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# CANONICAL SEMI-INVARIANTS AND THE PLANCHEREL FORMULA FOR PARABOLIC GROUPS

BY

RONALD L. LIPSMAN[1] AND JOSEPH A. WOLF[2]

ABSTRACT. A parabolic subgroup of a reductive Lie group is called "good" if the center of the universal enveloping algebra of its nilradical contains an element that is semi-invariant of weight proportional to the modular function. The "good" case is characterized here by invariance of the set of simple roots defining the parabolic, under the negative of the opposition element of the Weyl group. In the "good" case, the unbounded Dixmier-Pukanszky operator of the parabolic subgroup is described, the conditions under which it is a differential operator rather than just a pseudodifferential operator are specified, and an explicit Plancherel formula is derived for that parabolic.

**1. Introduction.** This paper continues our work on the Plancherel formula for parabolic subgroups of reductive Lie groups. Earlier, we considered maximal parabolic subgroups ([7] and [11]), minimal parabolic subgroups ([4] and [6]), and parabolic subgroups whose unipotent radical has square-integrable representations modulo its center ([7] and [12]). In each case we described the representation theory, the Plancherel measure, and the Dixmier-Pukanszky operator that appears in the Plancherel formula to compensate for nonunimodularity. We now consider arbitrary parabolic subgroups.

We isolate the common features of all previously treated cases which enabled us to describe the fundamental semi-invariant polynomial that gives rise to the Dixmier-Pukanszky operator. That is our first main result here. The second main result is a simple characterization of which parabolics have the requisite property, and the third is the construction in that case of the canonical semi-invariant polynomial. We explicitly compute the Dixmier-Pukanszky operator and the Plancherel formula for parabolics that have the requisite property.

Here is a brief summary of the contents of this paper.

In §2 we describe the key property (see Definition 2.1) for parabolic groups that guarantees the existence of a unique appropriate semi-invariant (Theorem 2.2) in the center of the universal enveloping algebra of the nilradical. We call such parabolics "good". Let $P$ be a good parabolic and $P = NAM$ its Langlands decomposition. We prove (Proposition 2.4) that the generic stability groups $S_i$ for

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the action of the Levi component $MA$ on $\hat{N}$ are unimodular, and (Theorem 2.7) we derive the Plancherel formula

$$f(1_P) = \sum_i \int_{\hat{S}_i} \text{trace } \pi_{i,\sigma}(Df)\, d\mu_i(\sigma) \tag{1.1}$$

where the sum runs over the (finite) set of generic $MA$-orbits on $\hat{N}$, the $\pi_{i,\sigma}$ are associated to the $i$th orbit by the Mackey machine, $D$ is the Dixmier-Pukanszky operator on $P$, and $\mu_i$ is ordinary Plancherel measure on the unimodular group $S_i$. Actually one might have to use cocycle representations here, but (1.1) is the same.

In §3 we characterize good parabolic subgroups as those whose defining set of simple roots is stable under a certain symmetry of the Dynkin diagram (Propositions 3.3 and 3.10). That stability says that a certain finite dimensional representation of the reductive group has a bilinear invariant. That bilinear invariant leads to an explicit construction (Proposition 3.12) of the canonical semi-invariant, and thus describes the Dixmier-Pukanszky operator $D$ in (1.1). Theorems 3.13 and 3.14 reformulate and summarize these results, and Proposition 3.15 applies them to characterize the tube domains among bounded symmetric domains.

In §4 we compile a collection of examples addressing the questions of whether $D$ is differential and whether the $S_i$ are reductive. With **[12]**, these examples incidentally complete the detailed discussion of the case where $N$ is abelian or 2-step nilpotent.

In §5 we consider the "domain problem", the problem of describing a subspace of the Schwartz space dense in $L_2$ for which (1.1) holds. There, one needs that $Df \in L_1(P)$ so that the $\pi_{i,\sigma}(Df)$ are defined, that the $\pi_{i,\sigma}(Df)$ are of trace class for $\mu_i$ almost all $\sigma$, and that $\sigma \mapsto \text{trace } \pi_{i,\sigma}(Df)$ is integrable against $\mu_i$. This problem has been with us throughout our work in this area. When the canonical semi-invariant lives on an abelian ideal of the nilradical $\mathfrak{n}$ the domain problem is tractable. We describe some conditions which guarantee the existence of such an ideal.

This paper completes the basic Fourier Inversion theory for good parabolics. There the recipe is complete: Given a good parabolic subgroup of a reductive Lie group, our results tell one how to write down all the ingredients of its Plancherel formula (1.1). We point out that most of this is valid on $p$-adic groups with only minor modifications.

We thank Tony Joseph for suggesting that we express the modular function in terms of the maximal set of strongly orthogonal roots. That is essential for §3.

**2. Good parabolic subgroups.** In this section we define the category of "good parabolics". Parabolic groups are always nonunimodular. Thus their Plancherel formula contains a Dixmier-Pukanszky operator–that is, an invertible, positive, selfadjoint unbounded operator $D$ on the Hilbert space of square-integrable functions on the group, which is semi-invariant of weight the modular function and affiliated with the left ring (see [7, Theorem 1.1] or [5, Theorem 6.4]). Another ingredient in the Plancherel formula is of course the Plancherel measure $\mu$. Neither $D$ nor $\mu$ is unique, but $D \otimes \mu$ is uniquely determined (see [7, Remark 2, p. 121]). A fundamental question in nonunimodular Plancherel theory arises: Is there a best possible choice of $D$ (and so also of $\mu$)? That quality of good parabolics that distinguishes them is that for these parabolics there is such a choice. We show in this section that for a good parabolic $P$ (see definition below) there is a canonical element $T$ in the center of the enveloping algebra of the nilradical and an integer $k > 0$ (eventually 1 or 2) such that $|T|^{1/k}$ is a Dixmier-Pukanszky operator on $P$. Then we prove that the generic stability group for the action of $P/N$ on the dual $\hat{N}$ of its nilradical is unimodular. This gives immediately a canonical choice of a measure $\mu_P$ on $\hat{P}$ in the Plancherel class. Then by an explicit computation we demonstrate that the pair $(|T|^{1/k}, \mu_P)$ occur together in the Plancherel formula. Before beginning, we remark that there is a domain problem for the operator $|T|^{1/k}$ that is not completely resolved. We shall comment on that in greater detail in §5.

2a. *Definition of a good parabolic.* Let $\underline{G}$ be a reductive algebraic group defined over the real number field $\mathbf{R}$. Then $G = \underline{G}(\mathbf{R})$, the set of all real points of $\underline{G}$, is a real reductive Lie group. We identify $\underline{G}$ with the set $\underline{G}(\mathbf{C})$ of complex points and view $\underline{G}$ as the complexification of $G$.

Suppose that $\underline{P}$ is a parabolic subgroup of the algebraic group $\underline{G}$. Then $P = \underline{P}(\mathbf{R})$ is a parabolic subgroup of the Lie group $G$. Suppose we have a Levi decomposition $\underline{P} = \underline{N}\underline{L}$ of $\underline{P}$, where $\underline{N}$ is the nilradical and $\underline{L}$ is a reductive Levi component. We set $N = \underline{N}(\mathbf{R})$ and $L = \underline{L}(\mathbf{R})$. Then $L$ splits canonically $L = AM$ so that $P = NAM$ is a Langlands decomposition of $P$.

We will need Calvin Moore's result that $\underline{P}$ has a Zariski-open orbit on $\underline{\mathfrak{n}}^*$, so there is a finite set of open $P$-orbits on $\hat{N}$ whose union is conull with respect to Plancherel measure. Moore presented this and a number of related results in a seminar at Berkeley in January 1972, but he did not publish it. In June 1973, Carmona circulated a slightly sharpened version, but that too seems not to have been published.

We set

$$\underline{\delta}(\underline{x}) = \det_{\mathbf{C}} \operatorname{Ad}(\underline{x})|_{\underline{\mathfrak{n}}}, \qquad \underline{\delta}\colon \underline{L} \to \mathbf{C}^*,$$
$$\delta(x) = \underline{\delta}|_L(x) = \det_{\mathbf{R}} \operatorname{Ad}(x)|_{\mathfrak{n}}, \qquad \delta\colon L \to \mathbf{R}^*.$$

Then

$$\delta_P(nx) = |\delta(x)|, \quad n \in N, x \in L, \qquad \delta_P\colon P \to \mathbf{R}_+^*$$

is the modular function of $P$. We also put $\underline{L}_0 = \operatorname{Ker} \underline{\delta}$, $\underline{P}_0 = \underline{N}\underline{L}_0$.

2.1. Definition. We say that $P$ is a *good* parabolic if $\underline{P}_0$ does not have an open orbit on $\underline{\mathfrak{n}}^*$. Otherwise we say that $P$ is *bad*. These definitions are justified by Theorems 2.7 and 3.13.

It is clear that in the case of a good parabolic, the dimension of the generic $\underline{P}_0$-orbits on $\underline{\mathfrak{n}}^*$ is $\dim_{\mathbf{C}} \underline{N} - 1 = \dim_{\mathbf{R}} N - 1$.

Now we set $P_0 = \underline{P}_0(\mathbf{R}) = NL_0$, where $L_0 = \underline{L}_0(\mathbf{R}) = \operatorname{Ker} \delta$. If $A_0 = L_0 \cap A$, then $A_0 = \operatorname{Ker} \delta_P|_A$ and $P_0$ is of finite index in $\operatorname{Ker} \delta_P$. Therefore, $\underline{P}_0$ has an open orbit on $\underline{\mathfrak{n}}^* \Leftrightarrow P_0$ has an open orbit on $\hat{N} \Leftrightarrow \operatorname{Ker} \delta_P$ has an open orbit on $\hat{N}$. Finally, let $\underline{\mathcal{O}}$ be the Zariski-open $\underline{P}$-orbit in $\underline{\mathfrak{n}}^*$. Then $\mathcal{O} = \underline{\mathcal{O}}(\mathbf{R})$ is a disjoint union $\mathcal{O} = \bigcup_{i=1}^r \mathcal{O}_i$ of open $P$-orbits in $\mathfrak{n}^*$. Let $S_i$ be the stability group in $L$ of a representation $\gamma_i \in \hat{N}$ that corresponds to a point $\phi_i \in \mathcal{O}_i$. For any $i$, $1 \leqslant i \leqslant r$, the conjugacy class of $S_i$ in $L$ is uniquely determined. Write $\mathfrak{Z}(\cdot)$ for the center of the universal enveloping algebra.

2b. *Canonical semi-invariants*. We prove the following

2.2. THEOREM. *Let $P$ be a good parabolic. Then there exist a positive integer $k$ and a nonzero $T \in \mathfrak{Z}(\underline{\mathfrak{n}})$, semi-invariant under $\underline{P}$ of weight $\underline{\delta}^k$, such that the operator $D = |T|^{1/k}$–considered as an operator on $L_2(P)$– is a Dixmier-Pukanszky operator. Furthermore $D$ is uniquely determined up to scalar; that is, if $T' \in \mathfrak{Z}(\underline{\mathfrak{n}})$, $k' > 0$, is another such pair, then*

$$|T|^{1/k} = c|T'|^{1/k'}$$

*for some scalar $c$.*

PROOF. By [2, Lemma 4.6], $\mathfrak{Z}(\underline{\mathfrak{n}})$ is finitely generated and hence is an affine algebra. Let $\underline{\Lambda}$ denote the corresponding affine variety of maximal ideals. The variety $\underline{\Lambda}$–which we think of as $\underline{\mathfrak{n}}^*/\underline{N}$–is an affine $\underline{L}$-space, and there is an open orbit. Restrict the action to $\underline{L}_0$. By assumption, the codimension of a generic $\underline{L}_0$-orbit in $\underline{\Lambda}$ is one. Now the group $\underline{L}_0$ is reductive. Hence there is a quotient. Let $\underline{\Omega}$ be the affine variety whose algebra of regular functions is $\mathbf{C}[\underline{\Omega}] \cong \mathfrak{Z}(\underline{\mathfrak{n}})^{\underline{L}_0}$. We now consider the action of $\mathbf{C}^*$ on $\mathbf{C}[\underline{\Omega}]$ given by factoring the action of $\underline{L}$ through $\underline{L}_0$ via $\underline{\delta}$–that is $\lambda \cdot f = \underline{x} \cdot f$ if $\underline{\delta}(\underline{x}) = \lambda \in \mathbf{C}^*, f \in \mathbf{C}[\underline{\Omega}] \cong \mathfrak{Z}(\underline{\mathfrak{n}})^{\underline{L}_0}$. Of course $\mathbf{C}^*$ has an open orbit on $\underline{\Omega}$.

Now we diagonalize the action of $\mathbf{C}^*$ on $\mathbf{C}[\underline{\Omega}]$. Let $m \in \mathbf{Z}$ and set

$$V_m = \{f \in \mathbf{C}[\underline{\Omega}] : \lambda \cdot f = \lambda^m f\}.$$

We assert that $V_m$ is a complex vector space of dimension zero or one. That it is a complex vector space is clear. Let $f_1, f_2 \in V_m$ where neither is identically zero. Choose a point $\omega \in \underline{\Omega}$ which is in the open $\mathbf{C}^*$-orbit. Then $\alpha = f_1(\omega) \neq 0$, $\beta = f_2(\omega) \neq 0$. Furthermore, for any $\lambda \in \mathbf{C}^*$ we have

$$\begin{aligned}(\beta/\alpha)f_1(\lambda \cdot \omega) &= (\beta/\alpha)(\lambda^{-1} \cdot f_1)(\omega) = (\beta/\alpha)\lambda^{-m}f_1(\omega) = \beta\lambda^{-m} \\ &= \lambda^{-m}f_2(\omega) = (\lambda^{-1} \cdot f_2)(\omega) = f_2(\lambda \cdot \omega).\end{aligned}$$

Therefore $(\beta/\alpha)f_1 = f_2$.

Choose $n \in \mathbf{Z}$ such that $|n|$ is minimal among the integers for which $V_n$ contains a nonconstant function. Then $n \neq 0$ because otherwise the existence of a nonconstant invariant $f$ would contradict the fact that $\mathbf{C}^*$ has an open orbit on $\underline{\Omega}$ (or that $\underline{L}$ has an open orbit on $\underline{\Lambda}$). Select $f_0 \in V_n$, $f_0 \neq 0$. Of course $f_0$ corresponds canonically to a nonscalar element $T \in \mathfrak{Z}(\underline{\mathfrak{n}})^{\underline{L}_0}$. But in fact $T$ is semi-invariant of weight $\underline{\delta}^n$. To show that, it is enough to demonstrate it for $f_0$. But that is true virtually by definition

$$\underline{x} \cdot f_0 = \underline{\delta}(\underline{x}) \cdot f_0 = \underline{\delta}(\underline{x})^n f_0, \qquad \underline{x} \in \underline{L}.$$

Therefore if we set $D = |T|^{1/n}$, then $D$ is semi-invariant of weight $|\delta| = \delta_P$ under the action of $P$. That it is actually a Dixmier-Pukanszky operator follows by the kind of reasoning employed in [7]. To wit, it is nonsingular by nilpotent Fourier analysis. Indeed any operator in $\mathfrak{Z}(\underline{\mathfrak{n}})$ (resp. a power of the absolute value of such an operator) defines a nonsingular operator on $L_2(N)$ because its Fourier transform is a polynomial (resp. power of the absolute value of a polynomial) on $\mathfrak{n}^*$. $D$ is in the left ring because any operator that lives on $N$, is right invariant there, and given by $Dh(nx) = D(h^x)(n)$, $h^x(n) = h(nx)$ is automatically so [7, §5a].

Next we show that $n$ must be positive. We choose a Cartan subgroup $\underline{H}$ of $\underline{G}$ inside $\underline{P}$ such that $\underline{H}(\mathbf{R}) \supseteq \underline{A}$. We also choose a collection of positive roots $\Sigma$ in $\mathfrak{h}^*$ so that $\alpha \in \Sigma \Leftrightarrow \alpha|_{\mathfrak{a}} \not\equiv 0$ and

$$\underline{\mathfrak{n}} = \sum_{\alpha \in \Sigma} \underline{\mathfrak{g}}^{\alpha}. \tag{2.3}$$

Then $\delta(a) = \exp(\Sigma_{\alpha\in\Sigma} n_\alpha \alpha(\log a))$, $a \in A$, where the $n_\alpha$ are the nonnegative integers $n_\alpha = \dim_{\mathbf{C}} \underline{\mathfrak{g}}^\alpha$. We can pick a basis for $\underline{\mathfrak{n}}$ that is compatible with the decomposition (2.3). Then any element of $\mathfrak{Z}(\underline{\mathfrak{n}})$ is a polynomial in those basis elements. In particular an element $T \in \mathfrak{Z}(\underline{\mathfrak{n}})^{\underline{L}_0}$ as above is a homogeneous polynomial. Since $T$ is semi-invariant of weight $\underline{\delta}^n$, we have

$$\delta^n(a) = \exp\left(\sum_{\alpha\in\Sigma} m_\alpha \alpha(\log a)\right), \qquad a \in A,$$

for some nonnegative integers $m_\alpha$–because $a \cdot X = e^{\alpha(\log a)}X$, $X \in \underline{\mathfrak{g}}^\alpha$. It follows that $n\Sigma_{\alpha\in\Sigma} n_\alpha\alpha = \Sigma_{\alpha\in\Sigma} m_\alpha\alpha$; and hence that $n > 0$.

Finally suppose $T' \in \mathfrak{Z}(\underline{\mathfrak{n}})$, $k' > 0$, is another pair. $T'$ corresponds to an element $f_0' \in \mathbf{C}[\underline{\Omega}]$ that must satsify $\underline{x} \cdot f_0' = \underline{\delta}(\underline{x}) \cdot f_0' = \underline{\delta}(\underline{x})^{k'} f_0'$. Consider $f_0^{k'}$ and $f_0'^{n}$. Then $\lambda \cdot f_0^{k'} = \lambda^{nk'} f_0^{k'}$ and $\lambda \cdot f_0'^{n} = \lambda^{k'n} f_0'^{n}$. Therefore $f_0^{k'} = c f_0'^{n}$ for some nonzero scalar $c$. This implies that

$$|T|^{1/n} = |c|^{1/nk'} |T'|^{1/k'}.$$

The proof of Theorem 2.2 is now complete.

If $P$ is a good parabolic, we shall always select $(T, k)$ from Theorem 2.2 so that $k$ is minimal. Then $T$ is uniquely determined up to scalar. We shall refer to $T$ as the *canonical semi-invariant*. Later we shall see that $D = |T|^{1/k}$ is the best choice of Dixmier-Pukanszky operator on $P$.

2c. *Unimodularity of the generic stability group*. Recall from 2a the definition of the stability groups $S_i$.

2.4. Proposition. *Let $P$ be a good parabolic. Then all the stability groups $S_i$ are unimodular.*

Proof. Let $dn$ be a Haar measure on $N$ and $d\mu_N$ the corresponding Plancherel measure on $\hat{N}$. As before we fix $\phi_i \in \mathcal{O}_i \subseteq \mathfrak{n}^*$, $\gamma_i \in \hat{N}$ the corresponding representation. Then we have a homeomorphism $S_i \setminus L \to L \cdot \gamma_i$. Now $dn$ is relatively invariant under the action of $L$ with modulus $\delta_P$. Therefore $d\mu_N$ is relatively invariant with modulus $\delta_P^{-1}$. Restricting to the open set $L \cdot \gamma_i$, we find that $S_i \setminus L$ carries a relatively invariant measure of modulus $\delta_P^{-1}$. But that can only happen if the modular function $\delta_{S_i}^{-1}$ extends to a homomorphism of $L$ into $\mathbf{R}_+^*$ and then

$$\delta_P | S_i = \delta_{S_i}. \tag{2.5}$$

Now consider the canonical semi-invariant $T$ constructed in Theorem 2.2, but viewed as a semi-invariant polynomial $f_0$ on $\underline{\mathfrak{n}}^*$. It must be that $f_0(\phi_i) \neq 0$–because the $\underline{P}$-orbit of $\phi_i$ is Zariski-open in $\underline{\mathfrak{n}}^*$. Let $s \in S_i$. Since $s \cdot \gamma_i = \gamma_i$, there must exist $u \in N$ such that $s \cdot \phi_i = u \cdot \phi_i$. That and the equation $s \cdot f_0 = \delta(s)^n f_0$ enable us to compute

$$f_0(\phi_i) = f_0(u \cdot \phi_i) = f_0(s \cdot \phi_i) = (s^{-1} \cdot f_0)(\phi_i) = \delta(s)^{-n} f_0(\phi_i).$$

Therefore $|\delta(s)| = 1$. But $\delta_P(s) = |\delta(s)| = 1$. Hence, by (2.5), $S_i$ is unimodular.

2d. *Computation of the Plancherel formula.* We now derive the Plancherel formula of a good parabolic. First we must describe the irreducible representations of $P = NL$. We shall do that by applying the Mackey machine to the group extension $N \lhd P$. Since we are only interested in the Plancherel formula, it is enough to restrict our attention to generic representations.

We start with the $N$-equivariant Kirillov map $\kappa$: $\mathfrak{n}^* \to \hat{N}$. Put $\mathfrak{U} = \kappa(\mathcal{O})$ and $\mathfrak{U}_i = \kappa(\mathcal{O}_i)$, $1 \leqslant i \leqslant r$. Then the conull set $\mathfrak{U} = \bigcup_{i=1}^r \mathfrak{U}_i$ is a disjoint union of open $L$-orbits. Let $\gamma_i \in \mathfrak{U}_i$ and set $S_i = L_{\gamma_i}$. Let $\tilde{\gamma}_i$ be an extension of $\gamma$ to $NS_i$. $\tilde{\gamma}_i$ may fail to be an ordinary representation–but the worst that can happen is that the obstruction, say $\omega_i$, is of order 2 [**1**]. Hence, according to Proposition 2.4, $\hat{S}_i^{\bar{\omega}_i}$ is either the unitary dual of a unimodular group, or the projective dual of a unimodular group with a fixed order 2 multiplier. In either case the Plancherel measure (ordinary or projective) of $\hat{S}_i^{\bar{\omega}_i}$ is unique up to scalar. We shall use that momentarily. But first, we observe that the generic irreducible unitary representations of $P$ are given by

$$\pi_{i,\sigma} = \operatorname{Ind}_{NS_i}^P \tilde{\gamma}_i \otimes \sigma, \qquad 1 \leqslant i \leqslant r, \quad \sigma \in \hat{S}_i^{\bar{\omega}_i}.$$

Now fix a Dixmier-Pukanszky operator $D = |T|^{1/k}$ according to Theorem 2.2. We fix choices of Haar measure $dn$ on $N$ and $dx$ on $L$, so that $dndx$ is right Haar measure on $P = NL$. Let $\gamma \in \hat{N}$. Then the scalar $\gamma_*(D)$ is well defined. Indeed $\gamma_*(T)$ is the infinitesimal character of $\gamma$ evaluated at $T \in \mathfrak{Z}(\underline{\mathfrak{n}})$ and $\gamma_*(D) = |\gamma_*(T)|^{1/k} > 0$. Alternatively, $D$ is diagonalizable–considered as an operator on $L_2(N)$–with respect to the spectral decomposition provided by the Plancherel Theorem–and $\{\gamma_*(D)\colon \gamma \in \hat{N}\}$ is its spectrum. Thus for $h \in L_2(N)$, we have $\gamma(Dh) = \gamma_*(D)\gamma(h)$ for $\mu_N$-a.a. $\gamma \in \hat{N}$.

Next note that for $x \in L$, the semi-invariance says that $x \cdot D = \delta_P(x)D$. Since $\delta_P(L) = \mathbf{R}_+^*$, we can choose for each $i = 1, 2, \ldots, r$ an element $\gamma_i \in \mathfrak{U}_i$ so that

$$(\gamma_i)_*(D) = 1.$$

We assume henceforth that such a choice has been made. We continue to write $S_i$ for $L_{\gamma_i}$; by Proposition 2.4, all the $S_i$ are unimodular. Furthermore, the choices of measures already made uniquely determine a Haar measure on $S_i$ as follows. The choice of $dn$ uniquely specifies a Plancherel measure $\mu_N$ on $\hat{N}$, therefore also on the open set $\mathfrak{U}_i$. The map $x \to x \cdot \gamma_i$, $L \to \mathfrak{U}_i$ factors to a Borel isomorphism

(2.6) $$S_i \setminus L \to \mathfrak{U}_i.$$

We put the unique Borel measure $d\bar{x}$ on $S_i \setminus L$ so that $\delta_P(x)\, d\bar{x} \to d\mu_N|_{\mathfrak{U}_i}$ under the map (2.6). $d\bar{x}$ is $L$-invariant; $\delta_P(x)\, d\bar{x}$ and $d\mu_N$ are relatively invariant. But then the measures $dx$ and $d\bar{x}$, on $L$ and $S_i \setminus L$ respectively, uniquely determine a Haar measure $ds_i$ on $S_i$ according to

$$\int_L f(x)\,dx = \int_{S_i\setminus L}\int_{S_i} f(s_i x)\,ds_i\,d\bar{x}.$$

Finally we pick $\mu_{S_i}$ to be the Plancherel measure on $\hat{S}_i^{\bar{\omega}_i}$ corresponding to $ds_i$ (see [5, Theorem 7.1 and Remark 2 following]). Our main result of this section is

2.7. THEOREM (PLANCHEREL FORMULA FOR THE GOOD PARABOLIC $P$).

$$h(1_P) = \sum_{i=1}^{r}\int_{\hat{S}_i^{\bar{\omega}_i}} \operatorname{Tr}\pi_{i,\sigma}(Dh)\,d\mu_{S_i}(\sigma).$$

PROOF. It has become standard in computations of this sort to use the formula for the character of an induced representation, namely [5, Theorem 3.2]. By (2.5) the $q$-function of $S_i \setminus L$ is identically 1. Hence [5, Theorem 3.2] gives

$$\begin{aligned}\operatorname{Tr}\pi_{i,\sigma}(Dh) &= \int_{S_i\setminus L}\delta_P^{-1}(x)\operatorname{Tr}\int_{NS_i}(Dh)\left(x^{-1}ns_i x\right)(\tilde{\gamma}_i\otimes\sigma)(ns_i)\,dn\,ds_i\,d\bar{x}\\ &= \int_{S_i\setminus L}\operatorname{Tr}\int_{NS_i}(Dh)\left(nx^{-1}s_i x\right)(\tilde{\gamma}_i\otimes\sigma)\left(xnx^{-1}s_i\right)dn\,ds_i\,d\bar{x}.\end{aligned}$$

Therefore

$$\begin{aligned}&\int_{\hat{S}_i^{\bar{\omega}_i}}\operatorname{Tr}\pi_{i,\sigma}(Dh)\,d\mu_{S_i}(\sigma)\\ &\quad= \int_{\hat{S}_i^{\bar{\omega}_i}}\int_{S_i\setminus L}\operatorname{Tr}\int_{NS_i}(Dh)\left(nx^{-1}s_i x\right)(\tilde{\gamma}_i\otimes\sigma)\left(xnx^{-1}s_i\right)dn\,ds_i\,d\bar{x}\,d\mu_{S_i}(\sigma)\\ &\quad= \int_{S_i\setminus L}\int_{\hat{S}_i^{\bar{\omega}_i}}\operatorname{Tr}\int_{NS_i}(Dh)\left(nx^{-1}s_i x\right)(\tilde{\gamma}_i\otimes\sigma)\left(xnx^{-1}s_i\right)dn\,ds_i\,d\mu_{S_i}(\sigma)\,d\bar{x}\\ (2.8)\qquad &\quad= \int_{S_i\setminus L}\operatorname{Tr}\int_N Dh(n)\gamma_i(xnx^{-1})\,dn\,d\bar{x}\\ &\quad= \int_{S_i\setminus L}\operatorname{Tr}\int_N Dh(n)\left(x^{-1}\cdot\gamma_i\right)(n)\,dn\,d\bar{x}\\ &\quad= \int_{S_i\setminus L}\operatorname{Tr}\left(x^{-1}\cdot\gamma_i\right)(Dh)\,d\bar{x} = \int_{S_i\setminus L}\left(x^{-1}\cdot\gamma\right)_*(D)\operatorname{Tr}\left(x^{-1}\gamma_i\right)(h)\,d\bar{x}\\ &\quad= \int_{S_i\setminus L}(\gamma_i)_*(x\cdot D)\operatorname{Tr}\left(x^{-1}\cdot\gamma_i\right)(h)\,d\bar{x} = \int_{S_i\setminus L}\delta_P(x)\operatorname{Tr}\left(x^{-1}\cdot\gamma_i\right)(h)\,d\bar{x}\\ &\quad= \int_{\mathfrak{U}_i}\operatorname{Tr}\gamma(h)\,d\mu_N(\gamma).\end{aligned}$$

Consequently

$$\begin{aligned}\sum_{i=1}^{r}\int_{\hat{S}_i^{\bar{\omega}_i}}\operatorname{Tr}\pi_{i,\sigma}(Dh)\,d\mu_{S_i}(\sigma) &= \sum_{i=1}^{r}\int_{\mathfrak{U}_i}\operatorname{Tr}\gamma(h)\,d\mu_N(\gamma)\\ &= \int_{\mathfrak{U}}\operatorname{Tr}\gamma(h)\,d\mu_N(\gamma) = h(1_N) = h(1_P).\quad\text{Q.E.D.}\end{aligned}$$

2e. *Remarks and observations*. (i) All the details implicit in the computation in the proof of Theorem 2.7 are straightforward except in two instances. The first is step (2.8); the second is the specification of the exact collection of functions $h$ for which it is valid. The latter is a delicate problem and we postpone its discussion to §5. The former is not serious. Step (2.8) is most easily understood as just a formal application of the Plancherel Theorem on the unimodular type I (perhaps projective) group $S_i$. The precise details are exactly the same as in [7, proof of Theorem 4.9].

(ii) We saw earlier (Proposition 2.4) that if $P$ is good, then the generic stability groups $S_i$ are unimodular. Actually, the following conditions are equivalent:

($\alpha$) $P = NL$ is good;

($\beta$) each generic stability group $S_i$ is contained in Ker $\delta_P|_L$;

($\gamma$) each generic stability group $S_i$ is unimodular.

PROOF. $(\alpha) \Rightarrow (\gamma)$ is already done.

$(\beta) \Leftrightarrow (\gamma)$. This follows immediately from (2.5).

$(\beta) \Rightarrow (\alpha)$. If $S_i \subsetneq \text{Ker}\, \delta_P|_L$, then the index of $S_i \cap L_0$ in $S_i$ is at most 2. Therefore $\dim(L_0 \cap S_i) \setminus L_0 = (\dim S_i \setminus L) - 1 = \dim \hat{N} - 1$. Hence $P_0$ cannot have an open orbit on $\mathfrak{n}^*$.

(iii) As an immediate corollary of (ii) we obtain: Minimal parabolics are good. That is true because any Lie subgroup $S_i \subseteq AM$ is reductive, and so unimodular.

(iv) If $P$ is good, the generic stability groups $S_i$ may or may not be reductive. See §4.

(v) The stability groups $S_i$ may not be mutually conjugate in $L$, but they are the real points of $\underline{L}$-conjugate complex groups. Here is an argument to demonstrate that. Consider the natural map $\Phi$: $\underline{\mathfrak{n}}^* \to \underline{\Lambda}$. As before, we denote by $\underline{\mathcal{O}}$ the open $\underline{P}$-orbit in $\underline{\mathfrak{n}}^*$. Of course $\Phi$ is $\underline{P}$-equivariant, so $\underline{\mathcal{U}} = \Phi(\underline{\mathcal{O}})$ is an open $\underline{P}$-orbit in $\underline{\Lambda}$. Define

$$\underline{\mathcal{V}} = \{x \in \underline{\mathfrak{n}}^*: \Phi(x) = \Phi(y) \Rightarrow y \in \underline{N} \cdot x\}.$$

By results of Rosenlicht [9], $\underline{\mathcal{V}}$ contains an open set. But note that $\underline{\mathcal{V}}$ is $\underline{P}$-invariant. Indeed if $x \in \underline{\mathcal{V}}$, $g \in \underline{P}$, then $\Phi(g \cdot x) = \Phi(y) \Rightarrow \Phi(x) = \Phi(g^{-1} \cdot y) \Rightarrow g^{-1} \cdot y \in \underline{N} \cdot x \Rightarrow y \in g\underline{N} \cdot x = \underline{N} \cdot g \cdot x$ (since $\underline{N} \lhd \underline{P}$). Therefore the open $\underline{P}$-orbit must be contained in $\underline{\mathcal{V}}$, i.e. $\underline{\mathcal{O}} \subseteq \underline{\mathcal{V}}$.

Now let $x_i \in \mathcal{O}_i$, $\lambda_i = \Phi(x_i)$. Then if we define

$$\underline{S}_{x_i} = \{g \in \underline{L}: g \cdot x_i \in \underline{N} x_i\}, \qquad \underline{S}_{\lambda_i} = \{g \in \underline{L}: g \cdot \lambda_i = \lambda_i\},$$

it is the case that these are equal. Indeed

$$g \cdot x_i = n \cdot x_i \Rightarrow g \cdot \lambda_i = g \cdot \Phi(x_i) = \Phi(g \cdot x_i) = \Phi(n \cdot x_i) = \Phi(x_i) = \lambda_i;$$

and conversely

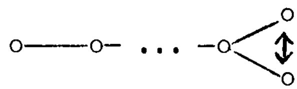
$$g \cdot \lambda_i = \lambda_i \Rightarrow \Phi(x_i) = g \cdot \Phi(x_i) = \Phi(g \cdot x_i),$$

whence $x_i \in \underline{\mathcal{V}} \Rightarrow g \cdot x_i \in \underline{N} \cdot x_i$. So $\underline{S}_{x_i} = \underline{S}_{\lambda_i}$. The conjugacy follows now because: $S_i = \underline{S}_{x_i}(\mathbf{R})$; and all the groups $\underline{S}_{\lambda_i}$ are $\underline{P}$-conjugate, since $\{\lambda_1, \ldots, \lambda_r\} \subseteq \bigcap_{i=1}^{r} \underline{\mathcal{O}}_i = \underline{\mathcal{O}}$, a single $\underline{P}$-orbit.

(vi) In at least one bad parabolic, the appropriate semi-invariant in the enveloping algebra has been computed **[14]**. Since it does not live on the nilradical, it does not correspond in any immediate way to an operator, and so its role in harmonic analysis is still obscure.

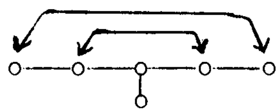
**3. Characterization and semi-invariants for good parabolics.** Every parabolic subgroup $P$ in a reductive Lie group $G$ is characterized by a subdiagram of the Dynkin diagram of $G$. The subdiagram is the Dynkin diagram of the reductive part $MA$ of $P = NAM$. The Weyl group of $G$ has a distinguished element $w_0$ that sends the positive chamber to its negative, so $-w_0$ induces an automorphism of the Dynkin diagram. We will see that $P$ is good if and only if its subdiagram is stable under $-w_0$, and in that case the stability will give us the required semi-invariant $T \in \mathfrak{Z}(\mathfrak{n})$. Since $-w_0$ preserves every component of the Dynkin diagram of $G$ and acts by the identity except in the cases

$A_\ell$, $\ell \geq 2$,



$D_{2n+1}$, $n \geq 1$

and



$E_6$

this specifies the good parabolics.

3a. *Preliminaries*: *strongly orthogonal roots*. Let $\mathfrak{g}$ be a real reductive Lie algebra, $\mathfrak{h}$ a Cartan subalgebra, and $\Delta = \Delta(\mathfrak{h}_{\mathbf{C}}, \mathfrak{g}_{\mathbf{C}})$ the system of $\mathfrak{h}_{\mathbf{C}}$-roots on $\mathfrak{g}_{\mathbf{C}}$. Choose a positive system $\Delta^+$, so the corresponding simple roots $\Psi = \{\alpha \in \Delta^+ \colon \alpha$ not a sum of two elements of $\Delta^+\}$ form the vertices of the Dynkin diagram.

We recall Kostant's "cascade construction", found in **[2]**, which is basic to the result of Moore mentioned in 2a. Two roots $\gamma, \gamma'$ are called *strongly orthogonal* if neither of $\gamma \pm \gamma'$ is a root. Then of course $\gamma \perp \gamma'$ under the (dual of the) Killing form, and the 3-dimensional simple subalgebras

$$\mathfrak{g}_{\mathbf{C}}[\gamma] = \mathfrak{g}_{\mathbf{C}}^{\gamma} + \mathfrak{g}_{\mathbf{C}}^{-\gamma} + [\mathfrak{g}_{\mathbf{C}}^{\gamma}, \mathfrak{g}_{\mathbf{C}}^{-\gamma}] \quad \text{and} \quad \mathfrak{g}_{\mathbf{C}}[\gamma'] = \mathfrak{g}_{\mathbf{C}}^{\gamma'} + \mathfrak{g}_{\mathbf{C}}^{-\gamma'} + [\mathfrak{g}_{\mathbf{C}}^{\gamma'}, \mathfrak{g}_{\mathbf{C}}^{-\gamma'}]$$

of $\mathfrak{g}_{\mathbf{C}}$ centralize each other. We denote this by $\gamma \perp\!\!\!\perp \gamma'$. The cascade construction produces a maximal strongly orthogonal family of roots $B = \{\beta\} \subset \Delta^+$ as follows.

*Level* 1. Decompose the Dynkin diagram into its components, i.e. decompose $\Psi = \Psi_1 \cup \cdots \cup \Psi_q$ into minimal mutually orthogonal subsets. That decomposes

$$\Delta^+ = \Delta_1^+ \cup \cdots \cup \Delta_q^+ \quad \text{where } \Delta_i^+ = \Delta^+ \cap (\mathbf{Z}\text{-span of } \Psi_i),$$

and defines a set $\{\beta_1, \ldots, \beta_q\}$ of mutually $\perp\!\!\!\perp$ roots by: $\beta_i$ is the maximal root in $\Delta_i^+$.

*Level* 2. For each $i$, $1 \leqslant i \leqslant q$, let $\Gamma_i^+ = \{\gamma \in \Delta_i^+ : \gamma \perp \beta_i\}$. Consider the subalgebra $\mathfrak{g}_{\mathbf{C}}[\Gamma_i^+]$ with root system $\Gamma_i^+ \cup -\Gamma_i^+$, and decompose its Dynkin diagram into components, i.e. decompose

$$\{\psi \in \Psi_i : \psi \perp \beta_i\} = \Psi_{i,1} \cup \cdots \cup \Psi_{i,q(i)}$$

into minimal mutually orthogonal subsets. That decomposes

$$\Gamma_i^+ = \Delta_{i,1}^+ \cup \cdots \cup \Delta_{i,q(i)}^+$$

where $\Delta_{i,j}^+ = \Delta^+ \cap (\mathbf{Z}\text{-span of } \Psi_{i,j})$, and defines a set $\{\beta_{i,1}, \ldots, \beta_{i,q(i)}\}$ of mutually $\underline{\perp\!\!\!\perp}$ roots by: $\beta_{i,j}$ is the maximal root in $\Delta_{i,j}^+$. Note that $\{\beta_i, \beta_{i,j}: 1 \leqslant i \leqslant q, 1 \leqslant j \leqslant q(i)\}$ are mutually $\underline{\perp\!\!\!\perp}$.

*Level* $k+1$. For each $I = (i_1, \ldots, i_k)$, $1 \leqslant i_j \leqslant q(i_1, \ldots, i_{j-1})$, decompose

$$\{\psi \in \Psi_I : \psi \perp \beta_I\} = \Psi_{I,1} \cup \cdots \cup \Psi_{I,q(I)}$$

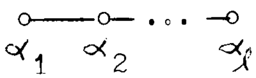
into minimal mutually orthogonal subsets. Let $\Delta_{I,i_{k+1}}^+$ denote $\Delta^+ \cup (\mathbf{Z}\text{-span of } \Psi_{I,i_{k+1}})$, and define: $\beta_{I,i_{k+1}}$ is the maximal root in $\Delta_{I,i_{k+1}}^+$. Note that $\{\beta_J : J = (i_1, \ldots, i_{k+1}), 1 \leqslant i_j \leqslant q(i_1, \ldots, i_{j-1})\}$ are mutually $\underline{\perp\!\!\!\perp}$.

The procedure eventually ends, giving us a maximal set $B = \{\beta\}$ of strongly orthogonal roots.

3.1. LEMMA. *Let $w_0$ be the Weyl group element such that $w_0(\Delta^+) = -\Delta^+$. Then $w_0(\beta) = -\beta$ for every $\beta \in B$.*

PROOF. First, $-w_0$ preserves each $\Delta_i^+$, so we may assume $\mathfrak{g}_{\mathbf{C}}$ simple. If $\mathfrak{g}_{\mathbf{C}}$ is not of type $A_l$, $D_l$ ($l$ odd) or $E_6$, then $-w_0$ is the identity and the assertion is clear.

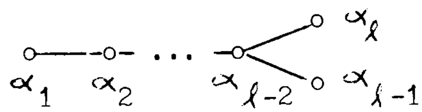
If $\mathfrak{g}_{\mathbf{C}}$ is of type $A_l$




then $-w_0(\alpha_i) = \alpha_{l+1-i}$ and $B$ consists of the roots

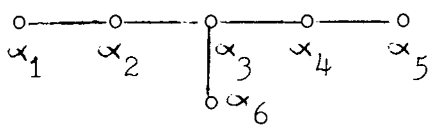
$$\alpha_i + \cdots + \alpha_{l+1-i}, \qquad 1 \leqslant i \leqslant \tfrac{1}{2}(l+1).$$

If $\mathfrak{g}_{\mathbf{C}}$ is of type $D_l$




with $l$ odd, then $-w_0$ interchanges $\alpha_{l-1}$ and $\alpha_l$ but fixes the other $\alpha_i$, and $B$ consists of the roots $\alpha_{2i-1}$ and $\alpha_{2i-1} + 2(\alpha_{2i} + \cdots + \alpha_{l-2}) + \alpha_{l-1} + \alpha_l$ for $1 \leqslant i \leqslant [l/2]$.

If $\mathfrak{g}_{\mathbf{C}}$ is of type $E_6$




then $-w_0$ interchanges $\alpha_1$ and $\alpha_5$, interchanges $\alpha_2$ and $\alpha_4$, fixes $\alpha_3$ and $\alpha_6$, and $B$ consists of the roots $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6$, $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$, $\alpha_2 + \alpha_3 + \alpha_4$, and $\alpha_3$. Q.E.D.

3b. *Necessary conditon for the semi-invariant.* Retain the notation of (3a) and let $\mathfrak{p}_{\mathbf{C}}$ be a parabolic subalgebra of $\mathfrak{g}_{\mathbf{C}}$ standard with respect to the choice of $\mathfrak{h}$ and $\Delta^+$. Thus we have a subset $\Phi \subset \Psi$ such that $\mathfrak{p} = \mathfrak{p}_\Phi$, the parabolic with

$$\text{reductive part} \quad \mathfrak{p}_\Phi^r = \mathfrak{h}_{\mathbf{C}} + \sum_{\gamma \in \langle \Phi \rangle} \mathfrak{g}_{\mathbf{C}}^\gamma,$$

$$\text{nilradical} \quad \mathfrak{p}_\Phi^n = \sum_{\Delta^+ \setminus \langle \Phi \rangle} \mathfrak{g}_{\mathbf{C}}^\gamma,$$

where $\langle \Phi \rangle$ denotes all roots that are linear combinations of elements of $\Phi$. Evidently, $\mathfrak{p}_\Phi$ acts on $\mathfrak{p}_\Phi^n$ with trace $\delta_\Phi = \Sigma_{\Delta^+ \setminus \langle \Phi \rangle} \gamma$. It is known (see the proof of [**10**, Theorem 8.5]) that $\delta_\Phi$ specifies $\mathfrak{p}_\Phi$, more precisely that if $\gamma \in \Delta$ then

$$(3.2) \qquad \langle \gamma, \delta_\Phi \rangle \text{ is } \begin{cases} \text{positive} & \text{for } \gamma \in \Delta^+ \setminus \langle \Phi \rangle, \\ \text{zero} & \text{for } \gamma \in \langle \Phi \rangle, \\ \text{negative} & \text{for } -\gamma \in \Delta^+ \setminus \langle \Phi \rangle. \end{cases}$$

In particular $\delta_\Phi$ is a dominant weight.

3.3. Proposition. *If there is a $\mathfrak{p}_\Phi$-semi-invariant polynomial of weight $k\delta_\Phi$ on $(\mathfrak{p}_\Phi^n)^*$ for some $k \neq 0$, then $-w_0(\Phi) = \Phi$.*

Proof. Any such semi-invariant polynomial corresponds to a $\mathfrak{p}_\Phi$-semi-invariant element $T$ of weight $k\delta_\Phi$ in the universal enveloping algebra $\mathfrak{U}(\mathfrak{p}_\Phi^n)$. Here $T$ is invariant under $\tilde{\mathfrak{n}} = \Sigma_{\gamma \in \Delta^+} \mathfrak{g}_{\mathbf{C}}^\gamma$, in particular under $\mathfrak{p}_\Phi^n$, so it is an $\tilde{\mathfrak{n}}$-invariant in the center $\mathfrak{Z}(\mathfrak{p}_\Phi^n)$ of that enveloping algebra. But A. Joseph showed [**2**, §§4.10–4.12] that the weights of $\mathfrak{h}_{\mathbf{C}}$ on $\mathfrak{Z}(\mathfrak{p}_\Phi^n)^{\tilde{\mathfrak{n}}}$ are those nonnegative integral linear combinations of the roots in $B$, which are dominant. Since $|k|\delta_\Phi = \pm k\delta_\Phi$ is dominant by (3.2), we now have an expression

$$(3.4) \qquad \pm k\delta_\Phi = \sum_{\beta \in B} n_\beta \beta, \qquad n_\beta \text{ integers} \geqslant 0.$$

Lemma 3.1 now tells us $-w_0(\delta_\Phi) = \delta_\Phi$, and from (3.2) we conclude that $-w_0(\Phi) = \Phi$. Q.E.D.

3c. *Construction of the semi-invariant.* Given $\nu \in \mathfrak{h}_{\mathbf{C}}^*$ dominant and integral, we denote

$\tau_\nu$: irreducible finite dimensional representation,

$V_\nu$: representation space of $\tau_\nu$

for highest weight $\nu$. Furthermore, define $\Phi_\nu = \{\alpha \in \Psi: \langle \alpha, \nu \rangle = 0\} \subset \Psi$. If $0 \neq v_\nu \in V_\nu$ is a highest weight vector, then the parabolic subalgebra $\mathfrak{p}_\nu = \mathfrak{p}_{\Phi_\nu}$ of $\mathfrak{g}_{\mathbf{C}}$ satisfies

$$(3.5) \qquad \mathfrak{p}_\nu = \mathfrak{p}_{\Phi_\nu} = \{\xi \in \mathfrak{g}_{\mathbf{C}}: \tau_\nu(\xi) v_\nu \in v_\nu \mathbf{C}\}.$$

We write $\mathfrak{p}_\nu^r$ for the reductive part $\mathfrak{p}_{\Phi_\nu}^r$, $\mathfrak{p}_\nu^n$ for the nilradical $\mathfrak{p}_{\Phi_\nu}^n$, and $\mathfrak{p}_\nu^{-n}$ for the "opposite" nilradical $\Sigma_{\Delta^+ \setminus \langle \Phi \rangle} \mathfrak{g}_{\mathbf{C}}^{-\gamma}$. Recall the symmetrization map from the symmetric algebra to the universal enveloping algebra,

$$(3.6) \qquad s: S(\mathfrak{p}_\nu^{-n}) \to \mathfrak{U}(\mathfrak{p}_\nu^{-n}) \text{ by } s(\xi_1 \cdot \cdots \cdot \xi_r) = \frac{1}{r!} \sum_\sigma \xi_{\sigma(1)} \cdots \xi_{\sigma(r)}.$$

Here the sum is over the symmetric group, $s(\xi_1 \cdot \cdots \cdot \xi_r)$ involves the product in $S(\mathfrak{p}_\nu^{-n})$, and $\xi_{\sigma(1)} \cdots \xi_{\sigma(r)}$ is the product in $\mathfrak{U}(\mathfrak{p}_\nu^{-n})$.

3.7. LEMMA. *Suppose that $\tau_\nu$ has a (nonzero) bilinear invariant $b_\nu$: $V_\nu \times V_\nu \to \mathbf{C}$. Then the function*

$$f_\nu: S(\mathfrak{p}_\nu^{-n}) \to \mathbf{C} \text{ by } \quad f_\nu(\Xi) = b_\nu(\tau_\nu(s\Xi)v_\nu, v_\nu)$$

*is a nonzero $\mathfrak{p}_\nu$-semi-invariant of weight $2\nu$.*

REMARK. $f_\nu(\Xi)$ is linear in $\Xi$, and to use it we identify $f_\nu \in S(\mathfrak{p}^{-n})^* = S((\mathfrak{p}^{-n})^*) = S(\mathfrak{p}^n) =$ (algebra of polynomial functions on $(\mathfrak{p}^n)^*$).

PROOF. By (3.5) we have $V_\nu = \tau_\nu(\mathfrak{U}(\mathfrak{p}_\nu^{-n})) \cdot v_\nu$, so $f_\nu$ is not identically zero.

Define $\phi$: $\mathfrak{p}_\nu^{-n} \times \cdots \times \mathfrak{p}_\nu^{-n} \to \mathbf{C}$ by

$$\phi(\xi_1, \ldots, \xi_r) = b_\nu(\tau_\nu\xi_1 \cdot \tau_\nu\xi_2 \cdot \cdots \cdot \tau_\nu\xi_r v_\nu, v_\nu).$$

If $\zeta \in \mathfrak{g}_{\mathbf{C}}$ with $\tau_\nu(\zeta)v_\nu = \rho v_\nu$ then we compute

$$\begin{aligned}(-\zeta \cdot \phi)(\xi_1, \ldots, \xi_r) &= \sum_1^r b_\nu(\tau_\nu\xi_1 \cdot \cdots \cdot \tau_\nu\xi_{i-1} \cdot [\tau_\nu\zeta, \tau_\nu\xi_i] \cdot \tau_\nu\xi_{i+1} \cdot \cdots \cdot \tau_\nu\xi_r v_\nu, v_\nu)\\
&= \sum_1^r b_\nu(\tau_\nu\xi_1 \cdot \cdots \cdot \tau_\nu\xi_{i-1} \cdot \tau_\nu\zeta \cdot \tau_\nu\xi_i \cdot \cdots \cdot \tau_\nu\xi_r v_\nu, v_\nu)\\
&\quad - \sum_1^r b_\nu(\tau_\nu\xi_1 \cdot \cdots \cdot \tau_\nu\xi_i \cdot \tau_\nu\zeta \cdot \tau_\nu\xi_{i+1} \cdot \cdots \cdot \tau_\nu\xi_r v_\nu, v_\nu)\\
&= b_\nu(\tau_\nu\zeta \cdot \tau_\nu\xi_1 \cdot \cdots \cdot \tau_\nu\xi_r v_\nu, v_\nu) - b_\nu(\tau_\nu\xi_1 \cdot \cdots \cdot \tau_\nu\xi_r \cdot \tau_\nu\zeta v_\nu, v_\nu)\\
&= -b_\nu(\tau_\nu\xi_1 \cdot \cdots \cdot \tau_\nu\xi_r v_\nu, \tau_\nu\zeta v_\nu) - b_\nu(\tau_\nu\xi_1 \cdot \cdots \cdot \tau_\nu\xi_r \cdot \tau_\nu\zeta v_\nu, v_\nu)\\
&= -2\rho\phi(\xi_1, \ldots, \xi_r).\end{aligned}$$

Thus the $r$-linear function $\phi$ on $\mathfrak{p}_\nu^{-n}$ is $\mathfrak{p}_\nu$-semi-invariant of weight $2\nu$. Since

$$f_\nu(\xi_1 \cdot \cdots \cdot \xi_r) = \frac{1}{r!} \sum_\sigma \phi(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(r)})$$

we conclude the same semi-invariance for $f_\nu$. Q.E.D.

In order to turn Lemma 3.7 around, starting with $\mathfrak{p}_\Phi$ and going to $\tau_\nu$ such that $\Phi = \Phi_\nu$, we recall the standard fact

(3.8) $\tau_\nu$ has a bilinear invariant if and only if $-w_0(\nu) = \nu$.

Now suppose that $-w_0(\Phi) = \Phi$ as in Proposition 3.3. Define

$$(3.9) \qquad \nu = \nu_\Phi = \begin{cases} \frac{1}{2}\delta_\Phi & \text{if } \frac{1}{2}\delta_\Phi \text{ is in the weight lattice,}\\ \delta_\Phi & \text{if } \frac{1}{2}\delta_\Phi \text{ is not in the weight lattice.}\end{cases}$$

Then $-w_0(\nu) = \nu$, so by (3.8) $\tau_\nu$ has a bilinear invariant $b_\nu$, and Lemma 3.7 gives a nonzero $\mathfrak{p}_\nu$-semi-invariant function $f_\nu$: $S(\mathfrak{p}_\Phi^{-n}) \to \mathbf{C}$ of weight $\nu$. According to the Remark, we have

3.10. PROPOSITION. *Suppose $-w_0(\Phi) = \Phi$. Then there is a $\mathfrak{p}_\Phi$-semi-invariant polynomial on $(\mathfrak{p}_\Phi^n)^*$ of weight $\delta_\Phi$ if $\frac{1}{2}\delta_\Phi$ is in the weight lattice, of weight $2\delta_\Phi$ if $\frac{1}{2}\delta_\Phi$ is not in the weight lattice.*

3d. *Degree of the semi-invariant.* Suppose that the parabolic $\mathfrak{p}_\Phi$ has a semi-invariant polynomial of some weight $k\delta_\Phi$ on $(\mathfrak{p}_\Phi^n)^*$. According to Propositions 3.3 and 3.10, we may take $k = 1$ or $k = 2$. Now, from (3.2) the sign in (3.4) is +, and (3.4) gives us an expression

$$(3.11)\qquad \begin{cases} \text{if } \frac{1}{2}\delta_\Phi \in \Lambda & \text{then } \delta_\Phi = \sum_{\beta\in B} n_\beta \beta = 2\nu, \\ \text{if } \frac{1}{2}\delta_\Phi \notin \Lambda & \text{then } 2\delta_\Phi = \sum_{\beta\in B} n_\beta \beta = 2\nu, \end{cases}$$

where $\Lambda$ is the weight lattice and the $n_\beta$ are integers $\geqslant 0$, and where $\tau_\nu$ gives rise to the semi-invariant of weight $2\nu$.

The bilinear invariant $b_\nu$: $V_\nu \times V_\nu \to \mathbf{C}$ pairs the highest weight space $v_\nu\mathbf{C}$ with the lowest weight space $v_{w_0(\nu)}\mathbf{C} = v_{-\nu}\mathbf{C}$. Thus the function $f_\nu$ of Lemma 3.7 has value $f_\nu(\Xi) \neq 0$ precisely when $\tau_\nu(s\Xi)v_\nu$ has nonzero component in $v_{-\nu}\mathbf{C}$.

3.12. Proposition. *If* $\gamma \in \Delta$ *choose a nonzero* $e_\gamma \in \mathfrak{g}_\mathbf{C}^\gamma$. *In the notation of* (3.11) *define*

$$\Xi = \prod_{\beta\in B} (e_{-\beta})^{n_\beta} \in S(\mathfrak{p}_\nu^{-n}).$$

*Then* $f_\nu(\Xi) \neq 0$, *and* $f_\nu$ *is a polynomial of degree* $n = \sum_{\beta\in B} n_\beta$ *on* $(\mathfrak{p}_\Phi^n)^*$.

Proof. Let $\mathfrak{q} = \bigoplus_{\beta\in B} \mathfrak{g}_\mathbf{C}[\beta]$, direct sum of the three-dimensional simple algebras $\mathfrak{g}_\mathbf{C}^\beta + \mathfrak{g}_\mathbf{C}^{-\beta} + [\mathfrak{g}_\mathbf{C}^\beta, \mathfrak{g}_\mathbf{C}^{-\beta}]$. Here we are using strong orthogonality of any pair $\beta, \beta' \in B$. Under $\tau_\nu$, each $\mathfrak{g}_\mathbf{C}[\beta]$ generates a cyclic module $W_\beta$ from $v_\nu$ and $\mathfrak{q}$ generates a cyclic module $W$. The representation theory of $\mathfrak{sl}(2)$ says that $W_\beta$ is irreducible under $\mathfrak{g}_\mathbf{C}[\beta]$, has dimension $(2\langle\beta,\nu\rangle/\langle\beta,\beta\rangle) + 1 = n_\beta + 1$, and has weight spaces $W_\beta^{\nu-p\beta} = \tau_\nu(e_{-\beta})^p v_\nu \mathbf{C}$, 1-dimensional $(0 \leqslant p \leqslant n_\beta)$. Evidently $W = \bigotimes_{\beta\in B} W_\beta$, so it is irreducible under $\mathfrak{q}$, has dimension $\prod_{\beta\in B}(n_\beta + 1)$, and has 1-dimensional weight spaces

$$W^{\nu-\Sigma p_i\beta_i} = \tau_\nu(e_{-\beta_1})^{p_1}\cdot \cdots \cdot \tau_\nu(e_{-\beta_t})^{p_t}\cdot v_\nu\mathbf{C}$$

where $B = \{\beta_1, \ldots, \beta_t\}$ and $0 \leqslant p_i \leqslant n_{\beta_i}$. In particular $0 \neq \tau_\nu(s\Xi)v_\nu\mathbf{C} = W^{\nu-\Sigma n_\beta\beta} = W^{-\nu}$, so $f_\nu(\Xi) \neq 0$ as asserted. In fact, this shows that the component $f_\nu^{(n)}$ of degree $n = \sum n_\beta$ of $f_\nu$ is nonzero, and the uniqueness in Theorem 2.2 shows $f_\nu = f_\nu^{(n)}$. Q.E.D.

3e. *Summary and example.* We summarize Propositions 3.3, 3.10 and 3.12 as follows.

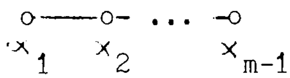
3.13. Theorem. *Let* $\mathfrak{g}_\mathbf{C}$ *be a complex reductive Lie algebra,* $\Psi$ *a simple root system corresponding to a choice of Cartan subalgebra* $\mathfrak{h}_\mathbf{C}$ *and positive* $\mathfrak{h}_\mathbf{C}$*-root system* $\Delta^+$, $B$ *the maximal set of strongly orthogonal roots produced by the cascade construction, and* $w_0$ *the Weyl group element that sends* $\Delta^+$ *to* $-\Delta^+$.

*Let* $\mathfrak{p}_\Phi$ *be the parabolic subalgebra corresponding to a subset* $\Phi \subset \Psi$, *and let* $\delta_\Phi = \sum_{\Delta^+\setminus\langle\Phi\rangle} \gamma$, *trace of* $\mathfrak{p}_\Phi$ *on its nilradical* $\mathfrak{p}_\Phi^n$.

1. *There is a* $\mathfrak{p}_\Phi$*-semi-invariant polynomial on* $(\mathfrak{p}_\Phi^n)^*$ *of some weight* $k\delta_\Phi$, $k \neq 0$, *if and only if* $-w_0(\Phi) = \Phi$.

2. *In that case, either* $\frac{1}{2}\delta_\Phi$ *is in the weight lattice and we may take* $k = 1$, $\nu = \frac{1}{2}\delta_\Phi$, *or* $\frac{1}{2}\delta_\Phi$ *is not in the weight lattice and we may take* $k = 2$, $\nu = \delta_\Phi$. *Then* $2\nu = \Sigma_{\beta \in B} n_\beta \beta$, $n_\beta$ *integers* $\geqslant 0$, *and the construction of Lemma* 3.7 *produces a* $\mathfrak{p}_\Phi$*-semi-invariant polynomial of weight* $k\delta_\Phi$ *and degree* $n = \Sigma_\beta n_\beta$ *on* $(\mathfrak{p}_\Phi^n)^*$.

Consider the case where $\mathfrak{g}_{\mathbf{C}}$ is $\mathfrak{gl}(m; \mathbf{C})$ or $\mathfrak{sl}(m; \mathbf{C})$ with simple roots



So $B = \{\beta_1, \ldots, \beta_{[m/2]}\}$ with $\beta_i = \alpha_i + \cdots + \alpha_{m-i}$ for $2i < m$, $\beta_i = \alpha_i$ in case $2i = m$. Let $\Phi$ be such that $\mathfrak{p}_\Phi$ is the $\mathfrak{g}_{\mathbf{C}}$-stabilizer of a "flag" $\mathbf{C}^p \subset \mathbf{C}^{m-p}$ inside $\mathbf{C}^m$, $2p < m$, or a subspace $\mathbf{C}^p$ in case $2p = m$. In the usual $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ notation,

$$\delta_\Phi = (m-p)\sum_1^p \varepsilon_i - (m-p)\sum_{m-p+1}^m \varepsilon_i = (m-p)\sum_1^p \beta_i.$$

If $m - p$ is even, now $\frac{1}{2}\delta_\Phi$ is in the weight lattice $\Lambda$, leading to a semi-invariant polynomial of weight $\delta_\Phi$ and degree $\frac{1}{2}p(m-p)$. If $m - p$ is odd, then $\frac{1}{2}\delta_\Phi \notin \Lambda$, and the semi-invariant polynomial has weight $2\delta_\Phi$ and degree $p(m-p)$. If we write $\mathfrak{p}_\Phi$ in block form matrices

$$\begin{bmatrix} a & x & z \\ 0 & b & y \\ 0 & 0 & c \end{bmatrix}$$

where $a$ and $c$ are $p \times p$, $b$ is $(m-2p) \times (m-2p)$, $x$ and ${}^t y$ are $p \times (m-2p)$, and $z$ is $p \times p$, then $\mathfrak{p}_\Phi^n$ is given by $x, y, z$ and the polynomial is a power of $\det(z)$. Compare [**12**, §10.3].

3f. *Reformulation for real parabolics*. We return to the situation of §2. $G$ is a real reductive Lie group and $P$ is a parabolic subgroup with Langlands decomposition $NAM$. Express its complexified Lie algebra $\mathfrak{p}_{\mathbf{C}} = \mathfrak{p}_\Phi$ as in Theorem 3.13. Then $P$ has modular function $\delta_P$ where $\delta_P\colon P \to \mathbf{R}^*$ is a quasi-character with differential $\delta_\Phi|_{\mathfrak{p}}$. As before, $w_0$ is the element of the complex Weyl group that sends $\Delta^+$ to its negative. Theorem 3.13 carries over to

3.14. THEOREM. *The parabolic* $P$ *is good if and only if* $\Phi = -w_0(\Phi)$. *In that case there are two possibilities as follows.*

(i) $\frac{1}{2}\delta_\Phi$ *is in the weight lattice* $\Lambda$, *we have an expression* $\delta_\Phi = \Sigma_{\beta \in B} n_\beta \beta$ *where* $B$ *is the set of strongly orthogonal roots from the cascade construction and the* $n_\beta$ *are integers* $\geqslant 0$, *and there is a* $P$*-semi-invariant polynomial of weight* $\delta_P$ *and degree* $n = \Sigma_B n_\beta$ *on* $\mathfrak{n}^*$.

(ii) $\frac{1}{2}\delta_\Phi \notin \Lambda$, $2\delta_\Phi = \Sigma_{\beta \in B} n_\beta \beta$ *with* $n_\beta$ *integers* $\geqslant 0$, *and there is a* $P$*-semi-invariant polynomial of weight* $\delta_P^2$ *and degree* $n = \Sigma_B n_\beta$ *on* $\mathfrak{n}^*$.

Consider the case where $G$ is an indefinite unitary group $U(k, l)$ or $SU(k, l)$, and let $P$ be the stabilizer of an isotropic subspace of dimension $p$ in the corresponding hermitian vector space $\mathbf{C}^{k,l}$. Now $\mathfrak{g}_{\mathbf{C}}$ is $\mathfrak{gl}(m; \mathbf{C})$ or $\mathfrak{sl}(m; \mathbf{C})$ where $m = k + l$, $1 \leqslant p \leqslant \min(k, l)$, and $\mathfrak{p}_{\mathbf{C}}$ is the parabolic $\mathfrak{p}_\Phi$ of the example at the end of §3e. So

$P$ is good, and we are in case (i) or case (ii) of Theorem 3.14 as $m - p$ is even or odd. Compare [**12**, §10.2]. One knows [**11**, §2] that $\mathfrak{n}$ has underlying real vector space structure ($p \times p$ skew hermitian matrices) + ($p \times (m - 2p)$ complex matrices) where (unless $m = 2p$) the first summand is the center of $\mathfrak{n}$, and the polynomial is a power of $(z, x) \mapsto \det(\sqrt{-1}\ z)$.

Finally, the correspondence from $\delta_\Phi$ to the semi-invariant polynomial $\phi$ on $\mathfrak{n}^*$ is not transparent. If $G = SL(2n + 1; \mathbf{R})$ (resp. $SL(2n + 2; \mathbf{R})$) and $P$ is the Borel subgroup with $N$ consisting of all strictly upper triangular matrices

$$\begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix},$$

then $\delta_\Phi = 2(\beta_1 + \cdots + \beta_n)$ (resp. $2(\beta_1 + \cdots + \beta_n) + \beta_{n+1}$), corresponding to $\phi = d_1^2 d_2^2 \cdots d_n^2$ (resp. $(d_1^2 \cdots d_n^2) d_{n+1}$). Here we view $\phi$ in $S(\mathfrak{n})$, and $d_j$ is the determinant of the $j \times j$ upper right-hand corner block submatrix. This particular formula for $\phi$ was known as a consequence of results [**13**] of Dixmier.

3g. *Remark on the tube domains.* Consider a hermitian symmetric space $G/K$ of noncompact type, and the complex flag manifold realization $G_{\mathbf{C}}/P$ of its compact dual symmetric space. If $G/K$ is irreducible, i.e. if $G_{\mathbf{C}}$ is simple, then for a Cartan subalgebra $\mathfrak{l} \subset \mathfrak{k}$ of $\mathfrak{g}$ there is a system $\Delta^+$ of positive $\mathfrak{l}_{\mathbf{C}}$-roots on $\mathfrak{g}_{\mathbf{C}}$ such that $\mathfrak{p} = \mathfrak{p}_\Phi$ where

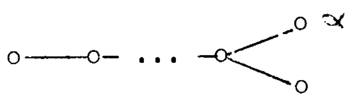
the system $\Psi$ of simple roots has just one noncompact root,

$\Phi$ consists of all the compact roots in $\Psi$.

One knows [**12**, Theorem 9.15] that there is a nonzero $\mathfrak{p}_\Phi$-semi-invariant polynomial on $(\mathfrak{p}_\Phi^n)^*$ if and only if the domain $G/K$ is a tube domain, i.e. a tube over a self-dual cone. In that case the semi-invariant is of the appropriate weight $k\delta_\Phi$. Now, from Theorem 3.13,

3.15. Proposition. *The noncompact type symmetric space $G/K$ is a tube domain if, and only if, the corresponding set (one for each simple factor of $G_{\mathbf{C}}$) of noncompact simple roots is $-w_0$ stable.*

Thus, for example, marking the simple root as $\alpha$, we immediately recover the well-known fact that $G/K = SO^*(2m)/U(m)$:




is of tube type if and only if $m$ is even.

**4. Examples.** In this section we consider good parabolic subgroups $P_\Phi = NAM$ and consider the questions of whether the operator $D$ can be taken to be differential and whether the isotropy subgroups of $MA$ on $\hat{N}$ are reductive. Examples show that one cannot decide these matters in an easy direct way from the root structure except under rather restrictive conditions.

4a. *Differentiality of D*. If $\frac{1}{2}\delta_\Phi$ belongs to the weight lattice $\Lambda$ then we construct a differential operator $D$ on $N$, semi-invariant of weight $\delta_P$. If one wants it positive he must take $|D|$, which need not be differential. If $\frac{1}{2}\delta_\Phi \notin \Lambda$ we construct an operator $E$ of weight $\delta_P^2$, and then $D = |E|^{1/2}$ is not differential. In any case the operator is constructed from a bilinear invariant $b_\nu$ of the representation $\tau_\nu$ where $\nu = \frac{1}{2}\delta_\Phi$ if $\frac{1}{2}\delta_\Phi \in \Lambda$ and $\nu = \delta_\Phi$ if $\frac{1}{2}\delta_\Phi \notin \Lambda$. There are two possibilities for $b_\nu$: It is symmetric or antisymmetric. Kostant suggested that we check whether symmetry of $b_\nu$ corresponds to $\frac{1}{2}\delta_\Phi \in \Lambda$. We now see that those conditions are independent.

If $G = F_{4,B_4}$ then the minimal parabolic subgroup $P = NAM$ has $N \cong \operatorname{Im} \mathcal{C} + \mathcal{C}$ where $\mathcal{C}$ is the Cayley division algebra and $(z, x)(z', x') = (z + z' + \operatorname{Im} x\bar{x}', x + x')$. The operator is $\Delta^{11/2}$ where $\Delta$ is Laplacian on $\operatorname{Im} \mathcal{C}$, i.e. $\frac{1}{2}\delta_\Phi \notin \Lambda$. But $b_{\delta_\Phi}$ is symmetric.

If $G = \operatorname{Sp}(2; \mathbf{R})$ then the Borel subgroup $P$ has $\frac{1}{2}\delta_\Phi \in \Lambda$, for half the sum of the positive roots always lies in the weight lattice, but there $b_{\delta_\Phi/2}$ is antisymmetric.

If $G = \operatorname{SL}(n; \mathbf{R})$ then the Borel subgroup $P$ has $\frac{1}{2}\delta_\Phi \in \Lambda$ and $b_{\delta_\Phi/2}$ is symmetric.

If $G = \operatorname{Sp}(1, 1)$ then the minimal parabolic subgroup $P$ has $\frac{1}{2}\delta_\Phi \notin \Lambda$ and here $b_{\delta_\Phi}$ is antisymmetric.

4b. *Square-integrable nilradical*. Suppose that $P = NAM$ is a good parabolic subgroup such that $N$ has square-integrable representations, i.e. representations with coefficients that are square-integrable on $N$ modulo its center. Going through **[12]** one sees that the generic isotropy subgroups of $MA$ on $\hat{N}$ are always symmetric subgroups of $MA$. In particular, they are reductive. Here note **[12**, §8**]** that $N$ is (abelian or) 2-step nilpotent.

4c. *Minimal parabolics*. Suppose that $P = NAM$ is a minimal parabolic subgroup. Then $M$ is compact, so the isotropy subgroups of $MA$ on $\hat{N}$ are reductive (and thus $P$ is good), even though $N$ generally does not (see **[3]**) have square-integrable representations.

4d. *Good parabolics with* 2-*step nilradical and nonreductive isotropy*. Let $G$ be the indefinite real orthogonal group $O(s + u, s + v)$ and let $P = NAM$ be the stabilizer of a totally isotropic $s$-dimensional subspace, where $s > 1$, $s$ is odd and $u + v > 0$. Then $P$ is the group $P_{s;u,v}(\mathbf{R})$ of **[11]**. It has structure $\mathfrak{n} = \operatorname{Im} \mathbf{R}^{s \times s} + \mathbf{R}^{s \times (u,v)}$ and $MA = \operatorname{GL}(s; \mathbf{R}) \times O(u, v)$ where the first summand of $\mathfrak{n}$ denotes antisymmetric $s \times s$ real matrices, and the second summand consists of all $s \times (u, v)$ real matrices $x = (x_1, x_2)$ in $s \times u$, $s \times v$ blocks, with

$$[(z, x), (z', x')] = \left(x_1 \cdot {}^t x_1' - x_1' \cdot {}^t x_1 - x_2 \cdot {}^t x_2' + x_2' \cdot {}^t x_2, 0\right).$$

Here $MA$ acts by $\operatorname{Ad}(\gamma, g)$: $(z, x) \mapsto (\gamma z \cdot {}^t\gamma, \gamma x \cdot {}^t g)$. $N$ is 2-step nilpotent but does not have square-integrable representations, and we are going to check that the generic isotropy subgroups of $MA$ on $\hat{N}$ are not reductive.

Identify $\mathfrak{n}$ to $\mathfrak{n}^*$ under $(z, x) \mapsto f_{z,x}$ where

$$f_{z,x}(z', x') = \operatorname{trace}(zz') + \operatorname{trace}\left(x_1 \cdot {}^t x_1' - x_2 \cdot {}^t x_2'\right).$$

Then $\operatorname{Ad}^*(N) \cdot f_{z,x} = \{f_{z,x+zx'} \colon x' \in \mathbf{R}^{s \times (u,v)}\}$, which has a distinguished element $f_{z,x''}$ with $zx'' = 0$, and the generic classes in $\hat{N}$ are those corresponding to the $f_{z,x}$

where $z$ has maximal possible matrix rank, $s-1$. $\mathcal{P}f$ denotes the Pfaffian on antisymmetric matrices; $\mathcal{P}f^2 = \det$. If we write out $x$ in columns as $(x^1, \ldots, x^{u+v})$ then

$$\psi(f_{z,x}) = \sum_{i=1}^{u}\left\{\mathcal{P}f\begin{pmatrix} z & x^i \\ -{}^tx^i & 0\end{pmatrix}\right\}^2 - \sum_{i=u+1}^{u+v}\left\{\mathcal{P}f\begin{pmatrix} z & x^i \\ -{}^tx^i & 0\end{pmatrix}\right\}^2$$

is a polynomial on $\mathfrak{n}^*$ that is $P$-semi-invariant of type $\det(\gamma)^2$ where $\gamma$ is the $GL(s;\mathbf{R})$-factor. Here $\delta_P = \det(\gamma)^{(s-1)(u+v)/2}$, so the Dixmier-Pukanszky operator is the Fourier transform of $|\psi|^{(s-1)(u+v)/4}$.

Let $[\pi] \in \hat{N}$ correspond to the functional $f_{z,x} \in \mathfrak{n}^*$ given by

$$z = \begin{bmatrix} J & & & \\ & \ddots & & \\ & & J & \\ & & & 0\end{bmatrix} \quad\text{and}\quad x = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ b_1 & \cdots & b_{u+v}\end{bmatrix}$$

such that ${}^tb \in \mathbf{R}^{u,v}$ has $\|{}^tb\|^2 \neq 0$ and $J = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$. Then $\psi(f_{z,x}) = \|{}^tb\|^2 \neq 0$, and the $MA = GL(s;\mathbf{R}) \times O(u,v)$ stabilizer of $\mathrm{Ad}^*(N)\cdot f_{z,x}$ consists of all $(\gamma, g)$ such that

$$\gamma = \begin{pmatrix} \gamma' & 0 \\ c & \varepsilon\end{pmatrix} \quad \text{with } \gamma' \in Sp\left(\tfrac{1}{2}(s-1);\mathbf{R}\right),\ c \in \mathbf{R}^{1\times(s-1)}$$

$$\text{and } \varepsilon = \pm 1 \text{ with } g\cdot{}^tb = \varepsilon\,{}^tb.$$

It has unipotent radical given by $\gamma = \begin{pmatrix} I & 0 \\ c & 1\end{pmatrix}$, $g = I$, isomorphic to $\mathbf{R}^{s-1}$. Thus the generic isotropy subgroups of $MA$ on $\hat{N}$ are not reductive.

4e. *Good parabolics with 2-step non-square-integrable nilradical and reductive isotropy*. Let $G = O(n,n)$, $n$ even, and let $P = NAM$ be the parabolic subgroup that stabilizes an isotropic flag (1-dim isotropic subspace) $\subset$ ($n$-dim isotropic subspace) inside $\mathbf{R}^{n,n}$. Take the usual basis in which the inner product has matrix $\begin{pmatrix} I & 0 \\ 0 & -I\end{pmatrix}$ in $n \times n$ blocks. Then $P$ has Lie algebra

$$\mathfrak{m} + \mathfrak{a} = \left\{\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & -{}^tA & 0 \\ 0 & 0 & 0 & -a\end{bmatrix} : a \in \mathbf{R}, A \in \mathfrak{gl}(n-1;\mathbf{R})\right\},$$

$$\mathfrak{n} = \left\{\begin{bmatrix} 0 & x & z & 0 \\ 0 & 0 & y & -{}^tz \\ 0 & 0 & 0 & -{}^tx \\ 0 & 0 & 0 & 0\end{bmatrix} : {}^tx, {}^tz \in \mathbf{R}^{n-1}, y \in \mathrm{Im}\,\mathbf{R}^{(n-1)\times(n-1)}\right\}.$$

The composition in $\mathfrak{n}$ is $[(z,y,x),(z',y',x')] = (xy' - x'y, 0, 0)$, and $MA = GL(1;\mathbf{R}) \times GL(n-1;\mathbf{R})$ acts on $\mathfrak{n}$ by $\mathrm{Ad}(a,A)(z,y,x) = (az\,{}^tA, Ay\,{}^tA, axA^{-1})$.

Identify $\mathfrak{n}$ to $\mathfrak{n}^*$ under $(z,y,x) \mapsto f_{z,y,x}$ where $f_{z,y,x}(z',y',x') = z\cdot z' - \mathrm{trace}(yy') + x\cdot x'$. Then $\mathrm{Ad}^*(N)\cdot f_{z,y,x} = \{f_{z,y+y',x'}\colon \mathbf{R}^{1\times(n-1)}y' = \mathbf{R}z\}$. The generic classes in $\hat{N}$ are those where $\begin{pmatrix} y & -{}^tz \\ z & 0\end{pmatrix}$ is nonsingular, and the $P$-semi-invariant of type $\delta_P$ on $\mathfrak{n}^*$ is just

$$\psi(f_{z,y,x}) = \det\begin{pmatrix} y & -{}^tz \\ z & 0\end{pmatrix} = \left\{\mathcal{P}f\begin{pmatrix} y & -{}^tz \\ z & 0\end{pmatrix}\right\}^2.$$

Thus the Dixmier-Pukanszky operator is the polynomial differential operator that is the Fourier transform of $\psi$, and is positive.

Let $[\pi] \in \hat{N}$ correspond to $f_{z,y,x} \in \mathfrak{n}^*$ given by $z = (0, \ldots, 0, b)$ with $b \neq 0$,

$$y = \begin{bmatrix} J & & & \\ & \ddots & & \\ & & J & \\ & & & 0 \end{bmatrix} \quad \text{with } J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and $x = (0, \ldots, 0)$. Then $\psi(f_{z,y,x}) = b^2 > 0$, and the $MA = \mathrm{GL}(1; \mathbf{R}) \times \mathrm{GL}(n-1; \mathbf{R})$ stabilizer of $\mathrm{Ad}^*(N) \cdot f_{z,y,x}$ consists of all $(a, A)$ with $A = \left(\begin{smallmatrix} A' & 0 \\ 0 & 1/a \end{smallmatrix}\right)$ and $A' \in \mathrm{Sp}(\frac{1}{2}(n-2); \mathbf{R})$. It is isomorphic to $\mathrm{Sp}(\frac{1}{2}(n-2); \mathbf{R}) \times \mathrm{GL}(1; \mathbf{R})$. Thus the generic isotropy subgroups of $MA$ on $\hat{N}$ are reductive.

4f. *The other good parabolics with* 2-*step nilradical*. The comments of 4b, 4d and 4e apply to good parabolic subgroups $P = NAM$ in a simple group $G$, such that $N$ is abelian or 2-step nilpotent. It is in fact easy to classify all good $P = P_\Phi$ such that

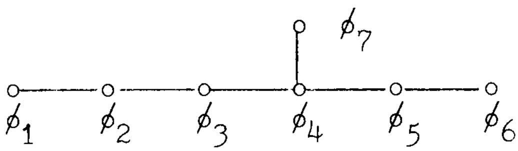
(i) $N$ is abelian or 2-step nilpotent,

(ii) $N$ does not have square-integrable representations, and

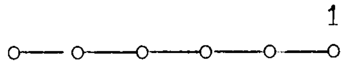
(iii) $P$ is good, i.e. $-w_0(\Phi) = \Phi$.

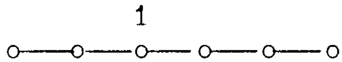
First, there are the parabolics $P \subset G$ considered in 4d and 4e, and their complexifications $P_{\mathbf{C}} \subset G_{\mathbf{C}}$. The parabolics $P_{s;1,1}(\mathbf{R}) \subset O(s+1, s+1)$, $s > 1$, $s$ odd, complexify and then intersect to give another such parabolic in $SO^*(2s+2)$, the real form of $O(2s+2, \mathbf{C})$ with maximal compact subgroup $U(s+1)$. These are the only cases inside classical groups.

Inside exceptional groups one only has $P$ in the (split) $E_{7,A_7}$ given by $P = P_\Phi$ where $\Phi = \{\phi_1, \ldots, \phi_6\}$ in




and its complexification $P_{\mathbf{C}}$ in the complex $E_7$. In this group $P = NAM$, $\mathfrak{n} = \mathfrak{z} + \mathfrak{r}$ where the center $\mathfrak{z}$ has dimension 7 and the complement $\mathfrak{r}$ has dimension 35. $M = (MA)_0 = \mathrm{GL}'(7; \mathbf{R}) = \mathrm{SL}(7, \mathbf{R}) \times \{\pm I\}$, where $\{\pm I\}$ is the center of $G = E_{7,A_7}$ and acts trivially, and $\mathrm{SL}(7; \mathbf{R})$ acts

on $\mathfrak{z} = (\mathbf{R}^7)^*$ by 


on $\mathfrak{r} = \Lambda^3(\mathbf{R}^7)$ by 


The multiplication in $\mathfrak{n}$ is $[(z, x), (z', x')] = (2x \wedge x', 0)$ where $x \wedge x' \in \Lambda^6(\mathbf{R}^7)$ which is identified to $(\mathbf{R}^7)^*$. In this notation, $N = \mathfrak{z} + \mathfrak{r}$ with $(z, x)(z', x') = (z + z' + x \wedge x', x + x')$, and the coadjoint action of $N$ on $\mathfrak{n}^* = \mathfrak{z}^* + \mathfrak{r}^*$ is

$$\mathrm{Ad}^*(z, x)^{-1} \cdot (\zeta, \xi) = (\zeta, \xi + \zeta \wedge x)$$

where $\zeta \wedge x \in \Lambda^4(\mathbf{R}^7)$ which is identified to $\Lambda^3(\mathbf{R}^7)^* = \mathfrak{r}^*$.

Define a polynomial $\psi$ on $\mathfrak{n}^*$ as follows. Let $(\zeta, \xi) \in \mathfrak{n}^* = \mathfrak{z}^* + \mathfrak{r}^* = \mathbf{R}^7 + \Lambda^3(\mathbf{R}^7)^*$. Now $\xi$ is a 3-form on $\mathbf{R}^7$, so the contraction $\zeta \lrcorner \xi$ is a 2-form on $\mathbf{R}^7$ that annihilates $\zeta$. If $\zeta \neq 0$ write $\{\zeta \lrcorner \xi\}$ for the 2-form induced on $\mathbf{R}^7/\zeta\mathbf{R} = \mathbf{R}^6$, and define $\psi(\zeta, \xi) = \mathcal{P}f\{\zeta \lrcorner \xi\}$. If $\zeta = 0$ then $\psi(\zeta, \xi) = 0$. The polynomial $\psi$ is well defined and $M$-invariant because, for $\zeta \neq 0$, the SL(7; $\mathbf{R}$)-stabilizer of $\zeta$ acts with determinant 1 on $\mathbf{R}^7/\zeta\mathbf{R}$, and it is $N$-invariant because $\zeta \lrcorner (\xi + \zeta \wedge x) = \zeta \lrcorner \xi$.

$A = \{aI: a > 0\}$ acts on $\mathfrak{n}$ by $(z, x) \mapsto (a^2 z, ax)$, on $\mathfrak{n}^*$ by $(\zeta, \xi) \mapsto (a^{-2}\zeta, a^{-1}\xi)$, so the modular function $\delta_P(aI) = a^{49}$. Also $aI$ sends $\psi$ to $a^7\psi$, so $|\psi|^7$ Fourier transforms to the Dixmier-Pukanszky operator on $P$.

Let $[\pi] \in \hat{N}$ correspond to $(\zeta, \xi) = (e_1, e_1^* \wedge e_2^* \wedge e_3^*) \in \mathfrak{n}^*$ where $\{e_i\}$ is the standard basis of $\mathbf{R}^7$ and $\{e_j^*\}$ is the dual basis of $(\mathbf{R}^7)^*$. Then $\psi(\zeta, \xi) = 1$, and $\mathrm{Ad}^*(N) \cdot (\zeta, \xi) = \{(e_1, e_1^* \wedge e_2^* \wedge e_3^* + \Sigma\, q_{ijk} e_i^* \wedge e_j^* \wedge e_k^*)\}$ where summation is over $1 < i < j < k \leqslant 7$ and $q_{ijk} \in \mathbf{R}$. The GL(7; $\mathbf{R}$)-stabilizer of this generic coadjoint orbit of $N$ consists of all

$$\begin{bmatrix} 1 & * & * \\ 0 & A & 0 \\ 0 & * & B \end{bmatrix} \quad \text{with } \det(A) = 1 = \det B,$$

where the diagonal blocks are $1 \times 1$, $2 \times 2$, $4 \times 4$ and the $*$'s give the unipotent radical. Thus the generic isotropy subgroups of $MA$ on $\hat{N}$ are not reductive.

4g. *Summary*. The examples in 4a show that differentiality of the Dixmier-Pukanszky operator $D$ for a good parabolic $P_\Phi$, i.e. the condition $\frac{1}{2}\delta_\Phi \in \Lambda$, is independent of whether the bilinear invariant is symmetric or antisymmetric.

The examples of 4b–4e, addressing the question of whether the generic isotropy groups of $MA$ on $\hat{N}$ are reductive for good parabolics $P = NAM$, give

from 4c: no bound on the degree of nilpotency of $N$ can prevent the generic isotropy subgroups from being reductive;

from 4b: if $N$ has square-integrable representations then $N$ is (abelian or) 2-step nilpotent and the generic isotropy groups are reductive;

from 4d: in some cases where $N$ is 2-step nilpotent the generic isotropy groups are not reductive;

from 4e: in some cases where $N$ is 2-step nilpotent but does not have square-integrable representations, the generic isotropy groups are reductive.

Finally, the results of **[12]**, 4d, 4e and 4f combine to give a complete discussion of the Plancherel formulae for good parabolics $P = NAM$ in which $N$ is (abelian or) 2-step nilpotent.

**5. The domain problem.** Recall the derivation of the Plancherel formula for a good parabolic $P$ that is carried out in §2d. It is assumed implicitly in that computation that $h \in \mathrm{Dom}\, D$. But since we also took $\pi(Dh)$ to mean the operator-valued integral $\int_P \pi(g) Dh(g)\, dg$ (when invoking **[5**, Theorem 3.2**]**), it is also necessary to have $Dh \in L_1(P)$. So we would like to know whether the space $\mathrm{Dom}\, D \cap D^{-1}L_1(P)$ is substantial–e.g. whether it is dense in $L_2(P)$. This question has arisen in every previous work on the Plancherel formula for specific nonunimodular groups, namely in **[4]**, **[7]**, **[6]**, **[12]**, **[8]**, **[5]**. A thorough treatment of the circle of ideas surrounding this question may be found in [7, especially §1]. The discussion there reveals that the property which is both desirable and reasonable to expect is the following:

(5.1) $\quad \text{Dom } D^t \cap D^{-t}L_1(P) \cap L_2(P)^{\mathfrak{L}}$ is dense in $L_2(P)$, $\forall t > 0$;

where $L_2(P)^{\mathfrak{L}}$ denotes the left bounded elements in $L_2(P)$.

In all previous cases (i.e. in the articles just cited), condition (5.1) has been verified. However, there are nonunimodular groups for which (5.1) fails. Phil Green has given such an example, a certain restricted direct product of $p$-adic groups. It seems quite likely that property (5.1) cannot fail for a Lie group, but we have not been able to prove (5.1) for an arbitrary good parabolic. On the other hand, we do have some partial results in that direction, and we conclude the paper by sketching them.

(i) We have seen in Theorem 2.2 that, for a good parabolic, the canonical Dixmier-Pukanszky operator "lives" on the nilradical. Thus we may replace the parabolic $P$ by its nilradical $N$ in equation (5.1).

(ii) Suppose that the nilradical $\underline{\mathfrak{n}}$ is abelian. Then the results of [7, §3] apply. If $f_0$ is the semi-invariant polynomial corresponding to the canonical semi-invariant $T$ given by Theorem 2.2, then the space $\mathcal{S}_{f_0}(N) = \{h \in \mathcal{S}(N) =$ Schwartz functions on $N$; $(\log h)\hat{} = 0$ near $f_0^{-1}(0)\}$ is a dense subspace of $L_2(N)$ which is contained in $\text{Dom } D^t \cap D^{-t}L_1(N) \cap L_2(N)^{\mathfrak{L}}$ for any $t > 0$.

(iii) Here is the observation which guarantees (5.1) in a large number of cases. Consider the condition

(5.2) $\quad$ There is an abelian subalgebra $\mathfrak{z} \subseteq \mathfrak{n}$ such that the canonical semi-invariant $T$ lies in the enveloping algebra $\mathfrak{U}(\mathfrak{z})$ of $\mathfrak{z}$.

If (5.2) is satisfied, then matters reduce to the abelian case described in (ii). Indeed, let $Z = \exp \mathfrak{z}$. We can find a complementary submanifold $V$ such that $N = Z \times V$ (as manifolds, not groups). Since $T$ lives on $Z$, once again the results of [7, §3] apply. If $f_0 \in S(\mathfrak{z}^*)$ is the invariant polynomial corresponding to $T$, then the space $\mathcal{S}_{f_0}(Z \times V)$, as defined in [7, 3.9], will be dense in $L_2$ and contained in the intersections described in (5.1).

(iv) Condition (5.2) is always satisfied for groups $G$ of type $A_n$ and $C_n$. Let $\underline{G}$ be SL($n$, C) or Sp($n$, C) and $\underline{B}$ a minimal parabolic (i.e. a Borel) subgroup. One knows the structure of the nilradical $\underline{\mathfrak{n}}_0$ of $\underline{B}$ very well, and condition (5.2) is satisfied. In fact, in these cases there is an abelian ideal $\underline{\mathfrak{z}} \subsetneq \underline{\mathfrak{n}}_0$ such that $\mathfrak{Z}(\underline{\mathfrak{n}}_0) \subsetneq \mathfrak{U}(\underline{\mathfrak{z}})$. Condition (5.2) is actually satisfied for any parabolic subgroup $\underline{P}$ of $\underline{G}$. The reason is the following. It is no loss of generality to assume that $\underline{P} \supsetneq \underline{B}$, so that the nilradical $\underline{\mathfrak{n}}$ of $\underline{\mathfrak{p}}$ is contained in $\underline{\mathfrak{n}}_0$. It does not follow that $\mathfrak{Z}(\underline{\mathfrak{n}}) \subseteq \mathfrak{Z}(\underline{\mathfrak{n}}_0)$. But the canonical semi-invariant of $\underline{P}$ must actually be in $\mathfrak{U}(\underline{\mathfrak{n}})^{\mathfrak{n}_0}$. The latter *is* contained in $\mathfrak{Z}(\underline{\mathfrak{n}}_0)$. Furthermore, since $\underline{\mathfrak{z}}$, $\underline{\mathfrak{n}}_0$ and $\underline{\mathfrak{n}}$ are all sums of root spaces, it is easy to see that the semi-invariant for $\underline{P}$ must lie in $\mathfrak{U}(\underline{\mathfrak{z}} \cap \underline{\mathfrak{n}})$; so once again we are in the abelian case.

(v) Unfortunately, condition (5.2) fails for the groups of 4d, 4e, and 4f, and others, such as the nilradical of a Borel subgroup of the exceptional group $G_2$. Thus we are led to the following interesting problem. Let $N$ be a simply connected nilpotent Lie group, $T \in \mathfrak{Z}(\mathfrak{n})$ nonzero, and $t > 0$. Consider the operator on $L_2(N)$ given by $D = |T|^t$. Can we find a partial Schwartz space that is dense in $L_2(N)$ and contained in Dom $D \cap D^{-1}L_1(N) \cap L_2(N)^{\mathcal{L}}$? The natural candidate seems to be the following:

$$\mathcal{S}_T(N) = \{h \in \mathcal{S}(N) = \text{Schwartz functions on } N; (\log h)^\wedge = 0 \text{ near } f_0^{-1}(0)\},$$

where $f_0$ is as usual the polynomial on $\mathfrak{n}^*$ corresponding to $T$. $\mathcal{S}_T(N)$ is dense in $L_2(N)$ and contained in Dom $D \cap L_2(N)^{\mathcal{L}}$. For $h \in \mathcal{S}_T(N)$, one has $Dh \in C^\infty(N)$, but we have not been able to verify the integrability of $Dh$.

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Department of Mathematics, University of Maryland, College Park, Maryland 20742 (Current address of R. L. Lipsman)

Department of Mathematics, University of California, Berkeley, California 94720 (Current address of J. A. Wolf)

Department of Mathematics, Harvard University, Cambridge, Massachusetts 02139