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Canonical transformations and exact invariants for time-dependent Hamiltonian systems

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Abstract. An exact invariant is derived for *n*-degree-of-freedom non-relativistic Hamiltonian systems with general time-dependent potentials. To work out the invariant, an *infinitesimal* canonical transformation is performed in the framework of the extended phase-space. We apply this approach to derive the invariant for a specific class of Hamiltonian systems. For the considered class of Hamiltonian systems, the invariant is obtained equivalently performing in the extended phase-space a *finite* canonical transformation of the initially time-dependent Hamiltonian to a time-independent one. It is furthermore shown that the invariant can be expressed as an integral of an energy balance equation.

The invariant itself contains a time-dependent auxiliary function $\xi(t)$ that represents a solution of a linear third-order differential equation, referred to as the auxiliary equation. The coefficients of the auxiliary equation depend in general on the explicitly known configuration space trajectory defined by the system's time evolution. This complexity of the auxiliary equation reflects the generally involved phase-space symmetry associated with the conserved quantity of a time-dependent non-linear Hamiltonian system. Our results are applied to three examples of time-dependent damped and undamped oscillators. The known invariants for time-dependent and time-independent harmonic oscillators are shown to follow directly from our generalized formulation.

Keywords: time-dependent Hamiltonian system, canonical transformation, invariant
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1 Introduction

The derivation of invariants – also referred to as conserved quantities or constants of motion – is a key objective studying analytically a given Hamiltonian system. If we succeed to isolate an invariant, we always learn about a fundamental system property. For instance, in the case of an autonomous Hamiltonian system the Hamilton function itself is easily shown to represent an invariant – which can be identified with the energy conservation law.

This direct way to an invariant no longer exists in the general case of explicitly time-dependent systems, where the Hamiltonian does not provide directly a conserved quantity. Various methods have been applied to attribute an invariant to time-dependent (non-autonomous) systems: time-dependent canonical transformations [1–5], general dynamical symmetry studies in conjunction with Noether's the-

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orem [6–10], and direct *ad hoc* approaches [11–15]. For the time-dependent *quadratic* Hamiltonian of the harmonic oscillator type, Leach [2] showed that a two-step time-dependent linear canonical transformation can be applied to find an invariant. The invariant itself was found to consist of both the canonical coordinates, given by the system's phase-space path, and an auxiliary function satisfying a non-linear second-order auxiliary differential equation. With regard to its physical interpretation, the invariant could be identified as the Hamiltonian of a related autonomous harmonic oscillator system.

In this article, we present a generalized method to determine invariants of n-degree-of-freedom non-relativistic Hamiltonian systems with non-linear and explicitly time-dependent potentials. The systems may include damping with forces linear in the velocities.

We first review in Sections 2 and 3 the formalism of canonical transformations in the framework of the extended phase-space – where the time and the negative Hamiltonian are taken as an additional pair of canonically conjugate coordinates. In Section 4, an *infinitesimal* canonical transformation is carried out for a particular class of Hamiltonian systems to directly isolate a conserved quantity. In a second approach, presented in Section 5, a *finite* canonical transformation in the frame of the extended phase-space is shown to map the time-dependent Hamiltonian of Section 4 onto a time-independent one. Expressing this new Hamiltonian in terms of the old coordinates, one immediately obtains an invariant in the original system.

In either case, the same invariant is obtained. It consists of both the canonical coordinates and an auxiliary function, which follows from a homogeneous, linear third-order auxiliary equation. Apart from isotropic linear systems, the coefficients of the auxiliary equation depend on all spatial particle coordinates. As the consequence, this differential equation can only be integrated in conjunction with the system's equations of motion. This enhanced complexity of the general auxiliary equation reflects – little surprisingly – the involved nature of a conserved quantity for time-dependent non-linear Hamiltonian systems. From the energy balance equation for time-dependent Hamiltonian systems, it is shown that the invariant can be interpreted as the sum of the system's time-varying energy content and the energy fed into or detracted from it.

As illustrative examples, we derive the invariant and the auxiliary differential equation for three specific systems in Section 7: the time-dependent damped harmonic oscillator, the time-dependent anharmonic undamped oscillator, and the n-dimensional anisotropic oscillator. For the harmonic oscillator, invariant and auxiliary equation of Sections 4 and 5 are shown to specialize to the known results [2]. For the non-linear oscillator, interesting insight is revealed from the detailed behavior of the auxiliary function. As expected, the phase-space symmetries of this non-linear system differ significantly from those of a linear system.

We will show in Appendix A that our invariant can also be derived from Noether's theorem in the context of the Lagrangian formalism. Our invariant thus corresponds to the Noether symmetry group that leaves the action integral invariant. Very similarly, the invariant may as well be worked out on the basis of a "direct approach", using an ansatz function with a quadratic dependence on the canonical momenta. This will be sketched in Appendix B.

2 Canonical transformations in the extended phase-space

We consider an n-degree-of-freedom system of classical particles that may be described in a $2n$-dimensional Cartesian phase-space by an – in general – explicitly time-dependent Hamilton function $H = H(\vec{q}, \vec{p}, t)$. Herein, $\vec{q}$ denotes the n-dimensional vector of the configuration space variables, and $\vec{p}$ the vector of conjugate momenta. The system's time evolution, also referred to as the phase-space path $(\vec{q}(t), \vec{p}(t))$, can be visualized as a unique curve in the $2n$-dimensional phase-space. If the system's state is known at two distinct instants of time t_1 and t_2, the system's actual phase-space path between these fixed times is known to obey Hamilton's variation principle

$$\delta \int_{t_1}^{t_2} \left[\sum_{i=1}^{n} p_i(t) \, \frac{\mathrm{d}q_i(t)}{\mathrm{d}t} - H(\vec{q}(t), \vec{p}(t), t) \right] \mathrm{d}t = 0 \; . \tag{1}$$

This variation integral (1) can easily be shown to vanish exactly if the "canonical equations"

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i} \, , \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q_i} \, , \qquad i = 1, \ldots, n \tag{2}$$

are satisfied. We observe that the time t plays the distinguished role of an external parameter which is contained in both the system path and the Hamilton function H itself. As will be worked out in the following, this distinguished role of the time t may not be desirable in the general case of explicitly time-dependent (non-autonomous) Hamiltonian systems. We therefore introduce an evolution parameter s that parameterizes the system's time evolution $t = t(s)$. With s the new integration variable, we may rewrite Hamilton's principle (1) as

$$\delta \int_{s_1}^{s_2} \left[\sum_{i=1}^{n} p_i(s) \, \frac{\mathrm{d}q_i(s)}{\mathrm{d}s} - H(\vec{q}(s), \vec{p}(s), t(s)) \, \frac{\mathrm{d}t(s)}{\mathrm{d}s} \right] \mathrm{d}s = 0 \, . \tag{3}$$

With this symmetric form of the integrand, it looks reasonable to conceive the negative Hamiltonian in conjunction with the time as an additional pair of canonically conjugate coordinates. We thus introduce the $(2n+2)$-dimensional "extended" phase-space by defining

$$q_{n+1} = t \, , \qquad p_{n+1} = -\mathcal{H}$$

as additional phase-space dimensions. In this notation, $\mathcal{H}$ is understood as an independent variable that may be identified with the actual energy content of the Hamiltonian system H. We furthermore introduce the quantity $\tilde{H}$ as an implicit function

$$\tilde{H} = \tilde{H}(q_1, \ldots, q_{n+1}; p_1, \ldots, p_{n+1}) = 0 \tag{4}$$

of the extended phase-space variables. Equivalently to the initial Hamiltonian H, the function $\tilde{H}$ must contain all information about the dynamics of the given system. With $\tilde{H}$ as defined by Eq. (4), the variation integral (3) – its summation index now ranging to $(n+1)$ – may be written in the equivalent form of

$$\delta \int_{s_1}^{s_2} \left[\sum_{i=1}^{n+1} p_i(s) \, \frac{\mathrm{d}q_i(s)}{\mathrm{d}s} - \tilde{H} \right] \mathrm{d}s = 0 \, . \tag{5}$$

Similar to the case of Eq. (1), the integral (5) vanishes exactly if the canonical equations

$$\frac{\mathrm{d}q_i}{\mathrm{d}s} = \frac{\partial \tilde{H}}{\partial p_i}\,, \qquad \frac{\mathrm{d}p_i}{\mathrm{d}s} = -\frac{\partial \tilde{H}}{\partial q_i}\,, \qquad i = 1, \ldots, n+1 \tag{6}$$

hold for the "extended" Hamiltonian $\tilde{H}$. Since $\tilde{H}$ by its definition (4) does not depend on s explicitly, its total s-derivative vanishes by virtue of Eqs. (6):

$$\frac{\mathrm{d}\tilde{H}}{\mathrm{d}s} = \sum_{i=1}^{n+1} \left(\frac{\partial \tilde{H}}{\partial q_i} \frac{\mathrm{d}q_i}{\mathrm{d}s} + \frac{\partial \tilde{H}}{\partial p_i} \frac{\mathrm{d}p_i}{\mathrm{d}s} \right) = 0\,.$$

The Hamiltonian $\tilde{H}$ thus describes an autonomous system in the $(2n+2)$-dimensional extended phase-space. Defining $\tilde{H}$ according to [16]

$$\tilde{H}(\vec{q}, \vec{p}, t, -H) = H(\vec{q}, \vec{p}, t) - \mathcal{H}\,, \tag{7}$$

the canonical equations (6) that follow from $\tilde{H}$ can be related in a simple way to the canonical equations (2) of the original Hamiltonian $H = H(\vec{q}, \vec{p}, t)$

$$\frac{\partial H}{\partial p_i} = \frac{\partial \tilde{H}}{\partial p_i}\,, \qquad \frac{\partial H}{\partial q_i} = \frac{\partial \tilde{H}}{\partial q_i}\,, \qquad \frac{\partial H}{\partial t} = \frac{\partial \tilde{H}}{\partial t}\,, \qquad \frac{\partial \tilde{H}}{\partial \mathcal{H}} = -1 \tag{8}$$

if we identify $t(s) = s$. With the Hamiltonian (7), we conclude from Eqs. (8) that the canonical variables $\vec{q}$ and $\vec{p}$ satisfy the original equations of motion. Then, the extension of the phase-space has no effect on the description of the system's dynamics. This means that the extended phase-space formalism can be applied to convert the description of a time-dependent Hamiltonian system to that of an autonomous system with two additional degrees of freedom.

Moreover, the extended phase-space formulation has the benefit to allow for more general canonical transformations that also include a transformation of time

$$(\vec{q}, \vec{p}, t, -H) \xrightarrow{\text{canon.transf.}} (\vec{q}\,', \vec{p}\,', t', -H')\,. \tag{9}$$

The transformation (9) is referred to as canonical if and only if Hamilton's variation principle (3) is maintained in the new (primed) set of canonical variables. The condition for an extended phase-space transformation (9) to be canonical can therefore be derived directly from Hamilton's principle in the form of Eq. (3)

$$\delta \int_{s_1}^{s_2} \left[\sum_{i=1}^{n} p_i \frac{\mathrm{d}q_i}{\mathrm{d}s} - H \frac{\mathrm{d}t}{\mathrm{d}s} \right] \mathrm{d}s = \delta \int_{s_1}^{s_2} \left[\sum_{i=1}^{n} p_i' \frac{\mathrm{d}q_i'}{\mathrm{d}s} - H' \frac{\mathrm{d}t'}{\mathrm{d}s} \right] \mathrm{d}s = 0\,.$$

This means that the integrands of the variation integrals may differ at most by a total differential in the extended phase-space

$$\sum_{i=1}^{n} p_i \,\mathrm{d}q_i - H\,\mathrm{d}t = \sum_{i=1}^{n} p_i' \,\mathrm{d}q_i' - H'\,\mathrm{d}t' + \mathrm{d}F_1(\vec{q}, \vec{q}\,', t, t')\,. \tag{10}$$

The function $F_1(\vec{q}, \vec{q}\,', t, t')$ is commonly referred to as the "generating function" of the canonical transformation. With the help of the Legendre transformation

$$F_2(\vec{q}, \vec{p}\,', t, H') = F_1(\vec{q}, \vec{q}\,', t, t') + \sum_{i=1}^{n} q_i' p_i' - t' H' ,$$

the generating function may be expressed equivalently in terms of the old configuration space and the new momentum coordinates. If we compare the coefficients pertaining to the respective differentials $\mathrm{d}q_i$, $\mathrm{d}p_i'$, $\mathrm{d}t$, and $\mathrm{d}H'$, we find the following coordinate transformation rules to apply for each index $i = 1, \dots, n$:

$$q_i' = \frac{\partial F_2}{\partial p_i'} , \qquad p_i = \frac{\partial F_2}{\partial q_i} , \qquad t' = -\frac{\partial F_2}{\partial H'} , \qquad H = -\frac{\partial F_2}{\partial t} . \tag{11}$$

In the following section, we will investigate the general properties of infinitesimal canonical transformations in the extended phase-space. In Section 5, the rules (11) will be applied to carry out a *finite* canonical transformation for a particular class of Hamiltonian systems.

3 Time-dependent symmetry mapping and corresponding infinitesimal canonical transformation

In the extended phase-space, the generating function F_2 of an infinitesimal canonical transformation is given by

$$F_2(\vec{q}, \vec{p}\,', t, H') = \sum_{i=1}^{n} q_i p_i' - tH' - \varepsilon G(\vec{q}, \vec{p}, t, H) , \tag{12}$$

with ε an infinitesimal parameter, and $G(\vec{q}, \vec{p}, t, H)$ the function that characterizes the deviation of the canonical transformation from the identity. Since the transformation generated by Eq. (12) is infinitesimal, to first order in ε the old (unprimed) canonical variables can be used calculating the derivatives of $G(\vec{q}, \vec{p}, t, H)$. From the transformation rules (11), we thus find

$$q_i' = q_i - \varepsilon \frac{\partial G}{\partial p_i} , \quad t' = t + \varepsilon \frac{\partial G}{\partial H} , \quad p_i' = p_i + \varepsilon \frac{\partial G}{\partial q_i} , \quad H' = H - \varepsilon \frac{\partial G}{\partial t} . \tag{13}$$

The variation of the function $G(\vec{q}, \vec{p}, t, H)$ is given by

$$\delta G = \sum_{i=1}^{n} \left(\frac{\partial G}{\partial q_i} \delta q_i + \frac{\partial G}{\partial p_i} \delta p_i \right) + \frac{\partial G}{\partial t} \delta t + \frac{\partial G}{\partial H} \delta H .$$

Inserting Eqs. (13), we get

$$\delta G = \varepsilon \left\{ \sum_{i=1}^{n} \left(-\frac{\partial G}{\partial q_i} \frac{\partial G}{\partial p_i} + \frac{\partial G}{\partial p_i} \frac{\partial G}{\partial q_i} \right) + \frac{\partial G}{\partial t} \frac{\partial G}{\partial H} - \frac{\partial G}{\partial H} \frac{\partial G}{\partial t} \right\} = 0 , \tag{14}$$

which means that $G(\vec{q}, t, \vec{p}, H)$ remains invariant by virtue of the canonical transformation. In other words, the infinitesimal part of Eq. (12) itself provides the quantity that is conserved performing the canonical transformation generated by (12).

We now define a general time-dependent infinitesimal symmetry mapping that is supposed to be consistent with Eq. (13)

$$t' = t + \delta t = t + \varepsilon\xi(t) \tag{15}$$

$$q_i'(t') = q_i(t) + \delta q_i = q_i(t) + \varepsilon\eta_i(\vec{q}, \vec{p}, t) \tag{16}$$

$$p_i'(t') = p_i(t) + \delta p_i = p_i(t) + \varepsilon\pi_i(\vec{q}, \vec{p}, t)\,. \tag{17}$$

Comparing the partial derivatives of the function $G(\vec{q}, \vec{p}, t, H)$, as given by Eq. (13), with the ansatz (15), (16), and (17) for the the time-dependent symmetry mapping, we find

$$\frac{\partial G}{\partial q_i} = \pi_i\,, \qquad \frac{\partial G}{\partial p_i} = -\eta_i\,, \qquad \frac{\partial G}{\partial H} = \xi(t)\,. \tag{18}$$

From the yet unused last expression of Eq. (13), we derive directly the condition for the partial time derivative of $G(\vec{q}, \vec{p}, t, H)$

$$H' - H = \delta H = \frac{\partial H}{\partial t}\,\delta t + \sum_{i=1}^{n}\left(\frac{\partial H}{\partial q_i}\,\delta q_i + \frac{\partial H}{\partial p_i}\,\delta p_i\right),$$

which reads with Eqs. (15), (16), and (17)

$$\frac{\partial G}{\partial t} = -\frac{\partial H}{\partial t}\,\xi(t) - \sum_{i=1}^{n}\frac{\partial H}{\partial q_i}\,\eta_i - \sum_{i=1}^{n}\frac{\partial H}{\partial p_i}\,\pi_i\,. \tag{19}$$

A necessary and sufficient condition for transformations to be canonical is to conserve the Poisson bracket $[u, v]$ of two arbitrary differentiable functions u, v. This requirement imposes conditions on the functions ξ, η_i, and π_i that ensure the canonical properties of Eqs. (15), (16), and (17). For the Poisson bracket of the pair of canonical variables $(t, -H)$, we have $[t, -H] = 1$, hence

$$[t', -H'] = (1 + \varepsilon\dot{\xi}(t))\left(1 - \varepsilon\,\frac{\partial^2 G}{\partial t\,\partial H}\right) \stackrel{!}{=} 1\,.$$

To first order in ε, this means

$$\frac{d\xi}{dt} = \frac{\partial}{\partial t}\left(\frac{\partial G}{\partial H}\right) \quad \text{or} \quad \frac{\partial G}{\partial H} = \xi(t)\,,$$

in agreement with the last expression in Eq. (18). In the same way, the Poisson bracket condition $[q_i, p_i] = 1 = [q_i', p_i']$ leads to the relation

$$[q_i', p_i'ht] = 1 + \varepsilon\left(\frac{\partial\eta_i}{\partial q_i} + \frac{\partial\pi_i}{\partial p_i}\right) + \varepsilon^2[\eta_i, \pi_i] \stackrel{!}{=} 1\,,$$

which is fulfilled to first order in ε if

$$\frac{\partial\eta_i}{\partial q_i} = -\frac{\partial\pi_i}{\partial p_i}\,. \tag{20}$$

The quantities δq_i and δp_i in Eqs. (16) and (17) stand for the variation of the canonical variables as a result of the canonical transformation at *different* instants of time. In order to separate the time shift from the coordinate transformation rules, we split Eqs. (16) and (17) into the coordinate transformation part at fixed time t', and the time shift part according to

$$\delta q_i = \varepsilon\eta_i = [q_i'(t') - q_i(t')] + [q_i(t') - q_i(t)]$$
$$\delta p_i = \varepsilon\pi_i = [p_i'(t') - p_i(t')] + [p_i(t') - p_i(t)] \,.$$

Since we are dealing with an *infinitesimal* transformation, the right brackets may be identified without loss of generality with the first order term of a Taylor series

$$q_i(t') - q_i(t) = \dot{q}_i\,\delta t\,, \qquad p_i(t') - p_i(t) = \dot{p}_i\,\delta t\,.$$

Solving for the terms in the left brackets, we finally have in conjunction with Eq. (15)

$$q_i'(t') - q_i(t') = \varepsilon\big(\eta_i - \xi(t)\,\dot{q}_i\big) \tag{21}$$
$$p_i'(t') - p_i(t') = \varepsilon\,(\pi_i - \xi(t)\,\dot{p}_i)\,. \tag{22}$$

With the requirement that the functions $\eta_i = \eta_i(\vec{q}, \vec{p}, t)$ and $\pi_i = \pi_i(\vec{q}, \vec{p}, t)$ fulfill condition (20), the coupled set of Eqs. (15), (21), and (22) provides the general form of the transformation rules for infinitesimal canonical transformations in the extended phase-space. In the following section, we will specialize this general transformation for a specific class of Hamiltonian systems. The invariant G can then be worked out in explicit form making use of Eq. (18). The conditional equation for $\xi(t)$ is finally established by Eq. (19).

4 Invariant for a class of Hamiltonian systems

We now consider a specific class of Hamiltonian systems, namely an n-degree-of-freedom system of particles moving in an explicitly time-dependent potential $V(\vec{q}, t)$ with time-dependent damping forces proportional to the velocity. A system of this class is described by the Hamiltonian

$$H = \tfrac{1}{2}\,\mathrm{e}^{-F(t)} \sum_{i=1}^{n} p_i^2 + \mathrm{e}^{F(t)}\,V(\vec{q}, t) \quad \text{with} \quad F(t) = \int_{t_0}^{t} f(\tau)\,\mathrm{d}\tau\,, \tag{23}$$

that provides the canonical equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = p_i\,\mathrm{e}^{-F(t)}\,, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} = -\frac{\partial V}{\partial q_i}\,\mathrm{e}^{F(t)}\,, \tag{24}$$

hence the equation of motion

$$\ddot{q}_i + f(t)\,\dot{q}_i + \frac{\partial V(\vec{q}, t)}{\partial q_i} = 0\,. \tag{24a}$$

In the following, we work out the invariant G of the Hamiltonian system (23) that corresponds to the symmetry mapping (15), (21), and (22) containing the more specific functions $\eta_i = \eta_i(q_i, t)$ and $\pi_i = \pi_i(q_i, p_i, t)$. We choose the connection between Eqs. (21) and (22) to be established by the first canonical equation of (24)

$$p_i'(t') - p_i(t') = \mathrm{e}^{F(t)} \frac{\mathrm{d}}{\mathrm{d}t} (q_i' - q_i)|_{t'} = \varepsilon\, \mathrm{e}^{F(t)} \frac{\mathrm{d}}{\mathrm{d}t} (\eta_i - \xi(t)\, \dot{q}_i)\,.$$

Hereby, we determine the invariant G to represent a conserved quantity along the system's evolution in time. With Eq. (22) and the first canonical equation of (24), we then obtain for π_i

$$\pi_i(q_i, p_i, t) = \left(\frac{\partial \eta_i}{\partial q_i} - \dot{\xi}(t) + \xi(t)\, f(t)\right) p_i + \frac{\partial \eta_i}{\partial t}\, \mathrm{e}^{F(t)}\,. \tag{25}$$

The function $\eta_i(q_i, t)$ can now be determined from Eq. (25) with the help of Eq. (20)

$$\frac{\partial \eta_i}{\partial q_i} = \frac{1}{2}\, \dot{\xi}(t) - \frac{1}{2}\, \xi(t)\, f(t)\,, \tag{26}$$

which can be integrated to give

$$\eta_i = \tfrac{1}{2}\, [\dot{\xi}(t) - \xi(t)\, f(t)]\, q_i + \psi_i(t)\,. \tag{27}$$

Herein, $\psi_i(t)$ denotes an arbitrary function of time only. Inserting Eq. (26) and the partial time derivative of Eq. (27) into Eq. (25), we eliminate its dependence on η_i

$$\pi_i(q_i, p_i, t) = -\tfrac{1}{2}\, p_i\, (\dot{\xi} - \xi f) + \tfrac{1}{2}\, q_i\, \mathrm{e}^{F(t)}\, (\ddot{\xi} - \dot{\xi} f - \xi \dot{f}) + \dot{\psi}_i\, \mathrm{e}^{F(t)}\,. \tag{28}$$

Now that η_i and π_i are specified by Eqs. (27) and (28), respectively, the invariant $G(\vec{q}, \vec{p}, t, H)$ can be deduced from its partial derivatives (18)

$$\begin{aligned} G(\vec{q}, \vec{p}, t, H) = \xi H - \tfrac{1}{2}\, (\dot{\xi} - \xi f) \sum_{i=1}^{n} q_i p_i + \tfrac{1}{4}\, \mathrm{e}^{F(t)}\, (\ddot{\xi} - \dot{\xi} f - \xi \dot{f}) \sum_{i=1}^{n} q_i^2 \\ + \sum_{i=1}^{n} (\dot{\psi}_i q_i\, \mathrm{e}^{F(t)} - \psi_i p_i)\,. \end{aligned} \tag{29}$$

The functions $\xi(t)$ and $\psi_i(t)$ are determined by the condition (19) for $G(\vec{q}, \vec{p}, t, H)$ to yield an invariant. Calculating the partial time derivative of Eq. (29) and making use of the explicit form of the Hamiltonian (23), Eq. (19) leads to the following linear differential equations for $\xi(t)$ and $\psi_i(t)$

$$\begin{aligned} \dddot{\xi} \sum_{i=1}^{n} q_i^2 + 4\dot{\xi} \left[V(\vec{q}, t) + \frac{1}{2} \sum_{i=1}^{n} q_i \frac{\partial V}{\partial q_i} - \frac{1}{2} \left(\dot{f} + \frac{1}{2} f^2\right) \sum_{i=1}^{n} q_i^2\right] \\ + 4\xi \left[\frac{\partial V}{\partial t} + f \left(V(\vec{q}, t) - \frac{1}{2} \sum_{i=1}^{n} q_i \frac{\partial V}{\partial q_i}\right) - \frac{1}{4}\, (\ddot{f} + f \dot{f}) \sum_{i=1}^{n} q_i^2\right] = 0\,, \end{aligned} \tag{30}$$

$$\ddot{\psi}_i q_i - \psi_i \ddot{q}_i + (\dot{\psi}_i q_i - \psi_i \dot{q}_i)\, f = 0\,, \qquad i = 1, 2, \ldots, n\,. \tag{31}$$

Since the $\psi_i(t)$ are arbitrary functions that do not depend on $\xi(t)$, the respective expressions must vanish separately. We thus obtain distinct differential equations for $\xi(t)$ and $\psi_i(t)$, as given by Eqs. (30) and (31). It turns out that the total time derivative of the ψ_i-terms in Eq. (29) vanish because of Eq. (31). Consequently, the related sum in Eq. (29) provides a separate invariant

$$\sum_{i=1}^{n} \left(\dot{\psi}_i(t)\, q_i\, \mathrm{e}^{F(t)} - \psi_i(t)\, p_i\right) = \tilde{I} = \text{const.},$$

which means that the invariant G can be written as a sum of two invariants

$$G(\vec{q}, \vec{p}, t, H) = I + \tilde{I}\,.$$

The terms associated with the function $\xi(t)$ thus form the invariant I

$$I = \xi H - \tfrac{1}{2}\,(\dot{\xi} - \xi f) \sum_{i=1}^{n} q_i p_i + \tfrac{1}{4}\,\mathrm{e}^{F(t)}\,(\ddot{\xi} - \dot{\xi} f - \xi \dot{f}) \sum_{i=1}^{n} q_i^2\,. \tag{32}$$

A discussion of the physical meaning of the invariant (32) and the solution $\xi(t)$ of the third-order equation (30) will be the subject of the following two sections.

5 Canonical transformation to the equivalent autonomous system

We will show in the following that the invariant I of Eq. (32) can be regarded as the Hamiltonian H' of an autonomous system, which is related to the Hamiltonian H of the original system by a finite canonical transformation in the extended phase-space. Since this transformation is unique, we may conceive the transformed system described by H' as the *equivalent* autonomous system of H.

The Hamiltonian $H(\vec{q}, \vec{p}, t)$ of Eq. (23) will be transformed by means of a canonical transformation into the new Hamiltonian

$$H'(\vec{q}\,', \vec{p}\,') = \tfrac{1}{2} \sum_{i=1}^{n} p_i'^2 + V'(\vec{q}\,')\,, \tag{33}$$

which is supposed to be independent of time explicitly. The canonical transformation in the extended phase-space be generated by

$$F_2(\vec{q}, \vec{p}\,', t, H') = \phi_2(\vec{q}, \vec{p}\,', t) - H' \int_{t_0}^{t} \frac{\mathrm{d}\tau}{\xi(\tau)}\,. \tag{34}$$

The function $\phi_2(\vec{q}, \vec{p}\,', t)$ contained herein be defined as the following generating function in the usual (non-extended) phase-space

$$\phi_2(\vec{q}, \vec{p}\,', t) = \sqrt{\frac{\mathrm{e}^{F(t)}}{\xi(t)}} \sum_{i=1}^{n} q_i p_i' + \frac{1}{4}\,\mathrm{e}^{F(t)} \left(\frac{\dot{\xi}(t)}{\xi(t)} - f(t)\right) \sum_{i=1}^{n} q_i^2\,. \tag{35}$$

For the moment, $\xi(t)$ may be regarded as an arbitrary differentiable function of time only. Working out the transformation rules (11) for the specific generating function F_2, as defined by Eq. (34), we find the following linear transformation between the old $\{q_i, p_i\}$ and the new set of coordinates $\{q_i', p_i'\}$

$$\begin{pmatrix} q_i \\ p_i \end{pmatrix} = \begin{pmatrix} \sqrt{\xi(t)/\mathrm{e}^{F(t)}} & 0 \\ \tfrac{1}{2}(\dot{\xi} - \xi f)\,\sqrt{\mathrm{e}^{F(t)}/\xi(t)} & \sqrt{\mathrm{e}^{F(t)}/\xi(t)} \end{pmatrix} \begin{pmatrix} q_i' \\ p_i' \end{pmatrix}. \tag{36}$$

Furthermore, the transformations of time t and Hamiltonian H are given by

$$t' = -\frac{\partial F_2}{\partial H'} = \int_{t_0}^{t} \frac{d\tau}{\xi(\tau)}\,, \qquad H = -\frac{\partial F_2}{\partial t} = -\frac{\partial \phi_2}{\partial t} + \frac{H'}{\xi(t)}\,. \tag{37}$$

From Eq. (37), the new Hamiltonian H' follows as

$$H' = \xi(t)\left(H + \frac{\partial \phi_2}{\partial t}\right). \tag{38}$$

The transformed Hamiltonian H' of Eq. (38) is obtained in the desired form of Eq. (33) if the old Hamiltonian H as well as the partial time derivative of Eq. (35) are expressed in terms of the new (primed) coordinates. Explicitly, the effective potential $V'(\vec{q}\,')$ of the transformed system evaluates to

$$V'(\vec{q}\,') = \tfrac{1}{4}\left[\ddot{\xi}\xi - \tfrac{1}{2}\dot{\xi}^2 - \xi^2\left(\dot{f} + \tfrac{1}{2}f^2\right)\right]\sum_{i=1}^{n} q_i'^2 + \xi\, e^{F(t)}\, V\left(\sqrt{\xi\, e^{-F}}\,\vec{q}\,', t\right). \tag{39}$$

The new potential V' consists of two components, namely a term related to the original potential V, and an additional quadratic potential that describes the linear forces of inertia occurring due to the time-dependent linear transformation (36) to a new frame of reference.

The required property of the new Hamiltonian (33) to describe an autonomous system is met if and only if the new potential $V'(\vec{q}\,')$ does not depend on time t explicitly. This means that the initially arbitrary function $\xi(t)$ – defined in the generating function (34) – is now tailored to eliminate an explicit time-dependence of V' at the location $\vec{q}\,'$. In order to set up the appropriate conditional equation for $\xi(t)$, we calculate the partial time derivative of Eq. (39)

$$\begin{aligned}\frac{\partial V'(\vec{q}\,')}{\partial t'} = \frac{1}{4}\,\xi\, e^{F}\Bigg\{&\left(\dddot{\xi} - 2\dot{\xi}\dot{f} - \xi\ddot{f} - \dot{\xi}f^2 - \xi f\dot{f}\right)\xi\, e^{-F}\sum_{i=1}^{n} q_i'^2 \\ &+ 4\dot{\xi}\left(V + \frac{1}{2}\sum_{i=1}^{n} q_i'\,\frac{\partial V}{\partial q_i'}\right) + 4\xi\left(\frac{\partial V}{\partial t} + fV - \frac{1}{2}f\sum_{i=1}^{n} q_i'\,\frac{\partial V}{\partial q_i'}\right)\Bigg\} = 0\,.\end{aligned} \tag{40}$$

Inserting the transformation rules that follow from Eq. (36)

$$q_i'^2 = \frac{e^F}{\xi}\, q_i^2\,, \qquad q_i'\,\frac{\partial V}{\partial q_i'} = q_i\,\frac{\partial V}{\partial q_i}\,,$$

we observe that Eq. (40) agrees with the auxiliary equation (30) for $\xi(t)$, derived in Section 4 from condition (19). With $\xi(t)$ a solution of Eq. (40), we thus have

$$\frac{\partial V'(\vec{q}\,')}{\partial t'} = 0\,,$$

which means that the Hamiltonian

$$H'(\vec{q}\,', \vec{p}\,') = \tfrac{1}{2}\sum_{i=1}^{n} p_i'^2 + V'(\vec{q}\,') = I \tag{41}$$

does not depend on time explicitly, hence constitutes indeed the invariant I in question. The Hamiltonian H' of Eq. (41) may be expressed as well in terms of the old coordinates $\{q_i, p_i\}$. We then encounter the invariant I in the form of Eq. (32), which was derived in Section 4 on the basis of an infinitesimal canonical transformation. The effect of the finite canonical transformation generated by Eq. (34) can be summarized as a transfer of the time-dependence from the original Hamiltonian $H(\vec{q}, \vec{p}, t)$ into the time-dependence of the frame of reference in the extended phase-space of the new Hamiltonian $H'(\vec{q}', \vec{p}')$. In other words, the autonomous system's Hamiltonian H' is *canonically equivalent* to the initial time-dependent Hamiltonian system H.

For $\xi(t) > 0$, the Hamiltonian H' represents a real physical system. Because of the uniqueness of the transformation rules (36) and (37), the Hamiltonian system H' may then be conceived as the autonomous system that is *physically equivalent* to the initial system described by H.

The instants of time t with $\xi(t) = 0$ mark the singular points where this transformation does not exist. For time intervals with $\xi(t) < 0$, the elements of coordinate transformation matrix (36) become imaginary, and – according to Eq. (37) – the flow of the transformed time t' with respect to t becomes negative. Under these circumstances, the transformed system does not possess a physical meaning anymore, which means that the equivalent autonomous system ceases to exist as a physical system. In other words, the particle motion within the time-dependent non-linear system can no longer be expressed as the linearly transformed motion within an autonomous system.

On the other hand, the invariant I in the representation of Eq. (32) exists for all $\xi(t)$ that are solutions of Eq. (30), including the regions with $\xi(t) \leq 0$. We may regard Eq. (32) as an implicit function $I = I(\vec{q}, \vec{p}, t)$ of the phase-space coordinates, visualized as a time-varying $(2n-1)$-dimensional surface in the $2n$-dimensional phase-space. In Section 7.2, we will show for a one-dimensional example that the times t with $\xi(t) = 0$ mark the transition points between different topologies of this phase-space surface.

To end this section, we finally note that a canonical transformation that maps an explicitly time-dependent Hamiltonian H into a time-independent H' can equivalently be formulated in the conventional (non-extended) phase-space. Presenting a two-step linear canonical transformation, this has been demonstrated for the damped *linear* oscillator by Leach [2]. A similar transformation that is followed by a rescaling of time [15] has been shown capable to work out invariants for more general cases of time-dependent *non-linear* Hamiltonian systems.

6 Discussion

The invariant (32) is easily shown to represent a time integral of Eq. (30) by calculating the total time derivative of Eq. (32), and inserting the canonical equations (24). Hence, Eq. (32) provides a conserved quantity exactly along the phase-space trajectory that represents the system's time evolution. This trajectory is given as a solution of the $2n$ first-order canonical equations (24) or, equivalently, of the n second-order equations of motion (24a). With these solutions, Eq. (30) must not be conceived as a conditional equation for the potential $V(\vec{q}, t)$. Rather, all q_i-dependent terms of Eq. (30) are in fact functions of the parameter t only – given along the system's phase-space trajectory. Correspondingly, the potential $V(\vec{q}(t), t)$ and its derivatives in Eq. (30) are time-dependent coefficients of an ordinary third-order differential equation for $\xi(t)$. According to the existence and uniqueness theorem for linear ordinary differential equations, a unique solution function $\xi(t)$ of Eq. (30) exists – and consequently the invariant I – if V and its partial derivatives are continuous along the system's phase-space path.

Vice versa, Eq. (30) together with the side-condition $I = \text{const.}$ from Eq. (32) may be conceived as a conditional equation for a potential $V(\vec{q}, t)$ that is consistent with a solution of the equations of motion (24). In other words, the invariant $I = \text{const.}$ in conjunction with the third-order equation (30) must imply the canonical equations (24). This can be shown inserting Eq. (30) into the total time derivative of Eq. (32). Since $\mathrm{d}I/\mathrm{d}t \equiv 0$ must hold for all solutions $\xi(t)$ of Eq. (30), the respective sums of terms proportional to $\dddot{\xi}(t)$, $\dot{\xi}(t)$, and $\xi(t)$ must vanish separately. For the terms proportional to $\dddot{\xi}(t)$, this means

$$\tfrac{1}{2}\, \ddot{\xi}\; \mathrm{e}^{F(t)} \sum_{i=1}^{n} q_i(\dot{q}_i - p_i\, \mathrm{e}^{-F}) \equiv 0\,. \tag{42}$$

The identity (42) must be fulfilled for *all* initial conditions $(\vec{q}(0), \vec{p}(0))$ and resulting phase-space trajectories $(\vec{q}(t), \vec{p}(t))$ of the underlying dynamical system. Consequently, the expression in parentheses must vanish separately for each index i, thereby establishing the first canonical Eq. (24). The remaining terms impose the following condition for a vanishing total time derivative of I

$$\mathrm{e}^{-F(t)} \sum_{i=1}^{n} \left(\xi p_i - \frac{1}{2}\, \mathrm{e}^{F}\, (\dot{\xi} - \xi f\right)\, q_i)\, \left(\dot{p}_i + \frac{\partial V}{\partial q_i}\, \mathrm{e}^{F}\right) \equiv 0\,.$$

Again, this sum must vanish for all $\xi(t)$ that are solution of Eq. (30). This can be accomplished only if the rightmost expression in parentheses, hence the second canonical Eq. (24), is fulfilled for each index $i = 1, \dots, n$. Summarizing, we may state that the triple made up by the canonical Eq. (24), the auxiliary Eq. (30), and the invariant I of Eq. (32) forms a logical triangle: if two equations are given at a time, the third can be deduced.

The physical interpretation of the invariant (32) can be worked out considering the total time derivative of the Hamiltonian (23). Making use of the canonical equations (24), we find

$$\frac{\mathrm{d}H}{\mathrm{d}t} + \frac{1}{2}\, f\, \mathrm{e}^{-F} \sum_{i=1}^{n} p_i^2 - \mathrm{e}^{F}\left(fV(\vec{q}, t) + \frac{\partial V}{\partial t}\right) = 0\,, \tag{43}$$

which represents just the explicit form of the general theorem $\mathrm{d}H/\mathrm{d}t = \partial H/\partial t$ for the Hamiltonian (23). Equation (43) can be interpreted as an energy balance relation, stating that the system's total energy change $\mathrm{d}H/\mathrm{d}t$ is quantified by the dissipation and the explicit time-dependence of the external potential. Inserting $\partial V/\partial t$ from the auxiliary Eq. (30) into Eq. (43), the sum over all terms can be written as the total time derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\xi H-\frac{1}{2}\,(\dot{\xi}-\xi f)\sum_{i=1}^{n}q_i p_i+\frac{1}{4}\,\mathrm{e}^{F}\,(\ddot{\xi}-\dot{\xi}f-\xi\dot{f})\sum_{i=1}^{n}q_i^2\right]=0\,.$$

The expression in brackets coincides with the invariant (32). As the function $\xi(t)$ is the solution of a homogeneous linear differential Eq. (30), it is determined up to an arbitrary factor. We are, therefore, free to conceive $\xi(t)$ as a dimensionless quantity.

With the initial conditions $\xi(0)=1$, $\dot{\xi}(0)=\ddot{\xi}(0)=0$ for the auxiliary Eq. (30), the invariant I can now be interpreted as the conserved *initial* energy H_0 for a non-autonomous system described by the Hamiltonian (23), comprising both the system's time-varying energy content H and the energy fed into or detracted from the system.

The meaning of $\xi(t)$ follows directly from the invariant (32) if the Hamiltonian H is treated formally as an independent variable $I=I(\vec{q},\vec{p},t,H)$. A vanishing total time derivative of the invariant I then writes

$$\frac{\mathrm{d}I}{\mathrm{d}t}=\left.\frac{\partial I}{\partial t}\right|_{\vec{q},\vec{p},H}+\frac{\partial H}{\partial t}\left.\frac{\partial I}{\partial H}\right|_{\vec{q},\vec{p},t}+\sum_i\left(\dot{q}_i\left.\frac{\partial I}{\partial q_i}\right|_{\vec{p},t,H}+\dot{p}_i\left.\frac{\partial I}{\partial p_i}\right|_{\vec{q},t,H}\right)=0\,.$$

Inserting q_i and p_i from the canonical Eq. (24), and making again use of the auxiliary Eq. (30) to eliminate the third-order derivative $\dddot{\xi}(t)$, we find the expected result

$$\left.\frac{\partial I}{\partial H}\right|_{\vec{q},\vec{p},t}=\xi(t)\,. \tag{44}$$

$\xi(t)$ thus quantifies how much the total energy I varies with respect to a change of the actual system energy H at fixed phase-space location $(\vec{q},\vec{p})$ and time t.

Some additional remarks are appropriate at this point. In the calculation of δp_i, we used the relation between p_i and $\dot{q}_i$, given by the first canonical Eq. (24). A similar situation is encountered in the framework of the Lagrangian formalism, where the Lagrange function $L(\vec{q},\dot{\vec{q}},t)$ depends on $\dot{\vec{q}}$, i.e. the time derivative of the coordinates $\vec{q}(t)$. As will be shown in Appendix A, the invariant (29) and the corresponding conditional Eqs. (30) and (31) can indeed be derived likewise in the Lagrangian context using Noether's theorem.

For a vanishing damping coefficient $(f(t)\equiv F(t)\equiv 0)$, Eqs. (30) and (32) are consistent with corresponding expressions presented earlier [14], where a direct approach to construct invariants for time-dependent Hamiltonian systems has been applied. As has been shown in [15], the direct approach can also serve as the basis to derive invariants for dissipative non-linear systems. We will review this approach briefly in Appendix B.

Equations (30) and (32) can be shown to agree with the corresponding results of Ref. [15] if the functions $c(t)$ and $f_2(t)$ used there are expressed in terms of our actual notation according to $c(t)=\mathrm{e}^{-F(t)}$ and $f_2(t)=\xi(t)\,\mathrm{e}^{-F(t)}$.

We further note that for the special case $f(t)\equiv\partial V/\partial t\equiv 0$, hence for autonomous systems, $\xi(t)\equiv 1$ is always a solution of Eq. (30). For this $\xi(t)$, the invariant

(32) reduces to $I \equiv H$, hence provides the system's total energy, which is a known invariant for Hamiltonian systems with no explicit time-dependence. Nevertheless, Eq. (30) also allows for solutions $\xi(t) \neq$ const. for these systems. We thereby obtain other non-trivial invariants for autonomous systems that exist in addition to the invariant representing the energy conservation law. This will be demonstrated for the simple case of the harmonic oscillator at the end of Section 7.1.

For general potentials $V(\vec{q}, t)$, the dependence of Eq. (30) on $\vec{q}(t)$ cannot be eliminated. Under these circumstances, the function $\xi(t)$ can only be obtained integrating Eq. (30) *simultaneously* with the equations of motion (24a). For the isotropic quadratic potential with $d(t)$ denoting an arbitrary continuous function of time

$$V(\vec{q}, t) = d(t) \sum_{i=1}^{n} q_i^2 , \tag{45}$$

Equation (30) may be divided by $\sum_i q_i^2$ and hereby strip its dependence on the single particle trajectories $q_i(t)$. Only for the particular linear system associated with Eq. (45), the third-order differential Eq. (30) for $\xi(t)$ decouples from the equations of motion (24a). Then, the solution functions $\xi(t)$, $\dot{\xi}(t)$, and $\ddot{\xi}(t)$ apply to all trajectories that follow from the equations of motion (24a) with Eq. (45). For the potential (45), Eq. (30) may be integrated to yield a non-linear second-order equation for $\xi(t)$

$$\ddot{\xi}\xi - \tfrac{1}{2}\,\dot{\xi}^2 + \xi^2[4d - \dot{f} - \tfrac{1}{2}f^2] = c\,, \tag{46}$$

$c =$ const. denoting the integration constant. With the help of Eq. (46), we may eliminate $\ddot{\xi}(t)$ in the expression (32) for the invariant I. After reordering, Eq. (32) may then be rewritten as

$$8\xi I = \mathrm{e}^{F(t)} \sum_{i=1}^{n} \left([2\xi\dot{q}_i - (\dot{\xi} - \xi f)\, q_i]^2 + 2cq_i^2 \right). \tag{47}$$

Obviously, for a positive integration constant $c > 0$, the quantities I and $\xi(t)$ must have the same sign. Thus, for $c > 0$, an invariant $I > 0$, and for the initial condition $\xi(0) > 0$, the function $\xi(t)$ remains non-negative at all times $t > 0$ for systems governed by the potential (45). On the other hand, $\xi(t)$ may change sign for general non-linear systems (23). As has been shown in Section 5, the condition $\xi(t) > 0$ provides a necessary criterion for the *physical* existence of an equivalent autonomous system of Eq. (23).

7 Examples

7.1 Time-dependent damped harmonic oscillator

As a simple example, we treat the well-known one-dimensional time-dependent harmonic oscillator with damping forces linear in the velocity. Its Hamiltonian is given by

$$H = \tfrac{1}{2}\,p^2\,\mathrm{e}^{-F(t)} + \tfrac{1}{2}\,\omega^2(t)\,q^2\,\mathrm{e}^{F(t)}\,, \tag{48}$$

which yields the equation of motion

$$\dot{q} = p\,\mathrm{e}^{-F(t)}\,, \qquad \ddot{q} + f(t)\,\dot{q} + \omega^2(t)\,q = 0\,. \tag{49}$$

Because of the quadratic structure of the potential $V(q,\,t) = \frac{1}{2}\,\omega^2(t)\,q^2$, the dependence of the auxiliary Eq. (30) on the particle coordinate q cancels. The differential equation for $\xi(t)$ thus simplifies to

$$\dddot{\xi} + \dot{\xi}(4\omega^2 - 2\dot{f} - f^2) + \xi(4\omega\dot{\omega} - \ddot{f} - f\dot{f}) = 0\,. \tag{50}$$

The invariant I is immediately found as the one-degree-of-freedom version of Eq. (32)

$$I = \xi H - \tfrac{1}{2}\,(\dot{\xi} - \xi f)\,qp + \tfrac{1}{4}\,\mathrm{e}^{F(t)}\,(\ddot{\xi} - \dot{\xi}f - \xi\dot{f})\,q^2\,, \tag{51}$$

with H standing for the Hamiltonian (48). For this particular case, the linear third-order equation (50) for $\xi(t)$ can be integrated once to yield a non-linear second-order equation

$$\xi\ddot{\xi} - \tfrac{1}{2}\,\dot{\xi}^2 + 2\xi^2(\omega^2 - \tfrac{1}{2}\,\dot{f} - \tfrac{1}{4}\,f^2) = 2c\,, \tag{52}$$

$c = \text{const.}$ denoting the integration constant. Using Eq. (52) to replace $\ddot{\xi}(t)$ in Eq. (51), we obtain the invariant I for the system (48) in the alternative form

$$I = \frac{c\,\mathrm{e}^F}{2\xi}\,q^2 + \frac{1}{2}\left(p\,\sqrt{\xi/\mathrm{e}^F} - \frac{1}{2}\,q[\dot{\xi} - \xi f]\,\sqrt{\mathrm{e}^F/\xi}\right)^2\,. \tag{51a}$$

As already concluded from Eq. (47), $\xi(t)$ cannot change sign if we define $c > 0$. The substitution $\xi(t) = \rho^2(t)$ then exists for real $\rho(t)$ at all times t, which means that Eqs. (52) and (51a) can be expressed equivalently in terms of $\rho(t)$. Setting $c = 1$, this leads to the auxiliary equation and the invariant in the form derived by Leach [2], who applied the method of linear canonical transformations with time-dependent coefficients.

The expression for the invariant (51) becomes particularly simple if expressed in terms of the coordinates of the canonically transformed system. Applying the related transformation rules (36), we find inserting Eq. (52)

$$I = \tfrac{1}{2}\,p'^2 + \tfrac{1}{2}\,cq'^2\,. \tag{53}$$

According to the canonical transformation theory of Section 5, the invariant I may regarded as the Hamiltonian H' of an autonomous system that is equivalent to (48)

$$H' = \tfrac{1}{2}\,p'^2 + V'(q')\,, \tag{54}$$

with the effective potential (39) in the transformed system simplifying to

$$V'(q') = \tfrac{1}{4}\,[\ddot{\xi}\xi - \tfrac{1}{2}\,\dot{\xi}^2 + 2\xi^2(\omega^2 - \tfrac{1}{2}\,\dot{f} - \tfrac{1}{4}\,f^2)]\,q'^2\,.$$

Comparing the expression in brackets with Eq. (52), we find that the new Hamiltonian (54) indeed agrees with the invariant (53). Figure 1 shows a special case of a numerical integration of the equation of motion (49). The results of the simultaneous numerical integration of the auxiliary Eq. (50) are included in this figure.

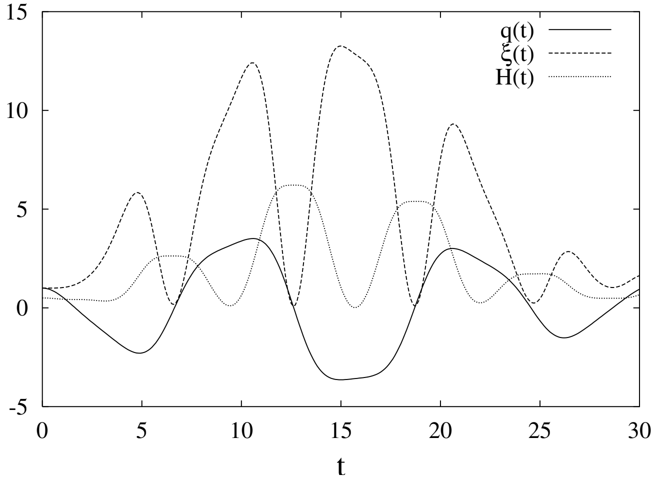

Fig. 1 Example of a simultaneous numerical integration of the equation of motion (49) and the auxiliary Eq. (50). In addition, $H(t)$ displays the actual system energy given by the Hamiltonian (48).

As coefficients of Eq. (49) we chose

$$\omega(t) = \cos(t/2)\,, \qquad f(t) = 1.8 \times 10^{-2} \sin(t/\pi)\,.$$

The initial conditions were set to $q(0) = 1$, $\dot{q}(0) = 0$, $\xi(0) = 1$, $\dot{\xi}(0) = 0$, and $\ddot{\xi}(0) = 0$. According to Eq. (51), we hereby define an invariant of $I = H(0) = 0.5$ for the sample particle.

In agreement with Eq. (47), the function $\xi(t)$ remains positive for this linear system. We furthermore observe that $\xi(t)$ is related to the energy transfer into the system according to Eq. (44): $\xi(t)$ becomes large for strong changes of the actual system energy $H(t)$ – and vice versa.

For the autonomous analog of Eq. (48), we have $\dot{\omega} \equiv f \equiv \dot{f} \equiv 0$, hence $\omega \equiv \omega_0$. The auxiliary Eq. (50) then further simplifies to

$$\dddot{\xi} + 4\omega_0^2 \dot{\xi} = 0\,. \tag{55}$$

Obviously, Eq. (55) has the special solution $\xi(t) \equiv 1$. The invariant (51) for this solution of Eq. (55) then coincides with the system's Hamiltonian $I \equiv H$, which represents the conserved total energy of the autonomous system. A further non-trivial invariant is obtained for the linear independent solution of Eq. (55) with $\xi(t) \neq \text{const.}$

$$\xi(t) = c_1 \cos 2\omega_0 t + c_2 \sin 2\omega_0 t\,.$$

Setting $c_1 = 1$ and $c_2 = 0$, this leads to the following representation of the second invariant for the time-independent harmonic oscillator

$$I = \tfrac{1}{2}\,(p^2 - \omega_0^2 q^2) \cos 2\omega_0 t + qp\omega_0 \sin 2\omega_0 t\,,$$

in agreement with the result obtained earlier by Lutzky [17]. Likewise, the third linear independent solution for $\xi(t)$ and the two solutions for $\psi(t)$ of Eq. (31) yield invariants. These five conserved quantities – of which only two are functionally independent – are correlated to the five-parameter Noether subgroup of the complete eight-parameter symmetry group SL(3, R) for the harmonic oscillator [17].

7.2 Time-dependent anharmonic undamped one-dimensional oscillator

As a second example, we investigate the one-dimensional non-linear system of a time-dependent anharmonic oscillator without damping, defined by the Hamiltonian

$$H = \tfrac{1}{2}\,p^2 + \tfrac{1}{2}\,\omega^2(t)\,q^2 + a(t)\,q^3 + b(t)\,q^4\,. \tag{56}$$

The associated equation of motion is given by

$$\dot{q} = p\,, \qquad \ddot{q} + \omega^2(t)\,q + 3a(t)\,q^2 + 4b(t)\,q^3 = 0\,. \tag{57}$$

Again, the invariant is immediately obtained writing the general invariant (32) for one dimension. With vanishing damping functions ($F(t) \equiv f(t) \equiv 0$), the invariant simplifies to

$$I = \tfrac{1}{2}\,\xi(t)\,[\dot{q}^2 + \omega^2(t)\,q^2 + 2a(t)\,q^3 + 2b(t)\,q^4] - \tfrac{1}{2}\,\dot{\xi}(t)\,q\dot{q} + \tfrac{1}{4}\,\ddot{\xi}(t)\,q^2\,. \tag{58}$$

For this particular case, the linear third-order equation for the auxiliary function $\xi(t)$ reads

$$\dddot{\xi} + 4\dot{\xi}\omega^2(t) + 4\xi\omega\dot{\omega} + 2q(t)\,[2\xi\dot{a} + 5\dot{\xi}a] + 4q^2(t)\,[\xi\dot{b} + 3\dot{\xi}b] = 0\,, \tag{59}$$

which follows from the general form of Eq. (30). We observe that – in contrast to the previous linear example – the particle trajectory $q = q(t)$ is explicitly contained in in the related auxiliary Eq. (59). Consequently, the integral function $\xi(t)$ can only be determined if Eq. (59) is integrated *simultaneously* with the equation of motion (57).

We may directly convince ourselves that I is indeed a conserved quantity. Calculating the total time derivative of Eq. (58), and inserting the equation of motion (57), we end up with Eq. (59), which is fulfilled by definition of $\xi(t)$ for the given trajectory $q = q(t)$.

The third-order differential Eq. (59) may be converted into a coupled set of first- and second-order equations. It is easily shown that the non-linear second-

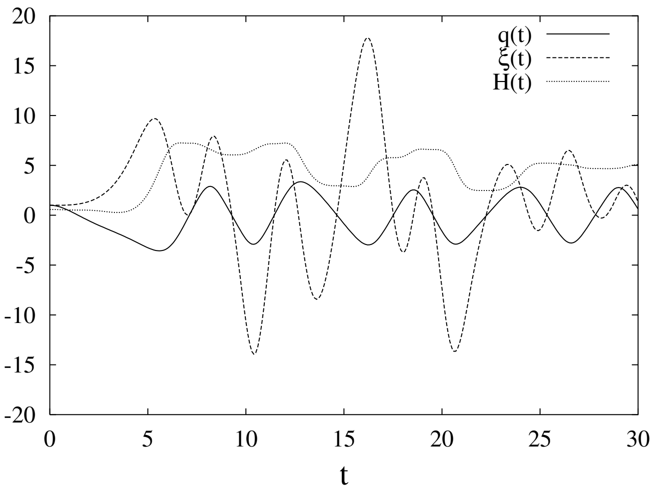

Fig. 2 Example of a simultaneous numerical integration of the equation of motion (57) and the coupled set (60), (61) for $\xi(t)$. In addition, $H(t)$ displays the time-dependent system energy given by the Hamiltonian (56).

order equation

$$\xi\ddot{\xi} - \tfrac{1}{2}\,\dot{\xi}^2 + 2\omega^2(t)\,\xi^2 = g(t) \tag{60}$$

is equivalent to Eq. (59), provided that the time derivative of the function $g(t)$, introduced in Eq. (60), is given by

$$\dot{g}(t) = -2q(t)\,\xi[2\xi\dot{a} + 5\dot{\xi}a] - 4q^2(t)\,\xi\big[\xi\dot{b} + 3\dot{\xi}b\big]\,. \tag{61}$$

With the help of the auxiliary equation in the form of Eq. (60), the invariant (58) may be expressed equivalently as

$$I = \frac{1}{2}\left[\xi\dot{q}^2 - \dot{\xi}q\dot{q} + \frac{\dot{\xi}^2}{4\xi}\,q^2 + 2\xi a q^3 + 2\xi b q^4\right] + \frac{g(t)}{4\xi}\,q^2\,. \tag{62}$$

In contrast to Eq. (59), the equivalent coupled set of equations (60) and (61) does not contain anymore the time derivative of the external function $\omega^2(t)$.

For the time-dependent harmonic oscillator $(a(t) = \dot{a}(t) = b(t) = \dot{b}(t) \equiv 0)$, Eq. (61) leads to $\dot{g}(t) = 0$, which means that $g(t) = g_0 = \text{const}$. For this particular case, g_0, and hence Eq. (60) no longer depends on the specific particle trajectory $q = q(t)$. Consequently, the solution function $\xi(t)$ applies to arbitrary trajectories.

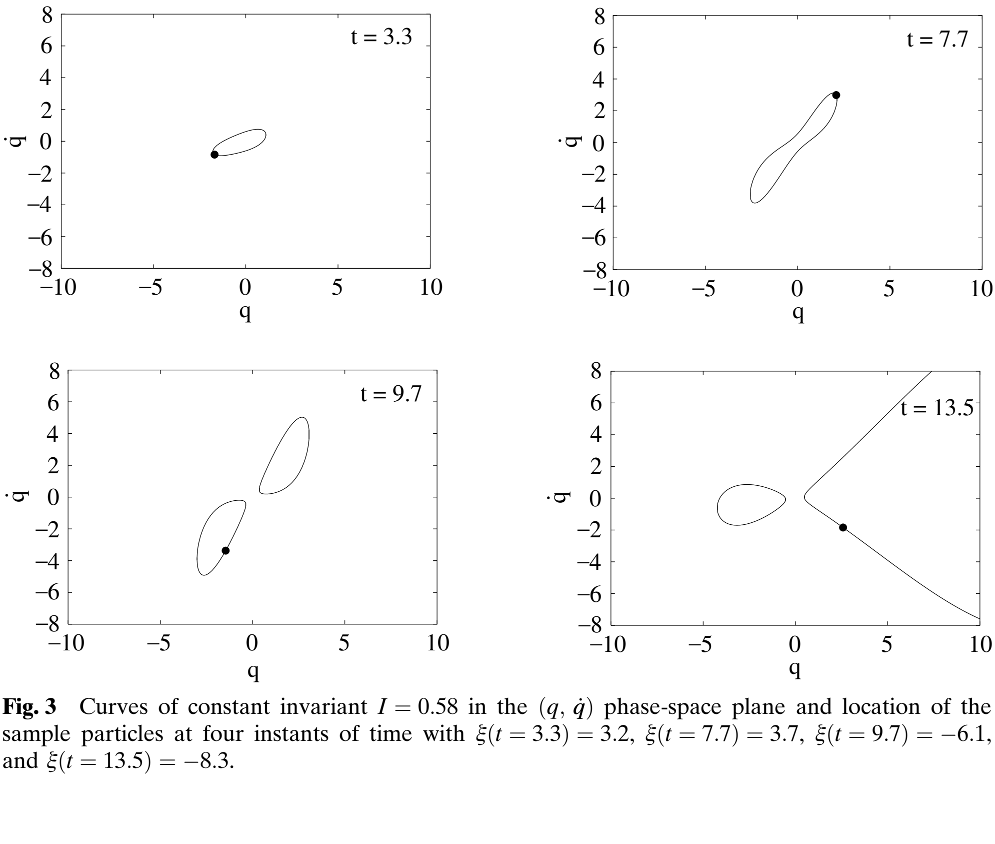

Fig. 3 Curves of constant invariant $I = 0.58$ in the $(q, \dot{q})$ phase-space plane and location of the sample particles at four instants of time with $\xi(t = 3.3) = 3.2$, $\xi(t = 7.7) = 3.7$, $\xi(t = 9.7) = -6.1$, and $\xi(t = 13.5) = -8.3$.

Setting $g_0 = 2$ and $\xi(t) = \rho^2(t)$ we obtain the well-known Lewis invariant [2]

$$I = \tfrac{1}{2}\left[\rho^{-2}q^2 + (\rho\dot{q} - \dot{\rho}q)^2\right].$$

For this linear case, we can rewrite Eq. (62) as

$$8I\xi(t) = \left[2\xi(t)\,\dot{q} - q\dot{\xi}(t)\right]^2 + 2q^2 g_0\,,$$

providing a special case of the general Eq. (47). For the initial condition $g_0 > 0$, and $I > 0$, we have $\xi(t) \geq 0$ for all $t > 0$. As a consequence, the linear canonical transformation (36) is regular for all times t, and the equivalent autonomous system represents a real physical system. In contrast, the function $g(t)$ is no longer a constant in the general non-linear case, and $\xi(t)$ may become negative. For $\xi(t) < 0$, the canonical transformation (36) becomes imaginary, which means that the equivalent system is no longer physical. Nevertheless, the invariant (58) exists as a real number for all solutions of the third-order differential equation (59), independently of the sign of $\xi(t)$.

Figure 2 shows a special case of a numerical integration of the equation of motion (57). Included in this figure, we see the result of the simultaneous numerical integration of Eqs. (60) and (61). The coefficients of Eq. (57) were chosen as

$$\omega(t) = \cos(t/2)\,,\quad a(t) = 5\times 10^{-2}\sin(t/3)\,,\quad b(t) = 8\times 10^{-2}\cos^2(t/3)\,.$$

The initial conditions were set to $q(0) = 1$, $\dot{q}(0) = 0$, $\xi(0) = 1$, $\dot{\xi}(0) = 0$, and $\ddot{\xi}(0) = 0$. According to Eq. (62), we hereby define an invariant of $I = H(0) = 0.58$ for the sample particle. Owing to the system's non-linear dynamics, $\xi(t)$ now becomes piecewise negative. Also, the relationship (44) between $\xi(t)$ and $H(t)$ appears more complicated compared to the linear case of Fig. 1. Interesting insight into the dynamical evolution of the sample particle can be obtained if the invariant (62) is regarded as an implicit representation of a phase-space curve $I = I(q, \dot{q}, t)$. Figure 3 displays snapshots of these curves at four different instants of time t. As expected, the particle lies exactly on these curves of constant I, thereby providing a numerical verification of Eq. (62). The two upper pictures display situations with $\xi(t) > 0$. Then, the phase-space curves are of closed elliptic type, being more or less deformed because of the non-linear terms in the Hamiltonian (56). When the function $\xi(t)$ becomes negative, as given for the lower two pictures, topological changes of the phase-space curves to more complex shapes are observed. For the special case of a conservative system, we have $\omega(t) = \omega_0 = \text{const.}$, $a(t) = a_0 = \text{const.}$, $b(t) = b_0 = \text{const.}$, and Eq. (59) reduces to

$$\dddot{\xi}(t) + \dot{\xi}(t)\left[4\omega_0^2 + 10q(t)\,a_0 + 12q^2(t)\,b_0\right] = 0\,. \tag{63}$$

Obviously, this equation has the special solution $\xi(t) \equiv 1$. For this case, Eq. (58) simplifies to

$$I = \tfrac{1}{2}\,\dot{q}^2 + \tfrac{1}{2}\,\omega_0^2 q^2 + a_0 q^3 + b_0 q^4 = H\,,$$

thus agrees with the system's Hamiltonian, which represents the conserved total energy. Another non-trivial invariant is obtained inserting a solution of the auxiliary Eq. (63) with $\xi(t) \neq \text{const.}$ into the expression for the invariant (58).

7.3 n-dimensional anisotropic oscillator with interaction

As the last example, we investigate the n-dimensional system of a time-dependent anisotropic oscillator with interaction between the particles. With an interaction potential $V(\vec{q})$ that depends on all configuration space variables, the Hamiltonian be defined as

$$H = \sum_{i=1}^{n} \left(\tfrac{1}{2}\, p_i^2 + \tfrac{1}{2}\, \omega_i^2(t)\, q_i^2\right) + V(\vec{q})\,. \tag{64}$$

The associated canonical equations follow as

$$\dot{q}_i = p_i\,, \qquad \dot{p}_i = -\omega_i^2(t)\, q_i - \frac{\partial V(\vec{q})}{\partial q_i}\,, \qquad i = 1, \ldots, n\,.$$

With H the Hamiltonian (64), the invariant is obtained from the general invariant (32) setting the damping functions to zero ($F(t) \equiv f(t) \equiv 0$)

$$I = \xi(t)\, H - \tfrac{1}{2}\, \dot{\xi}(t) \sum_{i=1}^{n} q_i p_i + \tfrac{1}{4}\, \ddot{\xi}(t) \sum_{i=1}^{n} q_i^2\,.$$

For this example, the auxiliary Eq. (30) for $\xi(t)$ specializes to

$$\sum_{i=1}^{n} \left(\dddot{\xi} + 4\dot{\xi}\omega_i^2(t) + 4\xi\omega_i\dot{\omega}_i\right) q_i^2(t) + 4\dot{\xi}\left(V(\vec{q}) + \frac{1}{2}\sum_{i=1}^{n} q_i\, \frac{\partial V}{\partial q_i}\right) = 0\,. \tag{65}$$

Defining $\omega^2(t)$ as the "average force function"

$$\omega^2(t) = \sum_{i=1}^{n} \omega_i^2(t)\, q_i^2 \Big/ \sum_{i=1}^{n} q_i^2\,,$$

the linear third-order auxiliary Eq. (65) may again be expressed equivalently in terms of a coupled set of a non-linear second-order equation for $\xi(t)$ and a first-order equation for $g(t)$

$$\begin{aligned} &\xi\ddot{\xi} - \tfrac{1}{2}\, \dot{\xi}^2 + 2\omega^2(t)\, \xi^2 = g(t) \\ &\dot{g}(t) = 4\left[\xi^2 \sum_{i=1}^{n} \left(\omega_i^2 - \omega^2\right) q_i p_i - \xi\dot{\xi}\left(V(\vec{q}) + \frac{1}{2}\sum_{i=1}^{n} q_i\, \frac{\partial V}{\partial q_i}\right)\right] \Big/ \sum_{i=1}^{n} q_i^2\,. \end{aligned} \tag{66}$$

We learn from Eq. (66) that $\dot{g}(t)$ is determined by two quantities of different physical nature: the interaction potential $V(\vec{q})$ and the system's anisotropy. In contrast to Eq. (52) of the one-dimensional example sketched in Section 7.1, the r.h.s. of the second-order equation for $\xi(t)$, i.e. the function $g(t)$, is generally *not* constant. Even in the linear case, which is obtained here for a vanishing interaction potential ($V(\vec{q}) \equiv 0$), the particle trajectories $q_i(t)$ are inevitably contained in the auxiliary equation (65). The dependence of Eq. (65) on the particle trajectories vanishes exclusively for the isotropic linear oscillator, hence for the particular case where $V(\vec{q}) \equiv 0$, and the oscillator frequencies agree in all degrees of freedom $\omega(t) \equiv \omega_i(t)$. With respect to the auxiliary function $\xi(t)$ – and hence the invariant I – an anisotropy thus induces an effective coupling of all particles.

8 Conclusions

For a fairly general class of time-dependent Hamiltonian systems, we have derived an invariant I, hence a quantity that is conserved along the system's phase-space trajectory. The representation of I was found to depend on a function $\xi(t)$, which in turn embodies a solution of a linear homogeneous third-order differential equation, referred to as the auxiliary equation.

The invariant I for time-dependent Hamiltonian systems H was derived in the framework of the extended phase-space, where time and the negative Hamiltonian itself are regarded as canonically conjugate coordinates. In this description, an invariant may be isolated directly from the generating function of an infinitesimal canonical transformation, hence from symmetry mappings of the canonical variables and time that leave the canonical equations invariant. The auxiliary equation for $\xi(t)$ then emerges from the requirement that the symmetry mapping be canonical. The invariant could be identified with the conserved total energy for non-autonomous systems, which is obtained if we add to the time-varying energy represented by the Hamiltonian H the energies fed into or detracted from the system.

We have furthermore shown that the invariant I is also obtained applying in the extended phase-space a finite canonical transformation to the initial Hamiltonian $H(\vec{q}, \vec{p}, t)$. Imposing the condition on the new Hamiltonian $H'(\vec{q}\,', \vec{p}\,')$ not to depend on time explicitly then coincides with the auxiliary equation for $\xi(t)$ – derived beforehand in the context of infinitesimal canonical symmetry mappings.

Expressed in terms of the old coordinates, the new Hamiltonian H' was found to agree with the invariant I for all $\xi(t)$ that follow as solutions of the auxiliary equation. On the other hand, the new Hamiltonian H' represents a real physical system only for time intervals with $\xi(t) > 0$. Then, H' may be interpreted as the Hamiltonian of the autonomous system that is equivalent to the non-autonomous system H.

In general, the auxiliary equation depends on the system's spatial coordinates. As the consequence, the auxiliary equation can only be integrated *in conjunction* with the equations of motion. From this viewpoint, the $2n$ first-order canonical equations – that determine uniquely the time evolution of the n particle system – form together with the three first-order equations of the auxiliary equation a closed coupled set of $2n + 3$ first-order equations that uniquely determine the invariant I.

For the particular case of isotropic quadratic Hamiltonians, the dependence of the auxiliary equation on the spatial coordinates cancels. The third-order auxiliary equation may then be analytically integrated to yield a well-known non-linear second-order equation that has been derived earlier for the time-dependent harmonic oscillator.

In the special case of autonomous systems – hence Hamiltonian systems with no explicit time-dependence – the function $\xi(t) \equiv 1$ is always a solution of the auxiliary equation. With this solution, the invariant I coincides with the invariant that is given by the system's Hamiltonian H itself. In view of this result, the familiar invariant $I = H$ just represents the particular case where the auxiliary equation possesses the special solution $\xi(t) \equiv 1$ – which is exactly given for autonomous Hamiltonian systems. In addition to the invariant $I = H$, another non-trivial invariant for autonomous systems always exists that is associated with a solution $\xi(t) \neq$ const. of the auxiliary equation. The dependence of the invariants of a Hamiltonian system on solutions $\xi(t)$ of an auxiliary equation thus constitutes a general feature – which disappears solely for the particular case of the invariant $I = H$ of an autonomous system.

For the case of explicitly time-dependent Hamiltonian systems, solutions of the auxiliary equation with $\xi(t) =$ const. do not exist. Therefore, the invariants for non-autonomous systems *always* depend on solutions $\xi(t)$ of the auxiliary equation. The additional complexity that arises for the invariants of non-isotropic linear and general non-linear Hamiltonian systems is that the auxiliary equation now depends on the system's spatial coordinates. The authors believe that this generalized viewpoint of the concept of an invariant is the main result of the present article.

It has been shown that the solution function $\xi(t)$ of the auxiliary equation remains non-negative for linear isotropic systems. In these cases, $\xi(t)$ may be interpreted as an amplitude function of the particle motion [18]. For all other Hamiltonian systems, the auxiliary function $\xi(t)$ may become negative. A connection between theses solutions of the auxiliary equation and the characteristics of the solutions of the equations of motion has not yet been established. Furthermore, the physical implications that are associated with an unstable behavior of $\xi(t)$ still await clarification.

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A Invariants derived from Noether's theorem

We will show in this Appendix that the invariant (29) can straightforwardly be derived on the basis of Noether's theorem [6]. This theorem relates the conserved quantities of a Lagrangian system $L(\vec{q}, \dot{\vec{q}}, t)$ to the one-parameter groups that leave the action integral invariant. The one-parameter first extended Lie group [19] is generated by

$$\mathbf{E} = \xi(t)\frac{\partial}{\partial t} + \sum_{i=1}^{n} \eta_i(q_i, t)\ \frac{\partial}{\partial q_i} + \sum_{i=1}^{n} (\dot{\eta}_i - \dot{q}_i\dot{\xi})\ \frac{\partial}{\partial \dot{q}_i} \tag{67}$$

with

$$\dot{\eta}_i(q_i, t) = \frac{\partial \eta_i}{\partial t} + \frac{\partial \eta_i}{\partial q_i}\ \dot{q}_i\,, \quad \text{and} \quad \xi = \xi(t)\,.$$

Noether's theorem then states that the solutions of the Euler-Lagrange equations

$$\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0\,, \qquad i = 1, 2, \ldots, n \tag{68}$$

admit the conserved quantity [20, 21]

$$I = \sum_{i=1}^{n} (\xi\dot{q}_i - \eta_i)\ \frac{\partial L}{\partial \dot{q}_i} - \xi L + f_0(\vec{q}, t)\,, \tag{69}$$

provided that the action integral is invariant under the group generator (67) – which is the case if the total time derivative of $f_0(\vec{q}, t)$ fulfills the condition

$$\frac{\mathrm{d}f_0(\vec{q}, t)}{\mathrm{d}t} = \mathbf{E}L + \dot{\xi}L \,. \tag{70}$$

The particular Lagrangian, whose Euler-Lagrange Eqs. (68) lead to the equations of motion (24a) is

$$L(\vec{q}, \dot{\vec{q}}, t) = \left(\tfrac{1}{2}\sum_{i=1}^{n} \dot{q}_i^2 - V(\vec{q}, t)\right) \mathrm{e}^{F(t)} \quad \text{with} \quad F(t) = \int_{t_0}^{t} f(\tau)\,\mathrm{d}\tau \,. \tag{71}$$

Inserting the Lagrangian (71) into Eq. (70) and equating coefficients of powers of $\dot{q}_i$ to zero, we obtain a set of equation for $\xi(t)$, $\eta_i(q_i, t)$, and $f_0(\vec{q}, t)$ which can be solved to find

$$\eta_i(q_i, t) = \tfrac{1}{2}\,[\dot{\xi}(t) - \xi(t)\,f(t)]\,q_i + \psi_i(t) \tag{72}$$

and

$$f_0(\vec{q}, t) = \sum_{i=1}^{n} [\tfrac{1}{4}\,q_i^2(\ddot{\xi}(t) - \dot{\xi}(t)\,f(t) - \xi(t)\,\dot{f}(t)) + \dot{\psi}_i(t)\,q_i]\,\mathrm{e}^{F(t)} \,. \tag{73}$$

The terms of Eq. (70) that do not depend on $\dot{q}_i$ sum up to

$$\frac{\partial f_0}{\partial t} + \mathrm{e}^{F(t)}\left[\sum_{i=1}^{n} \eta_i\,\frac{\partial V}{\partial q_i} + \xi\left(\frac{\partial V}{\partial t} + f(t)\,V\right) + \dot{\xi}V\right] = 0 \,. \tag{74}$$

Inserting Eq. (72) and the partial time derivative of Eq. (73) into Eq. (74), we get the third-order differential equation for $\xi(t)$ of Eq. (30), together with the n differential equations for the arbitrary functions $\psi_i(t)$, given by Eq. (31). Using the relation

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \dot{q}_i\,\mathrm{e}^{F(t)} \,,$$

we finally find the invariant (29) inserting Eqs. (71), (72), and (73) into Eq. (69).

B Invariants derived from a "direct approach"

Finally, we will show that the invariant (29) can as well be derived using a direct ansatz function being quadratic in the canonical momentum. This approach has been used earlier by Lewis and Leach [13]. In our actual case of an n-degree-of-freedom system with the canonical equations given by (24), this ansatz for the invariant may be defined as

$$I = \sum_{i=1}^{n} \tfrac{1}{2}\,\xi(t)\,p_i\dot{q}_i - \sum_{i=1}^{n} \eta(q_i, t)\;p_i + f_1(\vec{q}, t) \,. \tag{75}$$

The quantity I embodies a constant of motion if its total time derivative vanishes

$$\frac{\mathrm{d}I}{\mathrm{d}t} \stackrel{!}{=} 0 \,.$$

Calculating the total time derivative of Eq. (75), and inserting the particular canonical equations (24), we may separately equate to zero the sums proportional to $\dot{q}_i^2$, $\dot{q}_i^1$, and $\dot{q}_i^0$ – similar to the procedure pursued in the approach based on Noether's theorem. Making use of the canonical equations means that the invariant I constitutes a quantity which is defined exactly on the phase-space path that represents the system's time evolution.

For the function $\eta(q_i, t)$ contained in Eq. (75), we find

$$\eta_i(q_i, t) = \tfrac{1}{2}\,[\dot{\xi}(t) - \xi(t)\,f(t)]\,q_i + \psi_i(t)\,, \tag{76}$$

which obviously agrees with Eq. (72). The function $f_1(\vec{q}, t)$ of Eq. (75) evaluates to

$$f_1(\vec{q}, t) = \mathrm{e}^{F(t)}\left[\xi V(\vec{q}, t) + \tfrac{1}{4}\,(\ddot{\xi} - \dot{\xi}f - \xi\dot{f})\sum_{i=1}^{n} q_i^2 + \sum_{i=1}^{n}\dot{\psi}_i q_i\right]. \tag{77}$$

Inserting Eqs. (76) and (77) into Eq. (75), one directly obtains the invariant (29). The auxiliary equation follows from the terms of $\mathrm{d}I/\mathrm{d}t = 0$ that do not depend on the $\dot{q}_i$

$$\frac{\partial f_1}{\partial t} + \mathrm{e}^{F(t)}\sum_{i=1}^{n}\eta_i\,\frac{\partial V}{\partial q_i} = 0\,. \tag{78}$$

The differential equations (30) and (31) for $\xi(t)$ and the $\psi_i(t)$ are found inserting Eq. (76) and the partial time derivative of Eq. (77) into Eq. (78). We hereby observe that the ansatz approach of Eq. (75) to derive the invariant I is equivalent to the strategy based on Noether's theorem.

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