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Canonical surfaces with $p_g = p_a = 5$ and $K^2 = 10$

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Canonical Surfaces with $p_g = p_a = 5$ and $K^2 = 10$ (*).

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Introduction.

It is well known that, in the theory of algebraic surfaces of general type, the problem of finding out the existence of surfaces with given invariants is still open, particularly referring to the construction of their canonical or pluricanonical models and to the description of their moduli space. We have recently dealt with regular surfaces with $p_g = 4$, $K^2 = 5, \ldots, 10$ over $\mathbb{C}$ (see [C]), obtaining exhaustive results for surfaces whose canonical models have ordinary singularities in $\mathbb{P}^3$.

Turning to regular surfaces with $p_g = 5$, it is known that, if the canonical map is birational, it is $K^2 \geqslant 8$. If $K^2 = 8, 9$, the canonical models of such surfaces are complete intersections in $\mathbb{P}^4$. In this paper we deal with surfaces with $p_g = p_a = 5$, $K^2 = 10$, such that the canonical map is a birational morphism. As far as we know, these surfaces have not yet been exhaustively studied. Our interest has been also attracted on this subject by some remarks of F. Enriques in [E], p. 289.

The main results we reach are the following:

(i) there exist surfaces with $p_g = p_a = 5$, $K^2 = 10$, such that the canonical map is a birational morphism: there exists a unique irreducible component $\mathscr{K}^{(1)}$ of the coarse moduli space of surfaces with the above invariants, containing points corresponding to classes of birational equivalence of such surfaces; $\mathscr{K}^{(1)}$ is unirational, of dimension 40;

(ii) there exists a not empty open Zariski subset of $\mathscr{K}^{(1)}$ whose points correspond to classes of birational equivalence of surfaces whose canonical models have isolated singularities: these are canonical surfaces of degree 10

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in $\mathbb{P}^4$, not contained in any quadric, with a unique nonnormal ordinary double point (see no. 6) and in general with no other singular points;

(iii) there are points in $\mathscr{K}^{(1)}$ corresponding to classes of birational equivalence of surfaces, special with respect to moduli, whose canonical models have a singular curve: this can either be a double line, iff the canonical model is not contained in a quadric, or a curve of higher degree, which we are able to describe, if the canonical model lies in a quadric; clearly particularizing moduli it could be possible to have canonical surfaces with irrelevant singularities beyond the aforementioned ones.

An affirmative answer to a conjecture of Enriques (see l.c.) follows from (i).

The paper consists of two parts. The first one (no. 1-4) has, in our opinion, some independent interest. We give an extension to algebraic varieties of any dimension of results about curves contained in [S], part I, concerning some graded modules over the polynomial ring associated to an algebraic curve, to a morphism of it to a $\mathbb{P}^r$, and to a given invertible sheaf on it. We get in this way some algebraic tools, useful for studying surfaces in $\mathbb{P}^4$.

These results, as well as other well known ones, are applied in part II (no. 5-11) to the study of canonical surfaces. In no. 5 we give a first application providing an equation of matricial type for canonical surfaces in $\mathbb{P}^4$ with $p_g = p_a = 5$, not lying in any quadric. In no. 6 we recall some classical theorems of F. Severi which are frequently used in what follows. In no. 7 we prove the existence of canonical surfaces with the aforesaid invariants and with isolated singularities: the method we use is the construction by liaison. In no. 8 we show that the equivalence classes of such surfaces fill up an open Zariski subset of an irreducible, unirational, 40-dimensional component $\mathscr{K}^{(1)}$ of the coarse moduli space. We develope here an argument similar to the one used in [S] in order to prove the unirationality of the coarse moduli space of curves of genus 12. Our methods, strongly based upon results of part I, give also an explicit description of the ideal of canonical surfaces we deal with. In nos. 9-10 we prove the existence of canonical surfaces with double line not contained in any quadric, and we show that for a general deformation the canonical model has isolated singularities. We liked to follow an hint of Enriques in order to prove the existence, which could also be proved in different ways. Finally in no. 11 we study canonical surfaces contained in a quadric and their general deformations. The results in nos. 1-7 hold not only on $\mathbb{C}$ but on any algebraically closed field of characteristic zero.

We do not deal here with surfaces whose canonical system has base points: this will be the object of another paper.

We believe the methods we use in nos. 7-8 could be applied to the study of canonical surfaces with $p_g = p_a = 5$, $K^2 > 10$ with isolated singularities. However here some nasty technical difficulties occur in constructions by liaison. It seems more difficult to get complete and detailed results about canonical surfaces with $p_g = p_a = 5$, $K^2 > 10$, with nonisolated singularities.

Part I

1. – Preliminaries.

Let S be an irreducible, nonsingular, projective variety of dimension $n \geqslant 1$, defined over an algebraically closed field $\mathbb{K}$ of characteristic zero. We shall denote by $\mathcal{O}_S$ the structure sheaf of S and by K_S a canonical divisor on S.

Suppose we are given a divisor D on S and a vector subspace V of $H^0(S, \mathcal{O}_S(D))$, of dimension $r + 1 \geqslant n + 1$, such that in no point of S all sections in V vanish. Let $p: S \to \mathbb{P}^r$ be a morphism of S in the projective r-dimensional space over $\mathbb{K}$, corresponding to V. In what follows p will be assumed to be a *finite birational morphism*. Then, if $F = p(S)$, F is a nondegenerate, n-dimensional subvariety of $\mathbb{P}^r$ and p is the normalization morphism of F.

For any $j \in \mathbb{N}$, consider the natural map:

$$\varrho_j \colon H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(j)) \to H^0(S, \mathcal{O}_S(jD)) .$$

We call F *j-normal* if ϱ_j is onto; if $j = 1$ we shall say *linearly normal* instead of 1-normal. F is *projectively normal* if it is j-normal for any $j \in \mathbb{N}$. Let us put $\mathscr{I}(F)_j = \operatorname{Ker} \varrho_j$ for any $j \in \mathbb{N}$; $\mathscr{I}(F) = \bigoplus_{j\in\mathbb{N}} \mathscr{I}(F)_j$ is the ideal of polynomials vanishing on F. We shall also write $\Sigma^r(j, F)$ to denote the linear system $\mathbb{P}(\mathscr{I}(F)_j)$ of hypersurfaces of degree j in $\mathbb{P}^r$ containing F, and $\Sigma^r(j)$ instead of $\Sigma^r(j, \emptyset)$.

Let $\mathfrak{V}$ be the linear subsystem of the complete linear system $|D|$ corresponding to V. By the assumptions, $\mathfrak{V}$ has dimension $r \geqslant n \geqslant 1$ and no base points on S. By Bertini's theorem, there are nonsingular divisors C in $\mathfrak{V}$ such that the restriction p_C of p to C is a finite birational morphism of C onto a section Δ cut out on F by a hyperplane π of $\mathbb{P}^r$. If $n = 1$ we can assume that p_C is one to one. In the above case we shall say that Δ is a *generic hyperplane section* of F and that π is a *generic hyperplane* for F.

If $n > 1$, let R be a divisor on C such that $\mathcal{O}_C(R) = \mathcal{O}_S(D)|_C$; the natural map:

$$r\colon H^0(S, \mathcal{O}_S(D)) \to H^0(C, \mathcal{O}_C(R))$$

can be considered. If $W = r(V)$, W is the vector subspace of $H^0(C, \mathcal{O}_C(R))$ corresponding to the morphism $p_C\colon C \to \pi$. Similarly we can consider *generic sections* of F with a $\mathbb{P}^s \subset \mathbb{P}^r$, $s \geqslant r - n$. In particular, if $s = r - n + 1$, we have *generic curve sections*; they are nonsingular and their genus will be denoted by $g(S, D)$, or simply by g, if no confusion arises. Obviously ti is $g(S, D) = g(C, R)$.

For any $r \in \mathbb{N}$ we set $A^{(r)} = \mathbb{K}[x_0, \ldots, x_r]$. If M is a graded $A^{(r)}$-module, the $\mathbb{K}$-vector space of homogeneous elements of degree $j \in \mathbb{Z}$ in M will be denoted by M_j, so that $M = \bigoplus_{j \in \mathbb{Z}} M_j$. In particular it is $A^{(r)} = \bigoplus_{j \in \mathbb{Z}} A_j^{(r)}$; $A_j^{(r)}$ is, in a natural way, isomorphic to the j-th symmetric power of V, or to $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(j))$. Any ideal of $A^{(r)}$ will be, as usual, considered as an $A^{(r)}$-module, and any homogeneous one as a graded module. Moreover any $A^{(s)}$-module M, with $s < r$, will be considered as an $A^{(r)}$-module setting $x_i M = 0, i = s + 1, \ldots, r$. As usual we shall set $M(m) = \bigoplus_{j \in \mathbb{Z}} M_{j+m}$, namely $M(m)_j = M_{j+m}$, for any graded $A^{(r)}$-module M and any $m \in \mathbb{Z}$. M^* will denote the dual of the $A^{(r)}$-module M, namely the module $\mathrm{Hom}_{A^{(r)}}(M, A^{(r)})$; $\tilde{M}$ will denote the sheaf of $\mathcal{O}_{\mathbb{P}^r}$-modules associated to M (see [H], chapt. II). Any homomorphism between graded $A^{(r)}$-modules will be homogeneous of degree zero.

Let $\mathscr{F}$ be an invertible sheaf on S. We shall put, as usual $\mathscr{F}(jD) = \mathscr{F} \otimes \mathcal{O}_S(jD)$. One can look in a natural way at:

$$\gamma(\mathscr{F}) = \bigoplus_{j \in \mathbb{Z}} H^0(S, \mathscr{F}(jD))$$

as a graded $A^{(r)}$-module, where $\gamma(\mathscr{F})_j = H^0(S, \mathscr{F}(jD))$. It is known that $\gamma(\mathscr{F})$ is of finite type (see [O]). Besides $\dim(\gamma(\mathscr{F})) = n + 1$, since:

$$\mathrm{Ann}_{A^{(r)}}(\gamma(\mathscr{F})) = \mathscr{I}(F)\,.$$

(1.1) PROPOSITION. *The following are equivalent*:

(i) $\gamma(\mathscr{F})$ *is a Cohen-Macaulay* (CM) *module*;

(ii) $h^i(S, \mathscr{F}(jD)) = 0$, $i = 0, \ldots, n - 1$, $j \in \mathbb{Z}$.

PROOF. (i) $\Rightarrow$ (ii) It is

$$pd_{A^{(r)}}(\gamma(\mathscr{F})) = r - n\,.$$

Thus:

$$\operatorname{Ext}^{r-n+i}_{A^{(r)}}(\gamma(\mathscr{F}), M) = 0$$

for any $A^{(r)}$-module M and any integer $i > 0$. In particular it is:

$$\operatorname{Ext}^{r-n+i}_{A^{(r)}}(\gamma(\mathscr{F})(j), A^{(r)}(-r-1)) = 0$$

for any integer j and any integer $i > 0$. Applying Serre duality, we have:

$$H^{n-i}(\mathbf{P}^r, \gamma(\mathscr{F})(j)^\sim) = 0$$

for any $j \in \mathbf{Z}$ and $i = 1, \ldots, n-1$. Since $\gamma(\mathscr{F})^\sim = p_*(\mathscr{F})$ we get the assertion.

(ii) $\Rightarrow$ (i) The assertion being true for $n = 1$ (see [S], lemma (1.1)), we make induction on n. From the long exact sequence of cohomology:

$$(1.2) \qquad \ldots \to H^i(S, \mathscr{F}(jD)) \to H^i(C, \mathscr{F}(jD)|_C) \to H^{i+1}(S, \mathscr{F}(j-1)D) \to \ldots$$

for any $j \in \mathbf{Z}$, we get:

$$h^i(C, \mathscr{F}(jD)|_C) = h^i(C, \mathscr{F}|_C(jE)) = 0$$

for any $j \in \mathbf{Z}$ and $i = 1, \ldots, n-2$. By induction, $\gamma(\mathscr{F}|_C)$ is a CM module. Assuming, without restriction, that π has equation $x_r = 0$, and using again the exactness of (1.2) and the hypothesis (ii), we get:

$$\gamma(\mathscr{F}|_C) \simeq \gamma(\mathscr{F})/x_r\gamma(\mathscr{F}).$$

Then:

$$\operatorname{depth}_{A^{(r-1)}}(\gamma(\mathscr{F}|_C)) = \operatorname{depth}_{A^{(r-1)}}(\gamma(\mathscr{F})/x_r\gamma(\mathscr{F})) \leqslant \operatorname{dept}_{A^{(r)}}(\gamma(\mathscr{F})) - 1.$$

Since $\gamma(\mathscr{F}|_C)$ is CM of dimension n, it is:

$$\operatorname{depth}_{A^{(r-1)}}(\gamma(\mathscr{F}|_C)) = n.$$

Hence:

$$\operatorname{depth}_{A^{(r)}}(\gamma(\mathscr{F})) \geqslant n + 1 = \dim(\gamma(\mathscr{F})) \qquad \text{q.e.d.}.$$

If $\gamma(\mathscr{F})$ is a CM module, there exists a free resolution of $\gamma(\mathscr{F})$ of lenght $r - n$:

$$(1.3) \qquad 0 \to F_{r-n} \xrightarrow{f_{r-n}} \ldots \xrightarrow{f_1} F_0 \to \gamma(\mathscr{F}) \to 0$$

where the F_i's, $i = 0, \dots, r-n$, are free graded $A^{(r)}$-modules of finite type. We shall frequently assume, in what follows, that (1.3) is also minimal; in this case it is well defined, up to isomorphisms.

(1.4) PROPOSITION. *If* (1.3) *is a free resolution of* $\gamma(\mathscr{F})$, *there exists a surjective homomorphism*:

$$\eta\colon F^*_{r-n}(-r-1) \to \gamma(\mathscr{F}^*(K_S))$$

such that:

(1.5) $$0 \to F^*_0(-r-1) \xrightarrow{f_1^*(-r-1)} \dots \xrightarrow{f^*_{r-n}(-r-1)} f^*_{r-n}(-r-1) \xrightarrow{\eta} \gamma(\mathscr{F}^*(K_S)) \to 0$$

is a free resolution of $\gamma(\mathscr{F}^*(X_S))$. *If* (1.3) *is minimal,* (1.5) *is minimal too.*

PROOF. Analogous to the proof of proposition (1.7) of [S]. q.e.d.

(1.6) REMARK. From proposition (1.4) it follows that $\gamma(\mathscr{F})$ is CM if and only if $\gamma(\mathscr{F}^*(K_S))$ is CM. This is also a direct consequence of proposition (1.1) and Serre duality.

(1.7) REMARK. From the proof of proposition (1.1) it follows that if $\gamma(\mathscr{F})$ is CM, the same occurs to $\gamma(\mathscr{F}|_C)$. The converse is not true: look, for instance, at the case $n = 2$. If $\gamma(\mathscr{F})$ is CM, it is natural to look for a minimal free resolution of $\gamma(\mathscr{F}|_C)$ related to (1.3), if this one is minimal. Assume the generic hyperplane π has equation $x_r = 0$. Tensorizing (1.3) by $A^{(r-1)} = A^{(r)}/(x_r)$ over $A^{(r)}$, by the flatness of $A^{(r-1)}$ over $A^{(r)}$ we have that:

(1.8) $$0 \to F'_{r-n} \xrightarrow{f'_{r-n}} \dots \xrightarrow{f'_1} F'_0 \to \gamma(\mathscr{F}|_C) \to 0$$

is a free resolution of $\gamma(\mathscr{F}|_C)$, where $F'_i = F_i \bigotimes_{A^{(r)}} A^{(r-1)}$, $i = 0, \dots, r-n$, are considered as $A^{(r-1)}$-modules in the natural way, and the f'_i's, $i = 1, \dots r-n$, are the obvious maps. We shall see later that (1.8) is minimal in some interesting cases.

2. – The minimal free resolution of $\gamma(\mathscr{F})$.

We define now some integers related to $\mathscr{F}$ and p; first the *level* of D *with respect to* $\mathscr{F}$. This is the integer:

$$\ell(D, \mathscr{F}) = \max\{m \in \mathbb{Z}\colon h^0(S, \mathscr{F}(mD - K_S)^*) > 0\}.$$

Let us consider the natural maps:

$$a_i(p, \mathcal{F})\colon V \otimes H^0(S, \mathcal{F}(i-1)D) \to H^0(S, \mathcal{F}(iD))$$
$$b_j(p, \mathcal{F})\colon V \otimes H^0\big(S, \mathcal{F}^*(K_S + (j-1)D)\big) \to H^0(S, \mathcal{F}^*(K_S + jD))\,.$$

It is $b_j(p, \mathcal{F}) = a_j(p, \mathcal{F}^*(K_S))$. We shall set:

$$\alpha_i(p, \mathcal{F}) = \dim\big(\operatorname{coker} a_i(p, \mathcal{F})\big)$$
$$\beta_j(p, \mathcal{F}) = \dim\big(\operatorname{coker} b_j(p, \mathcal{F})\big)$$

for any integers i, j, and write $\ell(\mathcal{F})$, $a_i(\mathcal{F})$, etc., for $\ell(D, \mathcal{F})$, $a_i(p, \mathcal{F})$, etc., if no confusion arises. We shall also write ℓ for $\ell(\mathcal{O}_S)$, a_i for $a_i(\mathcal{O}_S)$, etc., if no confusion arises; ℓ will be simply called the *level* of D.

(2.1) REMARK. By definition of level, it is:

$$\alpha_i(\mathcal{F}) = \begin{cases} 0\,, & \text{if } i < -\ell\big(\mathcal{F}^*(K_S)\big) \\ h^0\big(S, \mathcal{F}(iD)\big)\,, & \text{if } i = -\ell\big(\mathcal{F}^*(K_S)\big)\,. \end{cases} \tag{2.2}$$

Observe that:

$$\ell\big(\mathcal{F}^*(K_S)\big) = \min\{m \in \mathbb{Z}\colon h^0\big(S, \mathcal{F}(mD)\big) > 0\}$$

so that:

$$\ell\big(\mathcal{O}_S(K_S)\big) = 0\,. \tag{2.3}$$

From (2.2) and (2.3) we get:

$$\alpha_i = \begin{cases} 0\,, & \text{if } i < 0 \\ h^0(S, \mathcal{O}_S) = 1\,, & \text{if } i = 0\,. \end{cases} \tag{2.4}$$

It is also:

$$\alpha_1 = h^0\big(S, \mathcal{O}_S(D)\big) - (r+1)\,. \tag{2.5}$$

Similarly it is:

$$\beta_j(\mathcal{F}) = \begin{cases} 0\,, & \text{if } j < -\ell(\mathcal{F}) \\ h^0\big(S, \mathcal{F}(-jD - K_S)^*\big)\,, & \text{if } j = -\ell(\mathcal{F})\,. \end{cases} \tag{2.6}$$

It is also clear that the integers $\ell\big(\mathcal{F}^*(K_S)\big)$ and $\ell(\mathcal{F})$ are completely determined by (2.2) and (2.6) respectively.

We now investigate the link between the numbers we have introduced above and the ranks of the modules appearing in the minimal free resolution (1.3) of $\gamma(\mathscr{F})$, if this is a CM module. Obviously, it is:

$$F_0 \simeq \bigoplus_{i\in\mathbb{Z}} A^{(r)}(-i)^{\alpha_i(\mathscr{F})} \tag{2.7}$$

where in fact it is $i \geqslant -\ell(\mathscr{F}^*(K_S))$ (see remark (2.1)). Similarly, by proposition (1.4) we have:

$$F_{r-n} \simeq \bigoplus_{j\in\mathbb{Z}} A^{(r)}(j-r-1)^{\beta_j(\mathscr{F})} \tag{2.8}$$

where in fact it is $j \geqslant -\ell(\mathscr{F})$. From (2.7) and (2.8) it is clear the importance of an explicit computation of the numbers $\alpha_i(\mathscr{F})$, $\beta_j(\mathscr{F})$, $i, j \in \mathbb{Z}$. In this direction we begin proving the:

(2.9) PROPOSITION. *If $\gamma(\mathscr{F})$ is a* CM *module and $n > 1$, it is:*

(i) $\alpha_i(\mathscr{F}) = \alpha_i(\mathscr{F}|_C)$, *for any* $i \in \mathbb{Z}$

(ii) $\beta_j(\mathscr{F}) = \beta_{j-1}(\mathscr{F}|_C)$, *for any* $j \in \mathbb{Z}$.

PROOF. Look at the commutative diagram:

$$\begin{tikzcd}
 & H^0(S, \mathscr{F}((i-1)D)) \arrow[dl, "\omega_{1,i}"'] \arrow[dr, "\omega_{2,i}"] & \\
V \otimes H^0(S, \mathscr{F}((i-1)D)) \arrow[rr, "a_i(\mathscr{F})"] \arrow[d, "\mu_i"] & & H^0(S, \mathscr{F}(iD)) \arrow[d, "\nu_i"] \\
W \otimes H^0(C, \mathscr{F}|_C((i-1)R)) \arrow[rr, "a_i(\mathscr{F}|_C)"] & & H^0(C, \mathscr{F}|_C(iR))
\end{tikzcd}$$

where μ_i, ν_i are the natural restriction maps, and $\omega_{1,i}$, $\omega_{2,i}$ are respectively given tensorizing and multiplying by a nonzero section $s \in H^0(S, \mathcal{O}_S(D))$ vanishing on C. The right column of the above diagram is exact and, since $\gamma(\mathscr{F})$ is CM, the maps μ_i, ν_i are onto for any $i \in \mathbb{Z}$, by proposition (1.1). Hence:

$$\operatorname{coker} a_i(\mathscr{F}) \simeq \operatorname{coker} a_i(\mathscr{F}|_C)$$

and (i) is proved. The proof of (ii) is analogous. q.e.d.

(2.10) COROLLARY. *If* $\gamma(\mathscr{F})$ *is a* CM *module, it is:*

$$\ell(D, \mathscr{F}) = \ell(R, \mathscr{F}|_C) - 1 .$$

PROOF. It follows from proposition (2.9), taking into account remark (2.1). q.e.d.

(2.11) REMARK. Suppose the resolution (1.3) is minimal. By proposition (2.9), the modules F'_0 and F'_{r-n} appearing in the resolution (1.8) have the right ranks in order that (1.8) is minimal. Hence (1.8) is minimal if (1.3) is minimal and $r - n = 1, 2$.

Let us finally prove the following:

(2.12) PROPOSITION. *If* $\gamma(\mathscr{F})$ *is a* CM *module, it is:*

(i) $\alpha_i(\mathscr{F}) = 0$, *for any integer* $i \geq \ell(\mathscr{F}) + n + 2$;
if $\mathscr{F}(\ell(\mathscr{F})D) = \mathscr{O}_S(K_S)$, *it is also* $\alpha_i(\mathscr{F}) = 0$ *for* $i = \ell(\mathscr{F}) + n + 1$;

(ii) $\beta_j(\mathscr{F}) = 0$, *for any integer* $j \geq \ell(\mathscr{F}^*(K_S)) + n + 2$;
if $\mathscr{F}^*(\ell(\mathscr{F}^*(K_S))D) = \mathscr{O}_S$, *it is also* $\beta_j(\mathscr{F}) = 0$ *for* $j = \ell(\mathscr{F}^*(K_S)) + n + 1$.

PROOF. The assertion is true if $n = 1$: in this case the proof, based on a well known Castelnuovo's lemma is analogous to the proof of proposition (2.6) of [S]. For $n > 1$, we make induction on n. Using corollary (2.10), (i) is easily proven. Part (ii) follows from (i), since $b_j(\mathscr{F}) = a_j(\mathscr{F}^*(K_S))$, taking into account remark (1.6). q.e.d.

3. – The case $\mathscr{F} = \mathscr{O}_S$. Subcanonical varieties.

From now on we shall restrict our attention to the case $\mathscr{F} = \mathscr{O}_S$. If $\gamma(\mathscr{O}_S)$ is CM, namely, by proposition (1.1), if and only if it is:

$$h^i(S, \mathscr{O}_S(jD)) = 0 , \quad i = 1, \dots, n-1,\ j \in \mathbb{Z} \tag{3.1}$$

applying proposition (2.12) we have:

$$\alpha_i = 0 , \quad i \geq \ell + n + 2 \tag{3.2}$$

and, taking also into account (2.3):

$$\beta_j = 0 , \quad j \geq n + 1 . \tag{3.3}$$

It is possible to say something more about β_n. Precisely:

(3.4) PROPOSITION. *If* $\gamma(\mathcal{O}_S)$ *is* CM, *it is*:

(i) $\beta_n \leq h^0(S, \mathcal{O}_S(D)) - (r+1)$, *if* $g > 0$;

(ii) $\beta_n = \deg(F) - 1$, *if* $g = 0$.

PROOF. The assertion is true for $n = 1$ (see proposition (2.6) of [S] for (i); case (ii) is trivial). The proof is easily achieved for $n > 1$, making induction on n and using (3.1). q.e.d.

(3.5) REMARK. If $\gamma(\mathcal{O}_S)$ is CM and $g > 0$, it is $\ell > -n$; if $g = 0$ and $\deg(F) > 1$, it is $\ell = -n$. This is true for $n = 1$ (see [S], no. 2). For $n > 1$ one makes induction on n, using corollary (2.10).

We assume now:

$$\mathcal{O}_S(K_S) = \mathcal{O}_S(dD)\,. \tag{3.6}$$

Then $\ell = d$ and F is called a *subcanonical variety of level d*. If $\gamma(\mathcal{O}_S)$ is CM and (3.6) holds, applying again proposition (2.12), we get:

$$\alpha_{d+n+1} = 0 \tag{3.7}$$

beyond (3.2) and (3.3). We have also more informations in this case about the resolution (1.3), if it is minimal and $\mathcal{F} = \mathcal{O}_S$:

(3.8) PROPOSITION. *If* $\gamma(\mathcal{O}_S)$ *is a* CM *module*, F *is subcanonical of level d and the resolution* (1.3) *is minimal*, *there are isomorphisms*:

$$F_i^*(-r-1-d) \simeq F_{r-n-i}\,, \qquad i = 0, \ldots, r-n\,. \tag{3.9}$$

PROOF. It is $\gamma(\mathcal{O}_S(K_S)) = \gamma(\mathcal{O}_S)(d)$, and the assertion follows applying proposition (1.4). q.e.d.

A trivial consequence of proposition (3.8) is the:

(3.10) COROLLARY. *In the same hypotheses of proposition* (3.8) *it is* $\alpha_i = \beta_{i-d}$ *for any* $i \in \mathbb{Z}$.

Proposition (2.5) of [S] shows that the converse of proposition (3.8) holds if $n = 1$. Similarly we can prove that:

(3.11) PROPOSITION. *If* $\gamma(\mathcal{O}_S)$ *is a* CM *module and there are isomorphisms* (3.9) *between the modules appearing in a free resolution* (1.3) *of* $\gamma(\mathcal{O}_S)$, *then* F *is a subcanonical variety of level* d.

PROOF. The assertion is true if $n = 1$ (see [S], proposition (2.5), and note that the hypothesis of minimality on the resolution (1.3) is not essential). We make induction on n. The isomorphism (3.9) gives rise to isomorphisms:

$$F'_i(r+1+d)^* \simeq F'_{r-n-i}, \qquad i = 0, \ldots, r-n$$

between the modules appearing in the free resolution (1.8) of $\mathcal{O}_S|_C = \mathcal{O}_C$ (see remark (1.7): here $\mathcal{F} = \mathcal{O}_S$). By induction Δ is a subcanonical variety of level $d+1$. Look at the short exact sequence of sheaves:

$$0 \to \mathcal{O}_S(X_S - (d+1)D) \to \mathcal{O}_S(X_S - dD) \to \mathcal{O}_S(K_S - dD)|_C = \mathcal{O}_C \to 0\,.$$

Since $h^0(S, \mathcal{O}_S(K_S - (d+1)D)) = 0$, being $\ell(D) = \ell(R) - 1 = d$ by corollary (2.10), and $h^1(S, \mathcal{O}_S(X_S - (d+1)D)) = 0$ by (3.1), we have:

$$h^0(S, \mathcal{O}_S(K_S - dD)) = h^0(C, \mathcal{O}_C) = 1\,.$$

Then, either (3.6) holds, or there exists an effective divisor L on S linearly equivalent to $K_S - dD$. Since D is ample on S, by Nakano-Moishezon criterion we should have:

$$L \cdot C \cdot D^{n-2} > 0$$

which is impossible. q.e.d.

(3.12) REMARK. If F is a subcanonical variety of level d, any its generic hyperplane section is subcanonical of level $d+1$. This is a consequence of the adjunction formula (see [GH], p. 196). Proposition (3.11) asserts that the converse is true if $\gamma(\mathcal{O}_S)$ is CM. If $\gamma(\mathcal{O}_S)$ is not CM, counterexamples are available. Look, for instance at the case F is a surface whose generic hyperplane section is an elliptic curve, which is subcanonical of level 0. It is well known that F is subcanonical of level -1, namely a Del Pezzo surface, if and only if it is regular; this is the same as saying that $\gamma(\mathcal{O}_S)$ is CM, as one can easily check.

(3.13) REMARK. We want to point out, like in [S] is done for the case $n = 1$, that proposition (3.11) implies that, if F is nonsingular and

$r = n + 2$, F is a complete intersection if and only if it is subcanonical and *projectively Cohen-Macaulay*, which means projectively normal and $\gamma(\mathcal{O}_S)$ CM. This is the extension to projective varieties of codimension 2 of a well known theorem about curves in $\mathbb{P}^3$ due to G. Gherardelli (see [GHE]).

4. – Fitting ideals and $\mathcal{I}(F)$.

If M is a finitely generated $A^{(r)}$-module, $\Phi(M)$ will denote the *0-th Fitting ideal* of M, which we shall simply call the *Fitting ideal* of M. Let us assume M graded and $\mathrm{Ann}\,(M) \neq 0$, which is the case we are interested in, and let us recall the definition of $\Phi(M)$. If $M \neq 0$, let:

$$\dots \to M_1 \xrightarrow{f} M_0 \xrightarrow{g} M \to 0 \tag{4.1}$$

be a free resolution of M, with M_0, M_1 of ranks m_0, m_1 respectively. Since $\mathrm{Ann}\,(M) \neq 0$, it is:

$$m_1 \geqslant m_0 \,. \tag{4.2}$$

Given minimal bases of homogeneous elements of M_0, M_1, the map f is represented, in the two bases, by an homogeneous matrix A of type $m_0 \times m_1$ (see [G], p. 190); here we are looking at the elements of M_0, M_1 as column vectors of polynomials. By definition, $\Phi(M)$ is the ideal generated in $A^{(r)}$ by the minors of order m_0 of A; by (4.2), these are the minors of maximal order of A. It can be proved that $\Phi(M)$ does not depend on the resolution (4.1) (see [MR]). If $M = 0$, by definition it is $\Phi(M) = \mathfrak{m}$, where $\mathfrak{m}$ is the maximal homogeneous ideal in $A^{(r)}$. $\Phi(M)$ is, in any case, an homogeneous ideal.

If A is an homogeneous matrix of elements of $A^{(r)}$, $\Phi(A)$ will denote the ideal generated in $A^{(r)}$ by the minors of maximal order of A. This is an homogeneous ideal, called the *Fitting ideal* of A. With the previous notations, it is $\Phi(M) = \Phi(A)$, if $M \neq 0$.

(4.3) PROPOSITION. *For any invertible sheaf on S, it is*:

$$\mathrm{Supp}\left(\mathrm{Proj}\, A^{(r)}/\Phi(\gamma(\mathcal{F}))\right) = F \,.$$

PROOF. For any module M, with $\mathrm{Ann}\,(M) \neq 0$, there exists an integer

$h > 0$ such that:

$$\text{(4.4)} \qquad \operatorname{Ann}(M)^h \subseteq \Phi(M) \subseteq \operatorname{Ann}(M)$$

(see [MR]). Hence we get:

$$\text{(4.5)} \qquad \operatorname{Rad}\big(\operatorname{Ann}(M)\big) = \operatorname{Rad}\big(\Phi(M)\big)\,.$$

Applying (4.5) to $M = \gamma(\mathscr{F})$, we get the assertion, since $\operatorname{Ann}\big(\gamma(\mathscr{F})\big) = \mathscr{I}(F)$. q.e.d.

(4.6) REMARK. Putting $M = \gamma(\mathscr{F})$ in (4.4) and (4.5), we get:

$$\text{(4.7)} \qquad \mathscr{I}(F)^h \subseteq \Phi\big(\gamma(\mathscr{F})\big) \subseteq \mathscr{I}(F), \quad \operatorname{Rad}\Big(\Phi\big(\gamma(\mathscr{F})\big)\Big) = \mathscr{I}(F)\,.$$

A natural question to ask is when it is:

$$\text{(4.8)} \qquad \Phi\big(\gamma(\mathscr{F})\big) = \mathscr{I}(F)\,.$$

If $M = \gamma(\mathscr{F})$ in (4.1), (4.8) holds if:

$$\text{(4.9)} \qquad r - n = m_1 - m_0 + 1\,.$$

This is a consequence of results of [BE$_1$]. If $r - n = 1$ and $\gamma(\mathscr{F})$ is CM we can take $m_0 = m_1$, thus (4.9), and hence (4.8), holds. For a direct discussion of this case, with $\mathscr{F} = \mathscr{O}_S$, see [C] (see also next remark (4.14)). If $\mathscr{F} = \mathscr{O}_S$ and F is projectively normal, (4.8) trivially holds.

(4.10) REMARK. If $\gamma(\mathscr{F})$ is CM, we shall denote by A_i the matrix representing the map f_i appearing in the free resolution (1.3) of $M = \gamma(\mathscr{F})$, in two homogeneous minimal bases of F_i, F_{i-1}, $i = 1, \dots, r - n$. If $\Phi(A_i) \neq 0$ for any $i = 1, \dots, r - n$, it is:

$$\text{(4.11)} \qquad \operatorname{Rad}\big(\Phi(A_i)\big) = \mathscr{I}(F)\,, \qquad i = 1, \dots, r - n$$

(see [BE$_2$], theorem 2.1). Moreover the ideals $\Phi^+ = \Phi(A_1)\cdot\Phi(A_3)\,\dots$, $\Phi^- = \Phi(A_2)\cdot\Phi(A_4)$. ... are isomorphic, namely there exists an $\alpha \in A^{(r)}$, $\alpha \neq 0$, such that $\Phi^+ = \alpha\Phi^-$. If $r = n + 2$, it is $\Phi(A_1) = \alpha\Phi(A_2)$, and in fact $\alpha \in \mathrm{K}$, since, by (4.11), $\operatorname{Rad}\big(\Phi(A_1)\big) = \operatorname{Rad}\big(\Phi(A_2)\big)$. Thus $\Phi(A_1) = \Phi(A_2)$. Applying this remark to projectively Cohen-Macaulay nonsingular varieties of codimension 2 in a projective space, it is possible to find all known results about the ideals defining these varieties (see [GA], [PS]).

(4.12) REMARK. Let $M = \gamma(\mathscr{F})$ in (4.1) and:

$$v = \left\| \begin{matrix} P \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{matrix} \right\| \in M_0$$

where v is a $m_0 \times 1$ matrix. Clearly $g(v) = 0$ if and only if $P \in \mathscr{I}(F)$. On the other hand, $g(v) = 0$ if and only if there exists a $m_1 \times 1$ matrix $w \in M_1$ such that $v = Aw$. This trivial observation gives a method for finding all polynomials in $\mathscr{I}(F)$, once we know the matrix A. If $a_1, \ldots, a_{m_0}$ are its rows, first we find all $m_1 \times 1$ matrices w such that:

$$a_i \cdot w = 0\,, \quad i = 2, \ldots, m_0\,.$$

For any such a w, $a_1 \cdot w \in \mathscr{I}(F)$, and in this way we get any element in $\mathscr{I}(F)$. It is obvious that the first row can be replaced by any other.

We now refer to the case $\mathscr{F} = \mathscr{O}_S$, and put $M = \gamma(\mathscr{O}_S)$ in (4.1). Then we can take:

$$M_0 \simeq A^{(r)} \oplus A^{(r)}(-1)^{\alpha_1} \oplus \ldots$$

(see (2.7)), so that $m_0 = \sum_{i=0}^{\infty} \alpha_i$. To get the matrix A, we choose the following basis of M_0:

$$e, e_{1,1}, \ldots, e_{1,\alpha_1}, \ldots$$

where e generates $A^{(r)}$, $e_{1,1}, \ldots, e_{1,\alpha_1}$ generate $A^{(r)}(-1)^{\alpha_1}$, etc. Hence we can write:

$$A = \left\| \begin{matrix} A_1 \\ A_2 \end{matrix} \right\|$$

here A_1 is the first row of A and A_2 is a $(m_0 - 1) \times m_1$ matrix. Let us give an interpretation of A_2. Consider the natural map:

$$r(S)\colon A^{(r)} \to \gamma(\mathscr{O}_S)$$

whose kernel is $\mathscr{I}(F)$ and whose image is the coordinate ring of F. If coker $r(S) = 0$, namely if and only if F is projectively normal, it is $M_0 \simeq A^{(r)}$

and A_2 does not exist. If $\operatorname{coker} r(S) \neq 0$, we have the following part of a free resolution of $\operatorname{coker} r(S)$:

$$\dots M_1 \xrightarrow{\bar{f}} M_0/eA^{(r)} \to \operatorname{coker} r(S) \to 0$$

where $\bar{f}$ is represented by A_2 in the two obvious bases of M_1 and $M_0/eA^{(r)}$. Therefore we have:

$$\Phi(A_2) = \Phi(\operatorname{coker} r(S)) . \tag{4.13}$$

When, in what follows, we shall assume $\gamma(\mathcal{O}_S)$ CM, we shall implicitely identify (4.1), in which $M = \gamma(\mathcal{O}_S)$, with the first part of (1.3), in which $\mathcal{F} = \mathcal{O}_S$. Thus $M_0 \simeq F_0$, $M_1 \simeq F_1$.

(4.14) REMARK. Let $\mathcal{E}_S$ be the sheaf of ideals of $\mathcal{O}_{\mathbf{P}^r}$ which is the inverse image of the sheaf of ideals conductor of $\mathcal{O}_S$ in $\mathcal{O}_F$, and let:

$$\mathcal{I}'(F) = \bigoplus_{i \geqslant 1} H^0(\mathbf{P}^r, \mathcal{E}_S(i))$$

be the corresponding homogeneous ideal of $A^{(r)}$. It is clearly:

$$\operatorname{Supp}(\mathcal{E}_S) = \mathcal{N}(F) \tag{4.15}$$

where $\mathcal{N}(F)$ is the closed Zariski subset of F formed by the points at which F fails to be normal. By our hypotheses $\mathcal{N}(F)$ coincides with the singular locus of F. From (4.15) we get:

$$\operatorname{Rad}(\mathcal{I}'(F)) = \mathcal{I}(\mathcal{N}(F)) . \tag{4.16}$$

Since $(\operatorname{coker} r(S))^{\sim} = p_*(\mathcal{O}_S/\mathcal{O}_F)$, and $\mathcal{E}_S$ is the sheaf of annihilators of $p_*(\mathcal{O}_S/\mathcal{O}_F)$, it is easy to check that there exists an integer $h > 0$ such that:

$$\mathcal{I}'(F)^h \subseteq \operatorname{Ann}(\operatorname{coker} r(S)) \subseteq \mathcal{I}'(F) .$$

By (4.4), (4.5) and (4.16), we get:

$$\mathcal{I}'(F)^h \subseteq \Phi(A_2) \subseteq \mathcal{I}'(F) , \quad \operatorname{Rad}(\Phi(A_2)) = \mathcal{I}(\mathcal{N}(F)) \tag{4.17}$$

for a suitable integer $h > 0$. Using the results of [BE$_1$], we can also state that, if:

$$r - \dim \mathcal{N}(F) = m_1 - m_0 + 2 \tag{4.18}$$

it is:

$$\Phi(A_2) = \mathrm{Ann}(\operatorname{coker} r(S))\,. \tag{4.19}$$

If $r - n = 1$ and $\mathcal{N}(F) \neq \emptyset$, (4.18), and then (4.19), holds (see [C]).

(4.20) PROPOSITION. *It is:*

$$\operatorname{Supp}\left(\operatorname{Proj} A^{(r)}/\Phi(A_2)\right) = \mathcal{N}(F)\,.$$

PROOF. Similar to the proof of proposition (4.3), taking into account remark (4.14). q.e.d.

Finally we want to compare $\operatorname{Proj} A^{(r)}/\Phi(\gamma(\mathcal{O}_S))$ and $F = \operatorname{Proj} A^{(r)}/\mathcal{I}(F)$ in order to get a more precise statement than the one of proposition (4.3), even if we are not going to make use of it in what follows.

First we remark that:

$$\Phi(\gamma(\mathcal{O}_S)) \subseteq \mathcal{I}(F) \subseteq \mathcal{I}'(F)\,. \tag{4.21}$$

Moreover, applying lemma (2.6) of [MR] to the exact sequence:

$$0 \to A^{(r)}/\mathcal{I}(F) \to \gamma(\mathcal{O}_S) \to \operatorname{coker} r(S) \to 0$$

and using (4.17), we get, for some integer $h > 0$:

$$\mathcal{I}(F)\cdot\mathcal{I}'(F)^h \subseteq \mathcal{I}(F)\cdot\Phi(\operatorname{coker} r(S)) \subseteq \Phi(\gamma(\mathcal{O}_S)) \tag{4.22}$$

$\mathcal{I}'(F)$ determines two not empty open subsets:

$$\Omega_1 = \{\mathfrak{p} \in \operatorname{Proj} A^{(r)}/\Phi(\gamma(\mathcal{O}_S)) : \mathfrak{p} \not\supseteq \mathcal{I}'(F)/\Phi(\gamma(\mathcal{O}_S))\}$$
$$\Omega_2 = \{\mathfrak{p} \in \operatorname{Proj} A^{(r)}/\mathcal{I}(F) : \mathfrak{p} \not\supseteq \mathcal{I}'(F)/\mathcal{I}(F)\}$$

which we consider as open subschemes of $\operatorname{Proj} A^{(r)}/\Phi(\gamma(\mathcal{O}_S))$ and $\operatorname{Proj} A^{(r)}/\mathcal{I}(F)$ respectively.

(4.23) THEOREM. *There exists a natural isomorphism between Ω_1 and Ω_2.*

PROOF. Let $\mathfrak{p}$ be any homogeneous prime ideal of $A^{(r)}$ containing $\Phi(\gamma(\mathcal{O}_S))$ but not containing $\mathcal{I}'(F)$. By (4.21) and (4.22), $\mathfrak{p}$ contains $\mathcal{I}(F)$. Therefore these exists a natual map $\varphi\colon \Omega_1 \to \Omega_2$. Using (4.22) it is easy to check that φ is an isomorphism. q.e.d.

(4.24) COROLLARY. *If* $\mathscr{N}(F) = \emptyset$ *it is*:

$$\operatorname{Proj} A^{(r)}/\Phi(\gamma(\mathcal{O}_S)) = F\,.$$

PROOF. If $\mathscr{N}(F) = \emptyset$, by (4.15) it is $\mathscr{I}'(F) \supseteq \mathfrak{m}^k$ for some integer $k > 0$. In this case it is $\Omega_1 = \operatorname{Proj} A^{(r)}/\Phi(\gamma(\mathcal{O}_S))$, $\Omega_2 = \operatorname{Proj} A^{(r)}/\mathscr{I}(F) = F$. By theorem (4.25) the assertion follows. q.e.d.

(4.25) REMARK. Many of the results stated up to here hold even if we drop the hypothesis S is nonsingular. Since we have only used Serre duality, it is sufficient for S to be a locally Cohen-Macauly variety (see [H], chapt. II, no. 8 chapt. III, no. 7). In particular, if $n = 2$ it is enough to suppose S normal (see [GR], example 7.19). We shall constantly use this fact in what follows. Moreover, often, and particularly in this no. 4, $\mathscr{F}$ could have been replaced by any locally free coherent sheaf of $\mathcal{O}_S$-modules.

Part II

5. – Canonical surfaces in $\mathbf{P}^4$.

Let $\tilde{S}$ be an irreducible, nonsingular, algebraic variety of dimension $n \geqslant 1$ defined over $\mathbf{K}$, and let $q: \tilde{S} \to F \subseteq \mathbf{P}^r$ be a birational morphism such that $q^*(H_F) \in |K_S - Z|$, where H_F is a hyperplane section of F and Z is the fixed part of $|K_{\tilde{S}}|$ on $\tilde{S}$. F is said to be a *canonical variety*. In what follows we shall deal with canonical surfaces, assuming, unless the contrary is explicitly stated, $|K_{\tilde{S}}|$ without base points. There is, a factorization of q:

$$\begin{tikzcd} & S \arrow[dr, "p"] & \\ \tilde{S} \arrow[ur, "\tilde{q}"] \arrow[rr, "q"'] & & F \end{tikzcd}$$

where $p: S \to F$ is the normalization of F. It is well known that S has only *irrelevant singularities*, also called *Du Val singularities*, and $\tilde{q}_*(\mathcal{O}_S(K_{\tilde{S}})) = \omega_S$ is the *dualizing sheaf* of S. We shall call ω_S the *canonical* sheaf of S, and *canonical divisor* on S any divisor of zeros of a section of ω_S. Then p is determined by a $(r+1)$-dimensional vector subspace $V \subseteq H^0(S, \omega_S)$, and F is a subcanonical surface of level 1 (see no. 3 and remark (4.25)), which will be for us a synonym of canonical surface. The generic hyperplane sections of canonical surfaces are subcanonical curves of level 2 (see remark (3.12)); these objects we shall call *half-canonical curves.* If $m = \deg(F)$, it is $m = K_S^2$ and, by adjunction formula, $g = K_S^2 + 1 = m + 1$.

(5.1) PROPOSITION. *If F is a canonical surface, $\gamma(\mathcal{O}_S)$ is a* CM *module if and only if $q(S) = h^1(S, \mathcal{O}_S) = 0$.*

PROOF. Similar to the proof of proposition (6.4) of [C]. q.e.d.

In order to apply the results of nos. 1-4 to canonical surfaces, we have, by proposition (5.1), to suppose, them regular. Moreover, since our goal is the study of linearly normal canonical surfaces in $\mathbf{P}^4$, we shall usually assume:

$$p_g(S) = p_a(S) = 5\,. \tag{5.2}$$

It is well known that, if (5.2) holds, it is $m = K_S^2 \geqslant 8$. Besides, if $m = 8, 9$, F is a complete intersection in $\mathbf{P}^4$ (see [E], cap. VIII). To avoid trivial cases we shall often assume $m \geqslant 10$.

(5.3) PROPOSITION. *If F is a canonical surface of degree m in $\mathbf{P}^4$, verifying* (5.2) *and not contained in any quadric, in the minimal free resolution* (1.3) *it is*:

$$\left\{\begin{array}{l} F_0 \simeq A^{(4)} \oplus A^{(4)}(-2)^{m-9} \\ F_1 \simeq A^{(4)}(-3)^{2m-16} \\ F_2 \simeq A^{(4)}(-6) \oplus A^{(4)}(-4)^{m-9} \end{array}\right. \tag{5.4}$$

if $\mathcal{F} = \mathcal{O}_S$.

PROOF. By (3.2), (3.3), proposition (3.4) and corollary (3.10), we get $\alpha_i = \beta_{i-1} = 0$ for any integer i different from $0, 1, 2$. By (2.5) and (2.6) we also get $\alpha_0 = \beta_{-1} = 1$ and $\alpha_1 = \beta_0 = 0$. Since no quadric contains F, we have $\alpha_2 = \beta_1 = h^0(S, \mathcal{O}_S(2K_S)) - 15 = m - 9$. Therefore, by (2.7) and (2.8)., F_0 and F_1 have the expression (5.4). Let us now put:

$$F_1 = \bigoplus_{j \in \mathbf{Z}} A^{(4)}(j-6)^{b_j}$$

and determine the integers b_j. By minimality it is $b_j = 0$ if $j \geqslant 6$. Besides, applying proposition (3.8), we have:

$$\bigoplus_{j \in \mathbf{Z}} A^{(h)}(j-6)^{b_j} = F_1 \simeq F_1^*(-6) = \bigoplus_{j \in \mathbf{Z}} A^{(4)}(-j)^{b_j}\,.$$

Hence $b_j = 0$ if $j \leqslant 0$ and $b_1 = b_5$, $b_2 = b_4$. But $b_5 = 0$, otherwise F should lie in a hyperplane; similarly $b_4 = 0$. So F_1 is like in (5.4). q.e.d.

(5.5) REMARK. We keep the hypotheses of proposition (5.3). In the minimal free resolution (1.3) there are two relevant maps: f_1, f_2. Choosing bases for F_0, F_1, F_2 in the usual way, f_1, f_2 are respectively represented by homogeneous matrices of the type:

$$(5.6)\qquad \begin{cases} A = \begin{array}{c} \left\| \begin{matrix} [3] \\ [1] \end{matrix} \right\| \begin{matrix} {\scriptstyle 1} \\ {\scriptstyle m-9} \end{matrix} \\ {\scriptstyle 2m-16} \end{array} \\[2ex] B = \begin{array}{cc} \|[3] & [1]\|\,{\scriptstyle 2m-16} \\ {\scriptstyle 1} & {\scriptstyle m-9} \end{array} \end{cases}$$

where $[i]$ stays for a matrix of homogeneous polynomials of degree i in $A^{(4)}$, and the numbers 1, $m-9$, $2m-16$ indicate the sizes of the matrices. Like in no. 4, A_2 will denote the matrix obtained from A erasing the first row. Clearly it is:

$$(5.7)\qquad A \cdot B = 0\,, \quad A_2 \cdot B = 0\,.$$

(5.8) REMARK. If F verifies the hypotheses of proposition (5.3), and $m = 10$, F must be singular. This follows from proposition (4.20), observing that $\Phi(A_2)$ is the ideal generated by four linear forms in $A^{(4)}$. If $\mathcal{N}(F)$ is one point, (4.18) holds, and so (4.19). Since $\Phi(A_2)$ is radical in this case, by (4.27) it is also $\Phi(A_2) = \mathcal{I}'(F)$. The only other chance for $\mathcal{N}(F)$ is to be a line.

Proposition (5.3) applies to many canonical surfaces. For instance if F has isolated singularities. In fact:

(5.9) PROPOSITION. *If F is a canonical surface of degree $m > 8$ in $\mathbb{P}^4$, with isolated singularities, then F does not lie on any quadric.*

PROOF. It is sufficient to show that if a quadric Q contains the generic hyperplane section Δ of F, which is a nonsingular half-canonical curve in $\mathbb{P}^3$, it must be $m = 8$. Assume Q nonsingular, and look at the exact sequence:

$$0 \to \mathcal{O}_Q(-4H_Q) \to \mathcal{O}_Q(\Delta - 4H_Q) \to \mathcal{O}_Q(\Delta - 4H_Q)|_\Delta \to 0$$

H_Q being the plane section of Q. Since:

$$h^1(Q, \mathcal{O}_Q(-4H_Q)) = h^1(Q, \mathcal{O}_Q(2H_Q)) = 0$$

the map:

$$H^0(Q, \mathcal{O}_Q(\Delta - 4H_Q)) \to H^0(\Delta, \mathcal{O}_Q(\Delta - 4H_Q)|_\Delta)$$

is an isomorphism. Being:

$$\mathcal{O}_Q(\Delta - 4H_Q)|_\Delta \simeq \mathcal{O}_Q(\Delta + K_Q)|_\Delta \otimes \mathcal{O}_Q(-2H_Q)|_\Delta \simeq \mathcal{O}_\Delta(K_\Delta) \otimes \mathcal{O}_\Delta(K_\Delta)^* \simeq \mathcal{O}_\Delta$$

it is $h^0(Q, \mathcal{O}_Q(\Delta - 4H_Q)) = 1$. Hence $\Delta \sim 4H_Q$, where $\sim$ denotes the linear equivalence. So it is $m = 8$. The case Q is a cone can be worked out in a similar way. q.e.d.

6. – Recalling properties of surfaces in $\mathbf{P}^4$.

Let F be a surface in $\mathbf{P}^4$. We shall say that F has *ordinary singularities* if it has at most a finite number of double points with tangent cone consisting of two planes spanning $\mathbf{P}^4$. Any such a double point we shall call a *node*. If F has ordinary singularities and $p: S \to F$ is the normalization S is nonsingular and the fiber of p is one point except at the nodes of F, whose fiber is formed by two distinct points.

If F is an irreducible surface of degree m in $\mathbf{P}^4$, with ordinary singularities, we can attach to F three numbers, which are projective invariants, but turn out to be related to the birational geometry of F. The three numbers are:

1) $d(F)$, the number of nodes of F;

2) $\omega_1(F)$, the degree of the ruled surface described by the tangent lines to a generic hyperplane section of F;

3) $\omega_2(F)$, the number of distinct tangent lines to F passing through a generic point of $\mathbf{P}^4$.

A theorem of F. Severi gives the connection between m, $d(F)$, $\omega_1(F)$, $\omega_2(F)$. Namely:

(6.1) THEOREM. *If F is an irreducible surface of degree m in $\mathbf{P}^4$ with ordinary singularities, it is*:

$$2d(F) = m(m-1) - \omega_1(F) - \omega_2(F)\,. \tag{6.2}$$

The proof of theorem (6.1) is in [SE$_1$]; in [SE$_2$] a more general result can be found. For a modern version, see [CA].

In order to give, a more intrinsic form to formula (6.2), it is necessary to relate $\omega_1(F)$, $\omega_2(F)$ to the birational invariants of S and to the invariants of the linear system $|D|$ on S, where, as usual, $\mathcal{O}_S(D) = p^*(\mathcal{O}_{\mathbf{P}^4}(1))$. The task is very easy for $\omega_1(F)$, being, as easily can be checked:

(6.3) $$\omega_1(F) = 2m + 2g - 2\,.$$

As for $\omega_2(F)$, we recall a theorem classically due to P. Bonnesen (see [B]), a proof of which can be found in [E], p. 173-176, or, for modern versions, in [KL], p. 162, and in [P], p. 115:

(6.4) THEOREM. *With the hypotheses of theorem* (6.1), *it is*:

$$\omega_2(F) = 2m + 8g - 12p_a(S) + 2K_S^2 - 20\,.$$

Putting together theorems (6.1), (6.4) and formula (6.3), we finally get:

(6.5) COROLLARY. *With the hypotheses of theorem* (6.1), *it is*:

$$d(F) = m(m-5)/2 - 5g + 6p_a(S) - K_S^2 + 11\,.$$

Besides, with few computations, we get:

(6.6) COROLLARY. *If F is a canonical surface of degree m in $\mathbf{P}^4$, with ordinary singularities, it is*:

$$d(F) = m(m-17)/2 + 6p_a(S) + 6$$

$p\colon S \to F$ being the normalization.

(6.7) REMARK. If F is a regular, linearly normal, canonical surface of degree m in $\mathbf{P}^4$, with ordinary singularities, by corollary (6.6) we get:

(6.8) $$d(F) = m(m-17)/2 + 36\,.$$

Applying corollary (6.6) and formula (6.8) to nonsingular surfaces, one checks that:

1) F is nonsingular and regular if and only if $m = 8, 9$, in these cases F is a complete intersection;
2) F is nonsingular and irregular only if $m = 12$, $p_a(S) = 4$, thus $q(S) = 1$.

It is an open problem to find out whether the latter surfaces exist.

(6.9) REMARK. If F is an irreducible surface of degree m in $\mathbf{P}^4$, we shall say that F has *quasi-ordinary singularities* if it is normal outside of finitely many nodes: of course a node is not a normal point. Analysing Severi's proof of theorem (6.1) one checks that it still holds if F has quasi-ordinary singularities. More generally if F has isolated singularities, F has non-normal singularities *equivalent*, in a sense that can be made rigorous, to $d(F)$ nodes, where $d(F)$ is given by (6.2). One has to be careful with theorem (6.4) if F has quasi-ordinary singularities: one has to remember indeed that isolated singularities affect $p_a(S)$, K_S^2. If F has only irrelevant singularities off the nodes, no problem arises, thus theorem (6.4), and its corollaries (6.5), (6.6) still hold.

Let us now assume that F, F' are irreducible surfaces in $\mathbf{P}^4$, with ordinary singularities, and that $F \cup F'$ is the complete intersection of two hypersurfaces Q, Q' of degrees, q, q' respectively, with at most isolated singularities. Since any hypersurface containing F (or F') has a singular point at any node of F (of F'), the same happens for Q and Q'. F and F' having ordinary singularities, Q and Q' have only double points at the nodes of F and F', which are the same points of $\mathbf{P}^4$. Moreover Q, Q' have at most a double point at any point of $F \cup F'$, since F, F' are nonsingular off their nodes. If V is any hypersurface in $\mathbf{P}^4$ containing F' but not F, the divisor $V \cdot F$ cut out by V on F is defined (see [H], p. 146). We define the divisor $\Gamma(F, F')$ on F to be the maximal divisor contained in all divisors $V \cdot F$, where V contains F' but not F. Similarly $\Gamma(F', F)$ is defined. If $p: S \to F$, $p': S' \to F'$ are the normalization morphisms, we put:

$$\gamma(F, F') = p^*(\Gamma(F, F')) , \qquad \gamma(F', F) = p'^*(\Gamma(F', F))$$

$\Gamma(F, F')$ and $\Gamma(F', F)$ are supported on the same curve of $\mathbf{P}^4$, namely the set-theoretic intersection of F and F'.

The first basic fact is that:

(6.10) PROPOSITION. *If F, F' are like above, it is:*

$$\mathcal{O}_S(tD - \gamma(F, F')) = \mathcal{O}_S(K_S) , \qquad \mathcal{O}_{S'}(tD' - \gamma(F', F)) = \mathcal{O}_{S'}(K_{S'})$$

where $t = q + q' - 5$ and D, D' have the usual meaning.

This statement, like the others in the reminder of this paragraph, is due to Severi; its proof is based upon the adjunction formula. For reference, see [SE$_3$].

By lemma (6.10), for any $j \in \mathrm{N}$ there is a natural map:

$$k_j\colon \mathscr{I}(F')_j/\mathscr{I}(F \cup F')_j \to H^0\big(S, \mathcal{O}_S(K_S + (j-t)D)\big)$$

and the similar map k'_j for S'. The main theorem, due to Severi (see [SE$_3$]), is the following:

(6.11) THEOREM. *The maps k_j, k'_j are isomorphisms for any $j \in \mathrm{N}$.*

An easy consequence of theorem (6.11), is the:

(6.12) COROLLARY. *If F, F' are like above, F is subcanonical of level d if and only if there exists a hypersurface of degree $t - d$ containing F'' but not F, cutting out $\Gamma(F, F')$ on F.*

7. – Canonical surfaces of degree 10 in $\mathbf{P}^4$ with isolated singularities: existence.

Let us prove the existence of canonical surfaces of degree 10 in $\mathbf{P}^4$ with isolated singularities, verifying (5.2). According to remark (5.8) and formula (6.8), we expect that any such a surface has a single node.

We start considering a rank 4 quadric cone Q in $\mathbf{P}^4$ with vertex P. On Q there are two linear pencils of planes $\mathscr{L}_1$, $\mathscr{L}_2$; any two distinct planes of the same pencil intersect only at P, and two planes of different pencils intersect in a line containing P. Let τ_1, τ_2 be two distinct planes in $\mathscr{L}_1$.

(7.1) LEMMA. *There are irreducible hypersurfaces in $\Sigma^4(4, \tau_1 \cup \tau_2)$, which are nonsingular off P, and have a double point at P with tangent cone of rank* 4, *intersecting Q in τ_1, τ_2 and in two distinct planes of $\mathscr{L}_2$.*

PROOF. Bertini's theorem can be applied, once we prove that $\Sigma^4(4, \tau_1 \cup \tau_2)$ contains:

(i) an irreducible hypersurface, nonsingular off P;

(ii) a hypersurface having an isolated double point at P, with the required behaviour of the tangent cone.

Since there are nonsingular quartic surfaces in $\mathbf{P}^3$ containing two skew lines, for instance the Fermat quartic, certainly there are quartic cones in $\Sigma^4(4, \tau_1 \cup \tau_2)$ verifying (i). To settle (ii), it is sufficient to take a reducible quartic formed by a quadric not containing P and a rank 4 quadric cone with vertex at P, intersecting Q in τ_1, τ_2 and in two distinct planes of $\mathscr{L}_2$. q.e.d.

The hypersurfaces in $\Sigma^4(4, \tau_1 \cup \tau_2)$ not containing Q, cut out on Q, off $\tau_1 \cup \tau_2$, a linear system $\mathscr{L}$ of surfaces of degree 6. Any surface in $\mathscr{L}$ is singular at P, having at least a node there.

(7.2) LEMMA. *There are irreducible surfaces in $\mathscr{L}$ with a node at P and no other singularity.*

PROOF. $\mathscr{L}$ contains all reducible surfaces formed by two distinct planes in $\mathscr{L}_2$ and in a quadric section of Q. The assertion follows by Bertini's theorem. q.e.d.

By lemmas (7.1), (7.2), it follows that there exists a not empty Zariski open subset $\mathscr{L}'$ of $\mathscr{L}$, such that:

1) any surface $F' \in \mathscr{L}'$ is irreducible, has a node at P and no other singularity;

2) any surface $F' \in \mathscr{L}'$ is the intersection of Q with an irreducible hypersurface $Q' \in \Sigma^4(4, \tau_1 \cup \tau_2)$, which is nonsingular outside of P and has a double point at P, with the properties described in lemma (7.1).

Remark that if $F' \in \mathscr{L}'$, the tangent cone of F' at P consists of two planes τ_1', τ_2' in $\mathscr{L}_2$.

Consider the linear pencils $\mathscr{L}_1'$, $\mathscr{L}_2'$ cut out on $F' \in \mathscr{L}'$ by the planes in $\mathscr{L}_1$, $\mathscr{L}_2$ respectively.

(7.3) LEMMA. *If $F' \in \mathscr{L}'$, it is:*

(i) *$\mathscr{L}_1'$ is a pencil of plane quartics, whose generic element has a node at P and no other singularity;*

(ii) *$\mathscr{L}_2'$ is a pencil of conics, whose generic element is irreducible.*

PROOF. Any plane $\tau \in \mathscr{L}_1 - \{\tau_1, \tau_2\}$, intersects F' where it intersects Q', since it has no curve in common with τ_1, τ_2. Thus (i) follows by the properties of the tangent cone of Q' at P and by Bertini's theorem. The proof of (ii) is similar. q.e.d.

(7.4) REMARK. Lemma (7.3), (ii) implies that F' is rational. Moreover the generic hyperplane section of F' is a nonsingular, hyperelliptic curve of degree 6 and genus 3: it is, in fact, a curve of type (4, 2) on a nonsingular quadric surface in $\mathbb{P}^3$ (see lemma (7.3)). Projecting F' into a $\mathbb{P}^3$ from any nonsingular point of F' one gets a quintic surface with a triple line. Hence there exists a birational map $F' \to \mathbf{P}^2$ mapping the linear system of hyperplane sections to the linear system of quintics with a triple base point and ten simple base points (see [CO], p. 482). F' is also a counterexample to a theorem stated in [SE_1], p. 37, whose proof is uncorrect.

Fix now any $F' \in \mathscr{L}'$. The hypersurfaces in $\Sigma^4(4, F')$ different from Q', cut out on Q', off F', a linear system $\mathscr{E}$ of surfaces of degree 10. Any surface in $\mathscr{E}$ is singular at P, with at least a node there.

(7.5) LEMMA. *There are irreducible surfaces in $\mathscr{E}$ with a node at P and no other singularity.*

PROOF. $\mathscr{E}$ contains the linear system of surfaces reducible in τ_1, τ_2 and in a quadric section of Q'. Moreover, the points of $\tau_1 \cup \tau_2 - \{P\}$ are not base points for $\mathscr{E}$: in fact the hypersurfaces in $\Sigma^4(4, F')$ not containing Q, cut out on Q any couple of planes of $\mathscr{L}_1$, since Q is projectively normal. The assertion follows by Bertini's theorem. q.e.d.

$\mathscr{E}$ cuts out on F' a linear system $\mathscr{L}''$ of curves.

(7.6) LEMMA. *$\mathscr{L}''$ is a linear system of curves of degree* 20, *whose generic element is irreducible, nonsingular outside P, where it has a quadruple point with four independent tangent lines.*

PROOF. The assertion follows from lemma (7.3), applying Bertini's theorem. q.e.d.

By lemmas (7.5), (7.6) and properties of $\Sigma^4(4, F')$, it follows that there exists a not empty Zariski open subset $\mathscr{E}'$ of $\mathscr{E}$ such that:

1) any surface $F \in \mathscr{E}'$ is irreducible, with a node at P, and no other singularity;

2) any surface $F \in \mathscr{E}'$ is the intersection of two irreducible hypersurfaces Q', $Q' \in \Sigma^4(4, F')$, off F'; Q', Q'' are nonsingular outside P, have a double point at P, with tangent cone of rank 4;

3) any surface $F \in \mathscr{E}'$ cuts out on F' an irreducible curve $c(F, F')$ of degree 20, with the properties listed in lemma (7.6).

We can finally prove the:

(7.7) THEOREM. *Any $F \in \mathscr{E}'$ is a canonical surface of degree* 10, *with ordinary singularities, verifying* (5.2).

PROOF. Since F is not contained in Q, Q cuts out on F a curve of degree 20, which is just $c(F, F')$. Keeping the notations of no. 6, it is $c(F, F') = \Gamma(F, F')$. This implies that F is canonical (see corollary (6.12)). Since F' is not the complete intersection of Q with a cubic (see remark (7.4)), no irreducible cubic hypersurface contains F'. Therefore, by theorem (6.11), it is $p_g(S) = 5$; here, as usual, $p\colon S \to F$ is the normalization of F. Finally, applying corollary (6.6), one checks that $p_a(S) = 5$, since $d(F) = 1$. q.e.d.

8. – Canonical surfaces of degree 10 in $\mathbb{P}^4$ with isolated singularities: unirationality of moduli space.

Once the existence of canonical surfaces of degree 10, with ordinary singularities, verifying (5.2), is proved, we want to describe their moduli space. In order to do this, we make the following considerations, assuming, from now on, $\mathbb{K} = \mathbb{C}$.

We introduce the indeterminates over $\mathbb{C}$:

$$\begin{array}{ll} x_h, & h = 0, \ldots, 4 \\ a_i^h,\ b_i^h, & h = 0, \ldots, 4,\ i = 1, \ldots, 4 \\ c_i^{hkl}, & h, k, l = 0, \ldots, 4,\ h \leqslant k \leqslant l,\ i = 1, \ldots, 4 \end{array}$$

and consider the ring $\tilde{R} = \mathbb{C}[x_h, a_i^h, b_i^h, c_i^{hkl}]$ as a graded algebra of polynomials over the ring $R = \mathbb{C}[a_i^h, b_i^h, c_i^{hkl}]$. We shall also put $R' = \mathbb{C}[a_i^h]$ and denote by $\mathbb{A}$, $\mathbb{A}'$ the affine spaces over $\mathbb{C}$ whose coordinate rings are R, R' respectively.

Let:

$$(8.1)\qquad \begin{cases} \mathscr{A}(x_h, a_i^h) = \|\mathscr{A}_i(x_h, a_i^h)\|_{i=1,\ldots,4} \\ \mathscr{B}(x_h, b_i^h, c_i^{hkl}) = \|\mathscr{B}_{ij}(x_h, b_i^h, c_i^{hkl})\|_{i=1,\ldots,4;\ j=1,2} \end{cases}$$

be matrices, of type 1×4 and 4×2 respectively, with elements in $\tilde{R}$ given by:

$$\mathscr{A}_i(x_h, a_i^h) = \sum_{h=0}^{4} a_i^h x_h$$

$$\mathscr{B}_{i1}(x_h, b_i^h, c_i^{hkl}) = \sum_{h,k,l=0}^{4} c_i^{hkl} x_h x_k x_l$$

$$\mathscr{B}_{i2}(x_h, b_i^h, c_i^{hkl}) = \sum_{h=0}^{4} b_i^h x_h .$$

If no confusion arises, we shall write:

$$\mathscr{A} = \|\mathscr{A}_i\|_{i=1,\dots,4}$$
$$\mathscr{B} = \|\mathscr{B}_{ij}\|_{i=1,\dots,4;\, j=1,2}$$

to denote the matrices (8.1).

Consider the matrix $\mathscr{A}\cdot\mathscr{B}$, of type 1×2, whose two entries are:

$$\mathscr{P}_j = \sum_{i=1}^{4} \mathscr{B}_{ij}\mathscr{A}_i\,, \quad j = 1, 2\,.$$

Let $\mathscr{I} \subseteq R$ be the ideal generated by the coefficients of $\mathscr{P}_1$, $\mathscr{P}_2$ considered as polynomials over R. Observe that the generators of $\mathscr{I}$ are homogeneous polynomials in the variables a_i^h, b_i^h, c_i^{hkl}, and separately are also homogeneous, and linear indeed, in each set of variables. Let I be the reduced, closed subvariety of $\mathbb{A}$ which is the support of $\operatorname{Spec} R/\mathscr{I}$, and consider the projection morphism $\Theta\colon I \to \mathbb{A}'$. By the properties of the ideal $\mathscr{I}$, the fibers of Θ are supported on affine subspaces of $\mathbb{A}$. More precisely, if we fix a point $\boldsymbol{\alpha} = \|\alpha_i^h\| \in \mathbb{A}'$, the support of $\Theta^{-1}(\boldsymbol{\alpha})$ has equation given by the vanishing of the coefficients b_i^h, c_i^{hki} of the two polynomials:

$$\mathscr{P}_j(x_h, \alpha_i^h, b_i^h, c_i^{hkl})\,, \quad j = 1, 2\,. \tag{8.2}$$

In this way one gets 85 linear homogeneous equations in the variables b_i^h, c_i^{hkl}, which are 160 indeterminates. The basic remark is that these 85 equations are not independent. In fact, for any $\boldsymbol{\alpha} = \|\alpha_i^h\| \in \mathbb{A}'$, there exists a nontrivial solution of the linear system:

$$\mathscr{A}(x_h, \alpha_i^h) = 0\,, \quad i = 1, \dots, 4 \tag{8.3}$$

in the variables x_h. Therefore the polynomials (8.2) both vanish at a certain point of the 4-dimensional projective space over $\mathbf{C}(\boldsymbol{\alpha})$. This means that there are at least two independent linear relations, with coefficients in $\mathbf{C}(\boldsymbol{\alpha})$, among the coefficients of the polynomials (8.2). Hence, for any $\boldsymbol{\alpha} \in \mathbb{A}'$ it is:

$$\dim\big(\Theta^{-1}(\boldsymbol{\alpha})\big) \geqslant 77$$

thus Θ is surjective.

Consider now the not empty Zariski open subset X of $\mathbb{A}'$, formed by points $\boldsymbol{\alpha} = \|\alpha_i^h\| \in \mathbb{A}'$ such that the linear system (8.3) has only one solution in the 4-dimensional projective space over $\mathbf{C}(\boldsymbol{\alpha})$. In other words,

X is the complement in $\mathbb{A}'$ of the locus defined by:

$$\operatorname{rank} \|a_i^h\|_{i=1,\dots,4;\, h=0,\dots,4} \leqslant 3 .$$

We want to prove the:

(8.4) LEMMA. *For any,* $\alpha \in X$, *it is*:

$$\dim\big(\Theta^{-1}(\alpha)\big) = 77 .$$

PROOF. The group $GL(5, \mathbb{C})$ acts in the natural way on the variables $x_0, \dots, x_4$; this induces an action of $GL(5, \mathbb{C})$ on $\mathbb{A}$ and $\mathbb{A}'$. It is obvious that:

(i) the action is transitive on X;

(ii) the action on $\mathbb{A}$ induces an action on I such that Θ is $GL(5, \mathbb{C})$-equivariant.

Hence it is sufficient to prove the assertion for a particular point in X. Let $F \in \mathscr{E}'$ be a canonical surface of degree 10, whose existence has been proved in theorem (7.7). Applying propositions (5.3), (5.9) to F, we get two matrices $\boldsymbol{A}$, $\boldsymbol{B}$, given by (5.6), with $m = 10$, verifying (5.7). We shall prove the assertion if α is the point of $\mathbb{A}'$ corresponding to the coefficients of $\boldsymbol{A}_2$, where $\boldsymbol{A}_2$ has the usual meaning (see remark (4.12)); α is in X by proposition (4.20). By remark (4.12) and the exactness of resolution (1.3) for $\mathscr{F} = \mathcal{O}_S$, it is:

$$\dim\big(\Theta^{-1}(\alpha)\big) = \big(\dim \mathscr{I}(F)_4 + 1\big) + \big(\dim \mathscr{I}(F)_6 + 16\big) .$$

Then, using theorem (6.11), remark (7.4), Riemann-Roch theorem and Kodaira vanishing theorem on S', we get:

$$\dim \mathscr{I}(F)_4 = h^0\big(S', \mathcal{O}_{S'}(K_{S'} + D')\big) + \dim \mathscr{I}(F \cup F')_4 = 5$$
$$\dim \mathscr{I}(F)_6 = h^0\big(S', \mathcal{O}_{S'}(K_{S'} + 3D)\big) + \dim \mathscr{I}(F \cup F')_6 = 55 \qquad \text{q.e.d.}$$

By lemma (8.4) and properties (i), (ii) of the action of $GL(5, \mathbb{C})$ on I and X listed in its proof, it follows that there exists a unique irreducible component Z of I such that the restriction Θ_Z of Θ to Z maps Z onto a subvariety of $\mathbb{A}'$ containing X. Z is an irreducible, reduced variety and, by lemma (8.4), it is rational, and:

$$\dim Z = 97 . \tag{8.5}$$

Consider now the Fitting ideal $\Phi(\mathscr{B})$ of the matrix $\mathscr{B}$ (see no. 4) and look at the commutative diagram of morphisms:

$$\begin{tikzcd} \operatorname{Supp}(\operatorname{Spec}\tilde{R}/\Phi(\mathscr{B})) \arrow[rr, "i"] \arrow[dr, "h"'] & & \mathbf{A}^5 \times \mathbf{A} \arrow[dl, "k"] \\ & \mathbf{A} & \end{tikzcd}$$

where i is the natural closed immersion, k is the projection on the second component, and h commutes the diagram. Since the ideal $\Phi(\mathscr{B})$ is homogeneous in the variables x_i^h, we can *projectify* $\operatorname{Supp}(\operatorname{Spec}\tilde{R}/\Phi(\mathscr{B}))$ in each fiber of h. In this way we get a new commutative diagram:

$$\begin{tikzcd} \mathscr{S} \arrow[rr, "i"] \arrow[dr, "k"'] & & \mathbf{P}^4 \times \mathbf{A} \arrow[dl, "h"] \\ & \mathbf{A} & \end{tikzcd}$$

where $\mathscr{S}$ is the variety obtained with the *partial projectification* of $\operatorname{Supp}(\operatorname{Spec}\tilde{R}/\Phi(\mathscr{B}))$ and i, k, h have the same meaning as above. Restricting everything to $Z \subseteq \mathbf{A}$, we get the commutative diagram:

$$\begin{tikzcd} \mathscr{S}_Z \arrow[rr, "i"] \arrow[dr, "h"'] & & \mathbf{P}^4 \times Z \arrow[dl, "k"] \\ & Z & \end{tikzcd}$$

where i, k, h still denote the natural morphisms. For any $\zeta \in Z$, $h^{-1}(\zeta)$, with its reduced structure is a subvariety in $\mathbf{P}^4 \times \{\zeta\}$. We want to show that:

(8.6) PROPOSITION. *There exists a not empty Zariski open subset Z' of Z, such that*:

(i) *for any canonical surface F of degree* 10 *in* $\mathbf{P}^4$ *with isolated singularities, verifying* (5.2), *there exists a point $\zeta \in Z'$ such that $F = h^{-1}(\zeta)$*;

(ii) *for any $\zeta \in Z'$, $h^{-1}(\zeta)$ is a canonical surface of degree* 10 *in* $\mathbf{P}^4$ *with isolated singularities, verifying* (5.2).

PROOF. Consider the subset Z'' of Z formed by all $\zeta \in Z$ such that $h^{-1}(\zeta)$ is a surface of degree 10 in $\mathbf{P}^4$ with one node and no other singularity. By standard arguments (see [H], p. 95, ex. 3.22), one checks that Z'' is a constructible subset of Z, namely a finite disjoint union of irreducible, locally closed subsets of Z. Let $Z''_1, \ldots, Z''_m$, be the components of Z'' containing a point ζ such that $h^{-1}(\zeta)$ is a canonical surface of degree 10: by theorem (7.7), remark (5.5) and proposition (5.9), it is $m \geqslant 1$. For any $i = 1, \ldots, m$, consider the restriction map $h: \mathscr{S}_{Z''_i} \to Z''_i$.

This is an algebraic family or surfaces over Z''_i, since, by constance of degrees, any fiber occurs with multiplicity one. Blowing up any fiber at the node, we get an algebraic family of nonsingular surfaces over Z''_i. By a theorem, of Igusa (see [I], p. 36) the arithmetic genus of the fiber is constant, hence is 5. Similarly one checks that the generic hyperplane section Δ of the fiber is a nonsingular curve of degree 10 and genus 11 in $\mathbb{P}^3$.

Let $\zeta \in Z''_i$, $F = h^{-1}(\zeta)$, $p: S \to F$ the normalization. It is $p_g(S) \geqslant p_a(S) = 5$ and $|K_S|$ cuts out on $C = p^*(\Delta)$ a g^r_{10} with:

$$r = p_g(S) - 1\,, \quad \text{if } K_S \nsim C$$
$$r = p_g(S) - 2\,, \quad \text{if } K_S \sim C\,.$$

The divisors of this g^r_{10} are residual, with respect to $|K_C|$ of the divisors of the linear series $|R|$, pull-back on C of the linear series cut out on Δ by the hyperplanes of $\mathbb{P}^3$. By a result of Comessatti (see [COM]; for a modern version see [BEA], no. 5), it is $\dim |R| = 3$ so that $r \leqslant 3$. Hence $p_g(S) = 5$ and $K_S \sim C$, namely F is canonical. Therefore, for each $i = 1, \ldots, m$, there exists a natural morphism $\varphi_i: Z''_i \to \mathscr{K}_i$, where $\mathscr{K}_i$ is a component of the coarse moduli space of surfaces with $p_g = p_a = 5$, $K^2 = 10$ (see [GI]). By propositions (5.3), (5.9), there exists an $i = 1, \ldots, m$, such that φ_i is dominant; say $i = 1$. Let us evaluate the dimension of Z''_i.

The basic remark is that there is a natural action on Z of the group $\mathscr{G} = GL(1, \mathbb{C}) \times GL(4, \mathbb{C}) \times G \times GL(5, \mathbb{C})$, where G is the group of matrices:

$$\boldsymbol{C} = \begin{Vmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{Vmatrix} \tag{8.7}$$

with $a_{11}, a_{22} \in \mathbb{C}^*$, $a_{21} \in A_2^{(4)}$. The action is defined in the following way:

$$(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}) \cdot (\mathscr{A}, \mathscr{B}) = \big(\boldsymbol{A} \cdot \mathscr{A}(\boldsymbol{D}) \cdot \boldsymbol{B}, \boldsymbol{B}^{-1} \cdot \mathscr{B}(\boldsymbol{D}) \cdot \boldsymbol{C}\big)$$

with $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}) \in \mathscr{G}$, $(\mathscr{A}, \mathscr{B}) \in Z$, and $\mathscr{A}(\boldsymbol{D})$, $\mathscr{B}(\boldsymbol{D})$ obtained letting $\boldsymbol{D}$ act on the variables x_h in the natural way (see proof of lemma (7.11)). This action induces an action of $\mathscr{G}$ on Z''_1 and, for any $\zeta \in Z''_1$, the fiber $\varphi_1^{-1}(\varphi_1(\zeta))$ is $\mathscr{G}$-stable. We claim that, for any $\zeta = (\mathscr{A}, \mathscr{B}) \in Z''_1$, the stabilizor $\mathscr{G}(\zeta)$ of ζ is a 2-dimensional subgroup of $\mathscr{G}$. First, remark that the projection of $\mathscr{G}(\zeta)$ in $PGL(5, \mathbf{C})$ via $GL(5, \mathbf{C})$ is a finite group, because the surface $h^{-1}(\zeta)$ has only a finite number of automorphisms. Hence, if $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}) \in \mathscr{G}(\zeta)^0$, $\mathscr{G}(\zeta)^0$ being the connected component of the unity of $\mathscr{G}(\zeta)$, $\boldsymbol{D}$ is a scalar matrix. Moreover $\mathscr{G}(\zeta)^0$ coincides with the group of matrices of the type $\big(\boldsymbol{A}(\lambda), \boldsymbol{B}(\lambda^{-1}\mu^{-1}), \boldsymbol{C}(\lambda), \boldsymbol{D}(\mu)\big)$, where:

$$\boldsymbol{A}(\lambda) = \|\lambda\|$$

$$\boldsymbol{B}(\lambda^{-1}\mu^{-1}) = \begin{Vmatrix} \lambda^{-1}\mu^{-1} & 0 & 0 & 0 \\ 0 & \lambda^{-1}\mu^{-1} & 0 & 0 \\ 0 & 0 & \lambda^{-1}\mu^{-1} & 0 \\ 0 & 0 & 0 & \lambda^{-1}\mu^{-1} \end{Vmatrix}$$

$$\boldsymbol{C}(\lambda) = \begin{Vmatrix} \lambda & 0 \\ 0 & \lambda \end{Vmatrix}$$

$$\boldsymbol{D}(\mu) = \begin{Vmatrix} \mu & 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & \mu \end{Vmatrix}$$

λ, $\mu \in \mathbf{C}^*$. In fact this 2-dimensional group is certainly contained in $\mathscr{G}(\zeta)^0$. Besides, if $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}) \in \mathscr{G}(\zeta)^0$, and $\boldsymbol{A} = \boldsymbol{A}(\lambda)$, $\boldsymbol{D} = \boldsymbol{D}(\mu)$, it must be $\boldsymbol{B} = \boldsymbol{B}(\lambda^{-1}\mu^{-1})$, since $\mathscr{A} \in X$. If C is given by (8.7), from the relation:

$$\mathscr{B}(\lambda^{-1}\mu^{-1}) \cdot \mathscr{B}(\boldsymbol{D}) \cdot \boldsymbol{C} = \mathscr{B}$$

we get:

$$(\lambda^{-1}a_{11} - 1)\mathscr{B}_{i1} + \lambda^{-1}a_{21}\mathscr{B}_{i2} = 0$$

$$\lambda^{-1}a_{22}\mathscr{B}_{i2} = \mathscr{B}_{i2}$$

for any $i = 1, \ldots, 4$. Since $\boldsymbol{\Phi}(\mathscr{B}) \neq 0$, it must be $a_{11} = a_{22} = \lambda$, $a_{21} = 0$. By the above considerations, we have $\dim \mathscr{G}(\zeta) = \dim \mathscr{G}(\zeta)^0 = 2$, thus for any $\zeta \in Z_1''$, it is:

$$\dim \varphi_1^{-1}\big(\varphi_1(\zeta)\big) \geqslant \dim \mathscr{G} - 2 = 57\,. \tag{8.8}$$

It is also:

$$\dim \mathscr{K}_1 \geqslant h^1(S, T_S) - h^2(S, T_S) = -2K_S^2 + 10\big(p_a(S) + 1\big) = 40 \tag{8.9}$$

where S is a surface whose class of birational equivalence is in $\mathscr{K}_1$, and T_S is its tangent sheaf (see [Z], appendix to chapt. V). Combining (8.8) and (8.9), we finally get:

$$\dim Z_1'' \geqslant 97 .$$

By (8.5), Z_1'' is a not empty Zariski open subset of Z. Let us consider the subset Z' of Z formed by all $\zeta \in Z$ such that $h^{-1}(\zeta)$ is a surface of degree 10 in $\mathbf{P}^4$ with a single node and no other nonnormal point. By the above considerations, Z' is a not empty Zariski open subset of Z. By arguments already used above, one ckecks that Z' fulfils the assertion. q.e.d.

(8.10) THEOREM. *There exists a unique component $\mathscr{K}^{(1)}$ of the coarse moduli space of surfaces with $p_g = p_a = 5$, $K^2 = 10$, containing points corresponding to equivalence classes of surfaces having a canonical model of degree* 10 *with isolated singularities. $\mathscr{K}^{(1)}$ is unirational of dimension* 40.

PROOF. Consider the restriction map $h\colon \mathscr{S}_{Z'} \to Z'$. This is an algebraic family of surfaces over Z', and each fiber has a single node, being normal outside it. Z' can be covered by a family $(Z_i')_{i\in\mathscr{I}}$ of open sets such that, in the induced family $h_i\colon \mathscr{S}_{Z_i'} \to Z_i'$, we can blow up in any fiber at the node. These new families are families of normal varieties and, provided we restrict to open subsets, they can be assumed to be flat and even smooth. Hence for any $i \in \mathscr{I}$ there is a morphism $\varphi_i\colon Z_i \to \mathscr{K}$, where $\mathscr{K}$ is the moduli space of surfaces with $p_g = p_a = 5$, $K^2 = 10$. Since for any $i, j \in \mathscr{I}$, φ_i and φ_j agree on $Z_i' \cap Z_j'$, there is a unique morphism $\varphi\colon Z' \to \mathscr{K}$ induced by the morphisms $(\varphi_i)_{i\in\mathscr{I}}$. By proposition (8.6), φ is dominant on an irreducible component $\mathscr{K}^{(1)}$ of $\mathscr{K}$, the unique one containing points corresponding to equivalence classes of surfaces having canonical model of degree 10 with isolated singularities. Moreover $\mathscr{K}^{(1)}$ is unirational, since Z' is rational. Argueing like in the proof of proposition (8.6), we have:

$$\dim \mathscr{K}^{(1)} \leqslant \dim Z' - \dim \mathscr{G} + 2 = 40.$$

But, since $\dim \mathscr{K}^{(1)} \geqslant 40$ (see (8.9)), we get the assertion. q.e.d.

(8.11) REMARK. It is a consequence of theorem (8.10) that:

(i) for any surface S with $p_g(S) = p_a(S) = 5$, $K_S^2 = 10$, whose canonical model has isolated singularities, its local moduli space has exactly dimension $40 = h^1(S, T_S) - h^2(S, T_S)$.

It is also possible to check that:

(ii) if F is the *generic* canonical surface of degree 10, verifying (5.2), with isolated singularities, it can be obtained as a residual intersection of two quartics containing a rational sextic surface with sections of genus 3 and isolated singularities.

This can be seen in many ways, for instance counting parameters. From the same proof of theorem (8.10) it also follows that:

(iii) the generic fiber of the morphism $\varphi\colon Z' \to \mathscr{K}^{(1)}$, contains an orbit for the action of $\mathscr{G}$ on Z' and has the same dimension of the orbit, namely $\dim \mathscr{G} - 2$. It seems likely that the fiber coincides with the orbit and it is certainly so if it is irreducible.

Let us conclude remarking that the arguments of this paragraph give, by remark (4.11), an explicit description of the ideal of canonical surfaces we have dealt with.

9. – Canonical surfaces of degree 10 in $\mathbf{P}^4$ with non isolated singularities, not contained in a quadric: existence.

In this paragraph we shall prove the existence of canonical surfaces of degree 10 in $\mathbf{P}^4$ with nonisolated singularities, verifying (5.2), not contained in any quadric. According to remark (5.8), the nonnormal points of such a surface lie on a line.

We begin recalling what is a *tacnode* of a surface in $\mathbf{P}^3$. This is an isolated singular point of a surface in $\mathbf{P}^3$, locally analytically isomorphic to the singularity of the surface of equation $z^2 = f(x, y)$ in $\mathbf{C}^3$ at the origin, where $f(x, y) = 0$ defines an analytic curve in $\mathbf{C}^2$ with an ordinary 4-ple point at the origin. The tangent cone at a tacnode is a plane counted twice; this plane is called *tacnodal plane*.

Let us prove the:

(9.1) LEMMA. *Given any five points $A_1, \dots, A_5$* in $\mathbf{P}^3$, *four by four linearly independent, and any plane α containing A_5 and not containing any of the other points, there exists a surface T of degree* 6 *in* $\mathbf{P}^3$ *with ordinary triple points at $A_1, \dots, A_4$, with a tacnode at A_5, tacnodal plane α, and no other singularities.*

PROOF. It is easy to check that there exist six nonsingular quadrics $Q_1, \dots, Q_6$ in $\mathbf{P}^3$ with the following properties:

(i) Q_1, Q_2, Q_3 contain $A_1, \dots, A_5$, are tangent to α at A_5, and intersect only at $A_1, \dots, A_5$;

(ii) $Q_1, \dots, Q_5$ generate the linear system of quadrics containing $A_1, \dots, A_5$;

(iii) $Q_1, \dots, Q_6$ generate the linear system of quadrics containing $A_1, \dots, A_4$.

$Q_1, \dots, Q_6$ will denote the quadrics and the quadric polynomials whose vanishing defines them. Consider the linear system of quartics:

$$\sum_{i,j=1}^{3} \lambda_{ij} Q_i Q_j = 0 \tag{9.2}$$

and let us show that the generic surface in (9.2) has:

(iv) an ordinary double point at $A_1, \dots, A_4$;

(v) a tacnode with tacnodal plane α at A_5;

(vi) no other singular point.

In fact the base points of (9.2) are just $A_1, \dots, A_5$ by (i). Applying again (i), it is easy to check that Q_1, Q_2, Q_3 have independent tangent planes at $A_1, \dots, A_4$. Hence, by Bertini's theorem, we get (iv) and (vi). As for (iv), let us assume $A_5 = (1, 0, 0, 0)$, α has equation $x_3 = 0$, and pass to affine coordinates $x = x_1/x_0$, $y = x_2/x_0$, $z = x_3/x_0$. Q_1, Q_2, Q_3 can be supposed to have affine equations $Q_i = z + f_i(x, y, z) = 0$, $i = 1, 2, 3$, with $f_i(x, y, z)$ homogeneous polynomial of degree 2 in x, y, z. Passing to affine coordinates in (9.2) we have:

$$\Big(\sum_{i,j=1}^{3} \lambda_{ij}\Big) z^2 + \Big(\sum_{i,j=1}^{3} \lambda_{ij}(f_i + f_j)\Big) z + \sum_{i,j=1}^{3} \lambda_{ij} f_i f_j = 0 \,. \tag{9.3}$$

For a generic choice of the λ_{ij}'s, the quartic of equation (9.3) has a double point at A_5 with tangent cone $z^2 = 0$. Besides its intersection with α has an ordinary 4-ple point at A_5. By Weistrass' preparation theorem, the equation (9.3) can be locally written:

$$z^2 + 2a_1(x, y) z + a_2(x, y) = 0 \tag{9.4}$$

around A_5. The double covering of the plane α defined by (9.4) has branch locus of equation $a_1(x, y)^2 - a_2(x, y) = 0$; hence it is locally analytically isomorphic around A_5 to the surface of equation (see [F_2]):

$$z^2 = a_1(x, y)^2 - a_2(x, y)$$

still having $z^2 = 0$ as tangent cone, so that the curve $a_1(x, y)^2 - a_2(x, y) = 0$ has an ordinary 4-ple point at A_5. Consider now the linear system of surfaces of degree 6:

$$\sum_{i,j=1}^{6} \mu_{ij} F_i Q_j = 0 \tag{9.5}$$

where $F_1, \dots, F_6$ are six quartics spanning the linear system (9.2) having the singularities described in (iv), (v), (vi). Taking into account (i), (ii), (iii), it is easy to check that the base locus of the linear system (9.5) is given by $A_1, \dots, A_5$, and that the generic surface in (9.5) has, by Bertini's theorem, the singularities required for T. q.e.d.

It is now possible to prove the:

(9.6) THEOREM. *There exist canonical surfaces F of degree* 10 *in* $\mathbf{P}^4$, *with a nodal line and no other singularities, verifying* (5.2).

PROOF. Let T be the sextic of lemma (9.1), and look at a minimal desingularization $\omega: S \to T$. If we set $\omega^*(A_i) = E_i$, $i = 1, \dots, 5$, the E_i's are nonsingular elliptic curves on S, and $E_i^2 = -3$, $i = 1, \dots, 4$, $E_5^2 = -2$. Moreover it is:

$$\mathcal{O}_S(K_S) = \omega^*(\mathcal{O}_{\mathbf{P}^3}(2)|_T) \otimes \mathcal{O}_S\Big(\sum_i^5 E_i\Big)^* \tag{9.7}$$

(see [E], cap. III; [F_1]); (9.7) reflects the well known fact that canonical divisors on S are proper transform by ω of divisors cut out on T, by quadrics containing $A_1, \dots, A_5$. Hence $p_g(S) = 5$ and the canonical map $p: S \to F \subseteq \mathbf{P}^4$ is a morphism, which fails to be injective exactly on E_5. Indeed $|K_S|$ cuts out on E_5 a complete g_2^1. Applying Riemann-Roch theorem and the Kodaira vanishing theorem to the linear system $|K_S + \omega^*(H)|$, whose divisors are proper transform by cubics containing $A_1, \dots, A_S$, one checks that $p_a(S) = 5$. q.e.d.

(9.8) REMARK. Let us explicitely point out that the canonical surfaces, whose existence has been proved in theorem (9.6), also contain four plane nonsingular cubics, corresponding to the four curves $E_1, \dots, E_4$.

The above canonical surfaces do not lie on any quadric. Indeed:

(9.9) PROPOSITION. *Let Δ be a half-canonical curve of degree* 10 *in* $\mathbf{P}^3$ *with a node or a cusp and no other singularity*; Δ *does not lie on any quadric.*

PROOF. Assume Δ on a quadric Q, and suppose Q is nonsingular. It is $\Delta \sim aA + bB$, where a, b are nonnegative integers and A, B are incident lines of Q. It is:

$$a + b = 10\,. \tag{9.10}$$

Let $p\colon C \to \Delta$ be the normalization of Δ, $P_1 + P_2$ the divisor on C corresponding to the double point of Δ, $s \in H^0(C, \mathcal{O}_C(P_1 + P_2))$, $s \neq 0$. If $A(C)$ is the vector subspace of $H^0\big(Q, \mathcal{O}_Q((a-2)A + (b-2)B)\big)$ given by sections vanishing at the double point of Δ, there exists a natural isomorphism:

$$\sigma \in A(C) \to \frac{p^*(\sigma|_\Delta)}{s} \in H^0(C, \mathcal{O}_C(K_C))$$

so that $2ab - 2(a + b) - 2 = 20$. Hence, by (9.10), it is $ab = 21$, and so $a = 3$, $b = 7$ or viceversa. But if it is so, Δ cannot be half-canonical, as easily can be seen. The case Q is a cone can be worked out in a similar way. q.e.d.

(9.11) REMARK. By proposition (9.9), it is possible to apply proposition (5.3) to canonical surfaces F of degree 10, verifying (5.2), with a nodal, or even cuspidal, line, and, may be, other isolated singularities. By remark (5.8) the latter singularities must be normal points, namely irrilevant singularities. Besides, with notations usual in no. 5, it is $\Phi(A_2) = \mathcal{I}'(F)$ (see remark (5.8)).

We conclude this paragraph showing the:

(9.12) PROPOSITION. *Let Δ be a half-canonical curve of degree* 10 *in* $\mathbf{P}^3$, *with a singular point P, not lying on a quadric. Then P is a node or a cusp.*

PROOF. Suppose P is not a node or a cusp. Then there exists a line r containing P, such that any plane through r has multiplicity of intersection at least 4 with Δ at P. If $p\colon C \to \Delta$ is the normalization, there are four, not necessarily distinct, points $P_1, \ldots, P_4$ on C, such that $p(P_i) = P$, $i = 1, \ldots, 4$, and $P_1 + \ldots + P_4$ is in the pull-back on C of any divisor cut out on Δ by a surface tangent to r at P. The quadrics tangent to r at P form a 7-dimensional linear system cutting out on Δ a 7-dimensional linear series of divisors, each one containing $P_1 + \ldots + P_4$. Thus there exists a g^7_{16} on C, which is complete by Clifford's theorem. This g^7_{16} is residual of

$P_1 + \ldots + P_4$ with respect to $|K_C|$; thus, by Riemann-Roch, $|P_1 + \ldots + P_4|$ is a g_4^1. Consider now the g_{10}^3 on C corresponding to the plane section of Δ. This is a complete series, by a theorem of Comessatti (see [COM]); moreover the g_4^1 is contained in it, and the residual series is a g_6^1, and viceversa. This implies that any divisor of each, the g_4^1 and the g_6^1, consists of collinear points. This is impossible. In fact, like in the proof of proposition (5.3), one checks that, in the minimal resolution (1.3), with $\mathscr{F} = \mathcal{O}_C$ it is $F_0 \simeq A^{(3)} \oplus A^{(3)}(-2)$, $F_1 \simeq A^{(3)}(-3)^4$, $F_2 = A^{(3)}(-6) \oplus A^{(3)}(-4)$. Therefore, by proposition (4.3), Δ is set-theoretically intersection of quartics, so that no sextuples of points of Δ lie on a line. q.e.d.

(9.13) REMARK. Let F be a canonical surface of degree 10 in $\mathbf{P}^4$, verifying (5.2), with nonisolated singularities, not lying on a quadric. Then any generic hyperplane section of F does not lie on a quadric (see [R], p. 151). Hence, by proposition (9.12), F has a nodal or cuspidal line. As observed in remark (9.11), there can be only irrelevant singularities of F beyond it.

10. – Canonical surfaces of degree 10 in $\mathbf{P}^4$ with nonisolated singularities, not contained in a quadric: general deformations.

Let F be any canonical surface of degree 10 in $\mathbf{P}^4$, verifying (5.2), with a nodal or cuspidal line. We aim to prove that the class of birational equivalence determined by F corresponds, in the moduli space of surfaces with $p_g = p_a = 5$, $K^2 = 10$, to a point in the component $\mathscr{K}^{(1)}$ containing all equivalence classes of surfaces whose canonical model has isolated singularities (see theorem (8.10)). In order to get this result, we need some preliminary considerations.

(10.1) LEMMA. *Let F be like above and $p: S \to F$ its normalization. There exists a flat family of surfaces containing S, such that its elements have $p_g = p_a = 5$, $K^2 = 10$, and its generic element has a canonical model with isolated singularities or with a nodal line and no other singularities.*

PROOF. A generic projection F_1 of F in $\mathbf{P}^3$ is a canonical surface of degree 10 in $\mathbf{P}^3$. Using standard arguments (see [E], p. 6; [GH], p. 611-618), it is easy to check that F_1 has the following singularities.

(i) a curve γ of degree 25, of double points; one component of γ is the projection r_1 of r;

(ii) finitely many multiple points, multiple for γ too, coming from multisecant lines of F passing through the center of projection;

(iii) the projection of the irrelevant singular points of F; these could, also lie on γ and, in particular, on r_1.

Let $\omega\colon \tilde{S} \to S$ be a minimal desingularization and Γ the proper transform on $\tilde{S}$ of γ via $p\circ\omega$. Since F_1 is a canonical surface in $\mathbf{P}^3$, there exists a surface F_1' of degree 5, containing γ, such that the pull-back on $\tilde{S}$, via $p\circ\omega$, of the divisor cut out by F_1' on F_1, is Γ. Hence F_1' does not contain any singular point of F_1 of type (iii), not lying on γ, and has a simple point at any such a point lying on γ. Consider the pencil of surfaces of degree 10:

$$\lambda F_1 + \mu (F_1')^2 = 0\,.$$

There exists a Zariski open subset X in the $\mathbf{P}^1$ parametrizing this pencil, containing the point $\lambda = 1$, $\mu = 0$, such that any surface corresponding to a point in X, except at most F_1, has singularities of type (i), (ii), r_1 as nodal curve, and no singularities of type (iii). This by Bertini's theorem. Provided we restrict to a suitable open subset of X containing $\lambda = 1$, $\mu = 0$, we can make a simultaneous normalization of the surfaces of the family, getting the required flat family. q.e.d.

By the previous lemma, we can restrict our attention to canonical surfaces F of degree 10 in $\mathbf{P}^4$, verifying (5.2), with a nodal line r and no other singularities. In this case, the normalization $p\colon S \to F$ is the canonical map for S, which is nonsingular. Let us put $E = p^*(r)$; E is nonsingular (by proposition (9.12)) and $p_E\colon E \to r$ is a double covering.

(10.2) Lemma. *If F, S are like above:*

(i) *the generic curve in $|K_S - E|$ is irreducible, nonsingular;*

(ii) *E is an irreducible curve of genus 1 and $E^2 = -2$ on S.*

Proof. E is a special divisor on S and it is $h^0(S, \mathcal{O}_S(K_S - E)) = 3$; all curves in $|K_S - E|$ are pull-backs on S of curves cut out on F, outside of r, by hyperplanes containing r. Hence $|K_S - E|$ has no base points on S. Thus, if (i) is not true, $|K_S - E|$ is composite by couples of curves of a rational pencil $\mathcal{P}$, without base points. Let $D \in \mathcal{P}$. Since $K_S \cdot E = 2$, it is $K_S \cdot D = 4$. Moreover $D^2 = 0$, thus all curves in $\mathcal{P}$ are mapped, via p, onto plane quartics. Let $\mathcal{P}'$ be the algebraic system of planes containing these quartics. Any couple of planes in $\mathcal{P}'$ is contained in a $\mathbf{P}^3$ through r, but $\mathcal{P}'$ is not contained in the same $\mathbf{P}^3$. Hence all planes in $\mathcal{P}'$ contain r.

But this is impossible, since F is a set-theoretical intersection of quartics (see remark (9.11)). Now, by Riemann-Roch, we have:

$$3 = h^0(S, \mathcal{O}_S(K_S - E)) = 5 + h^1(S, \mathcal{O}_S(K_S - E)) - (K_S - E)\cdot E/2$$

thus:

$$[(K_S - E)\cdot E/2] - 2 = h^1(S, \mathcal{O}_S(K_S - E)) \geqslant 0 .$$

Therefore it is:

$$(10.3) \qquad E^2 \leqslant -2 .$$

Remark now that it must be:

$$(10.4) \qquad (K_S - E)^2 = E^2 + 6 > 2$$

since $(K_S - E)^2$ is even and the generic curve in $|K_S - E|$ is not hyperelliptic. From (10.3) and (10.4) it follows $E^2 = -2$ since E^2 is even. By adjunction formula we have that E is elliptic. q.e.d.

(10.5) REMARK. By lemma (10.3), the generic curve in $|K_S - E|$, which is irreducible, nonsingular, has genus 7. Moreover $(K_S - E)\cdot E = 4$. Besides, since E is elliptic, there are four cuspidal points on r. If $P \in r$ is not a cuspidal point, there are two tangent planes at P to the two branches of F passing through r. Take the $\mathbf{P}^2$ consisting of all planes containing r, and look at the curve Θ described in this $\mathbf{P}^2$ by all couples of tangent planes to F at a point of r. Clearly there is a map $\varphi\colon E \to \Theta \subseteq \mathbf{P}^2$; φ corresponds to the linear series g_4^d, $d \leqslant 2$, cut out on E by $|K_S - E|$. Since, by adjunction, it is $\mathcal{O}_S(K_S - E)|_E = \mathcal{O}_S(2K_S)|_E$, the g_4^d contains the g_2^1 cut out by $|K_S|$ on E, corresponding to the map p_E. The residual series of the g_2^1 with respect to the g_4^d is contained in the g_2^1. Since $|K_S - E|$ has no base points, it is $2g_2^1 \subseteq g_4^d$, thus $d = 2$, and θ can either be an irreducible quartic or an irreducible conic, counted twice. Consider now the generic section of F with a hyperplane through r, outisde of r. This is an irreducible, nonsingular curve Δ_1 of degree 8 and genus 7, corresponding to a curve of $|K_S - E|$. Δ_1 cuts r in four distinct points $A_1, \ldots, A_4$; let $r_1, \ldots, r_4$ be the tangent lines to Δ_1 at $A_1, \ldots, A_4$ respectively. If θ is an irreducible quartic, no couple of lines $r_1, \ldots, r_4$ lies in a plane: the generic tangent plane to F at a point r is tangent at a single point. If θ is a conic, any tangent plane to F at a point of E is tangent at another point too. The g_4^2 is composite with a $\bar{g}_2^1$ such that couples of points of this $\bar{g}_2^1$ correspond to points on r with the same tangent plane. Remark that, by this reason, it must be

$\bar{g}_2^1 \neq g_2^1$ since r is nodal. The four points $A_1, \ldots, A_4$ can be arranged in two couples, say A_1, A_2 and A_3, A_4 such that r_1, r_2 are coplanar, and similarly r_3, r_4 are coplanar, the two planes being distinct.

(10.6) LEMMA. *Let Δ be a half-canonical curve of degree* 10 *in* $\mathbf{P}^3$ *with a singular point P, not lying on any quadric. The projection of Δ in a plane from P is either a double covering of a nonsingular quartic, or is birational and the image curve has at most double points.*

PROOF. The assertion follows from this property of Δ: there exists no line r through P which is at least a trisecant of Δ outside of P. Namely if $p: C \to \Delta$ is the normalization and $P_1 + P_2$ is the divisor corresponding to P (see proposition (9.12)), there are no triple of points P_3, P_4, P_5 on C such that $P_1 + \ldots + P_5$ is in the pull-back on C of any divisor cut out on Δ by a surface containing r and not containing Δ. If this happens indeed, look at the linear 6-dimensional system of quadrics through r. It cuts out on Δ a 6-dimensional linear series with $P_1 + \ldots + P_5$ as fixed divisor. Thus there exists a g_{15}^6 on C, residual of $|P_1 + \ldots + P_5|$ with respect to $|K_C|$. By Riemann-Roch it is $\dim |P_1 + \ldots + P_5| \geqslant 1$. In fact the equality holds, since C is neither birational to a plane quintic, nor hyperelliptic. The $g_5^1 = |P_1 + \ldots + P_5|$ is contained in the g_{10}^3 corresponding to plane sections of Δ; the residual series is still a g_5^1. Therefore both these g_5^1 consist of divisors of collinear points on Δ; this is impossible, since Δ is a set-theoretical intersection if quartics (see remark (9.11)). q.e.d.

(10.7) REMARK. Let F be like above. By lemma (10.6), the projection π of F from a noncuspidal point P of r in a $\mathbf{P}^3$ can a priori be:

1) birational; in such a case its image F_1 is an irreducible surface of degree 8 in $\mathbf{P}^3$ with no more than double points outside of the point $P_1 = \pi(r)$;

2) a double covering of a quartic surface.

The second case does not happen if P is suitably chosen on r. Let P be indeed such that the two tangent planes α_1, α_2 to F at P cut F in finitely many points off r. Such points do exist by lemma (10.2) and fill an open Zariski subset r' of r. Let now Δ be a hyperplane section of F through P, having a node at P, no other singularity and missing the points of intersection of α_1, α_2 with F off r. If π is not birational, π_Δ is not birational. Thus, by lemma (10.6) and by the above, the two branches of Δ at P are two flexes; hence α_1, α_2 are *inflexional* tangent planes. Then any $\mathbf{P}^3$

containing α_i, cuts F, outside of r, in a curve having a double point at P on the branch of F with tangent plane α_i, $i = 1, 2$. Since this cannot happen for any point of r' by Bertini's theorem applied to $|K_S - E|$, the conclusion is that there exists a Zariski open set r^* of r' such that for any $P \in r^*$ the projection of F from P on a $\mathbf{P}^3$ is birational. Let us examine the projection π from an abstract point of view, assuming it is birational. First we look at the normalization $p: S \to F$ of F, which is nonsingular. The fiber $p^{-1}(P)$ consists of two distinct points $P^{(1)}$, $P^{(2)}$ on S. We blow up these points, getting a new surface S and a morphism $\varphi: \tilde{S} \to S$ such that $\varphi^{-1}(P^{(i)}) = E_i$, $i = 1, 2$, E_1, E_2 being exceptional curves of the first kind; φ is an isomorphism off $P^{(1)}$, $P^{(2)}$. For any divisor H on S, we denote by $\tilde{H}$ its proper transform via φ, which is a divisor on $\tilde{S}$. Let us consider the 3-dimensional linear system $|K_S - E_1 - E_2|$ on $\tilde{S}$. It determines a map $\psi: \tilde{S} \to \mathbf{P}^3$, and the following diagram commutes:

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\varphi} & S \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle p} \\ \mathbf{P}^3 & \xleftarrow[\pi]{} & F \end{array}$$

Therefore it is sufficient to study the map ψ and its image F_1. Since $(\tilde{K}_S - E_1 - E_2)\cdot\tilde{E} = 0$, ψ contracts $\tilde{E}$ in one point, namely P_1. Besides $(\tilde{K}_S - E_1 - E_2)\cdot E_i = 1$, $i = 1, 2$, thus E_1, E_2 are birationally mapped onto two lines r_1, r_2 in $\mathbf{P}^3$. Using lemma (10.2), formula (10.4) and remark (10.5) one easily checks that P_1 is a 4-ple point. Being $K_{\tilde{S}} = \tilde{K}_S + E_1 + E_2$, it is $K_{\tilde{S}} - (\tilde{K}_S - E_1 - E_2) = 2(E_1 + E_2)$. If γ is the double curve of F_1 and Γ is the corresponding divisor on $\tilde{S}$, there exists a unique adjoint surface F_1' to F_1 of degree 3, cutting out on F_1 the divisor whose pull-back via ψ is $\Gamma + 2(E_1 + E_2)$. Thus F_1' has to contain γ and to touch F_1 along r_1, r_2. Moreover, since $p_g(S) = 5$, there exists some adjoint surface to F_1 of degree 4, distinct from F_1' plus a plane, containing r_1, r_2. Since the generic plane section of F_1 has 10 double points, which can be distinct or *infinitely near* it follows that $\gamma \cup r_1 \cup r_2$ is the set-theoretic complete intersection of F_1' with an adjoint quartic surface. Finally, P_1 being a 4-ple point for F_1, any adjoint surface has at least a double point at P_1 (see [E], p. 76). It is now easy to check that γ, considered as a subscheme of $\mathbf{P}^3$, complete intersection of a cubic and a quartic off r_1, r_2, has matricial equation:

$$\text{rank}\begin{Vmatrix} f_1^{(2)} & f_2^{(2)} & f_3^{(3)} \\ g_1^{(1)} & g_2^{(1)} & g_3^{(2)} \end{Vmatrix} < 2 \tag{10.8}$$

where the $f_i^{(j)}$'s, $g_i^{(j)}$'s are homogeneous polynomials in $x_0, \ldots, x_3$ of degree j, the quadric $f_1^{(2)}$ cuts the plane $g_1^{(1)}$ in $r_1 \cup r_2$ and $f_2^{(2)}$, $f_3^{(3)}$, $g_2^{(1)}$, $g_3^{(2)}$ contain the point P_1 (see [GA], [PS]).

We are now ready to prove the main theorem of this paragraph:

(10.10) THEOREM. *If S is like above, its local moduli space has dimension* 40 *and, for a general deformation, the canonical image is a canonical surface of degree* 10 *in* $\mathbf{P}^4$ *with a single node and no other singularity.*

PROOF. It is $h^1(S, T_S) - h^2(S, T_S) = 40$ (see formula (8.9)). By Kuranishi's theorem (see [K]), there exists an analytic subspace X in a neighbourhood of 0 in $H^1(S, T_S)$, defined by $h^2(S, T_S)$ equations, hence of dimension at least 40, and a family $f: \mathscr{S} \to X$ which is the universal deformation of S. We shall denote by S_x the fiber $f^{-1}(x)$, $x \in X$. Let $\omega_{\mathscr{S}|x}$ be the relative dualizing sheaf on $\mathscr{S}$. By Grauert's theorem (see [H], p. 288) we can assume that $f_*(\omega_{\mathscr{S}|x})$ is a trivial, rank 5, vector bundle over X. Fix five independent sections of this bundle. They determine five independent sections of $\omega_{\mathscr{S}|x}$ which, restricted to S_x, span $H^0(S_x, \mathcal{O}_{S_x}(K_{S_x}))$. for any $x \in X$. We may also assume that these five sections do not have common zeroes on $\mathscr{S}$, since this happens for their restriction to $S = S_0$. Thus a morphism is defined $k: \mathscr{S} \to \mathbf{P}^4 \times X$, such that the following diagram commutes:

$$\begin{tikzcd} \mathscr{S} \arrow[rr, "k"] \arrow[dr, "f"'] & & \mathbf{P}^4 \times X \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

g being the projection onto the second component; φ_{S_x} is the canonical map for S_x. Then $\varphi(S_x) = F_x$ is a canonical surface of degree 10 in $\mathbf{P}^4$, verifying (5.2), and we may assume it is not contained in any quadric, since F_0 is not. Suppose the theorem is not true. It can be assumed that for any x in a component $\tilde{X}$ of X, F_x has a nodal line r_x and no other singularity, since this is true for F_0 (see remark (9.13)). Keep now all notations of remark (10.7). Fix a hyperplane $\alpha \subseteq \mathbf{P}^4$ cutting r_0 in a point of r_0^*; we can suppose that α cuts each r_x in a point of r_x^*, if $x \in \tilde{X}$.

Fix a $\mathbf{P}^3$ not containing any point $P_x = \alpha \cap r_x$ and project F_x in the $\mathbf{P}^3$ from P_x. By the same remark (10.7), we get an abvious morphism $\mu: \tilde{X} \to \Sigma^3(8)$ whose fibers are finite. Let $\mathscr{P}$ be the, may be reducible, locally closed subset of $\Sigma^3(8)$ formed by all surfaces of degree 8 in $\mathbf{P}^3$ which are projection of a canonical surface of degree 10 in $\mathbf{P}^4$, verifying (5.2), with a nodal line r and no other singularity, from a point of r^*. Clearly μ

induces a morphism $\mu' : \tilde{X} \to \mathscr{P}$. Moreover $PGL(4, \mathbf{C})$ acts in a natural way on $\mathscr{P}$, with finite stabilizors, and μ' induces a morphism $\mu'' : \tilde{X} \to \mathscr{P}/PGL(4, \mathbf{C})$ whose fibers are still finite. Hence it must be:

$$(10.11) \qquad \dim \tilde{X} \leqslant \dim \mathscr{P} - 15 .$$

Let $\tilde{\mathscr{P}}$ be any irreducible component of $\mathscr{P}$. By remark (10.7), there exists a natural morphism $\nu : \tilde{\mathscr{P}} \to \mathscr{H}$, $\mathscr{H}$ being the component of the Hilbert scheme of curves of degree 10 in $\mathbf{P}^3$ having matricial equation (10.8). With easy computations one checks that:

$$(10.12) \qquad \dim \nu(\tilde{\mathscr{P}}) \leqslant 38 .$$

Let us fix $\gamma \in \nu(\tilde{\mathscr{P}})$, double curve of $F_1 \in \tilde{\mathscr{P}}$. The fiber $\nu^{-1}(\gamma)$ is contained in the linear system $\Sigma(F_1)$ of surfaces of degree 8 which are *biadjoint* surfaces to F_1(see [E], p. 89). They cut out on F_1, off γ, bicanonical divisors. Hence:

$$(10.13) \qquad \dim \nu^{-1}(\gamma) \leqslant \dim \Sigma(F_1) = 16 .$$

Finally, by (10.11), (10.12), (10.13), it is $\dim \tilde{X} \leqslant 39$, which is impossible. q.e.d.

(10.14) COROLLARY. *If F is a canonical surface of degree* 10 *in* $\mathbf{P}^4$, *verifying* (5.2), *not contained in any quadric, the class of birational equivalence corresponding to F gives, in the moduli space of surfaces with $p_g = p_a = 5$, $K^2 = 10$, a point of the component $\mathscr{K}^{(1)}$ containing all equivalence classes of surfaces whose canonical model has isolated singularities.*

PROOF. Easily follows from theorem (8.10), lemma (10.1) and theorem (10.10). q.e.d.

(10.15) REMARK. It could be perhaps possible to analyze more closely the subvariety $\mathscr{K}_0^{(1)}$ of $\mathscr{K}^{(1)}$ corresponding to classes of equivalence of surfaces whose canonical model has a double line. In order to study $\mathscr{K}_0^{(1)}$ one should analyse the family of projections in $\mathbf{P}^3$ of canonical surfaces of degree 10 in $\mathbf{P}^4$ with a double line from a point of this line. We do not explore this here, but we guess that $\mathscr{K}_0^{(1)}$ is an unirational subvariety of codimension 2 of $\mathscr{K}^{(1)}$. Unfortunately the methods used in no. 8, strongly based upon the results of part I, do not work in this case.

(10.16) REMARK. It seems unlikely that canonical surfaces of degree 10 in $\mathbf{P}^4$, verifying (5.2), with a cuspidal line, do in fact exist. Suppose indeed F

is such a surface with a cuspidal line r and without other singularities. If $p\colon S \to F$ is the normalization, let us put $E = p^*(r)$. E is rational and $E^2 = -3$. Argueing like in lemma (10.2), one shows that the generic curve in $|K_S - E|$ is irreducible, nonsingular. To this generic curve corresponds an irreducible, nonsingular curve of degree 8 and genus 8, Δ_1 on F, lying in a hyperplane through r, cutting r in four points. Using Riemann-Roch one checks that Δ_1 has to lie on a quadric, which contains r. Hence Δ_1 would be a complete intersection of a quadric and a quartic, thus of genus 9, which is impossible.

11. – Canonical surfaces of degree 10 in $\mathbb{P}^4$ contained in a quadric.

In order to work out a complete discussion about surfaces with $p_g = p_a = 5$, $K^2 = 10$, for which the canonical map is a birational morphism, we shall finally examine the case in which the canonical image is a surface of degree 10 in $\mathbb{P}^4$ contained in a quadric. By propositions (5.9) and (9.9) it follows that the generic hyperplane section has at least two nodes. As a matter of fact we have the following:

(11.1) LEMMA. *Let Δ be a half-canonical curve of degree* 10 *in* $\mathbb{P}^3$ *lying on an irreducible quadric Q. If Q is a cone, Δ is a complete intersection of Q with a quintic surface, and has five coplanar nodes or equivalent singularities. If Q is nonsingular, either Δ is like above or is a* (4,6)*-curve with four nodes or equivalent singularities, two by two lying on lines of the ruling of Q cutting out the g_4^1 on Δ.*

PROOF. If there exists a quintic surface containing Δ but not Q, Δ is a complete intersection. Assume Q is nonsingular. The canonical series is cut out on Δ by the adjoint curves in $|3H_Q|$. Since Δ is half-canonical, there exists a single curve in $|H_Q|$ which is adjoint to Δ and cuts on Δ the 0 divisor off the singularities. Thus Δ has five coplanar nodes, or equivalent singularities. The case Q is a cone can be worked out in a similar way. Suppose now there is no quintic surface containing Δ but not containing Q. Let us check that there exist irreducible sextics through Δ. If it were not so, any surface in $\Sigma^3(6, \Delta)$ should contain Q, thus $\dim \Sigma^3(6, \Delta) = 34$ and the linear series cut out by $\Sigma^3(6)$ on Δ were a g_{60}^{48}. This would imply that Δ has no more than one node or one cusp, which is impossible by proposition (9.9). Let $Q_1 \in \Sigma^3(6, \Delta)$ be irreducible. Q_1 cuts Q in Δ and in a residual curve of degree 2. By standard arguments, this curve has to be the union of two skew lines of the same ruling of Q, which is non singular. The reminder of the statement is easy to prove. q.e.d.

(11.2) REMARK. Let us keep all notations of the above proof and describe how Δ can be done. Assume Δ is a complete intersection of Q with a quintic, and denote by Δ' the unique adjoint curve to Δ in $|H_Q|$. Then:

1) if Δ' is nonsingular, Δ can have no more than double points, which can be nodes or cusps and can become *infinitely near along* Δ';

2) if Δ' is the union of two distinct lines intersecting at a simple point $P \in Q$, Δ has a triple point at P and two double points, nodes or cusps, one on each of the two lines of Δ'; clearly these double points could become *infinitely near to the triple point*; in this case Q is nonsingular;

3) if Q is a cone and Δ' is the union of two lines intersecting at the vertex P of Q, Δ has a double point at P and four double points, two by two lying on each line of Δ'; these double point are nodes or cusp and can become *infinitely near*;

4) if Δ' is a line counted twice Δ can either have a triple point and a tacnode on Δ, or a double point, node or cusp, at the vertex of Q on Δ' and two tacnodes on Δ'. The tacnodes could become cusps of second order.

If Δ is a (4.6)-curve then the four nodes of Δ can become cusps, or be *infinitely near*, e.g. giving tacnodes with tacnodal tangent a line on Q. Finally Δ could also have two tacnodes or cusps of second order, on a line of Q.

Let now F be a canonical surface of degree 10 in $\mathbf{P}^4$, verifying (5.2), contained in a quadric Q.

If Q is nonsingular, by lemma (11.1), F is a complete intersection of Q with a quintic hypersurface Q_1. F has a curve γ of singular points which we want to study. Consider the adjoint surfaces to F contained in Q: these are defined like in [Z], p. 71 is done for surfaces in $\mathbf{P}^3$. Argueing like in the proof of lemma (11.1), one checks that there exists a hyperplane π cutting out on Q a quadric F' which is the unique adjoint surface to F in $|H_Q|$. If $p: S \to F$ is the normalization and Γ is the divisor corresponding on S to γ, it is $\Gamma = p^*(F_1 \cdot F)$, where $F_1 \cdot F$ denotes the divisor cut out by F_1 on F. Hence:

$$\Gamma \sim K_S . \tag{11.3}$$

Assume γ has degree 5, namely the generic point of any component of γ is nodal or cuspidal for F (see remark (11.2)). In this case γ is a contact curve of F' with the quintic Q_1' cut out by Q_1 on π. Thus F' is a cone and γ contains the vertex P of F'. Applying remark (11.2) to the generic hyperplane section of F with a hyperplane through P, it is easy to check that P is a triple point for F and for γ too. Hence any adjoint surface to F in Q has a double point at P. Remark now that there are no irreducible adjoint surfaces to F in $|2H_Q|$, but there are irreducible adjoint surfaces to F in $|3H_Q|$, cutting out on F, off γ, curves whose pull-back via p is in $|2K_S|$. Hence γ belongs to the linear system $\mathscr{L}(\pi, F')$ cut out on F' by cubic surfaces in π having a double point at P and containing a line of F'. Thus:

$$\dim \mathscr{L}(\pi, F') = 9 . \tag{11.4}$$

If γ has not degree 5, namely if γ has some nonnodal or noncuspidal component, one can extend the above arguments, making few obvious changes. In any case γ can be seen as a curve in $\mathscr{L}(\pi, F')$. For sake of brevity we shall not go through these details.

We can now prove the:

(11.5) THEOREM. *Let S be like above. Then its local moduli space has dimension* 40 *and for a general deformation, the canonical image is a canonical surface of degree* 10 *in* $\mathbb{P}^4$ *with a single node and no other singularity.*

PROOF. The proof is similar to that of theorem (10.10); we shall keep all notations introduced there. Let $f: \mathscr{S} \to X$ be the universal deformation of S. If for some x in any component of X the canonical image F_x of S_x does not lie in a quadric the assertion follows from corollary (10.14). Assume that for any x in a component $\tilde{X}$ of X, F_x lies in a quadric, which can be supposed to be nonsingular, since this happens for F_0. Thus there exists a natural morphism $\mu: \tilde{X} \times PGL(5, \mathbb{C}) \to \Sigma^4(2) \times (\mathbb{P}^4)^*$; μ associates, to any couple $(x, \omega) \in \tilde{X} \times PGL(5, \mathbb{C})$ the couple $(Q_x, \pi_x) \in \Sigma^4(2) \times (\mathbb{P}^4)^*$, where Q_x is the quadric containing $\omega(F_x)$ and π_x is the hyperplane cutting out on Q_x the adjoint surface $\omega(F'_x)$ to $\omega(F_x)$. Let (x, ω) be any point in $\tilde{X} \times PGL(5, \mathbb{C})$, and let $\mu_{x,\omega}$ be the fiber $\mu^{-1}(\mu(x, \omega))$. A natural map $\nu_{x,\omega}$: $\mu_{x,\omega} \to \mathscr{L}(\pi_x, \omega(F'_x))$ is defined. If $(y, \omega') \in \mu_{x,\omega}$, let $\gamma_{y,\omega'} = \nu_{x,\omega}(y, \omega')$ be the singular curve of $\omega'(F_y)$. Using (11.3) it is easy to check that for any $(z, \omega'') \in \nu_{x,\omega}^{-1}(\gamma_{y,\omega'})$, $\omega''(F_z)$ cuts out on $\omega'(F_y)$ off $\gamma_{y,\omega'}$ a divisor whose pull-back on S_y is in $|3K_{S_y}|$. Thus:

$$\dim \nu_{x,\omega}^{-1}(\gamma_{y,\omega'}) \leqslant h^0(S, \mathcal{O}_S(3K_S)) = 36 .$$

Hence, by (11.4), it is:

$$\dim \mu_{x,\omega} \leqslant \dim \mathscr{L}(\pi_x, \omega(F'_x)) + \dim \nu_{x,\omega}^{-1}(\gamma_{y,\omega'}) \leqslant 45$$

and finally:

$$\dim \tilde{X} \leqslant \dim \left(\Sigma^4(2) \times (\mathbf{P}^4)^*\right) + \dim \mu_x - \dim PGL(5, \mathbf{C}) = 39$$

which is impossible since $\dim \tilde{X} \geqslant 40$. q.e.d.

If Q is a cone of rank 4, by lemma (11.1) and remark (11.2) either F is a complete intersection with a quintic hypersurface or is a section of Q with a sextic hypersurface outside of a couple of planes of one, say $\mathscr{L}_1$, of the two rulings of Q, and has two conics of double points γ_1, γ_2, in two planes of the other ruling $\mathscr{L}_2$. In the first case the singular curve of F can either be like the case Q is nonsingular, or can be reducible in a triple line r and two double lines r_1, r_2 both coplanar with r: the adjoint quadric F' to F should be the couple of planes spanned by r, r_1 and r, r_2. In the latter case the two conics γ_1, γ_2 can become tacnodal lines, or one tacnodal conic, etc. (see remark (11.2)). However, with suitable counts of parameters, similar to the one worked out in the proof of theorem (11.5), an analogous statement for the normalizations S of F can be proved. We do not reproduce these counts for sake of brevity.

If Q is a cone of rank 3, by lemma (11.1) and remark (11.2), F is complete intersection of Q with a quintic hypersurface. Here the singular curve can be either like the above cases, or reducible in a double line coinciding with the double line of Q, and in two conics of double points lying in two planes of Q, or reducible in a triple line and in a tacnodal line in the same plane of Q, etc. (see remark (11.2)). Here too, with suitable counts of parameters one can prove a similar statement to that of theorem (11.5). We do not reproduce these computations, which do not offer any difficulty.

Collecting the above results and taking into account corollary (10.14), we finally have:

(11.6) THEOREM. *If F is a canonical surface of degree* 10 *in* $\mathbf{P}^4$, *verifying* (5.2), *the class of birational equivalence corresponding to F gives, in the moduli space of surfaces with $p_g = p_a = 5$, $K^2 = 10$, a point of the component $\mathscr{K}^{(1)}$ containing all equivalence classes of surfaces whose canonical model has isolated singularities.*

(11.7) REMARK. Surfaces we have dealt with in this paragrph do exist. Let us show that, for instance, there exist canonical surfaces of degree 10 in $\mathbf{P}^4$, verifying (5.2), contained in a nonsingular quadric. Let Φ be a generic projection in $\mathbf{P}^3$ of a Del Pezzo surface of degree 5. Φ has degree 5 and has a nodal curve γ which is irreducible (see [E], p. 8), of degree 5, lies on a quadric cone ψ, and has a triple point at the vertex of ψ with three noncoplanar tangent lines, and no other singular points. Let Π be a plane in $\mathbf{P}^3$ cutting ψ in an irreducible conic γ'. The generic surface in the pencil:

$$\lambda\Phi\Pi^5 + \mu\psi^5 = 0$$

has degree 10 and, by Bertini's theorem, has γ as nodal curve, the conic γ' as quintic curve and no other singular points. It is easy to check that such surfaces are just projections in $\mathbf{P}^3$ of canonical surfaces we are looking for, from a point of the quadric in which they lie. Similarly it can be proved that there exist surfaces of degree 10 in $\mathbf{P}^3$ with two coplanar lines r_1, r_2 of multiplicity 4 and 6 respectively, and two double conics in two distinct planes through r_2. These are projections in $\mathbf{P}^3$ of canonical surfaces of degree 10 in $\mathbf{P}^4$ lying in a quadric cone of rank 4, which are not complete intersection of the cone with a quintic.

(11.8) Remark. Let us point out that certainly there exists a second component $\mathscr{K}^{(2)}$ of the moduli space of surfaces with $p_g = p_a = 5$, $\mathscr{K}^2 = 10$. Look, for instance at the family of double coverings of a $\mathbf{P}^2$ branched on a curve of degree 12 with three 4-ple points and two triple points each having an infinitely near triple point. Here the canonical system has two base points and is composite with an involution of order two. The canonical images of these surfaces are double coverings of Segre surfaces of degree 4 in $\mathbf{P}^4$. The component of the moduli space containing all classes of birational equivalence of these surfaces has at least dimension 40, thus is different from $\mathscr{K}^{(1)}$. The above example, which should be interesting to investigate carefully, is due to F. Enriques (see [E], p. 289).

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