(h_C))$. Since the sections h_i have only two zeros, the previous equation implies that

$$s \in H^0(C, \mathcal{O}_C(h_C))$$

and thus $K_C = k_C$. This contradicts the fact that $\dim_k H^0(C, \mathcal{O}_C(K_C)) = 1$.

Thus we have proved that if $[C]$ is a line bundle on S with $[C] \sim K$, $[C] \neq K$ we have:

$$2[C] = 2K,$$

therefore the torsion group of S has exponent 2. Since its order is $\leqq 3$ by Theorem 15, Corollary, it must be either (0) or $\mathbf{Z}_2$. Q.E.D.

12. Birational maps by projective methods.

Theorem 16. — Let S *be a minimal surface of general type, with*

$$K^2=3, \quad \chi(\mathcal{O})=1.$$

Then $X\to X^{[3]}$ *is a birational map.*

Proof. — We have $P_2=4$, $P_3=10$ and the 3-canonical model $X^{[3]}$ is embedded in $\mathbf{P}^9(k)$ but not in a projective space of lower dimension. Since $|3K|$ has no base points, the degree of the surface $X^{[3]}$ must be a divisor of $(3K)^2=27$, therefore if $\Phi_{3K}: S\to X^{[3]}$ is not a birational map we must have

$$\deg X^{[3]}=9.$$

We shall prove that in this case $X^{[3]}$ is a projective plane $\mathbf{P}^2(k)$ embedded in $\mathbf{P}^9(k)$ by means of the sections of the sheaf $\mathcal{O}_{\mathbf{P}^2}(3)$.

$X^{[3]}$ cannot contain a 1-dimensional algebraic system of lines. Otherwise, if Λ were such a line, $L=\Phi_{3K}^{-1}(\Lambda)$ would be a divisor on S with $3KL=3$, hence $KL=1$. Since $K^2=3$, the Algebraic Index Theorem gives $L^2\leqq -1$ and there would be only a finite number of divisors L.

Now we consider general projections

$$X^3=V_9\to V_8\to\ldots\to V_3$$

as follows. V_h is a surface of degree h in $\mathbf{P}^h(k)$ without an algebraic system of lines, and V_{h-1} is a general projection of V_h into $\mathbf{P}^{h-1}(k)$, from a general simple point of V_h. We end up with a cubic surface $V_3\subset\mathbf{P}^3(k)$ with 6 exceptional curves of the first kind on it, in general position. Hence V_3 is a non-singular cubic surface and V_9 is also non-singular. Clearly, since the mapping $V_h\to V_{h-1}$ blows up the point of projection but does not contract anything, we have

$$b_2(V_3)=b_2(V_9)+6,$$

where b_2 is the second Betti number of the surface. However, $b_2(V_3)=7$, therefore

$$b_2(V_9)=1$$

and it follows easily from this that V_9 is a projective plane $\mathbf{P}^2(k)$, embedded in $\mathbf{P}^9(k)$ by means of $\mathcal{O}_{\mathbf{P}^2}(3)$.

The linear system of lines on $\mathbf{P}^2(k)$ gives a linear system of skew cubics on V_9, of dimension 2 and degree 1. Hence we get on S, by lifting up this linear system, a linear system $|[L]|$ of curves with

$$KL=3, \quad L^2=3$$

and

$$\dim_k|[L]|\geqq 2.$$

Clearly $|[L]|$ is not composed of a pencil and if s_1, s_2, s_3 are linearly independent sections of $[L]$ we have that $s_i s_j$, $1 \leqq i \leqq j \leqq 3$, are linearly independent sections of $2[L]$. Hence

$$\dim_k H^0(S, \mathcal{O}(2[L])) \geqq 6.$$

On the other hand, $K^2=3$, $KL=3$, $L^2=3$ implies, using the Algebraic Index Theorem, that

$$[L] \sim K.$$

Now Theorem B and the Riemann-Roch theorem show that

$$\dim_k H^0(S, \mathcal{O}(2[L])) = P_2 = 4,$$

a contradiction. Q.E.D.

13. Conclusions, comments and problems.

In order to prove our Main Theorem, we note that: statements (i) and (ii) follow from Theorem 3; statement (iii) follows from Theorems 4, 6, 7, 8, 11, 12, 14 Corollary, 15 and 16 and the analysis of the cases $K^2=1$, $p_g=2$ and $K^2=2$, $p_g=3$ in Kodaira [8]; using Theorems 10 and 11, this can be obtained also from Enriques [5]. Finally statement (iv) follows from Theorem 5.

The question of whether the exceptions with $K^2=1$, $p_g=0$ and $K^2=2$, $p_g=0$ really occur is still open. We have proved that *if* $K^2=1$, $p_g=0$ *then* $X \to X^{[4]}$ *is a birational map*; the main difficulty here is to show that a surface S with $K^2=1$, $p_g=0$ does not contain a pencil of curves of genus 2 (¹).

We can also show that if $K^2=2$, $p_g=0$, then $X \to X^{[3]}$ is a birational map, except possibly if S contains a pencil of curves of genus 2 or 3 or if S can be represented as a double covering

$$S \to \mathbf{P}^2$$

with branching locus a curve of degree 10. Such surfaces have been constructed by Campedelli [4] (we note, however, that a similar construction proposed in [4] for surfaces with $K^2=1$, $p_g=0$ is not correct) but I have been unable so far to prove that $X \to X^{[3]}$ is a birational map for these surfaces. We note also that the analysis in Enriques [5] of surfaces with $K^2=1$ or 2, $p_g=0$ is oversimplified. Enriques' statement that $|3K|$ has no base points if $K^2=1$, $p_g=0$ is not correct, counterexamples being known (I can show that this is related to the torsion of the surface S), and his treatment of surfaces with $K^2=2$, $p_g=0$ holds only if the 2-canonical system $|2K|$ has no fixed parts.

It would be interesting to elucidate the structure of surfaces with $K^2=1$, $p_g=0$. Examples constructed by Godeaux (see [3] for a modern treatment) have $\mathbf{Z}_5$ as torsion group. I can show that the order of the torsion group is always $\leqq 5$, and the question is whether there are surfaces with $K^2=1$, $p_g=0$ other than the Godeaux surfaces.

(¹) Our proof is too long to be inserted here; we hope to return to this argument in another paper.

Concerning surfaces with $K^2=2$, $p_g=0$, the only examples I know of are those of Godeaux with torsion group $\mathbf{Z}_8$ and a double plane of Campedelli [4]; it is not clear what the torsion group of Campedelli's surface is.

From (iii) of our Main Theorem, we see that there are finitely many algebraic families of surfaces S for which Φ_{2K} is not birational, but S does not contain a pencil of curves of genus 2; it would be of interest to have other examples of these surfaces, beyond those given here.

We have proved that $X^{[5]}$ is always a normal model of S, and Kodaira [8] has proved that $X^{[8]}$ is a projectively normal model of S. It is still an open question whether $X^{[5]}$ is a projectively normal model.

Other interesting problems about surfaces of general type are the problem of the structure of the canonical map $X \to X^{[1]}$ and the problem of inequalities between K^2 and $\chi(\mathcal{O})$. From Theorem 9 we have

$$\chi(\mathcal{O}) \leqq \frac{1}{2}K^2+3$$

and this is best possible; it is conjectured that

$$K^2 \leqq 9\chi(\mathcal{O})$$

but this remains unsolved (see Van de Ven [16] for results in this direction).

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