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# Canonical structure of dynamical fluctuations in mesoscopic nonequilibrium steady states

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**Abstract** – We give the explicit structure of the functional governing the dynamical density and current fluctuations for a mesoscopic system in a nonequilibrium steady state. Its canonical form determines a generalised Onsager-Machlup theory. We assume that the system is described as a Markov jump process satisfying a local detailed balance condition such as typical for stochastic lattice gases and for chemical networks. We identify the entropy current and the traffic between the mesoscopic states as extra terms in the fluctuation functional with respect to the equilibrium dynamics. The density and current fluctuations are coupled in general, except close to equilibrium where their decoupling explains the validity of entropy production principles.

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Understanding the microscopic origin of equilibrium thermodynamics proceeds through fluctuation formulæ that appeared about a century ago. Consider for example a gas in large volume  $V$  in thermal equilibrium at inverse temperature  $\beta$  and with chemical potential  $\mu$ . Away from the phase coexistence regime the density fluctuations satisfy the asymptotic law

$$\mathbf{P} \left( \frac{N}{V} \simeq n \right) \sim e^{-\beta V [\Omega(\mu, n) - \Omega(\mu)]}, \quad (1)$$

where the variational functional is  $\Omega(\mu, n) = F(n) - \mu n$  with  $F(n)$  the free energy, and  $\Omega(\mu)$  is the grand potential. There thus exists an important relation between the structure of equilibrium fluctuations and the thermodynamics of the system. In particular, the variational principles characterising equilibrium can be understood as an immediate consequence of its fluctuation theory and response relations can be derived from expanding (1) around the equilibrium density  $n_0$ .

In order to include dynamics in the fluctuation theory, Onsager and Machlup derived the generic structure of small time-dependent equilibrium fluctuations and explained how their dynamics relates to the return to equilibrium [1]. The ensuing linear response theory formalised

the general relation between equilibrium current fluctuations and the response in driven systems in a first-order perturbation theory around equilibrium. To go beyond and challenged by *e.g.* the fast progress in nonequilibrium experiments on nanoscale, one soon realises a lack of general principles. Moreover we still feel very uncertain about what the physical quantities are that such a non-equilibrium physics should be based upon, surely not only on those that are typical to close-to-equilibrium descriptions. Yet, more recently there has been progress, too. One well-known approach to dynamical (and especially current) fluctuations in open systems adds to the models fields representing the various reservoirs that count the long-time statistics of associated “charges” by master equation or stochastic path methods, see *e.g.* [2–4]. The hydrodynamic fluctuations for some stochastic lattice gas models have been studied in *e.g.* [5,6]. For some standard lattice gas models the large deviations can in fact be explicitly calculated, see the review [7]. Up to now, special emphasis was put on the fluctuations of the current, also because of relations with a celebrated fluctuation symmetry of the entropy production [8,9].

In the present letter we come back to the basic question whether there is at all any systematics in the fluctuations beyond equilibrium or close to equilibrium. Can one develop a formalism that would —similarly to the equilibrium scheme— establish a link between the dynamical

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fluctuations and mean (thermo-)dynamical properties of a system, possibly with the entropy production playing a role similar to the entropy at equilibrium? And could that also explain the appearance and limitations of the entropy production variational principles on a fluctuation basis? As we have shown before [10], the (mesoscopic) minimum entropy production principle close to equilibrium follows from the fluctuation theory for the occupation (or residence) times, which are the relative times spent at different states of the system. This supports both the relevance of dynamical fluctuation theory for understanding the status and the validity of various nonequilibrium variational principles, and the importance of time-symmetric observables in these considerations. Indeed our results here strongly suggest that only by treating jointly the time-symmetric and -antisymmetric sectors do model-specific results make place for a unique fluctuation structure. As far as we know, that is one of the rare occasions on which the nonequilibrium world can be seen submitted to general laws.

To address the above questions and in the context of a stochastic network we propose to study the joint dynamical fluctuations of the occupation times (time-symmetric) and currents (time-antisymmetric). We show that these joint fluctuations have an explicit and general structure, with a fluctuation functional derived from the entropy current and the so-called traffic measuring the mean dynamical activity in the system. Formally, the traffic is the counterpart of the entropy or grand potential in the equilibrium static fluctuation theory, cf. (1). Only close to equilibrium there emerges a simple relation between that traffic and the entropy production. Together with a decoupling between small time-symmetric and time-antisymmetric fluctuations in the close-to-equilibrium domain, this lies behind the approximate validity of the entropy production principles. This substantially extends the argument in [10]. Our main results are relations (13), (19), and (29) below. More details and the application of our formalism for driven diffusions can be found in [11]. Our emphasis here on jump processes makes the analysis also suitable to the statistics of quantum transport as *e.g.* in [3,12]; particular examples will follow separately.

In contrast with full counting statistics methods, the reservoirs are not made explicit in our approach. Instead, we assume that the changes in all reservoirs or leads are mutually distinguishable and can be read off from the trajectory of the system. Another remark concerns the meaning of the occupation times. This is a dynamical observable and its fluctuations fundamentally differ from static fluctuations, which evaluate the plausibility that the system obeys some statistics given that the system was in its typical stationary state far in the past. These static fluctuations are governed by an effective action (or potential) yielding a nonequilibrium extension of the equilibrium free energy and providing a Lyapunov function for the nonequilibrium system; it has been extensively studied in the regime of weak noise [13–15]. Recently there

has been an important progress in the analysis of these static functionals for lattice gases in the hydrodynamic limit [16,17]. As a matter of principle, one expects that one could recover the static from the dynamical fluctuations that are studied here.

The mathematics involved is the theory of large deviations and stems from the work of Donsker and Varadhan [18,19]. A useful and repeatedly exploited technique in this approach is to compute the fluctuation functionals on more coarse-grained levels from constrained minimisations of a fine-grained functional. That is called the contraction principle.

**General formalism.** – We consider a nonequilibrium system modelled as a stationary Markov process running in continuous time and making jumps on a discrete set of states,  $\{x, y, \dots\}$ . We are given transition rates  $w(x, y)$  on ordered pairs  $x \rightarrow y$ , and we assume that the process has a unique stationary distribution  $\rho$ ,  $\rho(x) > 0$  over all states (ergodicity assumption). As is typical for a thermodynamic formalism it is not essential whether the process represents a single-particle random walk or perhaps a many-body system. However, for an easy interpretation we ask that whenever a transition  $x \rightarrow y$  is possible,  $w(x, y) > 0$ , then also  $w(y, x) > 0$ . Although that excludes certain singular yet relevant cases (*e.g.*, the totally asymmetric exclusion process), it is essential for the local detailed balance to be applicable: we assume a sufficiently fine-grained level of description so that  $\log [w(x, y)/w(y, x)]$  reads the entropy change in the environment (possibly made of several distinct reservoirs) per single event  $x \rightarrow y$ . Tracing the whole trajectory of the system,  $(\omega_t; t \geq 0)$ , all currents as well as the total entropy exchange with the environment can be determined. This is an essential assumption that will allow for a physical interpretation of the fluctuation formulæ below.

We start from that fine-grained level of description and we consider as dynamical observables the occupation times

$$p_T(x) = \frac{1}{T} \int_0^T \chi(\omega_t = x) dt \quad (2)$$

(with  $\chi$  equal to one or zero, indicating whether the event in brackets occurs, respectively, does not occur) jointly with the jump fractions  $x \rightarrow y$ ,

$$C_T(x, y) \delta t = \frac{1}{T} \int_0^T \chi(\omega_t = x) \chi(\omega_{t+\delta t} = y) dt \quad (3)$$

counting the number of jumps  $x \rightarrow y$ , both defined for each realisation of the process  $(\omega_t; 0 \leq t \leq T)$ . The occupation times  $p_T(x)$  form a random distribution that asymptotically approaches the stationary distribution,  $\lim_{T \rightarrow \infty} p_T(x) = \rho(x)$ , with probability one by the ergodic theorem. Similarly, the jump fractions have the almost-sure asymptotics  $\lim_{T \rightarrow \infty} C_T(x, y) = \rho(x)w(x, y)$ .

The question about dynamical fluctuations concerns the long-time asymptotics of possible deviations of  $p_T$  and

$C_T$  from their typical values: to compute the probability  $\mathbf{P}_T(p, k)$  to observe for all  $x$  and  $y$ ,

$$p_T(x) = p(x), \quad C_T(x, y) = p(x)k(x, y). \quad (4)$$

We must add here the stationarity condition  $\sum_y [p(y) \times k(y, x) - p(x)k(x, y)] = 0$  since  $\lim_{T \rightarrow \infty} \sum_y [C_T(x, y) - C_T(y, x)] = 0$  for every realisation of the process. The distribution of the process is

$$\mathbf{P}_T(\omega) = \rho(x_0) e^{-\lambda(x_0)t_1} w(x_0, x_1) dt_1 e^{-\lambda(x_1)(t_2-t_1)} \dots \dots w(x_{n-1}, x_n) dt_n e^{-\lambda(x_n)(T-t_n)} \quad (5)$$

on realisations  $\omega = (x_0, 0; x_1, t_1; \dots; x_n, t_n \leq T)$  with jumps  $x_{k-1} \rightarrow x_k$  at times  $t_k$  and with  $\lambda(x) = \sum_y w(x, y)$  the escape rates. To determine for that process the probability  $\mathbf{P}_T(p, k)$  of (4), we use a standard trick of the theory of large deviations, see [18,19], and we compare the path distribution of the original stationary process (5) to a fictitious stationary process  $\mathbf{P}_T^*$  having rates  $k$  and occupation fractions  $p(x)$ . In other words, the values  $p$  and  $k$  become *typical* under the fictitious process  $\mathbf{P}_T^*$  when  $T \rightarrow \infty$ . The path probabilities  $\mathbf{P}_T^*(\omega)$  under the fictitious process are obtained as in (5) by replacing  $w$  with  $k$ , and  $\lambda$  with the escape rates  $\sum_y k(x, y)$ . The crucial observation is that for any trajectory  $\omega$  such that the constraints (4) are satisfied, the density of  $\mathbf{P}_T$  with respect to the  $\mathbf{P}_T^*$  equals

$$\frac{d\mathbf{P}_T}{d\mathbf{P}_T^*}(\omega) = \frac{e^{-\sum_{x,y} p(x)w(x,y)}}{e^{-\sum_{x,y} p(x)k(x,y)}} \prod_{x,y} \left[ \frac{w(x,y)}{k(x,y)} \right]^{p(x)k(x,y)} \quad (6)$$

and hence only depends on the time averages  $p(x)$  and on the old ( $w(x, y)$ ), respectively new ( $k(x, y)$ ) transition rates. Considering now that the constraints (4) are indeed satisfied by the fictitious process  $\mathbf{P}_T^*$  when  $T \rightarrow \infty$ , we find directly from (6) that

$$\mathbf{P}_T(p, k) = \int d\mathbf{P}_T^*(\omega) \frac{d\mathbf{P}_T}{d\mathbf{P}_T^*}(\omega) \chi(\text{constraints (4)}) = \frac{e^{-\sum_{x,y} p(x)w(x,y)}}{e^{-\sum_{x,y} p(x)k(x,y)}} \prod_{x,y} \left[ \frac{w(x,y)}{k(x,y)} \right]^{p(x)k(x,y)} \quad (7)$$

asymptotically for  $T \rightarrow \infty$ . In other words,  $\mathbf{P}_T(p, k) \sim \exp[-T\mathcal{I}(p, k)]$  with the fluctuation functional

$$\mathcal{I}(p, k) = \sum_{x,y} p(x) \left[ k(x, y) \log \frac{k(x, y)}{w(x, y)} - k(x, y) + w(x, y) \right] \quad (8)$$

(remember that  $\mathcal{I}(p, k) = \infty$  whenever  $p$  is not stationary with respect to the transition rates  $k$ ). The above calculation is not entirely original and its variants can be found in the literature on large deviations. Yet the important lesson for nonequilibrium applications is that it clearly identifies the one-point and the two-point functions  $p_T$  and  $C_T$  as the natural and in a sense the complete collection of dynamical observables, cf. (6). Formula (8) is our starting point towards a systematic generation of various other fluctuation laws by contraction, in both the time-symmetric and the time-antisymmetric domains.

Their physical interpretation then follows by invoking the local detailed balance principle.

**Occupation-current fluctuations.** – The observed time-averaged currents correspond to the antisymmetric part of the jump fractions,  $C_T(x, y) - C_T(y, x)$ . The joint fluctuation law for the currents and the occupation times

$$\mathbf{P}_T(p, j) \sim e^{-T\mathcal{I}(p, j)} \quad (9)$$

can thus be derived from (8) by solving the minimisation problem

$$I(p, j) = \inf_k \{ \mathcal{I}(p, k) \mid p(x)k(x, y) - p(y)k(y, x) = j(x, y) \}. \quad (10)$$

For stationary currents,  $\sum_y j(x, y) = 0$ , to which we restrict ourselves from now on (otherwise  $\mathcal{I}(p, j) = \infty$ ), the solution is  $I(p, j) = \mathcal{I}(p, k^*)$  with  $k^*(x, y) = w(x, y) e^{\Delta(x, y)/2}$ , the Lagrange multipliers  $\Delta(x, y) = -\Delta(y, x)$  being determined from the constraints

$$p(x)k^*(x, y) - p(y)k^*(y, x) = j(x, y) \quad (11)$$

or, explicitly,

$$\Delta(x, y) = 2 \log \left\{ \frac{1}{2p(x)w(x, y)} [j(x, y) + \sqrt{j^2(x, y) + 4p(x)p(y)w(x, y)w(y, x)}] \right\}. \quad (12)$$

As a result,

$$I(p, j) = \frac{1}{4} \sum_{x,y} \Delta(x, y) j(x, y) - \frac{1}{2} \sum_{x,y} [t_p^*(x, y) - t_p(x, y)] \quad (13)$$

in which

$$t_p(x, y) = p(x)w(x, y) + p(y)w(y, x) \quad (14)$$

and

$$t_p^*(x, y) = p(x)k^*(x, y) + p(y)k^*(y, x) \quad (15)$$

measure the mean dynamical activities; we call them *traffic* and they yield the symmetric counterpart to the expected currents. The second term in (13) is therefore an *excess* in the overall traffic needed to create the fluctuation or to make it typical. Similarly, by the local detailed balance principle, the first term corresponds to an excess in the entropy flow to the environment which amounts to  $\dot{S} = \frac{1}{2} \sum_{x,y} j(x, y) \log[w(x, y)/w(y, x)]$  under the original process and analogously for the modified one.

Next, being motivated by the equilibrium fluctuation theory, cf. (1), we reveal a canonical structure in the dynamical fluctuations. Any nonequilibrium process can be related to a reference detailed balanced one with rates  $w_0(x, y)$ , so that  $w(x, y) = w_0(x, y) e^{\sigma(x, y)/2}$  with some driving  $\sigma(x, y) = -\sigma(y, x)$ . (For example, the rates  $w_0(x, y) = \sqrt{w(x, y)w(y, x)} e^{s(y) - s(x)}$  and  $s$  an arbitrary state function can serve as such a reference.) Having fixed  $w_0$ , the rates  $w(x, y) \equiv w_\sigma(x, y)$  are now parametrised

by the driving  $\sigma(x, y)$ , and we introduce the potential function

$$H(p, \sigma) = 2 \sum_{x, y} p(x) [w_\sigma(x, y) - w_0(x, y)] \quad (16)$$

equal to the excess in the overall traffic with respect to that reference. It is a potential for the expected transient currents  $j_{p, \sigma}(x, y) = p(x)w_\sigma(x, y) - p(y)w_\sigma(y, x)$  in the sense that

$$\delta H(p, \sigma) = \frac{1}{2} \sum_{x, y} j_{p, \sigma}(x, y) \delta \sigma(x, y) \quad (17)$$

(with the  $p$  kept fixed in the variation). The Legendre transform of  $H$  with respect to driving  $\sigma$  is

$$G(p, j) = \sup_{\sigma'} \left[ \frac{1}{2} \sum_{x, y} \sigma'(x, y) j(x, y) - H(p, \sigma') \right] \quad (18)$$

and we observe that the supremum (taken over all anti-symmetric matrices) is attained at  $\sigma' = \sigma^*$  such that  $j_{p, \sigma^*} = j$ , which means that the driving  $\sigma$  and the current  $j$  are canonically conjugated variables. Furthermore, the function  $\Delta$  that specifies the  $k^*$  in (11) equals  $\sigma^* - \sigma$ , hence the fluctuation functional  $I(p, j) = I_\sigma(p, j)$  of eq. (13), obtains the final form

$$I_\sigma(p, j) = \frac{1}{2} \left[ G(p, j) + H(p, \sigma) - \dot{S}(\sigma, j) \right] \quad (19)$$

with

$$\dot{S}(\sigma, j) = \frac{1}{2} \sum_{x, y} \sigma(x, y) j(x, y) \quad (20)$$

the observed entropy current into the environment. That is our main result, giving the fluctuation functional entirely in terms of the entropy current and of the potential function  $H(p, \sigma)$  (*i.e.* in terms of the overall traffic) and derived quantities. The functional  $G(p, j)$  directly gives the reference equilibrium dynamical fluctuations as  $I_0(p, j) = \frac{1}{2} G(p, j)$ , hence (19) specifies the nonequilibrium correction to that equilibrium. Remark also that the anti-symmetric part of the functional  $I_\sigma$  under time reversal equals  $I_\sigma(p, -j) - I_\sigma(p, j) = \dot{S}(\sigma, j)$ , compare [9,20], which is just the steady-state fluctuation symmetry. However, more important is that (19) also in a generic way specifies the time-symmetric component. That is why (19) represents a generalised (far-from-equilibrium) Onsager-Machlup Lagrangian describing steady fluctuations, the generalised dissipation functions being  $G$  and  $H$ . At the same time, one recognises the mathematical structure of equilibrium fluctuations; the grand potential  $\Omega(\mu)$  and the variational functional  $\Omega(\mu, n)$  of (1) get replaced here by  $-H(p, \sigma)/2$  and  $[G(p, j) - \dot{S}(\sigma, j)]/2$ , respectively. This should not really come as a surprise since such a mathematical structure is characteristic of the large deviation framework, equilibrium statistical thermodynamics just being its most prominent example and its guide for physical interpretation. In the nonequilibrium domain, related canonical structures like (17)–(19) have been obtained before, see *e.g.* [13,17] for static fluctuations. In contrast

with their work, we concentrate on the generic structure of steady fluctuations, with time  $T$  being the only large parameter of the asymptotic theory. This is well suited for the discussion of the validity of stationary variational principles already at the mesoscopic level (*i.e.*, for stochastic processes), as shortly demonstrated at the end of this letter.

To our knowledge, the introduction of an equilibrium reference as in (17)–(19) has not been considered before for a far-from-equilibrium dynamics. Note that while the potentials  $G$  and  $H$  do depend on the choice of the reference equilibrium, the resulting functional  $I(p, j)$  is of course independent of that.

Fluctuation laws on a more coarse-grained level, *e.g.*, the fluctuations of a single selected current, can be obtained by further contractions starting from (8) or (19). Then, depending on the particular level description, a modified canonical formalism can be expected.

There is a trivial yet important generalisation of the above results to systems in which a transition  $x \rightarrow y$  can go via multiple channels, each possibly corresponding to the interaction with different reservoirs. For these systems the formulæ (8), (13), (17), (18) etc. remain valid if the ordered pairs  $x, y$  in the sums get replaced with  $x, y, \alpha$ , the  $\alpha$  labelling the channels. A simple example of such a multi-channel model comes in the next section.

**Example.** – A paradigmatic model serving general considerations in full counting statistics is the one of bidirectional transport over a single-level “quantum dot.” The model is well known, see, *e.g.*, [2], and it is simple enough to allow explicit comparisons with other examples and methods. We demonstrate the above formalism in this particular example.

There are two configurations  $x = 0, 1$  corresponding to the level being empty, respectively, occupied, and it is coupled to the left ( $L$ ) and the right ( $R$ ) reservoirs. Using the notation  $V_L$  and  $V_R$  for the potential gradients between that level and the reservoirs, both oriented in the  $L \rightarrow R$  direction, the local detailed balance principle restricts the possible transition rates corresponding to each channel to the following general form:

$$\begin{aligned} w_L(0, 1) &= \Gamma_L e^{\beta V_L/2}, & w_L(1, 0) &= \Gamma_L e^{-\beta V_L/2}, \\ w_R(0, 1) &= \Gamma_R e^{-\beta V_R/2}, & w_R(1, 0) &= \Gamma_R e^{\beta V_R/2}. \end{aligned} \quad (21)$$

For simplicity, we consider here only the case  $\Gamma_L = \Gamma_R = \Gamma$ . Writing the occupation times as  $p(0) = (1 - v)/2$  and  $p(1) = (1 + v)/2$ , the expected transient currents (also both oriented in the  $L \rightarrow R$  direction) and traffic separately for each channel equal

$$j_v^{L,R} = \Gamma \sinh \frac{\beta V_{L,R}}{2} \mp \Gamma v \cosh \frac{\beta V_{L,R}}{2}, \quad (22)$$

$$t_v^{L,R} = \Gamma \cosh \frac{\beta V_{L,R}}{2} \mp \Gamma v \sinh \frac{\beta V_{L,R}}{2}. \quad (23)$$

As a reference equilibrium we take the dynamics (21) for  $V_L = V_R = 0$  (with the symmetric part  $\Gamma$  kept unchanged).

The current potential (16) is determined from the overall traffic:

$$H(v, V_L, V_R) = 2\Gamma \left( \cosh \frac{\beta V_L}{2} - v \sinh \frac{\beta V_L}{2} + \cosh \frac{\beta V_R}{2} + v \sinh \frac{\beta V_R}{2} - 2 \right). \quad (24)$$

One checks that  $\partial H / \partial V_{L,R} = \beta j_v^{L,R}$  which is an instance of (17). The Legendre transform of  $H$  at  $j^L = j^R = j$  gives the occupation-current fluctuation functional  $G(v, j) = I_0(v, j)/2$  for the reference equilibrium dynamics, cf. (18):

$$G(v, j) = \sup_{V_L, V_R} [\beta j(V_L + V_R) - H(v, V_L, V_R)] = 4j \log \left[ \frac{1}{\sqrt{1-v^2}} \left( \frac{j}{\Gamma} + \sqrt{1-v^2 + \frac{j^2}{\Gamma^2}} \right) \right] + 4\Gamma \left[ 1 - \sqrt{1-v^2 + \frac{j^2}{\Gamma^2}} \right]. \quad (25)$$

This extends to the nonequilibrium dynamics by the generalised Onsager-Machlup formula (19). *E.g.*, in the  $L$ - $R$  symmetric case  $V_L = V_R = V$ , the entropy current is  $\dot{S} = 2\beta V j$ , and the nonequilibrium fluctuation functional becomes

$$I_V(v, j) = 2j \log \left[ \frac{1}{\sqrt{1-v^2}} \left( \frac{j}{\Gamma} + \sqrt{1-v^2 + \frac{j^2}{\Gamma^2}} \right) \right] - \beta V j + 2\Gamma \left[ \cosh \frac{\beta V}{2} - \sqrt{1-v^2 + \frac{j^2}{\Gamma^2}} \right]. \quad (26)$$

Due to the ‘‘particle-hole’’ symmetry  $I(-v, j) = I(v, j)$ , the (marginal) current fluctuations correspond to the rate  $\mathfrak{J}_V(j) = I_V(0, j)$ , which is

$$\mathfrak{J}_V(j) = 2j \log \left[ \frac{j}{\Gamma} + \sqrt{1 + \frac{j^2}{\Gamma^2}} \right] - \beta V j + 2\Gamma \left[ \cosh \frac{\beta V}{2} - \sqrt{1 + \frac{j^2}{\Gamma^2}} \right]. \quad (27)$$

Again by contraction, the fluctuation functional for the occupation times is  $\mathfrak{J}_V(v) = I_V(v, j^*)$  where  $j^* = \Gamma \sqrt{1-v^2} \sinh(\beta V/2)$  is the most probable value of the stationary current given  $v$ . As a result,

$$\mathfrak{J}_V(v) = 2\Gamma \cosh \left( \frac{\beta V}{2} \right) (1 - \sqrt{1-v^2}). \quad (28)$$

**Regime of small fluctuations.** – The main features of the joint occupation-current fluctuations already become manifest in the leading order around the nonequilibrium steady state. For our original dynamics with stationary distribution  $\rho$ , steady current  $\bar{j}$  and steady traffic  $\bar{t}$ , we write  $p = \rho(1 + \epsilon u_1)$ ,  $j = \bar{j} + \epsilon j_1$ . Standard perturbation theory applied to (13), up to quadratic order

in  $\epsilon$ , gives as a final result  $I(p, j) = \epsilon^2 I^{(2)}(u_1, j_1)$ , where

$$I^{(2)}(u_1, j_1) = \frac{1}{4} \sum_{x,y} \left[ \frac{1}{2\bar{t}} j_1^2 + \frac{\bar{t}}{2} (\nabla^- u_1)^2 - \frac{\bar{j}}{\bar{t}} j_1 \nabla^+ u_1 + \frac{\bar{j}^2}{2\bar{t}} (\nabla^+ u_1)^2 \right] (x, y) \quad (29)$$

with the shorthand  $\nabla^\pm u_1(x, y) = [u_1(x) \pm u_1(y)]/2$ . This formula demonstrates how the occupation times and current become coupled away from equilibrium. That coupling is proportional to the stationary current and is inversely proportional to the stationary traffic. The coupling vanishes in the close-to-equilibrium regime where  $\bar{j} = O(\epsilon)$ .

The appearance/disappearance of the occupation-current correlation is deeply related with the validity/breaking of the entropy production principles. The expected value of the (transient) entropy production rate  $\mathcal{E}(p)$  at a given distribution  $p$  is the sum of the expected entropy current  $\frac{1}{2} \sum_{x,y} j_p(x, y) \log[w(x, y)/w(y, x)]$  and the rate of increase of the system’s entropy  $\sum_{x,y} j_p(x, y) \log p(x)$ , see [20]. In the same quadratic approximation as above but now close to equilibrium so that  $w = w_0[1 + O(\epsilon)]$  with  $w_0$  again a detailed balanced reference, that entropy production rate equals

$$\mathcal{E}(p) = \sum_{x,y} \left[ \frac{\epsilon^2 \bar{t}}{2} (\nabla^- u_1)^2 + \frac{\bar{j}^2}{2\bar{t}} \right] (x, y) \quad (30)$$

with  $\bar{j} = O(\epsilon)$ . On the other hand, from (29) the marginal distribution of the occupation times for  $\bar{j} = O(\epsilon)$  corresponds to the functional  $\mathfrak{J}^{(2)}(u_1) = \frac{1}{8} \sum_{x,y} \bar{t} (\nabla^- u_1)^2(x, y)$ , and hence  $\mathfrak{J}(p) = \epsilon^2 \mathfrak{J}(u_1)$  equals

$$\mathfrak{J}(p) = \frac{1}{4} [\mathcal{E}(p) - \mathcal{E}(\rho)] \quad (31)$$

see [10,11] for more details. Hence, the stationary distribution  $\rho$  is a minimiser of the entropy production rate and the latter governs the occupation fluctuations. A similar argument reveals a direct link between the current fluctuations and the maximum entropy production principle, [11].

The relation between Onsager-Machlup functionals and entropy production principles has been known for a long time, see, *e.g.*, [21]. Our results outlined in this section extend those arguments to a class of stochastic *driven* systems and *mesoscopic* variational principles.

On the other hand, these relations are no longer true beyond the close-to-equilibrium regime since there the occupation-current correlation becomes relevant, and the traffic and its changes do not derive from the entropy changes. Then the traffic appears essential in determining dynamical fluctuations as we have demonstrated before.

**Conclusions and remarks.** – We have derived an explicit formula for the functional governing the joint dynamical fluctuations of transition intensities and occupation times in a steady-state regime described by a Markov jump process (8). In the occupation-current form (19), it gets a remarkable canonical structure: the

(reference) equilibrium functional is corrected by its Legendre transform which is just a potential for the expected currents, and by the entropy flow. These functionals form a natural starting point towards the study of fluctuations for any selected collection of observables that can be expressed in terms of transitions/currents and occupations, via the contraction principle. That provides an alternative to the existing approaches.

As a new and crucial quantity, unseen in close-to-equilibrium considerations, the traffic measures the time-symmetric dynamical activity in the system. This observable naturally enters beyond the linear response theory, *e.g.*, in determining the ratchet current [22] and in the escape rate theory [23]. The overall traffic yields the current potential, and its excess together with an excess in the entropy flow directly determine the joint occupation-current fluctuations (13).

The time-symmetric and time-antisymmetric fluctuations mutually couple even for small fluctuations around the nonequilibrium state (29). Their decoupling in leading order around equilibrium is a fundamental reason for the known stationary variational principles to be approximately valid.

For extended systems with a large number of degrees of freedom, phase transitions may become visible through singularities of the fluctuation functionals [24]. It should indeed not escape the attention that the analysis from (13) to (19) requires some strict convexity arguments and uniqueness of solutions. That is certainly one of the most fascinating possibilities that can be discussed within our general framework.

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