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by TORSTEN EKEDAHL

Recall that if X is a smooth variety then the canonical bundle, ω_X, is the sheaf of highest order differentials; we will use K_X to denote the corresponding divisor class. Recall further that a smooth, proper and connected surface X is said to be of general type if the dimension of the space of sections of $\omega_X^n := \omega_X^{\otimes n}$ grows quadratically with n when n tends to infinity and that X is said to be minimal if it contains no smooth rational curves with self-intersection -1.

The proof of the following result is then the principal aim of the present article.

Main theorem. — Let X *be a minimal surface of general type over a field* $\mathbf{k}$, *algebraically closed of characteristic* $p > 0$. *Then*

(i) *For any* $i > 0$, $H^1(X, \omega_X^{-i}) = 0$ *except possibly when* $i = 1$, $p = 2$, $\chi(\mathcal{O}_X) = 1$ *and* X *is (birationally) an inseparable double cover of a* K3*-surface or a rational surface. In any case* $h^1(X, \omega_X^{-1}) \leqslant 1$.

(ii) *If* $m \geqslant 3$ *or* $m = 2$ *and* $K^2 \geqslant 2$, *then the linear system* $|(m+1)K_X|$ *is base-point free.*

(iii) *If* $m \geqslant 4$ *or* $m = 3$ *and* $K^2 \geqslant 2$, *then* $|(m+1)K_X|$ *is very ample considered as a linear system on the canonical model of* X.

In characteristic zero the corresponding result was proved by Bombieri ([Bo]). It should be noted that there are minimal surfaces X of general type with $H^1(X, \omega_X^{-1}) \neq 0$ as we will see.

The technique of proof is very similar to *(loc. cit).* and let us quickly review Bombieri's basic idea. To begin with (the analogue of) i) was proved by appealing to a vanishing result of Mumford, whereas the rest was reduced to a vanishing result of Ramanujan. This was done, for instance in ii) by noting that it is equivalent to the natural quotient map $\omega_X^{(m+1)} \to \omega_X^{(m+1)}/m_x\,\omega_X^{(m+1)}$ inducing a surjective map on global sections, where x is any closed point on X and m_x is the ideal of functions vanishing at x. Blowing up x this is equivalent to the natural map $\pi^*\,\omega_X^{(m+1)} \to \pi^*\,\omega_X^{(m+1)}{}_{|E}$ inducing a surjection on sections, where $\pi : \widetilde{X} \to X$ is the blowing up map and E the exceptional curve for π.

Using the long exact sequence of cohomology this is certainly implied by the vanishing of $H^1(\widetilde{X}, \pi^* \omega_X^{(m+1)}(-E))$ which by duality is equivalent to the vanishing of $H^1(\widetilde{X}, (\pi^* \omega_X^m(-2E))^{-1})$. After some work Bombieri shows that Ramanujan's vanishing criterion can be applied to $\pi^* \omega_X^m(-2E)$. The main difference between Bombieri's proof and the one to be given here is that there are no vanishing results for cohomology in positive characteristic of strength comparable to those that exist in characteristic zero. We will instead obtain various consequences of non-vanishing and combining these we will eventually reach a contradiction (to the assumption of non-vanishing). The starting point is the construction, given an example of non-vanishing, of a purely inseparable covering of degree p, which locally is obtained by taking the p-th root of a function. (Throughout this paper p will denote the characteristic of the base field which will assumed to be positive.) The above mentioned consequences are then obtained by investigating the somewhat paradoxical properties of this covering. By considering the canonical bundle one is usually able to show that the covering is ruled. Then, if the Albanese variety of X is non-trivial, the Albanese map has 1-dimensional image and much information is obtained by considering the fibers of this map. If the Albanese variety is trivial then we get by Noether's formula that $\chi(\mathcal{O}_X) \geqslant 1$ and this gives the needed additional strength to certain inequalities involving the self intersection and intersection number with the canonical bundle of the line bundle in question.

Besides the extra work involved in showing vanishing our strategy will differ from that of Bombieri in one other important respect. In order to construct the degree p covering we need to know—*a priori*—vanishing for a high power of the line bundle. This forces us to consider only numerically positive line bundles. The line bundles for which we need vanishing are the same as those used by Bombieri and it is by no means clear, and no doubt false in general, that they are numerically positive. Hence some preliminary work has to be done to show the existence of numerically positive line bundles such that if vanishing holds for them, then we get the theorem. The idea is that if one of the original line bundles is not numerically positive, then some curve must have negative intersection with it. The index theorem shows that this curve must be of a very special type and the existence of such a curve allows us to construct a new line bundle which can be tested for numerical positivity and so on. It should be noted, however, that if $m \geqslant 8$ then already the original line bundles are numerically positive and this extra argument is not needed. From the point of view of characteristic zero this part could be looked upon as a proof of Bombieri's result assuming only vanishing for numerically positive line bundles.

In his paper Bombieri proves corresponding results for smaller values of m. If one tries to apply the methods of the present paper to these cases it will be seen that the reduction to a numerically positive line bundle needs far stronger bounds on K_X^2 than are used by Bombieri. Hence, even though one gets the expected results for all but a bounded family of surfaces, some new idea seems to be needed to obtain the results known to be true by Bombieri's result, namely that—as we are dealing with a bounded family of exceptions—his results as stated will be true for all but a finite of number of characteristics.

As can be seen already from the theorem, characteristic two plays a special role. In the proofs this comes from the fact that 2 is the smallest prime number and the inequalities which involve p become less powerful the smaller p is. Indeed, we are saved from allowing even further exceptions by the fact that when $p = 2$ a cover of degree p is a double cover and that, with a slight twist, double covers are as simple in characteristic 2 as in any other characteristic. In the construction of examples showing that the exceptions really occur, 2 appears as the characteristic for which one can have more ordinary nodes than usual. For instance, we use that a K3-surface in characteristic 2 may have 21 disjoint -2-curves which is impossible in any other characteristic.

Finally, it is my hope that the techniques developed here will be useful in other situations as well. Therefore the results obtained along the way are often stronger than needed to simply prove the theorem.

The contents of this paper are as follows. First we recall some terminology and prove some preliminary results. In section I we will make a general study of inseparable maps of degree p between smooth and proper surfaces, giving for example formulas for how the canonical bundle changes under such a map. We will also introduce a special kind of such cover which is interesting because a failure of Kodaira vanishing will give rise to this type of cover. Then, some special features of characteristic 2 are considered. These results are then applied to obtain Theorem I:2.3 which is the main theorem of section I. It is a result giving very strong consequences from the assumption that there exist a line bundle contained in the tangent bundle which fulfills some positivity conditions. Already here we see that $p = 2$ appears as a special case as several of the exceptions occur only in characteristic 2. We finish the section by showing that the exceptions to Theorem I:2.3 do indeed occur.

In section II we apply the results of section I to obtain different consequences of the non-vanishing of the first cohomology group $H^1(X, \mathscr{L}^{-1})$ where X still is a smooth and proper surface and $\mathscr{L}$ is a numerically positive line bundle. The main general result is Theorem II:1.3 which gives a rather odd assortment of such consequences. It seems, unfortunately, necessary to obtain many different consequences of non-vanishing as usually one such consequence is not enough to obtain a contradiction (and hence a vanishing result). A rather direct corollary of this result is that Kodaira vanishing for a surface not of general type (for numerically positive line bundles) is true except essentially for those counterexamples already found by Raynaud. We are, however, mostly interested in surfaces of general type and in Theorem II:1.7 we prove i) of the Main theorem. As a corollary we obtain (Corollary II:1.8) some inequalities between the numerical invariants of a minimal surface of general type.

Section III is the section where the rest of the Main theorem is proved. A large part of the section is concerned with treating numerous special cases occuring when m is small and is the result of the author's urge to push his methods to their limit. The reader is referred to the introduction of that section where first the case $m \geqslant 8$ is discussed (a case which is relatively simple to follow) and then some indications as to the complications occurring when m becomes smaller are given.

At last I would like to thank Miguel Ibáñez for enabling me to brush up many of the notions to be found in this article.

CONTENTS

0. Notation and preliminaries

Let us begin by recalling the following standard definitions, where X is a proper (and smooth, where it is needed to make sense) $\mathbf{k}$-variety and $\mathbf{k}$ is an algebraically closed field.

$$\begin{aligned} b_i(\mathrm{X}) &:= \dim_{\mathbf{Q}_\ell} \mathrm{H}^i_{\acute{e}t}(\mathrm{X}, \mathbf{Q}_\ell), \\ h_{ij}(\mathrm{X}) &:= \dim_{\mathbf{k}} \mathrm{H}^j(\mathrm{X}, \Omega^i_{\mathrm{X}/\mathbf{k}}), \\ \chi(\mathrm{X}) &:= \Sigma(-1)^i h^{0i}(\mathrm{X}), \\ \chi_{\acute{e}t}(\mathrm{X}) &:= \Sigma(-1)^i b_i(\mathrm{X}), \\ \mathrm{P}_i(\mathrm{X}) &:= \dim_{\mathbf{k}} \mathrm{H}^0(\mathrm{X}, \omega^i_{\mathrm{X}/\mathbf{k}}), \\ p_g &:= \mathrm{P}_1. \end{aligned}$$

Let us also recall that the sheaf B_1, for X smooth and $\mathbf{k}$ of positive characteristic, is defined by the exactness of the following sequence

$$(1.1) \qquad 0 \to \mathcal{O}_{\mathrm{X}} \to \mathrm{F}_* \mathcal{O}_{\mathrm{X}^{(1)}} \to \mathrm{B}_1 \to 0,$$

where $\mathrm{F} : \mathrm{X}^{(1)} \to \mathrm{X}$ is the Frobenius map, $\mathbf{k}$-linearised by letting $\mathrm{X}^{(1)}$ be the pullback of X by the Frobenius map on $\mathbf{k}$. In particular, as $\mathbf{k}$ is algebraically closed, we get

$$\ker(\mathrm{F} : \mathrm{H}^1(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) \to \mathrm{H}^1(\mathrm{X}, \mathcal{O}_{\mathrm{X}})) = \mathrm{H}^0(\mathrm{X}, \mathrm{B}_1)$$

if X is proper.

By *general point*, *fiber*, etc., we will mean a closed point in some unspecified everywhere dense open subset of the base scheme. The use of these terms is usually meant to imply that the existence of this open subset follows from standard results. By *curve* we will mean an integral 1-dimensional projective scheme, whereas effective divisor will mean a purely 1-dimensional closed subscheme, without embedded points, of some considered surface. The latter notion will usually be confused with the corresponding formal linear combination, with multiplicities, of its component curves. The genus, $g(C)$, of a curve C is defined to be $\dim_{\mathbf{k}} H^1(C, \mathcal{O}_C)$ (and not as the genus of its normalisation). A rational (elliptic) curve is a curve whose normalisation has genus 0 (resp. 1)·

At several points we will need to use the results of [Mu1], [Bo-Mu1] and [Bo-Mu2]. We will use the catch-phrase " by the classification of surfaces " to refer to any of these articles.

Proposition **1.2.** — *Let* X *be a smooth and proper surface and* C *a* 2*-connected* (*cf.* [Bo:§3]) *effective divisor on* X. *If* C *is not a smooth rational curve, then* $|\omega_C|$ *is base-point free.*

Proof. — Let $x \in C$. It will suffice to show that $h^1(m_x \omega_C) = 1$, where m_x is the ideal of x. By duality this is equivalent to showing that $\dim_{\mathbf{k}} \mathrm{Hom}_{\mathcal{O}_C}(m_x \mathcal{O}_C, \mathcal{O}_C) = 1$. Suppose this dimension $\geqslant 2$. Let $\varphi : m_x \mathcal{O}_C \to \mathcal{O}_C$ be an $\mathcal{O}_C$-morphism and let $C_1 \leqslant C$ be maximal for the condition that $\varphi_{|C_1}$ vanishes and put $C_2 := C - C_1$. Hence we have an injective map with cokernel of finite support; $\varphi : m_x \mathcal{O}_{C_2} \to \mathcal{O}_{C_2}(-C_1)$. Taking degrees we get $-1 \leqslant -(C_1, C_2)$ and 2-connectedness gives C_1 or C_2 equal to zero. Suppose now all such φ vanishes at x. This means that the finite $\mathbf{k}$-algebra $\mathrm{End}_{\mathcal{O}_C}(m_x \mathcal{O}_{C_2}, m_x \mathcal{O}_C)$ has dimension at least 2. As any non-zero φ is injective with cokernel of finite support it is an isomorphism for degree reasons and so any non-zero φ is invertible, but $\mathbf{k}$, being algebraically closed, has no non-trivial finite division algebras. Hence there is a φ which does not vanish at x and so, by Nakayama's lemma, is surjective at x. This means that $m_x \mathcal{O}_C$ is invertible at x and thus that C is smooth at x and that $|(m_x \mathcal{O}_C)^{-1}|$ is a degree 1 linear system of dimension at least 1 without basepoints. If there is a component D of C on which x does not lie then $|(m_x \mathcal{O}_C)^{-1}|$ is trivial on D and so has sections vanishing on D contrary to assumption. Hence C is irreducible and, having a smooth point, also integral and $|(m_x \mathcal{O}_C)^{-1}|$ gives an isomorphism with $\mathbf{P}^1$. □

We will give a proof of the following well known result mainly for lack of appropriate references. (Note that here $\mathcal{L}^2$ for instance means the self-intersection number of $\mathcal{L}$ and not $\mathcal{L} \otimes \mathcal{L}$; to avoid confusion we will throughout use $\mathcal{L}^{\otimes 2}$ for $\mathcal{L} \otimes \mathcal{L}$ whenever there is risk of such.)

Proposition **1.3.** — *Let* $\mathcal{L}$ *be a numerically positive line bundle on a smooth, proper and connected surface, let* D *be the divisor of base curves of* $|\mathcal{L}| \neq \emptyset$.

(i) *If* $\mathcal{L}(-D)^2 = 0$, *then*

$$h^0(\mathcal{L}) \leqslant \frac{\mathcal{L}^2}{b} + 1,$$

where

$$b := \min\{(\mathcal{L}, C) : |kC| \text{ is basepoint free for some } k > 0 \text{ and } C^2 = 0\}.$$

(ii) *If the rational map* $X \dashrightarrow |H^0(X, \mathscr{L})|$ *has an image of dimension* $\leqslant 1$ *and* $\mathscr{L}(-D)^2 \neq 0$, *then*

$$h^0(\mathscr{L}) \leqslant \sqrt{\frac{\mathscr{L}^2}{c}} + 1,$$

where

$$c := \min\{ C^2 : |kC| \text{ has only isolated basepoints, } X \dashrightarrow |kC| \text{ has an image of dimension} \leqslant 1 \text{ for some } k > 0 \text{ and } C^2 > 0 \}.$$

(iii) *If* $f : X \dashrightarrow |H^0(X, \mathscr{L})|$ *has a 2-dimensional image then*

$$h^0(\mathscr{L}) \leqslant \frac{\mathscr{L}^2}{2m} + 2 \quad \textit{if } \operatorname{Im} f \textit{ is not ruled,}$$

$$h^0(\mathscr{L}) \leqslant \frac{\mathscr{L}^2}{m} + 2 \quad \textit{in general.}$$

where $m := \deg f$. *If equality holds, then* $|\mathscr{L}|$ *has no base points.*

Proof. — If $h^0(\mathscr{L}) = 1$ there is nothing to prove. Assume $X \dashrightarrow |H^0(X, \mathscr{L})|$ has 1-dimensional image. Then $|H^0(X, \mathscr{L})| = |aC| + D$ and $h^0(\mathscr{L}) \leqslant a + 1$. Thus $\mathscr{L}^2 = a(\mathscr{L}, C) + (\mathscr{L}, D) \geqslant a(\mathscr{L}, C)$ and $(\mathscr{L}, C) = aC^2 + (C, D) \geqslant aC^2$. If $C^2 = 0$ then $(\mathscr{L}, C) \geqslant b$ and we get i). If $C^2 \neq 0$ then $C^2 \geqslant c$ and we get ii).

If $X \dashrightarrow |H^0(X, \mathscr{L})|$ has 2-dimensional image, let Y be (the closure of) the image and let $d := \deg Y \leqslant \mathscr{L}^2/m$. In general, as Y is not contained in a hyperplane, $d \geqslant \dim |H^0(X, \mathscr{L})| - 1 = h^0(\mathscr{L}) - 2$. If T is not ruled, let $Y' \to Y$ be a resolution and H the inverse image on Y' of a general hyperplane. Thus H is a curve on Y' and the exact sequence

$$0 \to \mathcal{O}_{Y'} \to \mathcal{O}_{Y'}(H) \to \mathcal{O}_H(H) \to 0$$

gives $h^0(\mathscr{L}) \leqslant h^0(\mathcal{O}_H(H)) + 1$. As Y' is not ruled $(\omega_{Y'}, H) \geqslant 0$ and Clifford's inequality, which is true not only for smooth curves, and the adjunction formula gives

$$h^0(\mathcal{O}_H(H)) \leqslant H^2/2 + 1 = d/2 + 1$$

since $\deg_H H \leqslant \deg \omega_H$ as $(\omega_{Y'}, H) \geqslant 0$. Finally, if $|\mathscr{L}|$ has basepoints, $m\,d < \mathscr{L}^2$. $\square$

Proposition **1.4.** ("Purity"). — *Let* X *be a smooth* k*-scheme and* $\mathscr{G}$ *a finite flat* X*-group scheme which is an extension of a finite étale* X*-group scheme by a finite flat infinitesimal* X*-group scheme.*

(i) *Let* $\rho : Y \to X$ *be a* $\mathscr{G}$*-torsor. If* ρ *has a section over an open dense subset of* X *then it has a section.*

(ii) *Let* $F \subset X$ *be a closed subset of codimension at least two. If* $\rho : V \to U := X \backslash F$ *is a* $\mathscr{G}_{|U}$*-torsor, then there is an extension of* ρ *to* X.

Proof. — Indeed, i) follows directly from Zariski's main theorem (and is thus true far more generally). As for ii), it is sufficient to find a finite flat $\pi : W \to X$ s.t. the pull-back of ρ to $W \backslash \pi^{-1} F$ is trivial and W is smooth. Indeed, if this is the case then ρ is described by descent data for the pullback of ρ to $W \backslash \pi^{-1} F$. All schemes involved in the descent data are finite and flat over $W \backslash \pi^{-1} F$ and admit finite, flat extensions to W. Hence everything is controlled by some morphisms over $W \backslash \pi^{-1} F$ between vector bundles over W. As $\pi^{-1} F$ has codimension at least two these morphisms extend to W and so give descent data for an extension of ρ. By finite flat descent of affine morphisms these data are effective. Let now $\mathcal{N}$ be the infinitesimal normal X-subgroup scheme of $\mathcal{G}$ which exists by assumption and consider the associated $(\mathcal{G}/\mathcal{N})_{|U}$-torsor. This being étale we can apply the usual purity theorem to extend it to a $Z \to X$. Pulling back V to Z, we can reduce the structure group to $\mathcal{N}_Z$ and hence we may assume that $\mathcal{G}$ is infinitesimal. This means that ρ is split by some power of the Frobenius map on U and as the Frobenius map for X is flat, X being smooth, we get the wanted $Z \to X$. □

Remark. — I do not know to which extent ii) can be generalised for instance to more general $\mathcal{G}$.

We will need a few remarks on the behaviour of some numerical invariants under resolution of surfaces. Let us begin by noting that the Leray spectral sequence shows that χ drops during resolution of singularities and that if it is unchanged, the singularities are rational. Slightly more involved is the following result.

Lemma **1.5.** — *Let* X *and* Y *be smooth, proper surfaces and* $\pi : Y \to X$ *a purely inseparable map. Then* $b_1(X) = b_1(Y)$, $\chi_{et}(X) \leqslant \chi_{et}(Y)$ *and if equality holds, then* π *is finite.*

Proof. — Indeed, on étale cohomology $\pi_* \pi^* = \deg \pi$ which shows that π^* is injective and so $b_i(X) \leqslant b_i(Y)$. On the other hand, as π is purely inseparable, π factors, birationally, through some power of the Frobenius map on X, so after blowing up X we get a surjective map $X \to Y$ which gives $b_1(X) \geqslant b_1(Y)$, as the first Betti number is invariant under blowing ups. This gives the first equality and inequality. Finally, π_* is zero on the cycle classes of curves contracted by π, so if we have equality, then there are no curves contracted by π and π is finite by Zariski's main theorem. □

Finally, for ease of reference we give the following well known rearrangement of the terms of Noether's formula,

$$(1.6) \qquad 10 + 12p_g = K^2 + b_2 + 8q + 12(h^{01} - q),$$

where the point, of course, is that all the terms are non-negative (except possibly K^2).

I. Foliations

In this section we will study maps between varieties of *exponent* 1 i.e. dominant maps $X \to Y$ for which $\mathbf{k}(X)^p \subseteq \mathbf{k}(Y)$ where $p > 0$ is the characteristic of $\mathbf{k}$. As a derivation vanishes on a p-th power such a map can be described by first order information on X and this is spelled out at the beginning (a more detailed discussion can be found in [Ek]). Then a particular class of such maps is considered. Starting from Y they are obtained locally by adjoining a p-th root of a function f which is not itself a p-th root. Changing f by an affine transformation $f \mapsto af + b$ leads to an isomorphic covering and globally X is obtained from local p-th root coverings related in this way by affine transformations of the functions of which one takes p-th roots. As global data one obtains a line bundle $\mathscr{L}$ from the multiplicative factor a and then the additive factors give rise to an $\alpha_{\mathscr{L}}$-torsor where $\alpha_{\mathscr{L}}$ is a certain group scheme of order p locally isomorphic to the group scheme α_p of p-th roots of zero. Finally, using the previous results on the relation between first order data (i.e. a subsheaf of the tangent bundle) and maps of exponent 1, a formula for the change of the canonical bundle in an exponent 1 covering of degree p between surfaces and the classification of surfaces many consequences are obtained from the assumption of a " sufficiently positive " line bundle contained in the tangent bundle of a smooth and proper surface. It should be compared with the far stronger results obtained by Miyaoka [Mi] in the case of characteristic 0. Some examples are given to show that the obtained results approach the optimal.

1. Let X be a variety over an algebraically closed field $\mathbf{k}$ of characteristic $p > 0$. We will denote by $X^{(n)}$, $n \in \mathbf{Z}$, the base change of X by the p^n-th power map $\mathbf{k} \to \mathbf{k}$. Hence the n-th iterated Frobenius morphism is a $\mathbf{k}$-morphism $F^n : X^{(n)} \to X$, n positive, or $F^n : X \to X^{(n)}$, n negative. *Unless otherwise mentioned* X *will, from now on, be smooth.* A first order integrable distribution, smooth 1-foliation for short, is a subbundle $\mathscr{F}$ of $T^1_{X/k}$ which is a sub-p-Lie algebra i.e. closed under Lie brackets and p-th powers. We let $Y = X/\mathscr{F}$ denote the scheme with the same underlying space as X and whose structure sheaf consists of those elements of $\mathcal{O}_X$ killed by the derivations of $\mathscr{F}$. By definition there is a $\mathbf{k}$-morphism $f : X \to Y$ and $F : X \to X^{(-1)}$ factors as

$$X \xrightarrow{f} Y \xrightarrow{g} X^{(-1)}$$

for some $g : Y \to X^{(-1)}$.

Proposition **1.1.** — i) *f and g are finite flat morphisms and* Y *is smooth.*

(ii) *Locally on* X *there are coordinates $t_1, t_2, \ldots, t_n$ such that*

$$\mathscr{F} = \sum_{i=1}^{n} \mathcal{O}_X \frac{\partial}{\partial t_i}.$$

(iii) *There is an exact sequence*

$$(1.2) \qquad 0 \to \mathscr{F} \to T^1_{X/k} \to f^* T^1_{Y/k} \to F^*(\sigma^* \mathscr{F}) \to 0,$$

where $\sigma : X^{(1)} \to X$ *is the base change morphism, and in particular*

$$(1.3) \qquad f^* \omega_Y = \omega_X \otimes (\det \mathscr{F})^{1-p}.$$

Proof. — This is proved in [Ek:§3]. □

If U is an open dense subset of X and $\mathscr{F} \subset T^1_{U/k}$ is a smooth 1-foliation, then there is a unique $\mathcal{O}_X$-submodule $\bar{\mathscr{F}} \subset T^1_{X/k}$ s.t. $\bar{\mathscr{F}}_{|U} = \mathscr{F}$ and $T^1_{X/k}/\bar{\mathscr{F}}$ is torsion free. Furthermore, $\bar{\mathscr{F}}$ is closed under Lie brackets and p-th powers. Such an $\bar{\mathscr{F}}$ will be called a 1-foliation. It is always smooth outside a closed subset of codimension at least 2. We will let $X/\bar{\mathscr{F}}$ denote the normal variety whose underlying space is that of X and such that the sections of $\mathcal{O}_X/\bar{\mathscr{F}}$ are those of $\mathcal{O}_X$ killed by the derivations of $\bar{\mathscr{F}}$. We will freely use the fact that a 1-foliation is determined by its restriction to a dense open subset and that giving a 1-foliation on X is equivalent to giving a finite map $Y \to X$ of exponent 1 i.e. $\mathbf{k}(Y)^p \subset \mathbf{k}(X)$.

Lemma **1.4.** — *Let $\mathscr{F} \subset T^1_{X/k}$ be a submodule.*

(i) *The Lie bracket induces an $\mathcal{O}_X$-linear map*

$$\bigwedge^2 \mathscr{F} \to T^1_{X/k}.$$

(ii) *If $\mathscr{F}$ is closed under Lie brackets then the p-th power map induces an $\mathcal{O}_X$-linear map*

$$F^*(\sigma^{-1*}\mathscr{F}) \to T^1_{X/k}/\mathscr{F}.$$

(iii) *In particular, if $T^1_{X/k}/\mathscr{F}$ is torsion free and*

$$\mathrm{Hom}_{\mathcal{O}_X}(\bigwedge^2 \mathscr{F}, T^1_{X/k}/\mathscr{F}) = \mathrm{Hom}_{\mathcal{O}_X}(F^*(\sigma^{-1*}\mathscr{F}), T^1_{X/k}/\mathscr{F}) = 0,$$

then $\mathscr{F}$ is a 1*-foliation.*

Proof. — See [Ek:4.2]. □

A particular kind of 1-foliation is obtained as follows. Let $\mathscr{L}$ be a line bundle and consider it and $\mathscr{L}^p$ as smooth X-group schemes. The relative Frobenius morphism is a surjective group scheme morphism $\mathscr{L} \to \mathscr{L}^p$. Let $\alpha_{\mathscr{L}}$ be its kernel so that we have an exact sequence (in the flat topology) of group schemes

(1.5) $$0 \to \alpha_{\mathscr{L}} \to \mathscr{L} \to \mathscr{L}^p \to 0.$$

Then $\alpha_{\mathscr{L}}$ is a finite flat group scheme, in fact

$$\alpha_{\mathscr{L}} = \mathbf{Spec}(\bigoplus_{i=0}^{p-1} \mathscr{L}^{-i})$$

where the multiplication is the obvious one.

Let $\beta : Y \to X$ be a non-trivial $\alpha_{\mathscr{L}}$-torsor. Taking cohomology of the exact sequence (1.5) shows that, locally in the Zariski topology, β is obtained by applying the boundary map to a section $s \in \mathscr{L}^p$. Thus

$$Y = \mathbf{Spec}(\bigoplus_{i=0}^{p-1} \mathscr{L}^{-i})$$

where multiplication $\mathscr{L}^{-i} \otimes \mathscr{L}^{-j} \to \mathscr{L}^{-i-j}$ equals the identification map when $i + j < p$ and the composite

$$\mathscr{L}^{-i} \otimes \mathscr{L}^{-j} = \mathscr{L}^{-i-j} \xrightarrow{id \otimes s} \mathscr{L}^{-i-j+p}$$

when $i + j \geqslant p$. Even more locally (still in the Zariski topology) $\mathrm{Y} = \mathbf{Spec}\, \mathcal{O}_{\mathrm{X}}[t]/(t^p - f)$ for some $f \in \mathcal{O}_{\mathrm{X}}$. If $t \in \mathscr{L}$ is a local generator and s is as above we obtain an element $d(s/t^p) \in \Omega^1_{\mathrm{X}}$. This glues together to a globally defined $\mathcal{O}_{\mathrm{X}}$-morphism $\mathscr{L}^p \to \Omega^1_{\mathrm{X}}$. This morphism is non-zero. In fact, if it were not, then over some dense open subset $\mathrm{U} \subset \mathrm{X}$ where $\mathscr{L}_{|\mathrm{U}} \cong \mathcal{O}_{|\mathrm{U}}$ and $\mathrm{H}^1(\mathrm{U}, \mathcal{O}_{\mathrm{U}}) = 0$ we would get $\mathrm{Y} \mid \mathrm{U} = \mathbf{Spec}\, \mathcal{O}_{\mathrm{U}}[t]/(t^p - f)$ for some $f \in \mathcal{O}_{\mathrm{U}}$. Then $\mathscr{L}^p_{|\mathrm{U}} \to \Omega^1_{\mathrm{U}}$ would be given by $1 \mapsto df$ and thus f would be a p-th power. Therefore β would have a section over U and by purity (0:1.4) so would β itself.

Hence the annihilator $\mathscr{M} \subset \mathrm{T}^1_{\mathrm{X}}$ of the image of $\mathscr{L}^p$ is a *saturated* (i.e. the quotient sheaf is torsion-free) subsheaf and, as we have just seen, it is locally the annihilator of an exact 1-form thence a 1-foliation. It is also clear that the morphism $\mathrm{X} \to \mathrm{X}/\mathscr{M}$ is birationally equivalent to $\mathrm{X} \to \mathrm{Y}^{(-1)}$ (they differ if Y is not normal).

The morphism $\beta : \mathrm{Y} \to \mathrm{X}$ can be described more concretely as follows. To give an $\alpha_{\mathscr{L}}$-torsor is, using (1.5), the same thing as giving the associated $\mathscr{L}$-torsor and a trivialisation of the associated $\mathscr{L}^p$-torsor. In other words, an extension

(1.6) $$0 \to \mathcal{O}_{\mathrm{X}} \to \mathscr{E} \to \mathscr{L}^{-1} \to 0$$

and a splitting $\rho : \mathscr{L}^{-p} = \mathbf{F}^* \mathscr{L}^{-1} \to \mathbf{F}^* \mathscr{E}$ of $\mathbf{F}^* \mathscr{E} \to \mathbf{F}^* \mathscr{L}^{-1}$, where $\mathbf{F} : \mathrm{X} \to \mathrm{X}$ is the *absolute* Frobenius morphism. From this we can construct the affine algebra of β: In the symmetric algebra $\mathrm{S}^{\cdot}(\mathscr{E})$ consider the ideal $\mathscr{I}$ generated by " 1 " $-$ 1, where " 1 " is the identity element of $\mathrm{S}^1(\mathscr{E}) = \mathscr{E}$ which contains $\mathcal{O}_{\mathrm{X}}$ and 1 is the identity element of $\mathrm{S}^0(\mathscr{E})$. Put $\mathscr{R} := \mathrm{S}^{\cdot}(\mathscr{E})/\mathscr{I}$. It is the affine algebra of the $\mathscr{L}$-torsor associated to (1.6) and has a filtration $0 \subset \mathscr{R}^0 \subset \mathscr{R}^1 \subset \mathscr{R}^2 \subset \ldots$, where $\mathscr{R}^i := \mathrm{Im}\, \mathrm{S}^i(\mathscr{E})$, by sub-$\mathcal{O}_{\mathrm{X}}$-modules s.t. $\mathscr{R}^i \mathscr{R}^j \subset \mathscr{R}^{i+j}$ and $\mathscr{R}^i/\mathscr{R}^{i-1} \cong \mathscr{L}^{-i}$. To get the affine algebra of β we now divide $\mathscr{R}$ by the ideal generated by the image of ρ. Hence we see that $\beta_* \mathcal{O}_{\mathrm{X}}$ has a filtration, the image of $\{ \mathscr{R}^i \}$, with succesive quotients $\mathscr{L}^{-i}, 0 \leqslant i < p$ (the rest being zero).

I also claim that Y, as a non-trivial torsor, is a reduced scheme. Indeed, we have seen that at any generic point Y is obtained by adding a p-th root of a non-p-th power element and hence it is reduced in codimension zero. On the other hand, by the local description we see that Y is everywhere a local complete intersection and in particular fulfills condition S_1. By Serre's criterion Y is then reduced. Finally, by using the adjunction formula twice, first to the $\mathscr{L}$-torsor inside $\mathbf{Spec}\, \mathrm{S}^{\cdot}(\mathscr{E})$ and then to Y inside the $\mathscr{L}$-torsor we get that $\omega_{\mathrm{Y}} = \beta_*(\omega_{\mathrm{X}} \otimes \mathscr{L}^{p-1})$.

Example. — Consider the case of smooth curves. Then it is easy to see that giving $\mathrm{F} : \mathrm{C}^{(1)} \to \mathrm{C}$ the structure of an $\alpha_{\mathscr{L}}$-torsor is the same as giving C the structure of a Tango curve (cf. [Ra]).

Summing up we get

Proposition **1.7.** — *Let* $\mathscr{L} \in \operatorname{Pic} X$ *and let* $\beta : Y \to X$ *be a non-trivial* $\alpha_{\mathscr{L}}$*-torsor. Then* Y *is reduced, there is a filtration*

$$0 \subset \mathcal{N}^0 \subset \mathcal{N}^1 \subset \mathcal{N}^2 \subset \ldots \subset \mathcal{N}^{p-1} = \beta_* \mathcal{O}_Y \tag{1.8}$$

with successive quotients $\mathcal{N}^i/\mathcal{N}^{i-1} = \mathscr{L}^{-i}$ *and*

$$\omega_Y = \beta^*(\omega_X \otimes \mathscr{L}^{p-1}). \tag{1.9}$$

Proof. — It is not true that any finite purely inseparable map $Y \to X$ of degree p is an $\alpha_{\mathscr{L}}$-torsor for a suitable $\mathscr{L}$. The situation is different for $p = 2$, however. In fact, by slightly generalising the construction of $\alpha_{\mathscr{L}}$ we will be able to give a uniform description of all double covers in characteristic two. Let therefore $\mathscr{L} \in \operatorname{Pic} X$ and $s \in \Gamma(\mathscr{L}^{p-1})$. Let $F : \mathscr{L} \to \mathscr{L}^p$ be the Frobenius morphism and define the X-group scheme α_s as the kernel of $F - s : \mathscr{L} \to \mathscr{L}^p$. Hence α_s is isomorphic to α_p at the geometric points where $s = 0$ and to $\mathbf{Z}/p\mathbf{Z}$ at the geometric points where $s \neq 0$. Therefore any α_s-torsor is ramified exactly where s is 0. Checking fiber by fiber one sees that $F - s$ is flat and we therefore have an exact sequence in the flat topology

$$0 \to \alpha_s \to \mathscr{L} \xrightarrow{F-s} \mathscr{L}^p \to 0. \tag{1.10}$$

We can now describe double covers in characteristic two (the situation is of course even simpler in any characteristic different from two). □

Proposition **1.11.** — *Any finite morphism of degree two* $\pi : Y \to X$, *where* Y *is Cohen-Macaulay and* $\operatorname{char}(\mathbf{k}) = 2$, *is an* α_s*-torsor for a suitable line bundle* $\mathscr{L}$ *and* $s \in \Gamma(\mathscr{L})$.

Proof. — As Y is Cohen-Macaulay, X is smooth and π finite, $\pi_* \mathcal{O}_Y$ is locally free. Furthermore, the natural map $\mathcal{O}_X \to \pi_* \mathcal{O}_Y$ is injective on fibers being a ring homomorphism. Hence there is a line bundle $\mathscr{L}$ on X and an exact sequence

$$\iota : 0 \to \mathcal{O}_X \to \pi_* \mathcal{O}_Y \to \mathscr{L}^{-1} \to 0.$$

Consider now the trace map $\operatorname{Tr} : \pi_* \mathcal{O}_Y \to \mathcal{O}_X$. As π is of degree p this map vanishes on $\mathcal{O}_X$ and hence gives a map $\mathscr{L}^{-1} \to \mathcal{O}_X$ i.e. an $s \in \Gamma(\mathscr{L})$. If $t \in \pi_* \mathcal{O}_Y$ then, by Cayley-Hamilton's theorem, $t^2 - \operatorname{Tr}(t)\, t \in \mathcal{O}_X$. Hence, the morphism $t \mapsto t^2 - \operatorname{Tr}(t)\, t$ gives a splitting of $(F - s)^* (\iota)$. By (1.10) we hence get an α_s-torsor and a concrete description of the affine algebra of an α_s-torsor, completely analogous to the one given above for $\alpha_{\mathscr{L}}$-torsors, shows that Y is indeed this α_s-torsor. □

Remark. — The proposition contains as a special case the well known fact that in characteristic two all smooth curves are Tango curves.

2. Let us return to a general 1-foliation. From now on we will assume that X is purely 2-dimensional. In this case it is possible to explicate what happens to the canonical divisor when we resolve the singularities of $X/\mathscr{F}$. This result will be of utmost importance to us in what will follow.

Proposition **2.1.** — (i) *Let* $\mathscr{F}$ *be a* 1*-foliation of rank* 1 *on a smooth variety* X *of pure dimension* 2, *let* $\pi : (X/\mathscr{F})^{(1)} \to X$ *be the natural map and let* $\tau : Z \to (X/\mathscr{F})^{(1)}$ *be a minimal resolution of singularities. Then there is an effective divisor* C *on* Z *contracted by* τ *such that*

$$\omega_Z^p = (\pi\tau)^* (\omega_X \otimes \mathscr{F}^{1-p}) (-C). \tag{2.2}$$

Furthermore, C *is characterised by the formula*

$$p(K_Z, E) = -(C, E)$$

for all curves E *contracted by* τ.

(ii) *Let* $\mathscr{L} \in \operatorname{Pic} X$, X *as in* (i), *let* $\beta : Y \to X$ *be a non-trivial* $\alpha_{\mathscr{L}}$*-torsor and* $\tau : Z \to Y$ *a minimal resolution of the normalisation of* Y. *Then there is an effective divisor* C *on* Z *such that*

$$\omega_Z^p = (\beta\tau)^* (\omega_X \otimes \mathscr{L}^{p-1}) (-C). \tag{2.3}$$

Proof. — Outside the singularities of $\mathscr{F}$, (i) follows from (1.3) by applying, in the notation of Theorem 1.1, $g^{(1)*}$. Hence we get (2.2) for some divisor C, possibly non-effective. However, if E is a curve contracted by τ then

$$\begin{aligned} p(\omega_Z, E) &= ((\pi\tau)^* (\omega_X \otimes \mathscr{F}^{1-p}), E) - (C, E) \\ &= (\omega_X \otimes \mathscr{F}^{1-p}, (\pi\tau)_* E) - (C, E) = -(C, E). \end{aligned}$$

By negative definiteness of the intersection form, C is characterised by this. As the resolution is minimal $(K_Z, E) \geqslant 0$ for every E exceptional for τ and so the effectiveness of C follows from [Bou: §3, Lemme 6]. As for (ii), we can apply the argument of (i) to the normalisation of Y. On the other hand we get the canonical bundle of Y from (1.9). It then only remains to note that to get the canonical sheaf of the normalisation one tensors with the conductor ideal so one just adds more to C (which thus may not be contracted by τ). □

To simplify announcements let us say that X is almost $\mathscr{P}$, where $\mathscr{P}$ is a class of surfaces, if there exists a surface $Y \in \mathscr{P}$ and a dominant inseparable rational map of degree p, $Y \dashrightarrow X$. (Almost rational surfaces are also known as Zariski surfaces.)

Remark. — Almost ruled (or rational) is stronger than uniruled or even " purely inseparably uniruled ". In fact, let $f(x, y, z)$ be a general form of degree $3p^2$ and $Y \to \mathbf{P}^2$ the p^2-th root fibration of f so that

$$Y = \mathbf{Spec}\Big(\bigoplus_{i=0}^{p^2-1} \mathcal{O}_{\mathbf{P}^2}(-3i)\Big).$$

Hence, the standard Čech cocycle (cf. [Ha: III, Thm. 5.1]) $1/xyz$ gives a non-zero element of $H^2(Y, \mathcal{O}_Y)$. The Frobenius map applied to it gives the Čech cocycle $1/x^p y^p z^p$ which again is non-zero. Hence, the Frobenius map is non-zero on $H^2(Y, \mathcal{O}_Y)$. As f is general, the only singularities of Y are A_{p^2-1}-singularities, which in particular are rational, and therefore a resolution $\widetilde{Y}$ also has a non-zero Frobenius map on H^2. This shows that $\widetilde{Y}$ can not be almost rational; on an almost rational surface the Frobenius map is zero on H^i, $i = 1, 2$, as the Frobenius morphism factors birationally through a rational surface which has zero H^i, $i = 1, 2$.

The following result shows that we get some geometric consequences of having a line bundle which is contained in the tangent sheaf and is in some sense positive enough. It should be compared with [Mi].

Theorem **2.4.** — *Let* $\mathscr{L}$ *be a line bundle on* X *such that* $H^0(X, \mathscr{L}^{-1} \otimes T^1_{X/k}) \neq 0$ *and* $\mathscr{M}$ *a line bundle on* X *such that* $(\mathscr{M}, C) \geqslant 0$ *for every curve* C *on* X. *Suppose also that* X *is not (birationally) ruled.*

(i) *If* $(\omega_X, \mathscr{M}) + (1-p)(\mathscr{L}, \mathscr{M}) < 0$ *then* X *is almost ruled.*

(ii) *If* $\mathscr{M}$ *is ample and* $(\omega_X, \mathscr{M}) + (1-p)(\mathscr{L}, \mathscr{M}) = 0$ *and* X *is not almost ruled then one of the following four conditions hold*

a) X *is minimal of Kodaira dimension* 1, *an abelian surface, hyperelliptic or quasi-hyperelliptic. There is a minimal hyperelliptic, quasi-hyperelliptic or abelian surface* Y *and a smooth* 1-*foliation* $\mathscr{F} \subset T^1_{Y/k}$ *such that* $X \tilde{\to} Y/\mathscr{F}$ *and* $\mathscr{L} \cong T^1_{X/Y^{(-1)}}$ *where* $X \to Y^{(-1)}$ *is the natural map. Furthermore,* $\omega_X^{12p} \cong \mathscr{L}^{12p(p-1)}$.

b) $p = 2$ *and* X *is a non-classical* (i.e. $p_g = 1$) *possibly blown up Enriques surface such that the canonical double cover is birational to a* K3-*surface. Furthermore,* $\mathscr{L}$ *is trivial.*

c) $p = 2$ *and* X *is of Kodaira dimension* 1, $\chi(X) = 1$ *and the minimal model of* X *has an* $\alpha_{\mathscr{L}}$-*torsor* $\pi : Y' \to X$ *with only rational double points as singularities, whose minimal resolution* Y *is a* K3-*surface. The map* π *induces a smooth* 1-*foliation on* Y *whose quotient admits* X *as a blowing down. Furthermore,* $\omega_X^2 \cong \mathscr{L}^2$.

d) $p = 2$ *and* X *is of general type,* $\mathscr{L}$ *is the canonical bundle,* $\chi(X) = 1$ *and the minimal model of* X *has an* $\alpha_{\omega_X^{-1}}$-*torsor* $\pi : Y' \to X$ *with only rational double points as singularities, whose minimal resolution* Y *is a* K3-*surface. The map* π *induces a smooth* 1-*foliation on* Y *whose quotient admits* X *as a blowing down. Furthermore,* $\omega_X \cong \mathscr{L}$.

(iii) *If there exists a pencil on* X *with a curve* F *as a general fibre then*

$$-2 \leqslant \frac{1}{p}(\omega_X \otimes \mathscr{L}^{1-p}, F) + F^2.$$

(iv) (F *as in* (iii)). *If*

$$\frac{1}{p}(\omega_X \otimes \mathscr{L}^{1-p}, F) + F^2 < 0 \quad (\leqslant 0)$$

then the normalisation of F *has genus* 0 *(resp. genus* 0 *or* 1*).*

Remark. — It is known that the general non-classical Enriques surface has a surface birational to a K3-surface as its canonical double cover, so case (ii) b) really occurs. We will see that cases (ii) a), c), d) also occur.

Proof. — If $\mathscr{N}$ is an ample line bundle on X we may, by replacing $\mathscr{M}$ in (i) by $\mathscr{M}^i \otimes \mathscr{N}$, $i \gg 0$, assume that $\mathscr{M}$ is ample throughout. As X is not ruled we also have $(\omega_X, \mathscr{M}) \geqslant 0$. Let us consider (i). By assumption there is an embedding $\mathscr{L} \hookrightarrow T^1_{X/k}$.

If $\mathscr{L}' \hookrightarrow T^1_{X/k}$ is its saturation (i.e. $\mathscr{L}'$ is the inverse image of the torsion in $T^1_{X/k}/\mathscr{L}$) then $\mathscr{L}' = \mathscr{L}(D)$ with $D \geqslant 0$. As $(\mathscr{M}, D) \geqslant 0$ we may replace $\mathscr{L}$ by $\mathscr{L}'$ and hence assume that $\mathscr{L}$ is saturated in $T^1_{X/k}$. I claim that $\mathscr{L}$ is a 1-foliation. By (1.4) (iii) it will suffice to prove that $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{L}^p, T^1_{X/k}/\mathscr{L}) = 0$. However, $T^1_{X/k}/\mathscr{L}$ embeds in its double dual which is computed by taking determinants to be $\omega_X^{-1} \otimes \mathscr{L}^{-1}$. Thus it will suffice to show that $(\mathscr{L}^{1+p} \otimes \omega_X)^{-1}$ can not have non-zero global sections. However, from $(\omega_X, \mathscr{M}) \geqslant 0$ and the condition of (i), we get $(\mathscr{L}, \mathscr{M}) > 0$ and so $((\mathscr{L}^{1+p} \otimes \omega_X)^{-1}, \mathscr{M}) < 0$.

Consider now $X/\mathscr{L}$, let $Y \to (X/\mathscr{L})^{(1)}$ be a minimal resolution of singularities and let $\pi : Y \to X$ be the composite $Y \to (X/\mathscr{L})^{(1)} \to X$. Then $\pi^* \mathscr{M}$ is numerically positive and by (2.2)

$$p(\omega_Y, \pi^* \mathscr{M}) = p(\omega_X \otimes \mathscr{L}^{1-p}, \mathscr{M}) - (\pi_* D, \mathscr{M}) = p(\omega_X \otimes \mathscr{L}^{1-p}, \mathscr{M}) < 0.$$

By the classification of surfaces Y is then (birationally) ruled, so X is almost ruled.

For (ii) we proceed similarly. First any embedding $\mathscr{L} \hookrightarrow T^1_{X/k}$ is saturated because if $\mathscr{L}(D) \hookrightarrow T^1_{X/k}$ with $D > 0$ then we could apply (i) to get X almost ruled. We are going to show that there is an embedding $\mathscr{L} \hookrightarrow T^1_{X/k}$ which is a 1-foliation. Assume not. Then $(\mathscr{L}^{1+p} \otimes \omega_X, \mathscr{M}) \leqslant 0$, and as $(\omega_X, \mathscr{M}) \geqslant 0$, we get $(\mathscr{L}, \mathscr{M}) \leqslant 0$. The assumptions then give $(\mathscr{L}, \mathscr{M}) = (\omega_X, \mathscr{M}) = 0$ and so any non-zero section gives an isomorphism $\mathscr{O}_X \to \mathscr{L}^{p+1} \otimes \omega_X$. As $(\omega_X, \mathscr{M}) = 0$, by the classification of surfaces X is then minimal of Kodaira dimension 0. The obstruction morphism $\mathscr{L}^p \to T^1_{X/k}/\mathscr{L}$ composed with the embedding $T^1_{X/k}/\mathscr{L} \hookrightarrow (\omega_X \otimes \mathscr{L})^{-1}$ gives a non-zero map $\mathscr{L}^p \to (\omega_X \otimes \mathscr{L})^{-1}$. This map is then necessarily an isomorphism and therefore $T^1_{X/k}/\mathscr{L} \xrightarrow{\sim} (\omega_X \otimes \mathscr{L})^{-1}$ and $\mathscr{L}$ is a subbundle of $T^1_{X/k}$. Taking Chern classes and using that $\mathscr{L}$ and ω_X are numerically trivial we get that $c_2(X) = 0$ and by the classification of surfaces X is abelian, hyperelliptic or quasi-hyperelliptic.

If X is abelian, then $T^1_{X/k} \cong \mathscr{O}_X \oplus \mathscr{O}_X$ and, as $\mathscr{L}$ is a subbundle of $T^1_{X/k}$ and numerically trivial, it is trivial. Hence there is another embedding of $\mathscr{L}$ in $T^1_{X/k}$ which is a 1-foliation.

If X is hyperelliptic or quasi-hyperelliptic then Alb X is 1-dimensional. If $\mathscr{L}$ is trivial then either Im $\mathscr{L}$ is contained in the saturation of $T^1_{X/\mathrm{Alb}\,X}$ and so is a 1-foliation or the composite $\mathscr{L} \to T^1_{X/k} \to \pi^* T^1_{\mathrm{Alb}\,X/k} \cong \mathscr{O}_X$ is non-zero and hence an isomorphism. In the latter case $T^1_{X/k} \cong \pi^* T^1_{\mathrm{Alb}\,X/k} \oplus T^1_{X/\mathrm{Alb}\,X}$ and either $h^0(T^1_{X/\mathrm{Alb}\,X}) \neq 0$ and $\mathscr{L}$ embeds in $T^1_{X/\mathrm{Alb}\,X}$ or $h^0(T^1_{X/k}) = 1$ in which case the unique embedding of $\mathscr{L}$ gives a foliation as the set of global vector fields are stable under the p-th power. If $\mathscr{L}$ is not trivial, then, as $\mathscr{L}^{-1}$ has no sections, the composite $\mathscr{L} \to T^1_{X/k} \to \pi^* T^1 \mathrm{Alb}\, X/\mathbf{k} \cong \mathscr{O}_X$ is zero and $\mathscr{L}$ is contained in $T^1_{X/\mathrm{Alb}\,X}$ and therefore gives a 1-foliation.

We may therefore assume that there is 1*-foliation* $\mathscr{L} \hookrightarrow T^1_{X/k}$. Let Y' be $(X/\mathscr{L})^{(1)}$, $\tau : Y \to Y'$ a minimal resolution of singularities and $\pi : Y \to X$ the composite $Y \to Y' \to X$. As before, $\pi^* \mathscr{M}$ is numerically positive and $(\pi^* \mathscr{M}, \omega_Y) = p(\mathscr{M}, \omega_X \otimes \mathscr{L}^{1-p}) = 0$. Hence Y has Kodaira dimension $\leqslant 0$ and as X is not almost ruled, Y has in fact Kodaira dimension 0. Let $\rho : Y \to \widetilde{Y}$ be the mapping of Y onto its minimal model. Then

$\omega_Y = \rho^*(\omega_{\check{Y}}(E))$ where $E \geqslant 0$ and whose support is precisely the exceptional set of ρ. As $\omega_{\check{Y}}$, and therefore $\rho^* \omega_{\check{Y}}$, is a torsion line bundle we get

$$0 = (\omega_Y, \pi^* \mathcal{M}) = (E, \pi^* \mathcal{M}) = (\pi_* E, \mathcal{M}).$$

As $\mathcal{M}$ is ample this means that the support of E is contained in the exceptional set of $Y \to Y'$. Thus the exceptional curves of ρ are contracted by τ and as τ is minimal this means that Y is minimal. The following lemma now shows that the singularities of $X/\mathcal{L}$ are rational double points.

Lemma **2.5.** — *Let* Y *be a smooth, proper and connected minimal surface of Kodaira dimension* 0. *If* $Y \to Y'$ *is the resolution of a singular normal surface* Y', *then* Y' *has only rational double points as singularities. Furthermore,* Y *is a* K3-*surface or an Enriques surface.*

Proof. — Indeed, if C is any exceptional curve for $Y \to Y'$ then, as ω_Y is numerically trivial, $(\omega_Y, C) = 0$. This condition characterises rational double points ([Ar]). If Y is not K3 or Enriques then it is abelian, hyperelliptic or quasi-hyperelliptic and then it contains no smooth rational curves C : If Y is abelian this is clear, if not C must lie in a fiber of the Albanese map, but any such fiber is irreducible of genus 1. As the exceptional curves for a resolution of a rational double point are smooth rational curves we conclude. □

Returning to the proof of the theorem, a first consequence of the lemma is that $\omega_Y = \pi^*(\omega_X \otimes \mathcal{L}^{1-p})$ by (2.2) (i). Now, by the classification of surfaces, ω_Y is a torsion line bundle and applying π_* so is $\omega_X \otimes \mathcal{L}^{1-p}$. *Let us now suppose that* $p = 2$. By (1.11) there is a line bundle $\mathcal{N}$ on X such that $\beta : Y' \to X$ is an $\alpha_{\mathcal{N}}$-torsor (note that a 2-dimensional normal scheme is Cohen-Macaulay). This gives us a morphism $\mathcal{N}^{-2} \to \Omega^1_{X/k}$, the annihilator of which is $\mathcal{L} \hookrightarrow T^1_{X/k}$. From the local description of β as $\mathbf{Spec}\, \mathcal{O}_X[t]/(t^p - f)$ we see that the zero set F of $\mathcal{N}^{-2} \to \Omega^1_{X/k}$ equals β (Sing Y'). As Y' is normal, F is finite and hence $\mathcal{N}^{-2} \to \Omega^1_{X/k}$ is saturated.

If now Y is abelian, hyperelliptic or quasi-hyperelliptic then by the lemma Y' is smooth and hence $\mathcal{N}^{-2} \to \Omega^1_{X/k}$ is a subbundle as it is not a subbundle exactly where Y' is singular. We are then in case a). We will prove the rest of the statements in a) below and we hence assume that Y is K3 or Enriques. As $\mathcal{L} \to T^1_{X/k}$ is the annihilator of $\mathcal{N}^{-2} \to \Omega^1_{X/k}$ which is saturated we get (by taking determinants outside F) $\mathcal{L} \cong \mathcal{N}^{-2} \otimes \omega_X^{-1}$. As we saw above ω_X differs from $\mathcal{L}$ only by a torsion line bundle and thus $\mathcal{N}$ and ω_X^{-1} also differ by a torsion line bundle. By (1.7) we have an exact sequence

$$0 \to \mathcal{O}_X \to \pi_* \mathcal{O}_Y \to \mathcal{N}^{-1} \to 0$$

and as $\mathcal{N}^{-1} \sim_{\text{num}} \omega_X$ we get the equality $\chi(\mathcal{O}_{Y'}) = \chi(\mathcal{O}_X) + \chi(\omega_X) = 2\chi(\mathcal{O}_X)$ and as Y' has only rational singularities, $\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_{Y'})$. This excludes the possibility that Y is an Enriques surface as then $\chi(\mathcal{O}_Y) = 1$. *Let us first suppose that* X *is of general type.* I claim that we may then suppose that X is minimal. Indeed, an open subset of X is isomorphic to the minimal model of X minus a finite number of points. Therefore $\mathcal{N}$ and hence $\alpha_{\mathcal{N}}$

extends to the minimal model. By purity (0:1.4) so does Y, as well as the equality of $\mathcal{N}$ and ω modulo torsion line bundles which is what is needed. Now, Y is a K3-surface and Y′ has only rational double points as singularities (which do " not affect adjunction ") so $\omega_{Y'} \cong \mathcal{O}_{Y'}$. From (1.9) it then follows that $\beta^* \mathcal{N} \cong \beta^* \omega_X$ and so $\mathcal{N}^2 \cong \omega_X^2$. We also get $\chi(X) = \frac{1}{2}\chi(Y) = 1$. The rest of d) will be proved in the next section (II:1.7).

We are then left with the case where X *is of Kodaira dimension* 0 *or* 1. Now as Y′ has only rational double points as singularities and is locally a square root fibration one gets the minimal resolution Y by successively blowing up points on X (including infinitely close points) lying under singularities of Y′ and normalising in $\mathbf{k}(Y')$ (cf. [B-P-V:III, §7] the case of characteristic 2 needs only minor modifications). Hence we get a finite morphism from Y to some blowing up $\varphi : X' \to X$ and by following each step one sees that if $\mathcal{L}' \to T^1_{X'/\mathbf{k}}$ is the corresponding foliation then $\mathcal{L}' \otimes \omega_{X'}^{-1} \cong \varphi^*(\mathcal{L} \otimes \omega_X^{-1})$ so $\mathcal{L}'$ fulfills the conditions of $\mathcal{L}$ with the extra property that the foliation is smooth. Finally, as Y is a K3-surface we get $\chi(\mathcal{O}_X) = \frac{1}{2}\chi(\mathcal{O}_Y) = 1$.

Let us now suppose that $p \neq 2$. Suppose that $\mathcal{L} \hookrightarrow T^1_{X/\mathbf{k}}$ has a zero at $x \in X$ so that $\mathcal{L} \hookrightarrow m_x T^1_{X/\mathbf{k}}$ where m_x is the maximal ideal of x. We are going to show that this leads to a contradiction. Let $\rho : X' \to X$ be the blowing up at x. As $\rho^*(m_x T^1_{X/\mathbf{k}}) \subset T^1_{X'/\mathbf{k}}$ we get $\rho^* \mathcal{L} \hookrightarrow T^1_{X'/\mathbf{k}}$. If $\mathcal{L} \hookrightarrow m_x^2 T^1_{X/\mathbf{k}}$ then we get $\rho^* \mathcal{L} \hookrightarrow \rho^* m_x^2 T^1_{X/\mathbf{k}} \hookrightarrow T^1_{X'/\mathbf{k}}(-E)$, where E is the exceptional divisor. Thence $\rho^* \mathcal{L}(E) \hookrightarrow T^1_{X'/\mathbf{k}'}$, but

$$\omega'_X \otimes \rho^* \mathcal{L}(E)^{1-p} = \rho^*(\omega_X \otimes \mathcal{L}^{1-p})\ ((2-p)\ E)$$

so by applying (i) to $\mathcal{L}' := \rho^* \mathcal{L}(E)$ and $\mathcal{M}' := \rho^*(\mathcal{M}^m)\ (-E)$ with $m \gg 0$ we get that X would be almost ruled. Hence the foliation has no double zero at a point on X (including infinitesimally close points as the argument shows). The same argument shows that, as long as we blow up zeroes of the foliation, the inverse image of $\mathcal{L}$ must be saturated in $T^1_{X/\mathbf{k}}$. The following lemma then gives a contradiction.

Lemma **2.6.** — *Let* U *be a smooth* $\mathbf{k}$*-surface* ($\operatorname{char} \mathbf{k} = p > 0$) *and* $\mathcal{L} \hookrightarrow T^1_{U/\mathbf{k}}$ *a non-smooth* 1*-foliation. Then there is a succession* $U_n \to U_{n-1} \to \ldots \to U$ *of blowing ups, each time at a zero of the induced* 1*-foliation, such that the induced* 1*-foliation on* U_n *contains* $\rho^* \mathcal{L}(E)$ *where* $\rho : U_n \to U$ *and* E *is the exceptional divisor for the last blowing up.*

Proof. — By picking a zero and choosing coordinates we may replace U by $T = \mathbf{Spec}\, \mathbf{k}[[x, y]]$ and assume that $\mathcal{L}$ has a zero at m, the maximal ideal of $\mathbf{k}[[x, y]]$. Now $m T^1_{T/\mathbf{k}}$ (where $T^1_{T/\mathbf{k}}$ is the module of *continuous* derivations) is a sub-p-Lie algebra of $T^1_{T/\mathbf{k}}$ and $m^2\, T^1_{T/\mathbf{k}}$ is a p-Lie-ideal in $m T^1_{T/\mathbf{k}}$. Furthermore,

$$m T^1_{T/\mathbf{k}} / m^2\, T^1_{T/\mathbf{k}} = m/m^2 \otimes (m/m^2)^* = \operatorname{End}_{\mathbf{k}}(m/m^2)$$

with the usual p-Lie algebra structure. Finally, a linear coordinate change induces the adjoint action on $\operatorname{End}_{\mathbf{k}}(m/m^2)$. We may, by the argument above, assume that $\mathcal{L}$ does not have a double zero at m. Then the image of $\mathcal{L}$ in $m T^1_{T/\mathbf{k}} / m^2\, T^1_{T/\mathbf{k}}$ is generated, as a $\mathbf{k}$-vector space, by a non-zero element $\overline{D}$ for which $\overline{D}^p = \lambda\, \overline{D}$ for some $\lambda \in \mathbf{k}$, which we may assume to be 0 or 1. Making a suitable change of coordinates we may assume that D, a lifting to $\mathscr{L}$ of $\overline{\mathrm{D}}$ has one of the forms

A) $\mathrm{D} \equiv -ix\,\partial/\partial x + y\,\partial/\partial y \pmod{m^2\,\mathrm{T}^1_{\mathrm{T/k}}}$ where $i \in \mathbf{F}_p$,

B) $\mathrm{D} \equiv x\,\partial/\partial x \pmod{m^2\,\mathrm{T}^1_{\mathrm{T/k}}}$,

C) $\mathrm{D} \equiv x\,\partial/\partial y \pmod{m^2\,\mathrm{T}^1_{\mathrm{T/k}}}$.

If we are in case A, then writing

$$\mathrm{D} = -ix\,\partial/\partial x + y\,\partial/\partial y + f(x,y)\,\partial/\partial x + g(x,y)\,\partial/\partial y$$

with $f, g \in m^2$, blowing up at m to get $\rho : \mathrm{T}' \to \mathrm{T}$ and looking at the patch where we have coordinates s, t with $y = s$, $x = st$ we get $\partial/\partial x = s^{-1}\,\partial/\partial t$ and $\partial/\partial y = \partial/\partial s - s^{-1}\,t\,\partial/\partial t$, and thus

$$\mathrm{D} \equiv (-(i+1)\,t + f(0,s)/s)\,\partial/\partial t + s\,\partial/\partial s \pmod{(s,t)^2\,\mathrm{T}^1_{\mathrm{T/k}}}.$$

After a linear coordinate change this becomes

$$\mathrm{D} \equiv -(i+1)\,t\,\partial/\partial t + s\,\partial/\partial s \pmod{(s,t)^2\,\mathrm{T}^1_{\mathrm{T/k}}},$$

so that if $i \neq -1$ in $\mathbf{F}_p$, then D generates the saturation of $\rho^*\mathscr{L}$ at $s = t = 0$ and the 1-foliation has a zero there. Blowing up at $s = t = 0$ and continuing thus we end up at the case $i = -1$, in which case we have

$$\mathrm{D} \equiv s'\,\partial/\partial s' \pmod{(s',t')^2\,\mathrm{T}^1_{\mathrm{T''/k}}}$$

after the next blowing up and then either D vanishes along the exceptional divisor $\{s' = 0\}$ or we are in case B). In case B), blowing up gives after a linear change of coordinates

$$\mathrm{D} \equiv s\,\partial/\partial s \pmod{(s,t)^2\,\mathrm{T}^1_{\mathrm{T'/k}}},$$

so that either D vanishes along the exceptional curve or we are again in case B). Hence we may assume that there is an infinite sequence of blowing ups for which we stay in case B). Now, $\mathscr{L}$, as a 1-foliation is the annihilator of a function f, and by [Gi] we may assume that, at some point in the blowing up process, f has the form $u^i\,v^j + g^p$ for local coordinates u and v. As we are in case B) we necessarily have i or $j \equiv 0 \pmod{p}$ and so D vanishes at some divisor, which is necessarily exceptional.

In case C) we get $\mathrm{D} \equiv \alpha s\,\partial/\partial t \pmod{(s,t)^2\,\mathrm{T}^1_{\mathrm{T'/k}}}$ for a scalar α. Hence we are again in the situation that we either get vanishing along an exceptional divisor or we stay in C. We then finish as in B). □

We have now shown that $\mathscr{L} \hookrightarrow \mathrm{T}^1_{\mathrm{X/k}}$ is a smooth 1-foliation. I claim that X is minimal. If not there is a curve E on X s.t. $(\omega_{\mathrm{X}}, \mathrm{E}) = -1$ and as $\omega_{\mathrm{X}} \otimes \mathscr{L}^{1-p} \sim_{\mathrm{num}} \mathscr{O}_{\mathrm{X}}$ we get $0 = (\omega_{\mathrm{X}}, \mathrm{E}) + (1-p)(\mathscr{L}, \mathrm{E}) = -1 + (1-p)(\mathscr{L}, \mathrm{E})$ which contradicts the fact that $p \neq 2$.

As $\mathscr{L} \hookrightarrow T^1_{X/k}$ is a subbundle we have an exact sequence

$$0 \to \mathscr{L} \to T^1_{X/k} \to \omega_X^{-1} \otimes \mathscr{L}^{-1} \to 0.$$

Taking Chern classes gives $c_2(X) = -(\mathscr{L}, \omega_X \otimes \mathscr{L}) = -p\mathscr{L}^2$, as $\omega_X \sim_{\text{num}} \mathscr{L}^{p-1}$. Now, X is minimal so that ω_X, and hence $\mathscr{L}$, has non-negative self-intersection and therefore $c_2(X) \leqslant 0$. As $Y \to X$ is finite, radicial it is a homeomorphism in the étale topology and the c_2's, the étale Euler characteristics, are equal so $c_2(Y) = c_2(X) \leqslant 0$ and by the classification of surfaces Y is abelian, hyperelliptic or quasi-hyperelliptic. As then also $c_2(X) = 0$ we get $\omega_X^2 = (p-1)^2 \mathscr{L}^2 = -(p-1)^2/pc_2(X) = 0$ and as X is minimal it has Kodaira dimension 0 or 1.

The same argument goes through in the case left open above i.e. $p = 2$, $\mathscr{L} \hookrightarrow T^1_{X/k}$ is a smooth 1-foliation and Y abelian, hyperelliptic or quasi-hyperelliptic, as soon as we can show that X is minimal. However, if $X \to X'$ is some blowing down, then if Y'' is the normalisation of X' in $\mathbf{k}(X)$ the map $Y \to Y''$ is not finite contradicting lemma 2.5.

We can now show that Y *cannot be quasi-hyperelliptic.* Indeed, considering the diagram

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle\lambda} \\ \text{Alb } Y & \longrightarrow & \text{Alb } X \end{array}$$

we see that the Albanese map of X has 0- or 1-dimensional image; but 0-dimensional is excluded as X maps onto $Y^{(-1)}$. Also the diagram shows that the fibers of the Albanese map of X are rational, being images of the fibers of the Albanese map of Y and they are all singular as X is not ruled. Hence as $\mathcal{O}_X \cong \lambda^* \Omega^1_{\text{Alb}\,X/k} \to \Omega^1_{X/k}$ is not a subbundle along the set of singularities of λ, if we let $\mathcal{O}_X(E) \to \Omega^1_{X/k}$ be its saturation then the effective divisor E contains a curve not lying in a fiber of λ. Dualising, we get a map $\gamma : T^1_{X/k} \to \mathcal{O}_X(-E)$ and composing with $\mathscr{L} \to T^1_{X/k}$ we get a map $\mathscr{L} \to \mathcal{O}_X(-E)$ which is zero as $\mathscr{L}^{p-1} \sim_{\text{num}} \omega_X$. Hence the image of $\mathscr{L}$ is contained in ker γ, in fact equals it being saturated. However, by taking determinants we see that ker $\gamma \cong \omega_X^{-1}(E)$ and thus $\omega_X^2 \sim_{\text{num}} \mathcal{O}_X(E)$, but this contradicts the fact that X is of Kodaira dimension 0 or 1 and that ω_X therefore has zero intersection with the fibers of λ whereas E has horizontal components.

Finally, by the classification of surfaces $\omega_Y^{12} \cong \mathcal{O}_Y$ and so $\pi^*((\omega_X \otimes \mathscr{L}^{1-p})^{12}) \cong \mathcal{O}_Y$ and, as π has degree p, we get $\omega_X^{12p} \cong \mathscr{L}^{12p(p-1)}$.

Let us now turn to (iii) and (iv). We begin by reducing to the case $F^2 = 0$. Indeed, we proceed by induction on F^2 supposing the result true for $F^2 = 0$. If $F^2 > 0$ then there is a basepoint $x \in X$ for the given pencil. Let $\pi : X' \to X$ be the blowing up at x and E the exceptional divisor. For some $r \geqslant 1$ we get $\pi^* F = F' + rE$, where F' is the strict transform of F and as such a general member of a pencil on X'. We also have

$$\pi^* \mathscr{L}(-E) = \pi^*(m_x \mathscr{L}) \hookrightarrow \pi^*(m_x T^1_{X/k}) \hookrightarrow T^1_{X'/k}$$

if we start with some embedding $\mathscr{L} \hookrightarrow T^1_{X/k}$. Hence $h^0(T^1_{X/k} \otimes \mathscr{L}'^{-1}) \neq 0$ with $\mathscr{L}' = \pi^* \mathscr{L}(-E)$. As $F'^2 = F^2 - r^2 < F^2$ and $\omega_{X'} = \pi^* \omega_X(E)$ we get by induction

$$-2 \leqslant \frac{1}{p}(\omega'_X \otimes (\mathscr{L}')^{1-p}, F') + F'^2$$

$$= \frac{1}{p}(\omega_X \otimes \mathscr{L}^{1-p}, F) + F^2 + r - r^2 \leqslant \frac{1}{p}(\omega_X \otimes \mathscr{L}^{1-p}, F) + F^2$$

and similarly for (iv) as F' is birational to F.

From now on we assume that $F^2 = 0$. Now $(\omega_X, F) \geqslant 0$ as X is not ruled and if $(\mathscr{L}, F) \leqslant 0$ there is nothing to prove in (iii), the first case of (iv) would be contradicted and in the second we would get $(\omega_X, F) = 0$. Hence $(\omega_X, F) + F^2$ would be zero and by the adjunction formula F would be of genus 1. Hence we may assume $(\mathscr{L}, F) > 0$. As then $(\omega_X^{-1} \otimes \mathscr{L}^{-1-p}, F) < 0$, arguing as above we may assume that there is a 1-foliation $\mathscr{L} \to T^1_{X/k}$. Let $\pi : Y \to (X/\mathscr{L})^{(1)} \to X$ be a resolution of $(X/\mathscr{L})^{(1)}$. Hence, by (2.2), $\omega_Y^p = \pi^*(\omega_X \otimes \mathscr{L}^{1-p})(-D)$ and as F is a general member of the given pencil, which has no basepoints as $F^2 = 0$, and π (Supp D) is finite, F is disjoint from π (Supp D). If we put $F' := \pi^{-1}(F)_{\text{red}}$, we therefore get that F' is irreducible (being homeomorphic to F) and that F' is disjoint from Supp D. Furthermore, as $F'^2 = 0$ we get from the adjunction formula

$$-2 \leqslant (\omega_Y, F') + F'^2 = \frac{1}{p}(\pi^*(\omega_X \otimes \mathscr{L}^{1-p})(-D), F')$$

$$= \frac{1}{p}(\omega_X \otimes \mathscr{L}^{1-p}, \pi_* F').$$

Now $\pi_* F' = F$ or pF, so we get $1/p(\omega_X \otimes \mathscr{L}^{1-p}, F) \geqslant -2$ or $-2/p \geqslant -2$ and hence (iii). If F' is not rational (resp. of genus > 1) then $F'^2 + (\omega'_X, F') \geqslant 0$ (> 0) and as $\pi : F' \to F$ is surjective we get (iv). □

We want to finish this section by showing that the exceptions in (ii) do occur. Let us first consider the obviously particular case of characteristic two and let us construct examples for cases (ii) b-d). We have in fact already discussed the case of Enriques surfaces, where any non-zero vector field gives us an example. Let us now construct examples of Kodaira dimension 1. Take a Lefschetz pencil of cubic curves on $\mathbf{P}^2$ and let $f : Y \to \mathbf{P}^1$ be the map obtained by blowing up the nine base points of the pencil. Then f has 12 singular points and we may, and shall, assume that they lie in different fibers of f. Blowing up those to get $Y' \to Y$, with exceptional curves F_i, and pulling back by the Frobenius on $\mathbf{P}^1$ and normalising to get $g : X \to \mathbf{P}^1$, a simple calculation shows that X is a K3-surface and that the 12 exceptional curves on Y' become -2-curves, E_i, say (the new -2-curves is the phenomenon peculiar to characteristic two; it is based on the fact that the Frobenius map is everywhere ramified, while in any other characteristic a map of degree two from $\mathbf{P}^1$ to $\mathbf{P}^1$ is ramified in two points and hence at most two of the exceptional curves can become -2-curves).

Lemma **2.7.** — *There is an* $\mathscr{M} \in \operatorname{Pic} X$ *s.t.* $\mathscr{M}^2 \cong \mathcal{O}_X(\Sigma E_i)$.

Proof. — As $\operatorname{Pic} X$ is torsionfree we may work in $\operatorname{Pic} X \otimes \mathbf{Q}$. Let $\rho : Y'^{(1)} \to X$ be induced from $X \to Y'$. Applying the relative Riemann-Roch formula [SGA6: Exp VIII, Thm 3.6] to it we get $c_1(\rho_* \mathcal{O}_{Y'^{(1)}}) = 1/2(\rho_* c_1(Y'^{(1)}))$ in $\operatorname{Pic} X \otimes \mathbf{Q}$ i.e. $\mathscr{L}^2 \cong \rho_* c_1(Y'^{(1)})$ where $\mathscr{L} := c_1(\rho_* \mathcal{O}_{Y'^{(1)}})$. Now, if $f'^{(1)} : Y'^{(1)} \to \mathbf{P}^1$ is the natural morphism and $F_i^{(1)}$ corresponds to F_i, then $\omega_{Y'^{(1)}} = f'^{(1)*} \mathcal{O}_{\mathbf{P}^1}(-1)(\Sigma F_i^{(1)})$ and applying ρ_* we get $\rho_* c_1(Y'^{(1)}) = g^* \mathcal{O}_{\mathbf{P}^1}(-2)(\Sigma E_i)$. This gives the existence of $\mathscr{M}$, in fact $\mathscr{M} := \mathscr{L} \otimes g^* \mathcal{O}_{\mathbf{P}^1}(1)$ will do. $\square$

Put now $\mathscr{N} := g^* \mathcal{O}_{\mathbf{P}^1}(1)$ and $Q_n := \mathscr{M} \otimes \mathscr{N}^n$.

Proposition **2.8.** — *If* $n > 0$, *the Zariski open subset of* $H^1(X, \alpha_{Q_n})$ *consisting of smooth torsors is non-empty. If* $\pi : Z \to X$ *is a smooth* α_{Q_n}*-torsor then* $\pi^{-1}(E_i)_{\mathrm{red}}$ *is a* -1*-curve for all* i. *If we contract those we get* Z' *a minimal surface of Kodaira dimension* 1, *in fact,* $P_m(Z') \sim mn$. *Furthermore,* Z' *is not almost ruled,* $\chi(Z') = 1$ *and* $h^{01}(Z') = 0$ *or* 1 *and both cases may occur.*

Proof. — Indeed, $H^1(X, \alpha_{Q_n})$ is naturally a finite dimensional $\mathbf{k}$-vector space, in fact, (1.5) is a sequence of $\mathcal{O}_X$-modules, where $\mathscr{L}$ has the natural $\mathcal{O}_X$-module structure, $\mathcal{O}_X$ acts on $\mathscr{L}^p$ by p-th powers and the $\mathcal{O}_X$-module structure on $\alpha_{\mathscr{L}}$ is defined by it being a subsheaf of $\mathscr{L}$ stable under $\mathcal{O}_X$. Hence it has a natural Zariski topology and it makes sense to speak about open subsets. We have seen that the singular locus of an α_{Q_n}-torsor is the zero set of a map $Q_n^{-2} \to \Omega^1_{X/\mathbf{k}}$ and it is easily seen that the associated map $H^1(X, \alpha_{Q_n}) \to \operatorname{Hom}_{\mathcal{O}_X}(Q_n^{-2}, \Omega^1_{X/\mathbf{k}})$ is linear and hence continuous. Hence the openness part is clear. To continue we will need the following lemma.

Lemma **2.9.** — (i) $h^0(Q_n) = 0$, $h^1(Q_n) = h^1(Q_n^{-1}) = 1$ *and* $h^0(Q_n^2) = 2n + 1$.

(ii) *We have an exact sequence*

$$0 \to H^0(X, Q_n^2) \to H^1(X, \alpha_{Q_n}) \to H^1(X, Q_n) \to 0. \tag{2.10}$$

Proof. — As $\chi(Q_n) = -6/2 + 2 = -1$, by the Riemann-Roch formula, we see that $h^1(Q_n) = 1$ if $h^0(Q_n) = 0$, as clearly $h^2(Q_n) = h^0(Q_n^{-1}) = 0$. Suppose $\Sigma C_k \in |Q_n|$, where the C_k are curves. Then as $(Q_n, F) = 0$, where F is a fiber of g, we see that the C_k are contained in fibers of g. However, it is clear that any E_i has even intersection number with any curve contained in a fiber, but $(Q_n, E_i) = -1$. Therefore $h^0(Q_n) = 0$. The equality $h^0(Q_n^2) = 2n + 1$ follows from the case $n = 0$ and induction using the exact sequence

$$0 \to \mathscr{M}^2 \otimes \mathscr{N}^i \to \mathscr{M}^2 \otimes \mathscr{N}^{i+1} \to \mathcal{O}_F \to 0 \tag{2.11}$$

and $n = 0$ follows from this exact sequence for $i = -1$. Now, (ii) follows from (i) and (1.5) if we can show that $H^1(X, Q_n) \to H^1(X, Q_n^2)$ is zero. Let therefore $s \in H^0(X, \mathscr{N})$ and consider the commutative diagram

$$\begin{array}{ccc} H^1(X, Q_n) & \longrightarrow & H^1(X, Q_n^2) \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s^2} \\ H^1(X, Q_{n+1}) & \longrightarrow & H^1(X, Q_{n+1}^2) \end{array}$$

If we can choose s s.t. $s : H^1(X, Q_n) \to H^1(X, Q_{n+1})$ is zero and

$$s^2 : H^1(X, Q_{2n}) \to H^1(X, Q_{2n+1})$$

is injective then we are through. Now using (2.11) for $i = n$, $n + 1$, we see that $s^2 : H^1(X, Q_n^2) \to H^1(X, Q_{n+1}^2)$ is injective for any $s \neq 0$ as no fiber is a base curve for $\mathcal{M}^2 \otimes \mathcal{N}^i$. On the other hand, the exact sequence, for $s \neq 0$ and $F := \{ s = 0 \}$,

(2.12) $$0 \to Q_n \to Q_{n+1} \to \mathcal{M}_{|F} \to 0$$

and (i) shows that $s : H^1(X, Q_n) \to H^1(X, Q_{n+1})$ is zero if $h^0(\mathcal{M}_{|F}) \neq 0$. As $\mathcal{M}_{|F}^2 \cong \mathcal{O}_F$ we get $h^0(\mathcal{M}_{|F}) = 1$ if F is a supersingular elliptic curve as Pic F then contains no 2-torsion. However, g is semistable and non-constant and thus every isomorphism class of elliptic curves occur as fibers and in particular a supersingular fiber occurs as supersingular elliptic curves exist (there is, up to isomorphism one in characteristic 2). □

Let now $F_1, F_2, \ldots, F_{2n}$ be distinct irreductible fibers of g and let $\pi : Z \to X$ be the α_{Q_n}-torsor associated to the canonical section of $\mathcal{O}_X(\Sigma E_i + \Sigma F_j) \xrightarrow{\sim} Q_n^2$. Assume now that for all choices of $F_1, F_2, \ldots, F_{2n}$, Z is singular. Let $\overline{Z} \to Z$ be the normalisation of Z. By (1.11) $\overline{Z} \to X$ is an $\alpha_{Q'}$-torsor for some Q' and as $\overline{Z}$ maps to Z, the functoriality of (1.6) shows that $Q' \hookrightarrow Q_n$ i.e. $Q' = Q_n(-D)$ for some $D \geqslant 0$. At points on $UE_i \cup UF_j$, Z is given as adjoining a square root of a function vanishing along a smooth curve and is therefore smooth and so D is disjoint from the E_i and the F_j. Hence the components of D lie in fibers (as $n > 0$) and being disjoint from the E_i they lie in integral fibers and hence are fibers. This means that $D \in |\mathcal{N}^i|$ for some $i \geqslant 0$. Assume, for the moment that we know that for a general choice of $F_1, F_2, \ldots, F_{2n}$, $i = 0$. We then quickly reach a contradiction as follows. Let $\tau : \hat{Z} \to Z$ be a resolution of singularities. Now Z is normal and $\chi(\mathcal{O}_Z) = 1$, the latter by (i). Also, as $b_1(\hat{Z}) = b_1(X) = 0$, $\chi(\mathcal{O}_{\hat{Z}}) \geqslant 1$. This implies, by (0:1.5), that $\chi(\mathcal{O}_{\hat{Z}}) = 1$ and that Z has only rational singularities, which then are rational double points as Z has only hypersurface singularities. Thus $\omega_{\hat{Z}} = \tau^* \omega_Z = \tau^* \pi^* Q_n$, the last by (1.9), and so $c_1^2(\hat{Z}) = -12$ and by Noether's formula $c_2(\hat{Z}) = 24$. On the other hand, $\hat{Z} \to X$ is purely inseparable and $\chi_{et}(\hat{Z}) = 24 = \chi_{et}(X)$ so Z is smooth by (0:1.5).

We therefore want to show that the assumption that $i > 0$ for general $F_1, F_2, \ldots, F_{2n}$ leads to a contradiction. Now $Z \to X$ represents an element $\alpha \in H^1(X, \alpha_{Q_n})$ and $\overline{Z} \to X$ represents an element $\alpha' \in H^1(X, \alpha_{Q_{n-i}})$. The relation between these elements is that $\alpha = s * \alpha'$ for some $s \in H^0(X, \mathcal{N}^i)$ with $D = \{ s = 0 \}$ and where

$$* : H^1(X, \alpha_{Q_{n-i}}) \otimes H^0(X, \mathcal{N}^i) \to H^1(X, \alpha_{Q_n})$$

is induced from

$$\alpha_{Q_n} = \alpha_{Q_{n-i}} \otimes \mathcal{O}_X \mathcal{N}^i.$$

Hence we see that $\alpha = t * \beta$ for some $t \in H^0(X, \mathcal{N})$ and $\beta \in H^1(X, \alpha_{Q_{n-i}})$, as s is the composite of i sections of $H^0(X, \mathcal{N})$. By counting dimensions we will show that this is not possible for a general choice of $F_1, F_2, \ldots, F_{2n}$. In fact, from (2.7) (i) it follows that $\dim H^0(X, \mathcal{N}) * H^0(X, Q^2_{n-1}) \leqslant 2n$. On the other hand I claim that if $t * \alpha \in H^0(X, Q^2_n)$ for some $\alpha \in H^1(X, \alpha_{Q_{n-1}}) \backslash H^0(X, Q^2_{n-1})$ then t belongs to a finite number of lines in $H^0(X, \mathcal{N})$. Indeed, if $t * \alpha \in H^0(X, Q^2_n)$ and $\alpha \notin H^0(X, Q^2_{n-1})$, then $t * : H^1(X, Q_{n-1}) \to H^1(X, Q_n)$ is not injective and it follows from (2.12) that $\mathcal{M}_{|\{t=0\}}$ has a non-zero section and it is sufficient to show that the restriction of $\mathcal{M}$ to a general fiber of g has no sections. Now, if F is an irreducible fiber, the restriction of $\mathcal{M}$ to F has degree zero so if it has a section it is trivial. Hence if it had a section for a general F we would have $\mathcal{M} \cong \mathcal{O}_X(rF + E)$ where E is contained in fibers. This gives a contradiction as any E_i has even intersection number with curves in fibers and intersection -1 with $\mathcal{M}$. Hence we see that

$$\dim(H^0(X, \mathcal{N}) * (H^1(X, \alpha_{Q_{n-1}}) \backslash H^0(X, Q^2_{n-1}))) \cap H^0(X, Q^2_n) \leqslant 2n$$

and so altogether

$$\dim(H^0(X, \mathcal{N}) * (H^1(X, \alpha_{Q_{n-1}})) \cap H^0(X, Q^2_n) \leqslant 2n < \dim H^0(X, Q^2_n).$$

We have now proved that there is an open dense subset U of $H^1(X, \alpha_{Q_n})$ consisting of smooth torsors and in the course of the argument that $U \cap H^0(X, Q^2_n) \neq \emptyset$. Let $\pi : Z \to X$ be a smooth α_{Q_n}-torsor. As $\deg Q_n \mid E_i = -1$ we see that $\pi^{-1}(E_i) \to E_i$ is an $\alpha_{\mathcal{O}(-1)}$-torsor and as $H^0(\mathbf{P}^1, \mathcal{O}(-2)) = H^1(\mathbf{P}^1, \mathcal{O}(-1)) = 0$ this torsor is trivial, so if $F'_i := \pi^{-1}(E_i)_{\mathrm{red}}$, then $2F'_i = \pi^* E_i$ as divisors and $\pi : F'_i \to E_i$ is an isomorphism. Hence $F'^2_i = 1/4(\pi^* E_i, \pi^* E_i) = -1$ and so the F'_i are -1-curves and can be blown down to get $\tau : Z \to Z'$. Now $\omega_Z = \pi^*(Q_n)$ and $(\tau^* \omega_{Z'})(\Sigma F'_i) = \omega_Z$, so we get $\tau^*(\omega'_Z)^{\otimes 2} \cong \pi^*(\mathcal{N}^n)^{\otimes 2}$. This means that if $h : Z' \to \mathbf{P}^1$ is the map s.t. $h \circ \tau = g \circ \pi$ (which exists as $g \circ \pi$ maps the F'_i to points), then $\omega_{Z'}$ and $h^* \mathcal{O}_{\mathbf{P}^1}(n)$ differ by a line bundle of order 1 or 2. This shows that Z' is minimal elliptic. Furthermore, the general fiber of h is integral because if it were not the Stein factorisation of h would be non-trivial, which means that $Z' \to \mathbf{P}^1$ factors through the pullback of g by the Frobenius map on $\mathbf{P}^1$, but this pullback is normal and therefore equals Z' and has 24 A_1-singularities, which Z' certainly doesn't have. Hence we get that $P_m(Z') \sim mn$. We have an exact sequence

$$0 \to \mathcal{O}_X \to \pi_* \mathcal{O}_Z \to Q_n^{-1} \to 0.$$

From (2.9) (i) it then follows that $h^1(\mathcal{O}_{Z'}) = 1$, $h^2(\mathcal{O}_{Z'}) = 2$ or $h^1(\mathcal{O}_{Z'}) = 0$, $h^2(\mathcal{O}_{Z'}) = 1$ depending on whether the boundary map $H^1(X, Q_n^{-1}) \to H^2(X, \mathcal{O}_X)$ is zero or not. This map is, by duality, dual to the classifying map $H^0(X, \mathcal{O}_X) \to H^1(X, Q_n) = \mathrm{Ext}^1_{\mathcal{O}_X}(Q_n^{-1}, \mathcal{O}_X)$. Hence $h^{01}(Z)$, and therefore $h^{01}(Z')$, equals 1 if and only if $\pi : Z \to X$ belongs to $H^0(X, Q^2_n)$ and as noted above there are smooth α_{Q_n}-torsors in $H^0(X, Q^2_n)$. Finally, let W be $(X')^{(1)}$ where X' is the pullback of Y by the Frobenius map on $\mathbf{P}^1$. Then there is a radicial map $\rho : W \to Z'$ of degree p and as W is normal, ρ is an $\alpha_{\mathcal{K}}$-torsor for some $\mathcal{K} \in \mathrm{Pic}\, Z'$. Furthermore, W has 12 A_1-singularities and has $X^{(1)}$ as its minimal resolution.

Hence $\omega_W \cong \mathcal{O}_W$ and so $\rho^*(\omega_{Z'} \otimes \mathcal{K})$ is trivial. Pushing down by ρ_* we get that $\omega_{Z'}$ and $\mathcal{K}^{-1}$ differ by a linebundle of order 1 or 2. We can now show that Z' is not almost ruled. If it were, then by (0:1.5) it is almost rational. Suppose that $\varphi : \mathbf{P}^2 \dashrightarrow Z'$ is a dominant rational map. We are going to show that φ factors through ρ and then $\deg \varphi > p$ as W is not rational. Suppose therefore that φ does not factor through ρ. Then by purity (0:1.4) $\varphi^*(\rho)$ is a nontrivial $\alpha_{\mathcal{K}'}$-torsor, where $\mathcal{K}'$ is the extension of $\varphi^*(\mathcal{K})$ to the points where φ is not defined. Now, as $\omega_{Z'}$ and $\mathcal{K}^{-1}$ differ by a torsion line bundle, a negative power of $\mathcal{K}'$ has sections so $\mathcal{K}' \cong \mathcal{O}_{\mathbf{P}^2}(n)$ for some $n \leqslant 0$, but the long exact sequence of (1.5) and the known cohomology of line bundles on $\mathbf{P}^2$ shows that there are no nontrivial $\alpha_{\mathcal{K}'}$-torsors. □

We now want to show that there are examples of general type. Consider therefore once again the surface X. The map $g : X \to \mathbf{P}^1$ has nine sections, G_j say, coming from the nine base points of the original pencil. These curves are -2-curves, all disjoint from each other and also from the E_i as no member of the original pencil has a singularity at a basepoint. Put $\mathcal{P}_s := \mathcal{N}^2(G_1 + \ldots + G_s)$ and $\mathcal{R}_s := \mathcal{M} \otimes \mathcal{N}(G_1 + \ldots + G_s)$.

Lemma **2.13.** — (i) *If $s \geqslant 1$, then $\mathcal{P}_s$ is numerically positive and* C *is a curve for which $(\mathcal{P}_s, C) = 0$ iff* C *is one of the E_i or G_j.*

(ii) *There is a smooth and irreducible curve in $|\mathcal{P}_s|$ if $s \geqslant 2$.*

(iii) *Let $s \geqslant 1$. Then $h^0(\mathcal{P}_s) = s + 2$, $h^0(\mathcal{R}_s) = h^2(\mathcal{R}_s) = h^0(\mathcal{R}_s^{-1}) = h^2(\mathcal{R}_s^{-1}) = 0$ and $h^1(\mathcal{R}_s) = h^1(\mathcal{R}_s^{-1}) = 1$.*

(iv) *There is an exact sequence*

$$0 \to H^0(X, \mathcal{R}_s^2) \to H^1(X, \alpha_{\mathcal{R}_s}) \to H^0(X, \mathcal{R}_s) \to 0.$$

Proof. — Suppose that C is a curve on X and that $(\mathcal{P}_s, C) \leqslant 0$. As $\mathcal{P}_s$ contains divisors of the form $F_1 + F_2 + G_1 + \ldots + G_s$, where F_1, F_2 are any two fibers of g, we see that C either lies in a fiber of g and is disjoint from the G_j or is one of the G_j. However, any G_j has intersection 0 with $\mathcal{P}_s$ and if C is disjoint from the G_j and lies in a fiber, it is one of the E_i which has zero intersection with $\mathcal{P}_s$. Finally $(\mathcal{P}_s, \mathcal{P}_s) = 2s > 0$. Hence $\mathcal{P}_s$ is numerically positive. Now, $h^0(\mathcal{P}_s) > 0$, $H^1(X, \mathcal{O}_X) = 0$ and $\mathcal{P}_s$ is numerically positive. Therefore, $h^1(\mathcal{P}_s) = h^1(\mathcal{P}_s^{-1}) = 0$ and so by the Riemann-Roch theorem $h^0(\mathcal{P}_s) = 2s/2 + 2 = s + 2$. Let us now prove that, if $s \geqslant 2$ then $|\mathcal{P}_s|$ has no basepoints. Indeed, if we apply (0.3) then we get into problems only in case (i) and if we can prove that $b \geqslant 2$ then we are clear. Let therefore C be a curve with $C^2 = 0$. If $(\mathcal{N}, C) > 0$, then $(\mathcal{P}_s, C) \geqslant 2$ and if not C is contained, and hence equals, a fiber of g but then $(\mathcal{P}_s, C) \geqslant 2$ as $s \geqslant 2$. We get that a general member of $|\mathcal{P}_s|$ is a curve from [Jo] and that we have just proved that $|\mathcal{P}_s|$ is not composed with a pencil. Let now C be a general member of $|\mathcal{P}_s|$ and suppose that C is not smooth. Now in any case the map from the normalisation of C to C is radicial and so, as C is not smooth, the Frobenius map on $H^1(C, \mathcal{O}_C)$ is not injective, as the F-semi simple part of $H^1(-, \mathcal{O})$ is unchanged under radicial maps. By a specialisation argument this is also true for any special member D of $|\mathscr{P}_s|$ for which $H^0(D, \mathcal{O}_D) = \mathbf{k}$. However, $F_1 + F_2 + G_1 + \ldots + G_s$ is a graph of $\mathbf{P}^1$'s with transversal intersection so for this member we have that F is semi simple on H^1. Hence (i) and (ii) are proved. Now as $\mathscr{R}_s$ has negative intersection with any G_j, it is a base curve of $|\mathscr{R}_s|$. Hence $h^0(\mathscr{R}_s) = 0$ follows from (2.9) (i). Now, $h^0(\mathscr{R}_s^{-1}) = 0$ is obvious, $h^2(\mathscr{R}_s) = 0$ and $h^2(\mathscr{R}_s^{-1}) = 0$ follows by duality and $h^1(\mathscr{R}_s) = h^1(\mathscr{R}_s^{-1}) = 1$ follows from this and the Riemann-Roch formula. This gives (iii). Finally, (iv) is now clear except for the surjectivity to the right. However, we have an exact sequence

$$0 \to Q_2 \to \mathscr{R}_s \to \mathscr{R}_{s|G_1+\ldots+G_s} \to 0,$$

and as $\deg \mathscr{R}_{s/G_j} = -1$ the long exact sequence of cohomology shows that

$$H^1(X, Q_2) \to H^1(X, \mathscr{R}_s)$$

is an isomorphism. As the map $Q_2 \to \mathscr{R}_s$ induces a map between the two sequences coming from (1.5) we conclude by (2.9). $\square$

Let us now embark on the construction of the examples, a construction altogether similar to the construction of special type examples. Let $C \in |\mathscr{P}_s|$ be a smooth, irreducible curve and let $\pi : Z \to X$ be the $\alpha_{\mathscr{R}_s}$-torsor associated to the image of 1 in the isomorphism $\mathcal{O}_X(C + \Sigma E_i + G_1 + \ldots + G_s) \xrightarrow{\sim} \mathscr{R}_s^2$. By (2.13) (i) C is disjoint from the E_i and the G_j and so their union T is smooth. Hence Z is smooth above T and, again by (2.13) (i), Z has only isolated singularities and hence is normal, being Cohen-Macaulay. As above one shows then that Z is smooth. Hence the smooth $\alpha_{\mathscr{R}_s}$-torsors form a dense open subset and intersects $H^0(X, \mathscr{R}_s^2)$ non-trivially. We now have

Proposition **2.14.** — *Let* $\pi : Z \to X$ *be a smooth* $\alpha_{\mathscr{R}_s}$*-torsor* $2 \leqslant s \leqslant 9$. *Then the inverse images of the* E_i *and* G_j *are disjoint* $-$ 1*-curves on* Z *and can therefore be blown down to get* Z'. *Then* Z' *is a smooth surface with ample canonical sheaf,* $\chi(Z') = 1$ *and* $h^{01}(Z') = 0$ *or* 1 *and both possibilities may occur. Furthermore,* $c_1^2(Z') = s$, $H^1(Z', \omega_{Z'}^{-1}) \neq 0$ *and* Z' *is not almost rational.*

Proof. — Indeed, everything is completely similar to (2.8) but the ampleness and $H^1(Z', \omega_{Z'}^{-1}) \neq 0$. The ampleness follows however from (2.13) (i). Also,

$$h^0(\omega_{Z'}^2) = h^0(\omega_Z^2) \geqslant h^0(\mathscr{R}_s^2) \geqslant h^0(\mathscr{P}_s) = s + 2,$$

where we used that $\omega_Z = \pi^* \mathscr{R}_s$. On the other hand, by the Riemann-Roch theorem, $\chi(\omega_{Z'}^2) = s + 1$ and so $h^1(Z', \omega_{Z'}^{-1}) = h^1(Z', \omega_{Z'}^2) \geqslant 1$. $\square$

Remark. — (i) The case of $s = 9$ is particularly interesting. In this case the map of degree 4 down to $\mathbf{P}^2$ is radicial so that Z' is homeomorphic in the étale topology to $\mathbf{P}^2$, hence deserves the name fake $\mathbf{P}^2$ in a very strong sense. As we have just constructed a 10-dimensional family of such examples we see that rigidity on the line $c_1^2 = 3c_2$ fails miserably in characteristic two.

(ii) The 1-foliation induced by $Z' \dashrightarrow X$ embeds $\omega_{Z'}$ in $T^1_{X/k}$, so Z' has vector fields if $p_g \neq 0$.

Let us now consider case (ii) a) (so that now p is arbitrary). It turns out that smooth foliations on abelian or hyperelliptic surfaces can be completely classified. The hyperelliptic case reduces to the abelian by descent (I leave to the reader to work out the details) so we will consider only the abelian case.

Proposition **2.15.** — *Let* A *be an abelian surface. Then the following data give rise to a smooth* 1*-foliation of rank* 1 *on* A *(the correspondence will be given in the proof):*

An abelian 1*-dimensional subvariety* E *of* A *and a non-zero rational* 1*-form* ω *on* A/E *s.t.* $C(\omega) = \lambda\omega$, *where* $\lambda = 1$ *if* E *is ordinary and* $\lambda = 0$ *if* E *is supersingular and* C *is the Cartier operator.*

Two such pairs (E, ω) *and* (E', ω') *give rise to the same* 1*-foliation iff* $E = E'$ *and* ω *and* ω' *are* **k***-proportional modulo global* 1*-forms and any smooth* 1*-foliation which is not an isogeny is obtained by this construction.*

Remark. — Rational 1-forms ω with $C(\omega) = 0$ (resp. ω) are exactly the 1-forms of the form df (resp. $f^{-1}\,df$) for some (non-zero) rational function f (cf. [Ill: 0:2.1.9, 2.1.17]). Hence non-isogeny smooth 1-foliations exist as soon as A is not simple.

Proof. — Let $\mathscr{L}^{-1} \hookrightarrow T^1_{A/k}$ be a smooth 1-foliation and excluding the case of an isogeny we may assume that $\mathscr{L}$ is nontrivial. As the 1-foliation is smooth we have an exact sequence

$$0 \to \mathscr{L}^{-1} \to T^1_{A/k} \to \mathscr{L} \to 0,$$

and as $T^1_{A/k}$ is trivial $\mathscr{L}$ is generated by two global sections. This means that there is, a unique, 1-dimensional abelian subvariety E of A s.t. $\mathscr{L}$ is the pullback of a linebundle on A/E. Indeed, $\mathscr{L}$ has sections but is not ample as it is generated by two sections. Hence by [Mu2: § 16, 2nd thm] $K(\mathscr{L})$ (in the notation of *loc. cit.*) is positive dimensional but not equal to A, as $\mathscr{L}$ has sections but is non-trivial. Put $E := K(\mathscr{L})^0_{\mathrm{red}}$, an abelian subvariety which, by assumption, is non-zero and different from A, hence 1-dimensional. The restriction of $\mathscr{L}$ to E has degree zero and is generated by sections and is hence trival. Therefore $\mathscr{L}$ descends to $\mathscr{L}'$ on $B := A/E$. We can choose a basis of the global **k**-derivations of A, D_1 and D_2, such that D_1 is an B-derivation and either

I) $D_1^p = \lambda\, D_1$ and $D_2^p = \mu\, D_2$, where $\lambda, \mu = 0$ or 1,

or

II) $D_2^p = D_1$.

This follows from the fact that $T^1_{A/B}$ is trivial, as $T^1_{A/k}/T^1_{A/B} = \rho^*\, T^1_{B/k}$, where $\rho : A \to B$, which is trivial, and hence that the global B-derivations form a 1-dimensional sub-p-Lie algebra of the global **k**-derivations. Now, as ρ equals its own Stein factorisation all global sections of $\mathscr{L}$ come from sections of $\mathscr{L}'$ and in particular we get two global sections f_1 and f_2 of $\mathscr{L}'$ corresponding to the images in $\mathscr{L}$ of D_1 and D_2. Both f_1 and f_2 are non-zero as $\mathscr{L}$ is non-trivial. Let $U \subset B$ be the open subset where f_2 is non-zero and put $f := f_1/f_2$ which then is a non-zero rational function on B, regular on U. Hence

$D_1 - fD_2$ as a vector field on $\rho^{-1}(U)$ belongs to the 1-foliation $\mathscr{L}^{-1}$ and thus so does its p-th power. As $f \in \mathbf{k}(B)$, D_1 and D_2 commute and D_1 is an B-derivation we get $(D_1 - fD_2)^p = D_1^p - (fD_2)^p$ and using the formula of [Ka: 5.3] (making a sign change to make the formula true) this equals $D_1^p - f^p D_2^p + f D_2^{p-1}(f^{p-1}) D_2$. Assume now that we are in case I) above and map this element into $\mathscr{L}$ to check that we stay in $\mathscr{L}^{-1}$. This gives us the equality $\lambda f_1 - \mu f^p f_2 + f D_2^{p-1}(f^{p-1}) f_2 = 0$ and dividing by ff_2, we get $\mu f^{p-1} - D_2^{p-1}(f^{p-1}) = \lambda$. Now as $T^1_{A/k}/T^1_{A/B} = \rho^* T^1_{B/k}$ is trivial, D_2 stabilises $\mathcal{O}_B$ and hence induces a global $\mathbf{k}$-derivation on B, D say. Let η be the 1-form dual to D and apply the identity (cf. [Ill: 0:2.1.12])

$$\langle D, C(f^{p-1}\eta)\rangle^p = \langle D^p, f^{p-1}\eta\rangle - D^{p-1}\langle D, f^{p-1}\eta\rangle.$$

This gives, as $D^p = \mu D$, $\langle D, C(f^{p-1}\eta)\rangle^p = \lambda$, which amounts to $C(f^{p-1}\eta) = \lambda\eta$. Multiplying this equality with f^{-1} and putting $\omega := f^{-1}\eta$ gives $C(\omega) = \lambda\omega$. We can also reverse each step, the only point being that by (1.4) (ii) it suffices to check closedness under p-th powers for one single non-zero rational section and that $D_1 - fD_2$ cannot be zero as D_1 and D_2 are two A-invariant $\mathbf{k}$-linearly independent vector fields. Case II is altogether similar, and by varying the choices made one gets the uniqueness assertion. □

II. Consequences of Kodaira non-vanishing

We will now develop a number of criteria for the vanishing of $H^1(X, \mathscr{L}^{-1})$, where X is still a smooth, proper and connected $\mathbf{k}$-surface and $\mathscr{L}$ is a numerically positive line bundle. Often, however, a more natural condition is the following

$$(*) \qquad H^1(X, \mathscr{L}^{-p^i}) = 0 \quad \text{for all } i > 0.$$

Let us note that by [Sz: Exp 2, Prop. 2], which gives vanishing for $i \gg 0$, and (I:1.5) this is equivalent, when $\mathscr{L}$ is numerically positive, to the non-existence of non-trivial $\alpha_{\mathscr{L}^{-p^i}}$-torsors for all $i > 0$. We will usually tacitly pretend, when assuming that $(*)$ is not fulfilled for a line bundle $\mathscr{L}$, that it is fulfilled for $\mathscr{L}^p$, as the argument becomes even simpler if $\mathscr{L}$ is replaced by $\mathscr{L}^p$.

Proposition **1.1.** — *Let $\mathscr{L}$ be a numerically positive line bundle on* X, *let* $\pi : Y \dashrightarrow X$ *be a birational map, where* Y *is smooth and proper, and let* $\pi^* \mathscr{L}$ *denote the extension over the points where* π *is not defined of the pullback of* $\mathscr{L}$. *Then* $(*)$ *for* $\mathscr{L}$ *and* $\pi^*\mathscr{L}$ *are equivalent and* $\pi^*\mathscr{L}$ *is also numerically positive.*

Proof. — By factoring π as a composite of blowing ups and downs one sees that $\pi^*\mathscr{L}$ is numerically positive. Hence we can conclude by the remark just made and an appeal to purity (0:1.4). □

Another useful preliminary result is the following.

Proposition **1.2.** — *Let* X *and* Y *be smooth, proper and connected* **k**-*surfaces,* $\pi : X \to Y$ *a surjective morphism and let* $\mathscr{L}$ *be a numerically positive line bundle on* Y. *Assume* $(*)$ *for* $\pi^* \mathscr{L}$.

(i) *There exists a smooth and proper* **k**-*surface* Z *and a factorisation* $X \dashrightarrow Z \xrightarrow{\rho} Y$ *s.t.* $(*)$ *is true for* $\rho^* \mathscr{L}$ *and* ρ *is purely inseparable.*

(ii) *If* π *is purely inseparable of degree* p *and* $(*)$ *is false for* $\mathscr{L}$ *then there is an* $i > 0$ *s.t.* $h^0(\mathscr{M} \otimes \mathscr{L}^{-p^i}) \neq 0$, *where* $\mathscr{M} \subset \Omega^1_{Y/k}$ *is the annihilator of the* 1-*foliation on* Y *induced by* π.

Proof. — Indeed, we may assume that there exists a non-trivial $\alpha_{\mathscr{L}^{-p^i}}$-torsor $\tau : Z' \to Y$ for some $i > 0$. By assumption its pullback to X is trivial which means that there exists a factorisation $\pi = \tau \circ \beta$. By normalising and resolving singularities we get a purely inseparable map $\rho : Z \to Y$ with Z smooth and proper and a factorisation as in (i). If $(*)$ is true for $\rho^* \mathscr{L}$ we get (i), if not we continue, which is possible as $\rho^* \mathscr{L}$ is also numerically positive, and as π has finite degree the process stops. As for (ii), for reasons of degree β is birational. This means that the 1-foliation on Y induced by Z′ coincides with the one induced by X. We saw in (I:1) that the 1-foliation on Y induced by Z′ is the annihilator of a map $\mathscr{L}^{p^i} \to \Omega^1_{Y/k}$, the image of which has to lie in $\mathscr{M}$. □

The following theorem is the first major result giving strong geometric consequences of non-vanishing for *a*, in some sense, sufficiently positive line bundle $\mathscr{L}$ as measured by its intersection number with different line bundles.

Theorem **1.3.** — *Let* X *be a smooth, proper and connected* **k**-*surface, let* $\mathscr{L}, \mathscr{M} \in \operatorname{Pic} X$ *and assume that* $H^1(X, \alpha_{\mathscr{L}^{-1}}) \neq 0$.

(i) *We have the following inequality*

$$\left(\binom{p}{3} + \frac{1}{2}\binom{p}{2}\right)\mathscr{L}^2 + \binom{p}{2}\left((\mathscr{L}, \mathscr{M}) - \frac{1}{2}(\mathscr{L}, K_X)\right) + (p-1)\chi$$
$$\leqslant \binom{p}{2}\mathscr{M}^2 + \frac{1}{2}(\mathscr{M}^2 - (K, \mathscr{M})) + h^1(\mathscr{M}^p)$$
$$- h^2(\mathscr{M}^p) + h^0(\mathscr{M}^{-p} \otimes \omega_X^p \otimes \mathscr{L}^{p(1-p)}).$$

(ii) $h^0(\mathscr{L}) \leqslant h^0(B_1)$ *and if* $h^0(B_1) < h^1(\alpha_{\mathscr{L}^{-1}})$, *then* $h^0(\mathscr{L}) = 0$ *or* $h^1(\alpha_{\mathscr{L}^{-1}}) = 1$.

(iii) *Suppose that* $(\mathscr{M}, C) \geqslant 0$ *for every curve* C *and that* X *is not ruled. If*

$$(p-1)(\mathscr{L}, \mathscr{M}) > (\omega_X, \mathscr{M}),$$

then X *is almost ruled.*

(iv) *If* X *is not almost ruled,* $\mathscr{M}$ *is ample and*

$$(p-1)(\mathscr{L}, \mathscr{M}) = (\omega_X, \mathscr{M}),$$

then either of the following is true

a) X *is minimal and abelian, hyperelliptic or of Kodaira dimension* 1 *and* X *is of the form* $Y/\mathscr{F}$ *where* $\mathscr{F}$ *is a smooth foliation on* Y, *a minimal abelian or hyperelliptic surface. Furthermore,* $\omega_X^{12p} \cong \mathscr{L}^{12p(p-1)}$.

b) *$p = 2$ and X is a non-classical (i.e. $p_g = 1$) possibly blown up Enriques surface such that the canonical double cover is birational to a* K3-*surface. Furthermore, $\mathscr{L}^2$ is trivial.*

c) *$p = 2$ and X is of Kodaira dimension* 1, *$\chi(X) = 1$ and the minimal model of X has an $\alpha_{\mathscr{L}}$-torsor $\pi : Y' \to X$ with only rational double points as singularities and whose minimal resolution Y is a* K3-*surface. The map π induces a smooth* 1-*foliation on Y whose quotient admits X as a blowing down. Furthermore, $\omega_X^{\otimes 2} \cong \mathscr{L}^{\otimes 2}$.*

d) *$p = 2$ and X is of general type, $\mathscr{L}$ is the canonical bundle, $\chi(X) = 1$ and the minimal model of X has an α_{ω_X}-torsor $\pi : Y' \to X$ with only rational double points as singularities and whose minimal resolution Y is a* K3-*surface. The map π induces a smooth* 1-*foliation on Y whose quotient admits X as a blowing down. Furthermore, $\omega_X^{\otimes 2} \cong \mathscr{L}^{\otimes 2}$.*

(v) *Let F be a curve which is the general fiber of a pencil on X. Then*

$$-2 \leqslant (\omega_X, F) + (1-p)(\mathscr{L}, F) + F^2$$

(resp. $\qquad 0 \leqslant (\omega_X, F) + (1-p)(\mathscr{L}, F) + F^2$

and $\qquad 0 < (\omega_X, F) + (1-p)(\mathscr{L}, F) + F^2)$

(resp. if the normalisation of F is not rational, not rational or elliptic).

(vi) *If $\mathscr{L}$ and $\mathscr{M}$ are numerically positive linebundles and there exists an embedding $\mathscr{L} \hookrightarrow \mathscr{M}$, then* (*) *for $\mathscr{L}$ implies* (*) *for $\mathscr{M}$.*

Proof. — A non-zero element of $H^1(X, \alpha_{\mathscr{L}^{-1}})$ gives a non-zero morphism $\mathscr{L}^p \to \Omega^1_{X/k}$ and so a non-zero morphism $\omega_X^{-1} \otimes \mathscr{L}^p \to T^1_{X/k}$ using that the wedge product gives an isomorphism $T^1_{X/k} = \Omega^1_{X/k} \otimes \omega_X^{-1}$. We then apply Theorem I:2.4 to get (iii)-(v). To get (i) we use (I:1.7), applied to a non-trivial $\alpha_{\mathscr{L}^{-1}}$-torsor Y and the two mappings $X^{(1)} \xrightarrow{\pi} Y \xrightarrow{\tau}$ for which $\tau \circ \pi$ is the Frobenius. As Y is integral and π dominant we get that $h^0(\tau^* \mathscr{M}) \leqslant h^0(\pi^* \tau^* \mathscr{M}) = h^0(\mathscr{M}^p)$. On the other hand, $h^2(\tau^* \mathscr{M})$ equals by (I:1.9) and duality $h^0(\tau^*(\mathscr{M}^{-1} \otimes \omega_X \otimes \mathscr{L}^{1-p}) \leqslant h^0(\mathscr{M}^{-p} \otimes \omega_X^p \otimes \mathscr{L}^{p(1-p)})$ as before. Hence $h^0(\mathscr{M}^p) + h^0(\mathscr{M}^{-p} \otimes \omega_X^p \otimes \mathscr{L}^{p(1-p)}) \geqslant \chi(\tau^* \mathscr{M})$. Now, tensoring (I:1.8) with $\mathscr{M}$ we get a filtration on $\mathscr{M} \otimes \tau_* \mathcal{O}_Y = \tau_* \tau^* \mathscr{M}$ and using additivity of χ and the Riemann-Roch formula we get (i). As for (ii), let us note (cf. [Ar-Mi]) that the exact sequence

$$0 \to \mathscr{L}^{-1} \to F_* \mathscr{L}^{-p} \to \mathscr{L}^{-1} \otimes B_1 \to 0 \tag{1.4}$$

obtained by tensoring the defining sequence for B_1 by $\mathscr{L}^{-1}$, shows that

$$H^0(X, \mathscr{L}^{-1} \otimes B_1) = H^1(X, \alpha_{\mathscr{L}^{-1}}). \tag{1.5}$$

Thus, a non-zero element of $H^1(X, \alpha_{\mathscr{L}^{-1}})$ gives an embedding $\mathscr{L} \hookrightarrow B_1$ and so $h^0(\mathscr{L}) \leqslant h^0(B_1)$. More precisely, if $\mathbf{P} := |H^0(X, \mathscr{L}^{-1} \otimes B_1)|$, we have a canonical section c of $\mathscr{L}^{-1} \otimes B_1(1)$ on $\mathbf{P} \times X$ and therefore an embedding $\mathscr{L}(-1) \to B_1$. Projecting down to $\mathbf{P}$ gives us an embedding

$$\mathcal{O}_{\mathbf{P}}(-1) \otimes_k H^0(X, \mathscr{L}) \to \mathcal{O}_{\mathbf{P}} \otimes_k H^0(X, B_1).$$

By the injectivity above this is a subbundle and so gives a mapping to G, the Grassman variety of $h^0(\mathscr{L})$-dimensional subspaces in $\mathrm{H}^0(\mathrm{X}, \mathrm{B}_1)$. By [Ta1: Cor. 3.2] and the assumption this is a constant map. Hence $\mathscr{O}_{\mathbf{P}}(-1)\otimes_{\mathbf{k}}\mathrm{H}^0(\mathrm{X}, \mathscr{L})$ is a trivial subbundle of $\mathscr{O}_{\mathbf{P}}\otimes_{\mathbf{k}}\mathrm{H}^0(\mathrm{X}, \mathrm{B}_1)$ and is, in particular, trivial. This implies that $\dim \mathbf{P} = 0$ or $h^0(\mathscr{L}) = 0$, which gives (ii). (I would like to give my thanks to E. Ballico who drew my attention to [Ta1] thus enabling me to improve on an earlier version of the theorem. Note also that in the limit case one can use [Ta2] to obtain further conclusions.) Finally, to get (vi) we note that $h^0(\mathscr{M}^{-1}\otimes \mathrm{B}_1) \neq 0$ implies $h^0(\mathscr{L}^{-1}\otimes \mathrm{B}_1) \neq 0$ and the same is true for any p^i-th power so we conclude by (1.5). □

We can now use all this to investigate Kodaira vanishing for surfaces of special type. It turns out that in that case we almost always have vanishing.

Theorem **1.6.** — *Let* X *be a smooth, proper and connected surface over* **k**, *not of general type. If* $\mathscr{L}$ *is a numerically positive line bundle on* X, *then* $\mathrm{H}^1(\mathrm{X}, \mathscr{L}^{-1}) = 0$ *except (possibly) when* X *is a quasi-elliptic surface of Kodaira dimension* 1. *If in the latter case* $\varphi : \mathrm{X} \to \mathrm{C}$ *is the canonical quasi-elliptic fibration then*

a) *if* $p = 2$ *then* $h^0(\mathscr{N}\otimes\mathscr{L}^{-1}) \neq 0$, *where* $\mathscr{N}$ *is the saturation in* $\mathrm{B}_{1,\mathrm{X}}$ *of* $\varphi^*\mathrm{B}_{1,\mathrm{C}}$. *Conversely, if* $h^0(\mathscr{N}\otimes\mathscr{L}^{-1}) \neq 0$ *then* $\mathrm{H}^1(\mathrm{X},\mathscr{N}^{-1}) \neq 0$.

b) *If* $p = 3$ *then* $h^0(\mathscr{M}\otimes\mathscr{L}^{-3}) \neq 0$, *where* $\mathscr{M}$ *is the saturation of* $\varphi^*\omega_{\mathrm{C}}$ *in* $\Omega^1_{\mathrm{X}/\mathbf{k}}$.

Proof. — Indeed, by (1.1) the problem is birational. If X has Kodaira dimension $-\infty$ we may therefore assume that there is a morphism $\mathrm{X} \to \mathrm{C}$ with $\mathbf{P}^1$ as general fiber. By the remark before (1.1) we may assume that there is a non-trivial $\alpha_{\mathscr{L}^{-p^i}}$-torsor for some $i > 0$ and then we get a contradiction by (1.3) (v). If X is of Kodaira dimension 0 or 1, we may assume that X is minimal. If X is abelian we use either (I: after 1.5) and the fact that the cotangent bundle is trivial or [Mu2: § 16]. If X is K3 we use (1.3) (ii) and the fact that $h^0(\mathscr{L}) \neq 0$ by the Riemann-Roch formula. If X is an Enriques surface we use (1.3) (ii) and the fact that $h^0(\mathscr{L}) \geqslant 2$ by the Riemann-Roch formula, whereas $h^0(\mathrm{B}_1) \leqslant h^{01} \leqslant 1$. If X is elliptic we use (1.3) (v) and the elliptic pencil as in the ruled case. Finally, if X is quasi-elliptic consider the canonical map $\varphi : \mathrm{X} \to \mathrm{C}$. Pulling back by the Frobenius map on C, normalising and resolving singularities gives us a ruled surface. Applying (1.2) and the result already obtained for ruled surfaces we get an inclusion of $\mathscr{L}^p$ in the annihilator in $\Omega^1_{\mathrm{X}/\mathbf{k}}$ of the associated 1-foliation. This annihilator is birationally equal to the image of $\varphi^*\Omega^1_{\mathrm{C}/\mathbf{k}}$, the quotient of this 1-foliation being (the pullback by σ^{-1} of) the relative Frobenius $\mathrm{F}_{\mathrm{X}/\mathrm{C}} : \mathrm{X}^{(1)} \to \mathrm{C}$, and hence equals its saturation on all of X. This gives the $p = 3$ case and we get similarly the $p = 2$ case using (I:1.11). □

Remark. — By [Ra] the exceptions do occur.

We have now come to one of the main results of this paper.

Theorem **1.7.** — *Let* X *be a minimal surface of general type and let* $\mathscr{L}$ *be a line bundle on* X, *numerically equivalent to* ω_X^i *for some* $i > 0$. *Then* $H^1(X, \mathscr{L}^{-1}) = 0$ *except possibly when* $p = 2$, $i = 1$ *and either* X *is almost rational or almost* K3, $\omega_X^2 \cong \mathscr{L}^2$, $h^1(\omega_X^{-1}) = h^1(\mathscr{L}^{-1}) = 1$ *and* $\chi = 1$ *or* $\omega_X \not\cong \mathscr{L}$, X *is almost ruled and* $\chi \leqslant 0$.

Proof. — We are clearly aiming to show that (*) is true for $\mathscr{L}$. Assume therefore that it is false. By replacing $\mathscr{L}$ by the last positive power of it for which $H^1 \neq 0$ we may assume that there is a non-trivial $\alpha_{\mathscr{L}^{-1}}$-torsor. As X is minimal, $\mathscr{L}$ is numerically positive and so all results obtained so far are applicable. In particular (1.3) (iii)-(iv) shows that X is almost ruled or almost K3 (the last is possible only when $i = 1$ and $p = 2$) where we take as $\mathscr{M}$, ω_X resp. any ample sheaf.

Assume now $b_1(X) \neq 0$. Then X is almost ruled and functoriality of the Albanese maps shows that X has 1-dimensional image in its Albanese variety. Hence we get a pencil on X without basepoints. Let F be a general fiber and let us assume that $p \neq 2$. Then by (1.3) (v) we get $(1 + i(1 - p))\,(K, F) \geqslant -2$. As $F^2 = 0$ we have $(K, F) \geqslant 2$ and we get a contradiction unless $i = 1$, $p = 3$ and $(K, F) = 2$, so we assume these equalities. Let $F : X \to C$ be the Albanese fibration and let $\pi : \overline{X} \to X$ be a non-trivial $\alpha_{\mathscr{L}^{-1}}$-torsor which exists by assumption. If $\eta = \mathbf{Spec}(\mathbf{k}(C)) \to C$ is the generic point of C, then $K := H^0(\eta, (\varphi\pi)_* \mathscr{O}_{\overline{X}}) \subset \mathbf{k}(\overline{X})$ and K is a field which is algebraic over $\mathbf{k}(C)$. As $[\mathbf{k}(\overline{X}) : \mathbf{k}(X)] = 3$ and $\mathbf{k}(C)$ is algebraically closed in $\mathbf{k}(X)$, we get $\dim_{k(C)} H^0(\eta, (\varphi\pi)_* \mathscr{O}_{\overline{X}}) \leqslant 3$. By generic base change we get $h^0((\pi_* \mathscr{O}_{\overline{X}})_{|F}) \leqslant 3$. The filtration $\{\mathscr{N}^i\}$ of (I:1.8) restricted to F gives exact sequences

$$0 \to \mathscr{O}_F \to \mathscr{N}^1_{|F} \to \mathscr{L}_{|F} \to 0$$

$$0 \to \mathscr{N}^1_{|F} \to \mathscr{N}^2_{|F} \to \mathscr{L}^{\otimes 2}_{|F} \to 0.$$

As $\mathscr{N}^2_{|F} = (\pi_* \mathscr{O}_{\overline{X}})_{|F}$, we get $h^0(\mathscr{N}^2_{|F}) \leqslant 3$. Now $g(F) = 2$ so the Riemann-Roch formula gives $h^0(\mathscr{N}^1_{|F}) - h^1(\mathscr{N}^1_{|F}) = 0$ and $h^0(\mathscr{L}^{\otimes 2}_{|F}) = 4$. The second exact sequence now gives $4 = h^0(\mathscr{L}^{\otimes 2}_{|F}) \leqslant h^1(\mathscr{N}^1_{|F}) - h^0(\mathscr{N}^1_{|F}) + h^0(\mathscr{N}^2_{|F}) \leqslant 3$.

Let us now assume $p = 2$ (*and still* $b_1(X) \neq 0$) and let φ and C have the meaning of above. We may also assume that $c_1^2(X) > 32$. In fact, if $\rho : Y \to X$ is an étale cover of degree prime to two with $H^1(Y, \rho^* \mathscr{L}^{-1}) = 0$ then, as $H^1(X, \mathscr{L}^{-1})$ is a direct factor of $H^1(Y, \rho^* \mathscr{L}^{-1})$ by the trace map, it is also zero and $\rho^* \mathscr{L}$ and Y fulfills the conditions of $\mathscr{L}$ and X. As $b_1(X) \neq 0$ such coverings with arbitrarily high $c_1^2(X)$ exist. Assume first that $\mathscr{L} \cong \omega_X^i$. From (1.3) (ii) we get $h^0(\mathscr{L}) \leqslant H^0(X, B_1)$ and as always $h^0(\omega_X^i) \geqslant h^0(\omega_X)$, we get $\dim(\ker F : H^1(X, \mathscr{O}_X) \to H^1(X, \mathscr{O}_X)) \geqslant p_g$. Clifford's theorem applied to $B_{1,C}$, which is a line bundle, gives, as has noted Serre [Se], $h^0(B_1, C) \leqslant (g + 1)/2$, $(g := g(C))$. As $H^1(C, \mathscr{O}_C) \hookrightarrow H^1(X, \mathscr{O}_X)$ we get

$$p_g \leqslant \dim(\ker F : H^1(X, \mathscr{O}_X)) \leqslant \dim(\ker F : H^1(C, \mathscr{O}_C)) + h^{01} - g,$$

and so $$(g - 1)/2 \leqslant h^{01} - p_g = -\chi(X) + 1.$$

Furthermore, as X is almost ruled, the general fibers of φ are rational (singular) and so $b_1(X) = 2g$, which gives $c_2(X) \geqslant -4(g-1)$ since $b_2(X) \geqslant 2$. As $c_1^2(X) > 0$, Noether's formula gives $\chi > (-g+1)/3$ and so $-\chi + 1 < (g+2)/3$, which together with the above gives $(g+2)/3 > (g-1)/2$ i.e. $g < 7$. This in turn gives $c_2(X) \geqslant -20$ and, as $h^{01} \geqslant h^0(B_1) \geqslant p_g$, $\chi \leqslant 1$. Therefore, $c_1^2 = 12\chi - c_2 \leqslant 12 + 20 = 32$.

If now only $\mathscr{L} \sim_{\text{num}} \omega_X^i$ (and so $i \geqslant 2$), then

$$h^0(\mathscr{L}) = \chi(\mathscr{L}) + h^1(\mathscr{L}) = \chi(\omega_X^i) + h^1(\mathscr{L}) = h^0(\omega_X^i) + h^1(\mathscr{L}),$$

the last equality as $h^1(\omega_X^i) = h^1(\omega_X^{-(i-1)}) = 0$ by what has just been proved. Hence we get as above $h^0(B_1) \geqslant h^0(\mathscr{L}) \geqslant h^0(\omega_X^i) \geqslant h^0(\omega_X)$ and we continue as before.

We are thus left with the case $b_1(X) = 0$. Noether's formula then gives $\chi \geqslant 1$. Applying (1.3) (i), with $\mathscr{M}$ trivial we get,

$$\left(i^2\binom{p}{3} + \frac{1}{2}(i^2 - i)\binom{p}{2}\right)K^2 + p\chi \leqslant 1 + h^0(\omega_X^p \otimes \mathscr{L}^{p(1-p)}).$$

As $h^0(\omega_X^p \otimes \mathscr{L}^{p(1-p)}) \leqslant 1$ for numerical reasons with equality iff $\omega_X^p \otimes \mathscr{L}^{p(1-p)}$ is trivial, in which case $i = 1$ and $p = 2$, we get a contradiction unless $i = 1$, $p = 2$, $\chi = 1$ and $\omega_X^{\otimes 2} \cong \mathscr{L}^{\otimes 2}$. *Let us therefore assume* $i = 1$, $p = 2$ *and* $\chi = 1$. Applying (1.3) (i), with $\mathscr{M} = \omega_X$, gives a contradiction unless $h^1(\omega_X^{\otimes 2}) \neq 0$ and for this we have only used $\chi \geqslant 1$. Let $\pi : Y \to X$ be a non-trivial $\alpha_{\mathscr{L}^{-1}}$-torsor and let $\tau : Z \to Y$ be the normalisation. By (I:1.11) Z is an $\alpha_{\mathscr{M}^{-1}}$-torsor for some $\mathscr{M} = \mathscr{L}(D)$, where D is an effective divisor. As Z is birational to a ruled surface or a K3-surface we can apply (1.2) and (1.6) to conclude that any non-trivial $\alpha_{\mathscr{L}^{-1}}$-torsor $Y' \to X$ has Z as its normalisation. This means that if $\rho : \mathscr{M} \to B_1$ is the map corresponding to Z through (1.5) then the map corresponding to Y' factors through ρ. Therefore

$$h^1(\mathscr{L}^{-1}) = h^1(\alpha_{\mathscr{L}^{-1}}) = h^0(\mathscr{L}^{-1} \otimes \mathscr{M}) = h^0(\mathcal{O}_X(D)).$$

Suppose we can show that $h^1(\mathcal{O}_D) = 0$. Then it follows that $h^1(\mathcal{O}_{D'}) = 0$ for any $0 \neq D' \leqslant D$ as we have a surjection of structure sheaves of 1-dimensional schemes. Therefore $(K, D') + D'^2 \leqslant -2$ by the adjunction formula and so $|D|$ can have no moving part, thence $h^0(\mathcal{O}_X(D)) = 1$. To show that $h^1(\mathcal{O}_D) = 0$ we note that by (I:1.9) ω_Y has order 1 or 2 and by (I:1.8) $\chi(Y) = 2$ and so $h^0(\omega_Y) \geqslant 1$, which means that ω_Y is trivial and hence $h^1(\mathcal{O}_Y) = 0$. This shows that $h^0(\tau_* \mathcal{O}_Z/\mathcal{O}_Y) = 0$. On the other hand, the following diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \pi_*\mathcal{O}_Y & \longrightarrow & \omega_X & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & (\pi\tau)_*\mathcal{O}_Z & \longrightarrow & \omega_X(D) & \longrightarrow & 0
\end{array}$$

shows that $\pi_*(\tau_* \mathcal{O}_Z/\mathcal{O}_Y) = \omega_X(D)_{|D} = \omega_D$. Therefore, $h^1(\mathcal{O}_D) = h^0(\omega_D) = 0$. □

Remark. — We have seen (I:2.14) that there exists minimal surfaces with $h^1(\omega_X^{-1}) \neq 0$ with K_X^2 taking any value between 2 and 9. This leaves open the case $K_X^2 = 1$. I do not know if $h^1(\omega_X^{-1}) \neq 0$ is possible in that case but can show that an example must fulfill extremely stringent conditions.

Corollary **1.8.** — *Let* X *be a minimal surface of general type. Then*

$$P_2 \geqslant K^2 + \chi \geqslant 2,$$

$$h^{01} \leqslant K^2 - 1 \quad \text{if } p_g = 0,$$

$$h^{01} \leqslant K^2 + 2 - p_g.$$

Proof. — Indeed, by the Riemann-Roch formula, $P_2 \geqslant K^2 + \chi$. If $\chi \geqslant 1$ then $K^2 + \chi \geqslant 1 + 1 = 2$. If $\chi \leqslant 0$ then $P_2 = K^2 + \chi$ by the theorem and by the classification of surfaces, $P_2 \geqslant 1$. The second inequality follows from the first. Also by the Clifford argument, $2(p_g - 1) \leqslant P_2 - 1 = K^2 + \chi - 1$ (resp. $K^2 + \chi$ if $h^1(\omega_X^{-1}) \neq 0$) as we have just seen, which gives the third inequality except when $h^1(\omega_X^{-1}) \neq 0$. Hence we are left with the possibilities that $K^2 + \chi = 1$ and then $\chi \leqslant 0$ and $P_2 = 1$ or that $h^1(\omega_X^{-1}) \neq 0$. *Assume the first possibility.* Hence $p_g \leqslant 1$ and by (0:1.6) this implies that $h^{01} \leqslant 2$. On the other hand, as $\chi \leqslant 0$, $c_2 < 0$ and this gives $b_1 \geqslant 4$, the possibility $b_1 = 2$ giving $b_2 \geqslant 2$ as the Albanese mapping has 1-dimensional image. This gives only the possibility that $h^{01} = q = 2$. If $X \to \operatorname{Alb} X$ is surjective we get $b_2(X) \geqslant b_2(\operatorname{Alb} X) = 6$ and then $c_2(X) \geqslant 0$. Hence the Albanese mapping gives a map $X \to C$ where C is a smooth curve of genus 2. Let $C' \to C$ be an étale connected covering of degree m, which is prime to p. The induced covering $X' \to X$ is also connected and $h^{01}(X') \geqslant g(C')$. As $g(C') - 1 = m(g(C) - 1)$, we get $h^{01}(X') - 1 \geqslant m(g(C) - 1) = m(h^{01}(X) - 1)$ and as $\chi(X') = m\chi(X)$, we get $mp_g(X) \leqslant p_g(X')$. Now $\chi(X') \leqslant 0$, so by what we have already proved, $h^{01}(X') \leqslant K^2_{X'} + 2 - pg(X')$ and as $K^2_{X'} = mK^2_X$, we get

$$m(h^{01}(X) - 1) \leqslant h^{01}(X') - 1 \leqslant K^2_{X'} + 1 - p_g(X') \leqslant mK^2_X + 1 - mp_g(X')$$

i.e. $h^{01} \leqslant K^2 + 1/m + 1 - p_g$. As m can be arbitrarily large, this gives $h^{01} \leqslant K^2 + 1 - p_g$ i.e. $2p_g \leqslant K^2 + \chi = 1$ which implies $p_g = 0$ and so $\chi = -1$. On the other hand as $b_1 = 4$, we get $c_2 \geqslant -5$ and so by Noether's formula $-1 \geqslant -5/12$.

We are therefore left with the possibility that $h^1(\omega_X^{-1}) \neq 0$ *and* $h^{01} = K^2 + 2 - p_g$, $h^{01} = p_g$ *and* $P^2 - 1 = 2(p_g - 1)$. Hence $p_g = K^2/2 + 3/2$ and $|2K| = 2|K|$, as we get equality from the Clifford argument. If $|K|$ is not composed with a pencil this gives a contradiction as then $X \dashrightarrow |2K|$ has 2-dimensional image and so the general member of the moving part is integral by Bertini's theorem [Jo] which contradicts $|2K| = 2|K|$. Therefore, by (0:1.3), $K^2 = 1$ and $|K|$ is a pencil with a single basepoint x and as $K^2 = 1$ all members of $|K|$ intersect transversally at x. Blowing up x to get $\mu : \widetilde{X} \to X$ we have a morphism $\pi : \widetilde{X} \to \mathbf{P}^1$. Furthermore, as x is a smooth point on all members of $|K|$, μ is an isomorphism on all fibers of π. By [Bo: 4.1] these fibers are 2-connected and so [Ram: Lemma 3], π is cohomologically flat and (0:1.2), we have an everywhere defined morphism $\tau : \widetilde{X} \to \mathbf{P}(\pi_* \, \omega_{\widetilde{X}/P^1})$. Now, by duality and cohomological flatness, $R^1 \pi_* \, \omega_{\widetilde{X}/P^1} = \mathcal{O}_{P^1}$ and by the formula $\omega_{\widetilde{X}} = \omega_{\widetilde{X}/P^1} \otimes \omega_{P^1}$ and the projection formula,

$$\pi_* \, \omega_{\widetilde{X}} = \pi_* \, \omega_{\widetilde{X}/P^1} \otimes \mathcal{O}_{P^1}(-2)$$

and
$$R^1 \pi_* \, \omega_{\widetilde{X}} = \mathcal{O}_{P^1}(-2).$$

Hence, putting $\mathscr{E} := \pi_* \, \omega_{\widetilde{X}/P^1}$ and using the Leray spectral sequence for π, we get $2 = h^0(\omega_X) = h^0(\mathscr{E}(-2))$ and $2 = h^1(\omega_X) = h^1(\mathscr{E}(-2)) + h^0(\mathcal{O}_{P^1}(-2)) = h^1(\mathscr{E}(-2))$. As $\mathrm{rk}\,\mathscr{E} = 2$, we can write $\mathscr{E} \cong \mathcal{O}_{P^1}(u) \oplus \mathcal{O}_{P^1}(v)$, with $u \geqslant v$ and so

$$2 = h^0(\mathcal{O}_{P^1}(u-2)) + h^0(\mathcal{O}_{P^1}(v-2))$$

and
$$2 = h^1(\mathcal{O}_{P^1}(u-2)) + h^1(\mathcal{O}_{P^1}(v-2)).$$

This immediately gives $u = 3$ and $v = -1$, as $h^0(\mathcal{O}_{P^1}(w))\, h^1(\mathcal{O}_{P^1}(w)) = 0$ for any w. Now, if $s \in \mathbf{P}^1$, then $\omega_{\widetilde{X}_s} = \mu^* \omega_{X/\widetilde{X}_s}$ and so τ contracts exactly the -2-curves, as no -2-curve on X contains x. In fact, if it did it would meet all members of $|\,K\,|$ and hence be contained in them having zero intersection with K. Hence we get a diagram

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "\lambda"] \arrow[rr, bend left, "\tau"] \arrow[dr, "\pi"'] & \hat{X} \arrow[r, "\hat{\tau}"] \arrow[d, "\hat{\pi}"] & \mathbf{P}(\mathscr{E}) \arrow[dl, "\rho"] \\
& \mathbf{P}^1 &
\end{tikzcd}$$

where λ blows down the -2-curves not meeting the exceptional divisor, $\widetilde{X}$ is the canonical model of X with a smooth point blown up and $\hat{\tau}$ now is a finite map. As $\hat{\tau}$ is fiber by fiber of degree 2 it is of degree 2 and as $\mathbf{P}(\mathscr{E})$ is regular and $\widetilde{X}$ is Cohen-Macaulay, we get, by (I:1.11), that $\hat{\tau}$ is an α_s-torsor for a line bundle $\mathscr{L}$ on $\mathbf{P}(\mathscr{E})$ and a global section s of $\mathscr{L}$. Hence we have an exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}(\mathscr{E})} \to \hat{\tau}_* \, \mathcal{O}_{\hat{X}} \to \mathscr{L}^{-1} \to 0,$$

giving rise to a long exact sequence

$$\begin{array}{ccccccccccccc}
0 \to & \rho_* \, \mathcal{O}_{\mathbf{P}(\mathscr{E})} & \to & \hat{\pi}_* \, \mathcal{O}_{\hat{X}} & \to & \rho_* \, \mathscr{L}^{-1} & \to & R^1 \rho_* \, \mathcal{O}_{\mathbf{P}(\mathscr{E})} & \to & R^1 \hat{\pi}_* \, \mathcal{O}_{\hat{X}} & \to & R^1 \pi_* \, \mathscr{L}^{-1} & \to 0 \\
& \| & & \| & & & & \| & & & & & \\
& \mathcal{O}_{P^1} & & \mathcal{O}_{P^1} & & & & 0 & & & & &
\end{array}$$

giving $\rho_* \mathscr{L}^{-1} = 0$ and $R^1 \rho_* \mathscr{L}^{-1} \cong R^1 \hat{\pi}_* \mathcal{O}_{\hat{X}} \cong R^1 \pi_* \mathcal{O}_{\tilde{X}}$, the last as λ only contracts -2-curves. On the other hand, by duality,

$$R^1 \pi_* \mathcal{O}_{\tilde{X}} = \mathcal{H}om_{\mathcal{O}_{P^1}}(\pi_* \omega_{\tilde{X}/P^1}, \mathcal{O}_{P^1}) \cong \mathcal{O}_{P^1}(-3) \oplus \mathcal{O}_{P^1}(1)$$

and if $\mathscr{L}^{-1} \cong \mathcal{O}_{P(\mathscr{E})}(t) \otimes \rho^* \mathcal{O}_{P^1}(r)$, then

$$R^1 \rho_* \mathscr{L}^{-1} \cong \begin{cases} 0 & \text{if } t \geqslant -1, \\ (S^{-t-2} \mathscr{E})^* \otimes \mathcal{O}_{P^1}(r-2) & \text{if } t \leqslant -2. \end{cases}$$

Hence we get $\mathscr{L}^{-1} \cong \mathcal{O}_{P(\mathscr{E})}(-3) \otimes \rho^* \mathcal{O}_{P^1}(2)$. By (I:1.10) we have $\alpha_s = \mathrm{Cone}(\mathscr{L} \xrightarrow{F-s} \mathscr{L}^2)$. Taking direct images and noting that $R^1 \pi_* \mathscr{L} = R^1 \pi_* \mathscr{L}^2 = 0$, we get

$$R\rho_* \alpha_s = \mathrm{Cone}(\rho_* \mathscr{L} \xrightarrow{F-s} \rho_* \mathscr{L}^2).$$

Now we have

$$\rho_* \mathscr{L} \cong \mathcal{O}_{P^1}(7)\, v^3 \oplus \mathcal{O}_{P^1}(3)\, v^2 w \oplus \mathcal{O}_{P^1}(-1)\, vw^2 \oplus \mathcal{O}_{P^1}(-5)\, w^3$$

and

$$\rho_* \mathscr{L}^2 \cong \mathcal{O}_{P^1}(14)\, v^6 \oplus \mathcal{O}_{P^1}(10)\, v^5 w \oplus \mathcal{O}_{P^1}(6)\, v^4 w^2 \oplus \mathcal{O}_{P^1}(2)\, v^3 w^3 \oplus \mathcal{O}_{P^1}(-2)\, v^2 w^4 \oplus \mathcal{O}_{P^1}(-6)\, vw^5 \oplus \mathcal{O}_{P^1}(-10)\, w^6,$$

where we have written $\mathscr{E} = \mathcal{O}_{P^1}(3)\, v \oplus \mathcal{O}_{P^1}(-1)\, w$ to keep track of fiberwise homogeneity. Hence we can write $s = av^3 + bv^2 w$. We now want to prove that the map induced by $F - s$, $\varphi : H^1(\mathbf{P}^1, \rho_* \mathscr{L}) \to H^1(\mathbf{P}^1, \rho_* \mathscr{L}^2)$ is injective. Indeed, $H^1(\mathbf{P}^1, \rho_* \mathscr{L}) = H^1(\mathbf{P}^1, \mathcal{O}_{P^1}(-5))\, w^3$ and the composite of φ with the projection on $H^1(\mathbf{P}1, \mathcal{O}_{P^1}(10))\, w^6$ is the Frobenius map as $s = av^3 + bv^2 w$ is divisible by v. However,

$$F : H^1(\mathbf{P}^1, \mathcal{O}_{P^1}(n)) \to H^1(\mathbf{P}^1, \mathcal{O}_{P^1}(pn))$$

is always injective, which is immediately seen on the standard Čech cocycles.

We thus see, using the Leray spectral sequence, that

$$H^1(\mathbf{P}(\mathscr{E}), \alpha_s) = \mathrm{coker}(\varphi : H^0(\mathbf{P}^1, \rho_* \mathscr{L}) \to H^0(\mathbf{P}^1, \rho_* \mathscr{L}^2)).$$

Let $r \in H^0(\mathbf{P}^1, \rho_* \mathscr{L}^2)$ be a representative for $\check{X}$. Writing $r = cv^6 + dv^5 w + ev^4 w^2 + fv^3 w^3$, the equation for $\check{X}$ is $z^2 + sz = r$. However, as v divides s and v^2 divides r, we see that $\tilde{X}$ is singular over $\{ v = 0 \}$ and so is non-normal, contrary to assumption. $\square$

Corollary **1.9.** — *Let* X *be a minimal surface of general type and* $\mathscr{L}$ *a numerically positive line bundle on* X. *If* $h^1(\mathscr{L}^{-1}) \neq 0$ *then*

$$\left(\binom{p}{3} + \frac{1}{2}\binom{p}{2}\right) \mathscr{L}^2 + \frac{1}{2}\binom{p}{2}(K_X, \mathscr{L}) + (p-1)\chi \leqslant \binom{p}{2} K_X^2 + \alpha,$$

where α *is zero if* $h^1(\omega_X^{-1}) = 0$ *and* 1 *if not. If we have equality, then there is a non-trivial* $\alpha_{\mathscr{L}^{-1}}$*-torsor* $\rho : Y \to X$ *s.t.* $|pK_X| = \tau^* |\rho^{(-1)*} K_{X^{(-1)}}|$ *where* $\tau : X \to Y^{(-1)}$ *and* $\rho^{(-1)} \circ \tau$ *is the Frobenius map.*

Proof. — This follows from (1.3) (i) applied to ω_X and (1.7) and the case of equality follows by analysing the proof of (1.3) (i). □

In the next section we will use the following result which, however, seems to fit in most naturally here.

Lemma **1.10.** — *Let* X *be a minimal surface of general type and assume that* $h^1(\omega_X^{-1}) \neq 0$ *and* $p_g = 1$. *Then the bicanonical system has no basepoints,* $H^0(X, \omega_X^2)_{|C} = H^0(C, \omega_C)$ *where* C *is a canonical curve and* $X \to |2K|$ *factors as* $X \to Y \to |2K|$ *where* $Y^{(1)} \to X$ *is a non-trivial* $\alpha_{\omega_X^{-1}}$*-torsor and* $X \to Y \to X^{(-1)}$ *the Frobenius map.*

Proof. — Pick $C \in |K_X|$ and consider the exact sequence

$$0 \to \omega_X \to \omega_X^2 \to \omega_C \to 0.$$

Now, $h^2(\omega_X^2) = h^0(\omega_X^{-1}) = 0$ and $h^1(\omega_C) = h^0(\mathcal{O}_C) = 1$ as C is 1-connected [Bo: 4.1], so the long exact sequence gives that $H^1(C, \omega_C) \to H^2(X, \omega_X)$ is surjective and hence injective. As $h^1(\omega_X^2) = 1 = h^1(\omega_X)$, the first by Theorem 1.7, the second also by Theorem 1.7 and the assumption $p_g = 1$, we get $H^1(X, \omega_X) \to H^1(X, \omega_X^2)$ surjective and hence injective and so $H^0(X, \omega_X^2)_{|C} = H^0(C, \omega_C)$. Now, C is 2-connected *(loc. cit.)* and so by (0:1.2) $|\omega_C|$ is without base points as is then $|\omega_X^2|$. The second statement has already been proved and as for the third we use (1.9). □

III. Pluricanonical systems on surfaces of general type

We are now going to apply our vanishing criteria to the study of pluricanonical systems on surfaces of general type. Quite some preliminary work will be needed to get numerically positive line bundles to which section II can be applied. A large part of this work is necessary only to cover the case of small m. Let us therefore begin with sketching a proof of (iii) (and consequently (ii)) of the Main theorem in the case when $m \geqslant 8$. Let us first prove that $|(m+1)K_X|$ separates 2 points x and y not lying on a -2-curve. If we blow up these two points to get $\pi : \widetilde{X} \to X$ with exceptional divisors L and M then the long exact sequence of cohomology shows that this result is implied by the vanishing of $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}((m+1)\pi^* K_X - L - M))$ which by duality is equivalent to the vanishing of $H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-(m\pi^* K_X - 2L - 2M)))$. As the pullback of $\mathcal{O}_{\widetilde{X}}(-(m\pi^* K_X - 2L - 2M))$ to X equals $\mathcal{O}_X(-mK_X)$ this vanishing is implied by (II:1.1) and (II:1.7) as soon as we know that $\mathcal{O}_{\widetilde{X}}(m\pi^* K_X - 2L - 2M)$ is numerically positive. If this is not the case then, as the self intersection of it certainly is positive, there is a curve $\widetilde{C}$ on $\widetilde{X}$ with negative intersection with $m\pi^* K_X - 2L - 2M$. Clearly $\widetilde{C}$ cannot be one of the exceptional curves L and M so it is the strict transform of some curve C on X and the condition that $\widetilde{C}$ have negative intersection with $m\pi^* K_X - 2L - 2M$ translates to $m(K, C) < 2(r+s)$ where $r := \operatorname{mult}_x C$ and $s := \operatorname{mult}_y C$. Using the index formula which gives $K^2 C^2 \leqslant (K, C)^2$ and the adjunction formula

$$-2 \leqslant 2g(\widetilde{C}) - 2 = (K, C) + C^2 - r(r-1) - s(s-1)$$

one quickly gets a contradiction (the appropriate inequalities are given in (1.1) (i)). One then proves in a similar fashion that $|(m+1)\,K_X|$ separates tangents at points not on a -2-curve. The case of points on -2-curves is a little more difficult. As in [Bo] to separate two different fundamental cycles Z_λ and Z_μ one needs to know that $H^1(X, \mathcal{O}_X(-(mK_X - Z_\lambda - Z_\mu))) = 0$. First one shows that $mK_X - Z_\lambda - Z_\mu$ is numerically positive. If not there is a curve C with negative intersection with $mK_X - Z_\lambda - Z_\mu$ and this time one reaches a contradiction by applying the index formula to the group in NS(X) spanned by K, $Z_\lambda + Z_\mu$ and C (the appropriate inequalities are to be found in (1.1) (iii)). In trying to show that $H^1(X, \mathcal{O}_X(-(mK_X - Z_\lambda - Z_\mu))) = 0$ one can now try to apply the same techniques we used to show the vanishing of $H^1(X, \omega_X^{-m})$. The case when $b_1(X) = 0$ is almost exactly the same and for the case when $b_1(X) \neq 0$ one notes that as Z_λ and Z_μ are unions of rational curves they necessarily lie in fibers of the Albanese mapping and so has zero intersection with a fiber; one then continues as in the mK_X-case. The rest of part (ii) of the Main theorem for $m \geqslant 8$ then proceeds in a similar way.

More complications arise when we want to consider the case of smaller m. The inequalities we obtain from the index theorem and the adjunction formula are then not enough to exclude the possibility of there being a curve C with negative intersection with one of the line bundles above; we get instead that there are at most a finite number of possibilities for the pairs (K, C), C^2 (the possibilities are enumerated in Definition 1.12 and the arguments leading up to that enumeration are given in 1.13). If we consider for instance the case of separating tangents of a point x not on a -2-curve, then we are saved by the fact that any such curve C contains x as a point of multiplicity 2 and that it hence suffices to show that $|(m+1)\,K_X|$ restricts to the complete linear system $|(\omega_X^{m+1})_{|C}|$ and that $|(\omega_X^{m+1})_{|C}|$ separates tangents of x on C. That the latter statement is true is proved in Lemma 1.16 and it then remains to investigate the truth of the former. Clearly it is implied by the vanishing of $H^1(X, \mathcal{O}_X((m+1)\,K_X - C))$ and so by duality by the vanishing of $H^1(X, \mathcal{O}_X(-(mK_X - C)))$. To show that this group vanishes it is as before desirable to have that $mK_X - C$ is numerically positive. In Proposition 1.14 it is shown that this is almost always true except for the ever present possibility that there is a -2-curve with (strictly) positive intersection with C. In any case we can add any curve D contradicting the numerical positivity of $mK_X - C$ to C and ask if $mK_X - C'$ is numerically positive where $C' := C + D$ which is the same problem only that C' is now an effective divisor. It is shown in (1.17) that this process eventually stops and so there is an effective divisor C such that to finish the proof it would suffice to show the vanishing of $H^1(X, \mathcal{O}_X(-(mK_X - C))$ and $mK_X - C$ is numerically positive. To show the required vanishing we apply the results of section II; this is done in (1.18) which gives almost all the results needed to finish the proof of the Main theorem. In Theorem 1.20 we finally gather together all the results and finish the few cases left open from (1.18) by more explicitly invoking the hypothesis that the Main theorem is false in order to arrive at a contradiction.

1. For $x \in \mathbf{R}$ put $[x] :=$ the largest integer $\leqslant x$, $\{x\} :=$ the largest integer $< x$ and $[\![x]\!] := |\, x -$ closest integer $|$.

Lemma **1.1.** — *Let* X *be a smooth, proper and connected surface with* $K^2 > 0$.

(i) *Let* $x, y \in X$ *and* D *be a curve on* X *s.t.* $(K, D) \geqslant 0$. *Put* $r := \text{mult}_x D$ *and* $\alpha := \text{mult}_x D + \text{mult}_y D$. *Let* $\lambda \in \mathbf{R}_+$ *have the property that* $(K, D) < \lambda\alpha$. *Then, if* $0 < \beta \leqslant K^2$,

$$2 + \{\alpha\lambda\} + \left[\frac{\{\alpha\lambda\}^2}{\beta}\right] - \frac{\alpha^2}{2} + \alpha \geqslant 2\left(r - \frac{\alpha}{2}\right)^2. \tag{1.2}$$

In particular,

$$2 + \{\alpha\lambda\} + \left[\frac{\{\alpha\lambda\}^2}{\beta}\right] - \frac{\alpha^2}{2} + \alpha \geqslant 2\left[\!\!\left[\frac{\alpha}{2}\right]\!\!\right]^2, \tag{1.3}$$

and if $1 > 2\lambda^2/\beta$,

$$\alpha \leqslant \left[\frac{\lambda + 1 + \sqrt{5 + \lambda^2(1 - 8/\beta) + 2\lambda}}{1 - 2\lambda^2/\beta}\right]. \tag{1.4}$$

(ii) *Let* $x \in X$, D *and* r *be as in* (i). *Let* $\lambda \in \mathbf{R}_+$ *be s.t.* $(K, D) < \lambda r$. *Then, if* $0 < \beta < K^2$,

$$2 + \{r\lambda\} + \left[\frac{\{r\lambda\}^2}{\beta}\right] - r^2 + r \geqslant 0 \tag{1.5}$$

and, if $1 > \lambda^2/\beta$,

$$r \leqslant \left[\frac{\lambda + 1 + \sqrt{5 + \lambda^2(1 - 4/\beta) + 2\lambda}}{2(1 - \lambda^2/\beta)}\right]. \tag{1.6}$$

(iii) *Let* C *be a divisor and* D *a curve on* X *s.t.* $(K, D) > 0$. *For any* $m \geqslant (K, C)/K^2$ *for which* $(C, D) \geqslant m(K, D) + 1$ *we have*

$$\begin{aligned} - (mK - C)^2 (K, D)^2 + ((K, C)^2 - K^2 C^2 + 2(K, C) - 2mK^2)\,(K, D) \\ + 2(K, C)^2 - 2K^2 C^2 - K^2 \geqslant 0. \end{aligned} \tag{1.7}$$

In particular, if

$$(mK - C)^2 > 0$$

and

$$1/2(K, C)^2 - C^2 + (2m + 1)\,(K, C) \leqslant (m^2 + m + 1/2C^2)\,K^2,$$

then

$$K^2(- m^2 - 3C^2 - 2m - 1) + 2(m + 1)\,(K, C) - C^2 + 3(K, C)^2 \geqslant 0. \tag{1.8}$$

If equality holds in (1.7), *then* $(C, D) = m(K, D) + 1$ *and either* $(K, D) + D^2 = -2$ *or* $K^2 C^2 = (K, C)^2$, *and* K, C *and* D *are in any case linearly dependent in* Num(X). *If equality holds in* (1.8), *then also* $(K, D) = 1$.

(iv) *Let* C *be a divisor and* D *a curve on* X. *Let* $x \in X$ *and* $r := \mathrm{mult}_x D$. *Suppose* $0 < \lambda \leqslant (K, D)$ *and* $m(K, D) < (C, D) + 2r$, *for some* λ *and* m. *Then either*

$$\left(m - \frac{(K, C)}{K^2}\right)(K, D) < 2r \tag{1.9}$$

or

$$\alpha^2 - \frac{1/\lambda((K, C)^2 - K^2 C^2) - 4(K, C) + 4mK^2}{4K^2 + (K, C)^2 - K^2 C^2}\alpha \leqslant \frac{(\lambda + 2)((K, C)^2 - K^2 C^2) + 2m\lambda^2(K, C) - K^2\lambda^2 m^2 - \lambda^2 C^2}{\lambda^2(4K^2 + (K, C)^2 - K^2 C^2)} \tag{1.10}$$

where $\alpha := r/(K, D)$.

Proof. — Indeed, to prove (i) we note that $(K, D) \leqslant \{\alpha\lambda\}$ and that by the index theorem $D^2 \leqslant [(K, D)^2/K^2] \leqslant [\{\alpha\lambda\}^2/\beta]$. If $s := \mathrm{mult}_y D$, the adjunction formula gives

$$\begin{aligned}(K, D) + D^2 &= r(r - 1) + s(s - 1) + g(D') - 2\\ &\geqslant r(r - 1) + s(s - 1) - 2,\end{aligned}$$

where D' is the strict transform of D on the blowing up at x and y. This gives $2 + \{\alpha\lambda\} + [\{\alpha\lambda\}^2/\beta] \geqslant r(r - 1) + (\alpha - r)(\alpha - r - 1)$ i.e. (1.2) and (1.3) follows as r is an integer. Estimating the right hand side of (1.3) by 0, $\{\alpha\lambda\}$ by $\alpha\lambda$ and $[\{\alpha\lambda\}^2/\beta]$ by $(\lambda^2/\beta)\,\alpha^2$ and completing the square gives (1.4). To get (ii) we simply let $y \notin D$ and hence $\alpha = r$ in (i). As for (ii), we first note that the index theorem applied to the space spanned by K, C and D gives, as $K^2 > 0$,

$$0 \leqslant \begin{vmatrix} K^2 & (K, C) & (K, D)\\ (K, C) & C^2 & (C, D)\\ (K, D) & (C, D) & D^2\end{vmatrix} = -D^2((K, C)^2 - K^2 C^2) + 2(K, C)(K, D)(C, D) - K^2(C, D)^2 - C^2(K, D)^2. \tag{1.11}$$

The adjunction formula gives $(K, D) + D^2 \geqslant -2$ and as $(K, C)^2 \geqslant K^2 C^2$, by the index theorem, we get

$$-D^2((K, C)^2 - K^2 C^2) \leqslant ((K, D) + 2)((K, C)^2 - K^2 C^2).$$

Furthermore, as $K^2 > 0$, $x \mapsto 2(K, C)(K, D)x - K^2 x^2$ has a maximum at $x = (K, C)/K^2(K, D)$ and decreases after this point. Hence the assumptions imply that

$$\begin{aligned}&2(K, C)(K, D)(C, D) - K^2(C, D)^2\\ &\qquad \leqslant 2(K, C)(K, D)(m(K, D) + 1) - K^2(m(K, D) + 1)^2.\end{aligned}$$

These two inequalities together with (1.11) give (1.7). The additional inequalities in (iii) imply that the left hand side of (1.7) has a single maximum as a function of (K, D) and that $(K, D) = 1$ lies to the right of it. Hence as $(K, D) \geqslant 1$, we can replace (K, D) by 1 in (1.7) and then we get (1.8). Finally, the conditions implied by equality are easily found by following the proof and noting that by the index theorem again, equality in (1.11) implies linear dependence in Num(X).

To prove (iv) we divide by $(K, D)^2$ in (1.11) and then use

$$(K, D) + D^2 \geqslant r(r-1) - 2,$$

which gives

$$-\frac{D^2}{(K, D)^2} \leqslant \frac{1}{\lambda} + \frac{2}{\lambda^2} - \alpha^2 + \frac{\alpha}{\lambda}.$$

We then proceed as in (iii). □

Definition **1.12.** — *Let* C *be a curve (effective divisor) on a minimal surface of general type. We say that* C *is a*

$$\begin{aligned}
&-3\text{-}\mathit{curve}\ (\mathit{divisor})\ \mathit{if}\ (K, C) = 1, \quad C^2 = -3,\\
&(-2, 2)\text{-}\mathit{curve}\ (\mathit{divisor})\ \mathit{if}\ (K, C) = 2, \quad C^2 = -2,\\
&-1\text{-}\mathit{curve}\ (\mathit{divisor})\ \mathit{if}\ (K, C) = 1, \quad C^2 = -1,\\
&0\text{-}\mathit{curve}\ (\mathit{divisor})\ \mathit{if}\ (K, C) = 2, \quad C^2 = 0,\\
&1\text{-}\mathit{curve}\ (\mathit{divisor})\ \mathit{if}\ (K, C) = 1, \quad C^2 = 1,\\
&2\text{-}\mathit{curve}\ (\mathit{divisor})\ \mathit{if}\ (K, C) = 2, \quad C^2 = 2,\\
&3\text{-}\mathit{curve}\ (\mathit{divisor})\ \mathit{if}\ (K, C) = 3, \quad C^2 = 3,\\
&4\text{-}\mathit{curve}\ (\mathit{divisor})\ \mathit{if}\ (K, C) = 2, \quad C^2 = 4.
\end{aligned}$$

We will call any such curve (divisor) distinguished.

Remark. — This disagrees with standard terminology as far as " $-$ 1-curve " is concerned. As our $-$ 1-curves are defined for minimal surfaces no confusion should arise. Note also that " $-$ 2-curve " will have its usual meaning.

Proposition **1.13.** — *Let* X *be a minimal surface of general type.*

(i) *Let* $x \neq y \in X$, *let* $\pi : \widetilde{X} \to X$ *be the blowing up of* x *and* y *and let* L *resp.* M *be the exceptional divisors. Suppose that* $m\pi^* K - 2L - 2M$ *is not numerically positive. Then, if* $m \geqslant 4$, *there is an elliptic or rational* $-$ 1-, 1- *or* 4-*curve passing through both* x *and* y *or a* $-$ 2-*curve on which* x *or* y *lies. If* $K^2 \geqslant 2$ *and* $m = 3$ *there is an elliptic or rational* $-$ 3-, $-$ 1-, 0-, 2- *or* 3-*curve passing through both* x *and* y *or a* $-$ 1-*curve having* x *or* y *as singular point, or a* $-$ 2-*curve on which* x *or* y *lies. In all cases the curve in question is a base curve of* $mK - 2x - 2y$.

(ii) *Let* $x \in X$, *let* $\pi : \widetilde{X} \to X$ *be the blowing up of* x *and let* L *be the exceptional divisor. Suppose* $m\pi^* K - 2L$ *is not numerically positive. Then if* $m \geqslant 3$ *there is an elliptic or rational* $-$ 1-, 1- *or a* $-$ 2-*curve passing through* x. *If* $K^2 \geqslant 2$ *and* $m = 2$ *there is a* $-$ 1- *or elliptic or rational* 2-*curve having* x *as singular point or a* $-$ 2-*curve on which* x *lies. In all cases the curve in question is a base curve of* $mK - 2x$.

(iii) *Let* $x \in X$, *let* $\pi : \widetilde{X} \to X$ *be the blowing up of* x *and let* L *be the exceptional divisor. Suppose* $m\pi^* K - 3L$ *is not numerically positive. Then if* $m \geqslant 4$ *there is an elliptic or rational* -1-, 1-, 2- *or* 4-*curve having* x *as singular point or a* -2-*curve passing through* x. *If* $K^2 \geqslant 2$ *and* $m = 3$ *there is a* -1- *or elliptic or rational* 2-*curve having* x *as singular point or a* -2-*curve on which* x *lies. In all cases the curve in question is a base curve of* $mK - 3x$.

(iv) *Let* Z_λ *and* Z_μ *be two disjoint fundamental cycles* (*i.e. maximal* -2-*divisors*). *If* $m \geqslant 4$ *or* $m = 3$ *and* $K^2 \geqslant 2$, *then* $mK - Z_\lambda - Z_\mu$ *is numerically positive.*

(v) *Let* Z *be a fundamental cycle. Then* $2K - Z$ *is numerically positive and so, in particular, if* $m \geqslant 4$, $mK - 2Z$ *is numerically positive. If* $K^2 \geqslant 2$ *and* $3K - 2Z$ *is not numerically positive then there is a* -3-*curve* C *with* $(Z, C) = 2$ *and* C *is a base curve of* $3K - 2Z$.

(vi) *Let* Z *be a fundamental cycle, let* $x \in X$, $x \notin a -2$-*curve, let* $\pi : \widetilde{X} \to X$ *be the blowing up of* x *and let* L *be the exceptional divisor. If* $m \geqslant 4$ *then* $m\pi^* K - \pi^* Z - 2L$ *is numerically positive. If* $K^2 \geqslant 2$ *then either* $3\pi^* K - \pi^* Z - 2L$ *is numerically positive or* x *lies on a* -1-*curve* C *with* $(C, Z) = 0$ *or* 1, *or on a* -3-*curve* C *with* $(C, Z) = 2$ *and* C *is a base curve of* $3K - Z - 2x$.

Remark. — (i) It follows from the proof that the curves contradicting numerical positivity must fulfill stronger properties than stated. For instance, in (i) a 4-curve contradicting numerical positivity would have x as a point of multiplicity 3 and y as a point of multiplicity 2 or *vice versa.*

(ii) It also follows from the proof that if $m \geqslant 8$ then $m\pi^* K - 2L - 2M$ and $m\pi^* K - 3L$ are always numerically positive.

Proof. — Let us start with (i) and $m \geqslant 4$. As $(m\pi^* K - 2L - 2M)^2 = m^2 K^2 - 8 > 0$, if $m\pi^* K - 2L - 2M$ is not numerically positive, there is a curve $\widetilde{C}$ on $\widetilde{X}$ s.t. $0 > (m\pi^* K - 2L - 2M, \widetilde{C})$. Now,

$$2 = (m\pi^* K - 2L - 2M, L) = (m\pi^* K - 2L - 2M, M)$$

so $\widetilde{C}$ is the strict transform of a curve C on X. If $r := \mathrm{mult}_x C$ and $s := \mathrm{mult}_y C$, then $r = (L, \widetilde{C})$ and $s = (M, \widetilde{C})$, so $m(K, C) < 2(r + s)$. We now apply (1.1) (i) with $\lambda = 1/2$ and $\beta = 1$. From (1.4) we then get $\alpha \leqslant 7$. On the other hand, as X is minimal of general type, $(K, C) \geqslant 0$ so $\alpha \geqslant 1$. Using (1.3) and (1.2) and going through the cases $\alpha = 1, 2, \ldots, 7$ we get (i) for $m \geqslant 4$. The case $m = 3$, $K^2 \geqslant 2$ is done in the same way, (ii) and (iii) is done using (1.1) (ii) (iv) and (v) using (1.1) (iii) or in some cases going back to (1.11) and, finally (vi) using (1.1) (iv) (1.11) and the technique used in (i) and (ii) to reduce the problem to conditions on a curve on X plus the fact that a fundamental cycle has non-positive intersection with any -2-curve. Finally, in each case one sees that C or $\widetilde{C}$ has genus 0 or 1 and C is certainly a base curve of the appropriate linear system as otherwise the condition of negative intersection would not be fulfilled. □

Proposition **1.14.** — *Let* X *be a minimal surface of general type.*

(i) *Let* C *be a* -1*-divisor on* X *and* $m \geqslant 3$ *or* $m = 2$, $K^2 \geqslant 2$. *If* $mK - C$ *is not numerically positive then* C *has positive intersection with some* -2*-curve* D *and* $h^0(mK - C) = h^0(mK - C - D)$.

(ii) *Let* C *be a* -3*-divisor on* X *and* $K^2 \geqslant 2$. *If* $3K - C$ *is not numerically positive then either* C *has positive intersection with a* -2*-curve or there is a* -3*-curve* D *with* $(C, D) = 4$ *and* $h^0(3K - C) = h^0(3K - C - D)$.

(iii) *Let* C *be a* 0*-divisor on* X. *If* $3K - C$ *is not numerically positive then* C *has positive intersection with a* -2*-curve* D *and* $h^0(3K - C) = h^0(3K - C - D)$.

(iv) *Let* C *be a* 1*-divisor on* X. *Then* $2K - C$ *is numerically positive.*

(v) *Let* C *be a* 2*-divisor on* X *and* $m \geqslant 4$ *or* $m \geqslant 2$, $K^2 \geqslant 2$. *If* $mK - C$ *is not numerically positive then* C *has positive intersection with a* -2*-curve* D, $K^2 = 1$ *and* $h^0(mK - C) = h^0(mK - C - D)$.

(vi) *Let* C *be a* 4*-divisor on* X. *Then* $3K - C$ *is numerically positive.*

(vii) *Let* C *be a* -1*-divisor on* X, $K^2 \geqslant 2$ *and* x *a point not on* C. *Let* $\pi : \widetilde{X} \to X$ *be the blowing up at* x *and* L *the exceptional divisor. If* $3\pi^* K - \pi^* C - 2L$ *is not numerically positive then there is a curve* D *passing through* x *s.t. either* D *is a* 2*-curve with* $(C, D) = 1$ *or a* -1*-curve with* $(C, D) = 0$, 1 *or* 2 *and* $h^0(3K - C - 2x) = h^0(3K - C - D)$.

(viii) *Let* C *be a* $(-2, 2)$*-divisor on* X *and* $K^2 \geqslant 2$. *Then* $3K - C$ *is numerically positive unless there is a* -3*-curve* D *with* $(C, D) = 4$ *and* $h^0(3K - C) = h^0(3K - C - D)$.

(ix) *Let* C *be a* 3*-divisor on* X *and* $K^2 \geqslant 2$. *Then* $3K - C$ *is numerically positive unless there is a* -2*-curve* D *with* $(C, D) = 1$ *and* $h^0(3K - C) = h^0(3K - C - D)$.

Proof. — Indeed, this is completely analogous to the proof of (1.13). □

Lemma **1.15.** — *Let* D *be a divisor on a minimal surface of general type. Then there is no infinite sequence* $Z_1, Z_2, Z_3, \ldots$ *of* -2*-curves s.t.*

$$\forall i \geqslant 1 : (D + Z_1 + Z_2 + \ldots + Z_{i-1}, Z_i) > 0.$$

Proof. — Assume such a sequence exists. As the basis dual to the basis consisting of -2-curves of the space spanned by those is negative [Bou: § 3, Lemme 6], there is an effective divisor E consisting of -2-curves s.t. $D + E$ has negative intersection with all -2-curves. We prove by induction that $Z_1 + Z_2 + \ldots + Z_i \leqslant E$. In fact, as $(D, Z_1) > 0$ and $0 \geqslant (D + E, Z_1)$ we get $(E, Z_1) < 0$ and so $Z_1 \leqslant E$. At the i-th step we have $(D + Z_1 + Z_2 + \ldots + Z_{i-1}, Z_i) > 0$ and

$$0 \geqslant (D + Z_1 + Z_2 + \ldots + Z_{i-1} + (E - (Z_1 + Z_2 + \ldots + Z_{i-1})), Z_i),$$

so $Z_i \leqslant E - (Z_1 + Z_2 + \ldots + Z_{i-1})$. □

Lemma **1.16.** — *Let* X *be a minimal surface of general type.*

(i) *Let* C *be an effective divisor on* X *with* $(K, C) \leqslant 3$. *Let* $\mathfrak{p} := \mathscr{I}_C + \mathfrak{m}$ *where* $\mathscr{I}_C$ *is the ideal defining* C *and* $\mathfrak{m}$ *is an ideal of codimension* 2 *if* $m \geqslant 4$ *or* $m = 3$, $K^2 \geqslant 2$ *and codimen-*

sion 1 *if* $m = 2$, $K^2 \geqslant 2$, $(K, C) \leqslant 2$ *or* $m = 3$ *and* $\mathrm{Supp}(\mathcal{O}_X/\mathfrak{m})$ *does not meet a* -2*-curve. Then the reduction map*

$$H^0(C, \mathcal{O}_C((m+1)K)) \to H^0(C, \mathcal{O}_C((m+1)K)/\mathfrak{p}\mathcal{O}_C((m+1)K))$$

is surjective.

(ii) *Let* C *be an effective divisor on* X *with* $(K, C) \leqslant 2$ *and let* Z_λ *and* Z_μ *be two fundamental cycles s.t.* $Z_\lambda + Z_\mu \leqslant C$. *Put* $\mathfrak{p} := \mathcal{I}_{Z_\lambda} . \mathcal{I}_{Z_\mu}$. *Suppose* $m \geqslant 4$ *or* $m = 3$, $K^2 \geqslant 2$. *Then*

$$H^0(C, \mathcal{O}_C((m+1)K)) \to H^0(C, \mathcal{O}_C((m+1)K)/\mathfrak{p}\mathcal{O}_C((m+1)K))$$

is surjective.

(iii) *Let* C *be an effective divisor on* X *with* $(K, C) \leqslant 2$ *and let* Z *be a fundamental cycle s.t.* $Z \leqslant C$. *Put* $\mathfrak{p} := \mathcal{I}_Z$. *Suppose* $m \geqslant 3$ *or* $m = 2$, $K^2 \geqslant 2$. *Then*

$$H^0(C, \mathcal{O}_C((m+1)K)) \to H^0(C, \mathcal{O}_C((m+1)K)/\mathfrak{p}\mathcal{O}_C((m+1)K))$$

is surjective.

(iv) *Let* C *be an effective divisor on* X *with* $(K, C) \leqslant 2$ *and let* Z *be a fundamental cycle s.t.* $Z \leqslant C$. *Put* $\mathfrak{p} := \mathcal{I}_Z + \mathfrak{m}$, *where* $\mathfrak{m}$ *is an ideal of codimension* 1 *s.t.* $\mathrm{Supp}(\mathcal{O}_X/\mathfrak{m})$ *does not meet a* -2*-curve. Suppose* $m \geqslant 4$ *or* $m = 3$, $K^2 \geqslant 2$. *Then*

$$H^0(C, \mathcal{O}_C((m+1)K)) \to H^0(C, \mathcal{O}_C((m+1)K)/\mathfrak{p}\mathcal{O}_C((m+1)K))$$

s surjective.

Proof. — Indeed, it will suffice to show that $H^1(C, \mathfrak{p}\mathcal{O}_C((m+1)K)) = 0$. By duality it will be enough to show that $\mathrm{Hom}_{\mathcal{O}_C}(\mathfrak{p}\mathcal{O}_C((m+1)K), \omega_C) = 0$. Suppose there is a non-zero $\mathcal{O}_C$-morphism $h : \mathfrak{p}\mathcal{O}_C((m+1)K) \to \omega_C$, let $C_2 \leqslant C$ be maximal for $h_{|C_2} = 0$ and let $C_1 := C - C_2$. Then h induces a non-zero map $\mathfrak{p}\mathcal{O}_{C_1}((m+1)K) \to \omega_C(-C_2)$. As $\omega_C(-C_2) = \omega_{C_1}$ and $(K, C_1) \leqslant (K, C)$, we may replace C by C_1 and so assume that h is injective and coker h has finite support. Hence $\deg_C(\mathfrak{p}\mathcal{O}_C((m+1)K)) \leqslant \deg_C \omega_C$ i.e.

$$\deg_C \mathfrak{p}\mathcal{O}_C + m(K, C) \leqslant C^2.$$

Let us now consider (i). Then $\deg_C \mathfrak{p}\mathcal{O}_C \geqslant -2$ (resp. -1) so we get $0 \leqslant C^2 + 2 - m(K, C)$ (resp. $C^2 + 1 - m(K, C)$). The index theorem gives $C^2 \leqslant (K, C)^2/K^2$ and $(K, C) \leqslant 1, 2$ or 3 gives by substitution a contradiction. If $(K, C) = 0$, C consists of -2-curves so by assumption $\deg_C \mathfrak{p}\mathcal{O}_C = 0$ and we again get a contradiction.

As for (ii) we have then $\deg_C \mathfrak{p}\mathcal{O}_C = -(Z_\lambda + Z_\mu, C)$. Assume first $Z_\lambda = Z_\mu =: Z$. From (1.11) it follows that

$$-2C^2K^2 - K^2(C, Z)^2 + 2(K, C)^2 \geqslant 0$$

i.e.
$$C^2 \leqslant \frac{1}{K^2}(K, C)^2 - \frac{1}{2}(C, Z)^2.$$

Hence we get

$$0 \leqslant \frac{1}{K^2}(K, C)^2 - \frac{1}{2}(C, Z)^2 + 2(C, Z) - m(K, C).$$

The function $(C, Z) \mapsto -1/2(C, Z)^2 + 2(C, Z)$ takes on its maximum for $(C, Z) = 2$, so we get

$$0 \leqslant \frac{1}{K^2}(K, C)^2 + 2 - m(K, C),$$

which gives a contradiction if $(K, C) = 1$ or 2. If $(K, C) = 0$ then C consists of -2-curves so $(C, Z) \leqslant 0$ as Z is a fundamental cycle, but $-(C, Z) = \deg_C \mathfrak{p}\mathcal{O}_C \leqslant C^2 < 0$. If instead $Z_\lambda \neq Z_\mu$ we put $Z := Z_\lambda + Z_\mu$ and use (1.11) again to get a contradiction. Similarly, we obtain (iii) and (iv). □

Lemma **1.17.** — (i) *Let* $x \in X$, $\pi : \widetilde{X} \to X$ *the blowing up at* x *and* $L := \pi^{-1}(x)$. *Suppose that* x *does not lie on a* -2*-curve. If* $m \geqslant 3$ *or* $m = 2$, $K^2 \geqslant 2$, *then either* $m\pi^* K - 2L$ *is numerically positive or there is a distinguished divisor* C *all of whose components are rational or elliptic s.t.* $x \in C$, $(K, C) \leqslant 2$ *and* $mK - C$ *is numerically positive. If* $m = 2$ *then* C *may be assumed to be either a* -2*-curve with* $\operatorname{mult}_x C \geqslant 3$, *a* -1*-divisor or a* 1*-divisor and in any case* $h^0(K - x) = h^0(K - C)$.

(ii) *With* x, π *and* L *as in* (i). *If* $m \geqslant 4$ *or* $m = 3$, $K^2 \geqslant 2$ *then either* $m\pi^* K - 3L$ *is numerically positive or there is a distinguished divisor* C *all of whose components are rational or elliptic s.t.* $\mathcal{I}_C \subset m_x^2$ *and* C *is a* -1-, 1-, 2- *or* 4*-divisor if* $m \geqslant 4$ *and a* -1- *or* 2*-divisor if* $m = 3$ *for which* $mK - C$ *is numerically positive and* $h^0(mK - 3x) = h^0(mK - C)$.

(iii) *Let* Z *be a fundamental cycle. If* $m \geqslant 4$ *or* $m = 3$, $K^2 \geqslant 2$, *then either* $mK - 2Z$ *is numerically positive or there is a distinguished divisor* C *all of whose components are rational or elliptic s.t.* $Z \leqslant C$, $(K, C) \leqslant 2$ *and* $mK - C$ *is numerically positive.*

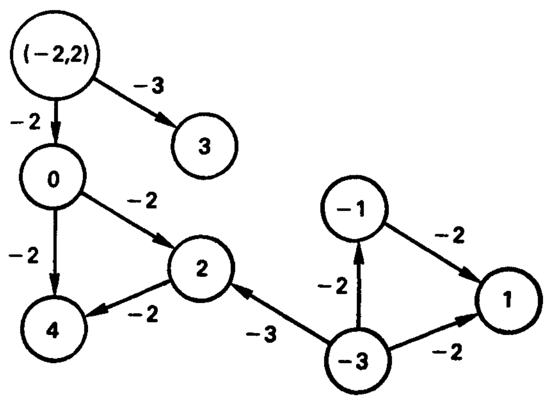
(iv) *Let* Z_λ *and* Z_μ *be fundamental cycles. If* $m \geqslant 4$ *or* $m = 3$, $K^2 \geqslant 2$, *then either* $mK - Z_\lambda - Z_\mu$ *is numerically positive or there is a distinguished divisor* C *all of whose components are rational or elliptic s.t.* $Z_\lambda + Z_\mu \leqslant C$, $(K, C) \leqslant 2$, $m = 3$ *and* $mK - C$ *is numerically positive.*

(v) *Let* $x \neq y \in X$, $\pi : \widetilde{X} \to X$ (*resp.* $\rho : \hat{X} \to X$) *the blowing up at* x *and* y (*resp. at* y) *and* $L := \pi^{-1}(x)$, $M := \pi^{-1}(y)$ (*resp.* $M := \rho^{-1}(y)$). *Suppose that* x *and* y *do not lie on a* -2*-curve. If* $m \geqslant 4$ *or* $m = 3$, $K^2 \geqslant 2$, *then either* $m\pi^* K - 2L - 2M$ *is numerically positive or there is a* -1*-curve* C *s.t.* $x \in C$ *and* $m\rho^* K - \rho^* C - 2M$ *is numerically positive or there is a distinguished divisor* C *all of whose components, except possibly for one* 2*-curve, are rational or elliptic s.t.* $x, y \in C$, C *is a* -1-, 1- *or* 4*-curve if* $m \geqslant 4$ *and* $mK - C$ *is numerically positive and* $h^0(mK - 2x - 2y) = h^0(mK - C - 2y)$ *resp.* $h^0(mK - 2x - 2y) = h^0(mK - C)$.

(vi) *Let* Z *be a fundamental cycle and let* $x \in X$, *where* x *does not lie on a* -2*-curve, let* $\pi : \widetilde{X} \to X$ *be the blowing up at* x *and* $L := \pi^{-1}(x)$. *If* $m \geqslant 4$ *or* $m = 3$, $K^2 \geqslant 2$, *then either* $m\pi^* K - \pi^* Z - 2L$ *is numerically positive or there is a distinguished divisor* C *all of whose components are rational or elliptic s.t.* $Z \leqslant C$ *and* $x \in C$, $(K, C) \leqslant 2$, $m = 3$ *and* $mK - C$ *is numerically positive.*

Proof. — Assume the lemma false. Then (1.13) provides us with a distinguished curve containing the required subscheme. Put C either equal to this distinguished curve or the distinguished curve plus the (sum of) fundamental cycles if we are in part (iii),

(iv) or (vi). Thus C is a distinguished divisor and as the lemma is false mK $-$ C is not numerically positive. Hence (1.14) gives us a new distinguished curve or a $-$ 2-curve. Adding this to C gives us a new distinguished divisor. We then have the following flow chart:



Here the type of the curve whose addition gives a new distinguished divisor is written beside the arrow. Also the possibility of the addition of a $-$ 2-curve giving the same type has not been displayed. Now (1.15) assures us that we can not add only $-$ 2-curves indefinitely and the chart shows us, as it contains no loops, that eventually we will reach a contradiction. We get the precisions on the types by using the conditions in (1.13) to see where we enter the flow chart in the different cases. Furthermore, by (1.13) we start with rational or elliptic curves and we see that those we add are rational or elliptic except for one possible 2-curve in part (v). Finally, the condition on the dimension of linear systems follows from (1.14). □

Proposition **1.18.** — *Let* X *be a minimal surface of general type.*

(i) *Let* C *be a* $-$ 3-, ($-$ 2, 2)-, $-$ 1-, 1-, 2-, 3- *or* 4-*divisor containing at most one non-rational or non-elliptic component which is a* 2-*curve, or an effective divisor with* $(K, C) = 0$, $C^2 = -4$ *or* -8 *and let* $m \geq 3$ *be s.t.* mK $-$ C *is numerically positive. Then* $H^1(X, \mathcal{O}_X(-(mK - C))) = 0$ *except possibly when*

(K, C)	C^2	K^2	m	$h(\omega_X^{-1})$
3	3	1	$\leqslant 5$	0,1
3	3	2	3	1
2	2	1	4	1
2	$\leqslant 0$	1	4	0,1
2	$\leqslant 2$	1	3	0,1
1	-1	1	3	1
1	-3	1	3	0,1
0	-8	1	3	1

and in any case X *is almost ruled,* $\chi \geqslant 1$ *and* C *contains no* 2-*curve if* $K^2 \geqslant 2$.

(ii) *Let* C *be a* -1- *or* 2-*divisor s.t.* $2K - C$ *is numerically positive and* $K^2 \geqslant 2$. *If* $H^1(X, \mathcal{O}_X(-(2K - C))) \neq 0$ *then* $p = 2$, $\chi(X) \leqslant 0$, $H^0(X, \mathcal{O}_X(K - C)) = 0$ *and* X *is almost ruled or* C *is a* 2-*divisor.*

Proof. — Assume that the proposition is false and first that $\chi(X) \leqslant 0$ and hence $q \geqslant 2$. Now, as $(K + (1-p)(mK - C), K) < 0$ (II:1.3) (iii) shows that X is almost ruled. Let $f: X \to T$ be the Albanese fibration so that T is a curve of genus q. Assume that all components of C are rational or elliptic. They then lie in fibers of f as $q \geqslant 2$ so if F is a fiber $(C, F) = 0$ and $(K + (1-p)(mK - C), F) = (K + (1-p)mK, F) < -2$ except when $m = 2$, $p = 2$, which contradicts (II:1.3) (v). If not, there is at most one component D of C which is not elliptic or rational and it is a 2-curve. As we may assume that $K^2 \geqslant 2$ a 2-curve is numerically equivalent to K and so we get

$$(m-1)\,K \sim_{\text{num}} mK - D \leqslant mK - C$$

and we obtain a contradiction from (II:1.3) (vi) and (II:1.7).

We are therefore left with the case $m = 2$, $p = 2$ (still assuming $\chi \leqslant 0$) and we have just shown that X is almost ruled and so it only remains to verify that $H^0(X, \mathcal{O}_X(K - C)) = 0$. If not, we get an embedding $\omega_X \hookrightarrow \mathcal{O}_X(2K - C)$ contradicting (II:1.3) (vi) and (II:1.7).

Assume now $\chi \geqslant 1$. From (1.9) and (1.10) we get

$$\textbf{(1.19)}\qquad \left(\binom{p}{3} + \frac{1}{2}\binom{p}{2}\right)(m^2 K^2 - 2m(K, C) + C^2) + \frac{1}{2}\binom{p}{2}(mK^2 - (K, C)) + (p-1)\,\chi \leqslant \binom{p}{2} K^2 + 1 + h(\omega_X^{-1}).$$

As $n \mapsto \binom{n}{k}$ is increasing we can put $p = 2$ in (1.19) if

$$1/2(m^2 K^2 - 2m(K, C) + C^2 + mK^2 - (K, C)) - K^2 \geqslant 0.$$

This inequality is easily seen to be fulfilled so we get

$$(m^2 + m - 2)\,K^2 + 2\chi \leqslant (2m+1)\,(K, C) - C^2 + 2 + 2h^1(\omega_X^{-1}).$$

A case by case study then gives the proposition, if we use the argument above to exclude containment of a 2-curve. □

We should now be amply prepared for the proof of the main theorem of this paper.

Theorem **1.20.** — *Let* X *be a minimal surface of general type and consider* $|(m+1)\,K_X|$ *as a linear system on the canonical model of* X.

(i) *If* $m \geqslant 3$ *or* $m = 2$ *and* $K^2 \geqslant 2$ *then* $|(m+1)\,K_X|$ *has no base points.*

(ii) *If* $m \geqslant 4$ *or* $m = 3$ *and* $K^2 \geqslant 2$ *then* $|(m+1)\,K_X|$ *is very ample.*

Proof. — Indeed, assume that $x \in \mathrm{X}$ is a base point of $|(m+1)\,\mathrm{K_X}|$ and assume first that x does not lie on a -2-curve. Let $\pi : \widetilde{\mathrm{X}} \to \mathrm{X}$ be the blowing up at x and L the exceptional divisor. The assumption just made implies that $\mathrm{H}^1(\widetilde{\mathrm{X}}, \mathcal{O}_{\widetilde{\mathrm{X}}}(-m\pi^*\mathrm{K} + 2\mathrm{L})) \neq 0$ and we thus get from (II:1.1) and (II:1.7) that $m\pi^*\mathrm{K} - 2\mathrm{L}$ is not numerically positive. This implies, by (1.17), (1.18) and (1.16) and using the factorisation

$$\mathrm{H}^0(\mathrm{X}, \mathcal{O}_{\mathrm{X}}((m+1)\,\mathrm{K})) \to \mathrm{H}^0(\mathrm{C}, \mathcal{O}_{\mathrm{C}}((m+1)\,\mathrm{K})) \to \mathcal{O}_{\mathrm{X}}((m+1)\,\mathrm{K})_x$$

that $m = 2$ and that there is a -1-divisor or a 2-curve C with $x \in \mathrm{C}$ s.t.

$$h^1(\mathcal{O}_{\mathrm{X}}(-(2\mathrm{K} - \mathrm{C}))) \neq 0,$$

$h^0(\mathrm{K} - \mathrm{C}) = h^0(\mathrm{K} - x) = h^0(\mathrm{K})$ as x is a base point of $|3\mathrm{K}|$ and $\mathrm{mult}_x\,\mathrm{C} \geqslant 3$ if C is a 2-curve. Now if C is a -1-divisor then, again by (1.18), $\chi \leqslant 0$ and so, by (0:1.6), $h^0(\mathrm{K}) \neq 0$ which contradicts (1.18) (ii). Hence we may assume that C is a 2-curve with $\mathrm{mult}_x\,\mathrm{C} \geqslant 3$. As $\mathrm{K}^2 \geqslant 2$ we get $\mathrm{K}^2 = 2$ and $\mathrm{K} \sim_{\mathrm{num}} \mathrm{C}$. We first show that $\mathrm{K} \equiv \mathrm{C}$. In fact, if $b_1(\mathrm{X}) \neq 0$, then as $\mathrm{mult}_x\,\mathrm{C} \geqslant 3$ and $g(\mathrm{C}) = 3$ implies that C is rational C lies in a fiber of the Albanese mapping and so $2 = \mathrm{C}^2 \leqslant 0$. Hence $b_1(\mathrm{X}) = 0$ and so $\chi \geqslant 1$. Now, as $\mathrm{C} \sim_{\mathrm{num}} \mathrm{K}$ we get $h^0(\mathrm{C} - \mathrm{K}) + h^0(2\mathrm{K} - \mathrm{C}) \geqslant \chi \geqslant 1$ and so either $\mathrm{C} \equiv \mathrm{K}$ or $h^0(2\mathrm{K} - \mathrm{C}) > 0$. In the latter case we get, by (II:1.3) (ii), $h^1(\mathrm{K}) > 0$ and thus $h^0(\mathrm{K} - \mathrm{C}) = h^0(\mathrm{K} - x) = h^0(\mathrm{K}) > 0$ and therefore $\mathrm{C} \equiv \mathrm{K}$ and $h^0(\mathrm{K}) = 1$. As then $h^1(\omega_{\mathrm{X}}^{-1}) \neq 0$ we get from (II:1.10) that $|2\mathrm{K}|$ is without base points and that $\mathrm{X} \to |2\mathrm{K}|$ factors as $\mathrm{X} \xrightarrow{\rho} \mathrm{Y} \to |2\mathrm{K}|$, where $\mathrm{Y} \xrightarrow{\tau} \mathrm{X}^{(-1)}$ is a non-trivial $\alpha_{\mathscr{L}}$-torsor where $\mathscr{L} := \omega^{-1}_{\mathrm{X}^{(-1)}}$ and $\tau \circ \rho$ is the Frobenius map. This means that exactly one of the maps $\mathrm{C} \to \rho(\mathrm{C})$ and $\rho(\mathrm{C}) \to \tau \circ \rho(\mathrm{C})$ is of degree 1 and the other of degree 2. As $(\mathrm{K}, \mathrm{C}) = 2$, the pullback of $\tau^{(1)}$ to the normalisation of $\mathrm{C}(\cong \mathbf{P}^1)$ is an $\alpha_{\mathcal{O}(-2)}$-torsor and hence is trivial. Therefore, $\rho(\mathrm{C}) \to \tau \circ \rho(\mathrm{C})$ is of degree 1 and so $\mathrm{C} \to \rho(\mathrm{C})$ is of degree 2. Furthermore, from (II:1.10) it follows that $\mathrm{X} \to |2\mathrm{K}|$ restricts to the canonical map $\mathrm{C} \to |\mathrm{K_C}|$. Therefore C is hyperelliptic and so all its singularities have multiplicity $\leqslant 2$, contradicting $\mathrm{mult}_x\,\mathrm{C} \geqslant 3$.

If x lies on a -2-curve (1.13) (v) and (1.18) give a contradiction and thus (i) is proved.

As for (ii) let first x and y be two distinct points none of which lie on a -2-curve and assume that $|(m+1)\,\mathrm{K}|$ does not separate them. Assume first $m \geqslant 4$. Then (1.16), (1.17) and (1.18) gives an immediate contradiction. If $m = 3$, then we get $\mathrm{K}^2 = 2$, $h^1(\omega_{\mathrm{X}}^{-1}) = 1$ and $h^0(3\mathrm{K} - 2x - 2y) = h^0(3\mathrm{K} - \mathrm{C}) \leqslant h^1(\mathcal{O}_{\mathrm{X}}) = h^0(\mathrm{K})$, the last as $\chi = 1$. Now, as x and y are not separated by $|4\mathrm{K}|$, $h^0(\mathrm{K} - x - y) \geqslant h^0(\mathrm{K}) - 1$ and $h^0(2\mathrm{K} - x - y) \geqslant h^0(2\mathrm{K}) - 1 = \mathrm{K}^2 + \chi + 1 - 1 = 3$. Therefore,

$$\begin{aligned} h^0(\mathrm{K}) \geqslant h^0(3\mathrm{K} - 2x - 2y) &\geqslant h^0(\mathrm{K} - x - y) + h^0(2\mathrm{K} - x - y) - 1 \\ &\geqslant h^0(\mathrm{K}) + 1. \end{aligned}$$

Let now x be a point not on a -2-curve and assume that $|(m+1)\,\mathrm{K}|$ does not separate tangents at x. This means that there is a codimension 2 ideal $\mathfrak{m}$ of $\mathcal{O}_{\mathrm{X}}$, with support at x s.t. the restriction map $\mathrm{H}^0(\mathrm{X}, \mathcal{O}_{\mathrm{X}}((m+1)\,\mathrm{K})) \to \mathcal{O}_{\mathrm{X}}((m+1)\,K)/\mathfrak{m}\mathcal{O}_{\mathrm{X}}((m+1)\,\mathrm{K})$ is not surjective. Assume that $m \geqslant 4$. We get a contradiction as before unless $m = 4$, $K^2 = 1$, $h^1(\omega_X^{-1}) = 1$ and $h^0(4K - 3x) \leqslant h^0(K)$. We also get

$$h^0(4K - 3x) \geqslant h^0(2K) + 2h^0(K) - 5 = 3 + 2h^0(K) - 5 \quad \text{i.e. } h^0(K) \leqslant 2.$$

On the other hand, $h^0(4K - 3x) \geqslant h^0(4K) - 6 = 1$ by the Riemann-Roch formula. Hence we see that $h^0(K - x) = h^0(K) - 1$ as otherwise the argument above would give $h^0(K) \leqslant 0$. This means that there is a canonical curve not passing through x and so $| 4K |$ cannot fill up everything up to order 3 at x, as $| 5K |$ does not fill up everything even up to order 2 by assumption. Hence, $h^0(4K - 3x) > h^0(4K) - 6 = 1$ and so $h^0(K) \geqslant 2$ contradicting (II:1.8). If instead $m = 3$, then, as $K^2 \geqslant 2$, we see from (1.17) (ii) and (1.18) (i) that there are no problems.

Suppose now that $|(m + 1) K|$ does not separate two singular points or fails to separate tangents at a singular point. Then from (1.17) (iv), (1.18) and [Bo: 5.10] we get a contradiction. Similarly for separation of a smooth and a singular point. □

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