Canonical transformations associated with second order problems in the calculus of variations (*).

by H.S.P. GRÄSSER

Department of Mathematics, University of South Africa (Pretoria, South Africa)

Summary. - *Our object is a systematic investigation of some of the properties of canonical transformations associated with second order problems in the calculus of variations. After the introduction of such transformations, together with the related concepts of Lagrange and Poisson brackets, the bracket relationships are found which characterize canonical transformations. This characterization is also achieved by means of so-called reciprocity relations between the original transformation and its inverse (which always exists). The effect of the canonical transformation on the underlying variational problem is discussed. It is also shown that the Jacobian of such a transformation always has the value unity.*

The special case when the canonical transformation is independent of the parameter (a generalization of the so-called time-independent canonical transformation of mechanics) is treated in some detail. Finally it is indicated how the present theory can be extended to problems of higher order.

1. - Introduction.

A so-called problem of the m-th order in the calculus of variations is one whose Lagrange function contains derivatives of up to the m-th order of the dependent variables. The concept of a *canonical transformation* has been investigated thoroughly by CARATHÈODORY ([1], chapter 6) (1), while the intimate relationship between these transformations and variational problems of the *first* order has recently again been demonstrated very clearly by RUND ([7], section 2.12). It is the object of this article to develop some aspects of a theory of *generalized canonical transformations* which are associated with the *second order problem in the calculus of variations,* and to indicate how these results can be extended to transformations associated with a problem of arbitrary order. To a great extent our approach is a generalization of that one of RUND ([7], *loc. cit.*).

(*) Some of the results of this paper are contained in a doctoral thesis ([2]) which was presented to the University of South Africa. The writer wishes to express his gratitude to his supervisor, Professor H. RUND, for his interest, encouragement and advice concerning this work.

(1) Numbers in square brackets refer to the literature listed at the end of this article.

Recently RUND ([5]) obtained a set of relations which characterize an extremal congruence of a parameter–invariant second order problem. We obtain the integrability conditions of these relations and show how they immediately lead to the concept of *Lagrange brackets* which are associated with the second order problem (section 2). After having introduced our generalized canonical transformation in terms of a generating function, we proceed to find *Lagrange bracket relations* which are characteristic of such a transformation (section 3), and which may immediately be applied to establish the existence of the *inverse* of a generalized canonical transformation (section 4).

The existence of the inverse then enables us to characterize our transformations in another significant manner, namely by means of a set of $16n^2$ so-called *reciprocity relations* between the partial derivatives of the original functions defining the canonical transformation and those of the inverse functions. An important consequence of these reciprocity relations is the fact that they give rise, in a very natural manner, to *Poisson brackets* associated with our theory. These Poisson brackets again allow us to characterize the generalized canonical transformations in terms of *Poisson bracket relations* (section 5).

As in the first order case, our canonical transformations *leave invariant the equations of the extremals of the underlying variational problems* as is shown in section 6. We then proceed with an investigation of some properties of the functional determinant of the transformation and prove, with the aid of associated *elementary canonical transformations, that this determinant always has the value unity* (section 7).

The penultimate section of this article is devoted to the so-called *t–independent canonical transformations*. Finally it is briefly indicated how the theory developed in this paper may be extended to canonical transformations corresponding to *variational problems of arbitrary order.*

It should be pointed out that a generalized canonical transformation has also been suggested by MEFFROY ([4]). However, this transformation is not at all connected with any variational problem, and does not, therefore, have any bearing on the theory developed in this paper.

2. - Lagrange brackets associated with the second order problem.

We consider the fundamental integral of a second order problem in the calculus of variations, *viz.*

$$I = \int_{C\, Q_1}^{Q_2} L(t,\ x^i,\ \dot{x}^i,\ \ddot{x}^i)\, dt, \tag{2.1}$$

where [2] $C: x^i = x^i(t)$, $i = 1, 2, ..., n$, is some curve of class C^2 in an $(n+1)$-dimensional differentiable manifold X_{n+1} with local coordinates (t, x^i). As regards differentiation with respect to the curve parameter t we employ the notation $\dot{x}^i = dx^i/dt$, $\ddot{x}^i = d^2x^i/dt^2$. It is also assumed that the Lagrangian function L is at least of class C^4.

The *canonical momenta* corresponding to this problem are defined as usual by

$$R_i = \frac{\partial L}{\partial \ddot{x}^i}, \quad P_i = \frac{\partial L}{\partial \dot{x}^i} - \frac{dR_i}{dt} \tag{2.2}$$

(cf. [9], p. 266), so that the *canonical coordinates* are a set of $4n$ independent quantities, namely x^i, $\dot{x}^i$, P_i, R_i.

A *congruence* of extremals of this second order problem can be characterized in a manner which is analogous to the method of RUND ([5], pp. 100-101) for the parameter-invariant second order problem: We assume that the differential equations

$$\dot{x}^i = \psi^i(t, x^j) \tag{2.3}$$

define an n-parameter family of curves

$$x^i = x^i(t, u^k) \tag{2.4}$$

of class C^2, which cover a finite region of X_{n+1} simply. Differentiation along the family (2.3) is denoted by

$$\dot{\psi}^i \overset{\text{def}}{=} \frac{\partial \psi^i}{\partial t} + \frac{\partial \psi^i}{\partial x^j} \psi^j, \quad \ddot{\psi}^i \overset{\text{def}}{=} \frac{\partial \dot{\psi}^i}{\partial t} + \frac{\partial \dot{\psi}^i}{\partial x^j} \psi^j. \tag{2.5}$$

If $S = S(t, x^k, \dot{x}^k)$ is a function of class C^2 which satisfies the HAMILTON-JACOBI equation of the theory, *the relations*

$$P_i(t, x^j, \psi^j(t, x^k), \dot{\psi}^j(t, x^k), \ddot{\psi}^j(t, x^k)) = \frac{\partial S(t, x^j, \psi^j(t, x^k))}{\partial x^i}, \tag{2.6}$$

$$R_i(t, x^j, \psi^j(t, x^k), \dot{\psi}^j(t, x^k)) = \frac{\partial S(t. x^j, \psi^j(t, x^k))}{\partial \dot{x}^i}, \tag{2.7}$$

ensure that the congruence (2.3) *is indeed a congruence of extremals.*

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[2] Throughout this article the indices i, j, h, k assume the values $1, 2, .., n$, and α, β run from 1 to $n-1$, while the customary summation convention is always applied.

The integrability conditions for the relations (2.6), (2.7) are easily derived. If we substitute from (2.4) into (2.3), and into the arguments of P_i, R_i according to (2.5), the congruence (2.4) determines the canonical coordinates

$$x^i = x^i(t,\ u^k),\ \dot{x}^i = \dot{x}^i(t,\ u^k),\ P_i = P_i(t,\ u^k),\ R_i = R_i(t,\ u^k) \tag{2.8}$$

in terms of the parameters u^k. The relations (2.6), (2.7) may thus be written

$$P_i(t,\ u^k) = \frac{\partial S(t,\ x^j(t,\ u^k),\ \dot{x}^j(t,\ u^k))}{\partial x^i},\quad R_i(t,\ u^k) = \frac{\partial S(t,\ x^j(t,\ u^k),\ \dot{x}^j(t,\ u^k))}{\partial \dot{x}^i}. \tag{2.9}$$

Let us put

$$\sigma(t,\ u^k) \overset{\text{def}}{=} S(t,\ x^j(t,\ u^k),\ \dot{x}^j(t,\ u^k)),$$

and evaluate $\partial\sigma/\partial u^k$, substituting into the resulting expression from (2.9). A further differentiation with respect to u^h then yields the equations

$$\frac{\partial^2 \sigma}{\partial u^h \partial u^k} = \frac{\partial P_i}{\partial u^h}\frac{\partial x^i}{\partial u^k} + P_i \frac{\partial^2 x^i}{\partial u^h \partial u^k} + \frac{\partial R_i}{\partial u^h}\frac{\partial \dot{x}^i}{\partial u^k} + R_i \frac{\partial^2 \dot{x}^i}{\partial u^h \partial u^k}. \tag{2.10}$$

Since, as a result of our assumptions, the functions (2.8) as well as σ are of class C^2, the required *integrability conditions* follow directly, *viz.*

$$\frac{\partial x^i}{\partial u^k}\frac{\partial P_i}{\partial u^h} - \frac{\partial x^i}{\partial u^h}\frac{\partial P_i}{\partial u^k} + \frac{\partial \dot{x}^i}{\partial u^k}\frac{\partial R_i}{\partial u^h} - \frac{\partial \dot{x}^i}{\partial u^h}\frac{\partial R_i}{\partial u^k} = 0. \tag{2.11}$$

This immediately suggests that we should define the generalized LAGRANGE bracket corresponding to the canonical coordinates of the second order problem as follows. Let

$$x^i = x^i(u,\ v,\ \ldots),\ \dot{x}^i = \dot{x}^i(u,\ v,\ \ldots),\ P_i = P_i(u,\ v,\ \ldots),\ R_i = R_i(u,\ v,\ \ldots)$$

be functions of class C^1 in r parameters u, v, ... Then the *generalized Lagrange bracket of x^i, $\dot{x}^i$, P_i, R_i with respect to any two of these parameters, say u, v, is defined by*

$$[u,\ v] \overset{\text{def}}{=} \frac{\partial x^i}{\partial u}\frac{\partial P_i}{\partial v} - \frac{\partial x^i}{\partial v}\frac{\partial P_i}{\partial u} + \frac{\partial \dot{x}^i}{\partial u}\frac{\partial R_i}{\partial v} - \frac{\partial \dot{x}_i}{\partial v}\frac{\partial R_i}{\partial u}. \tag{2.12}$$

The integrability conditions (2.11) therefore become

$$[u^k,\ u^h] = 0, \tag{2.13}$$

which is the form in which they appear in the first order theory (e.g. [7], p. 29).

The LAGRANGE brackets have properties which are analogous to those of the latter theory. In particular, they are anti-symmetric, while admitting a geometric interpretation in terms of the underlying extremal congruence, for the details of which the reader is referred to [2]. We merely remark that conditions (2.13) are not in general sufficient to ensure the existence of a function $S = S(t, x^k, \dot{x}^k)$ for which conditions (2.6), (2.7) hold (cf. [2], p. 236).

3. - Generalized canonical transformations and Lagrange bracket relations.

Consider the following transformation to a new set of canonical variables:

$$\bar{x}^i = \bar{x}^i(x^j, \dot{x}^j, P_j, R_j), \quad \dot{\bar{x}}^i = \dot{\bar{x}}^i(x^j, \dot{x}^j, P_j, R_j), \tag{3.1a}$$

$$\bar{P}_i = \bar{P}_i(x^j, \dot{x}^j, P_j, R_j), \quad \bar{R}_i = \bar{R}_i(x^j, \dot{x}^j, P_j, R_j). \tag{3.1b}$$

This is said to constitute a *generalized canonical transformation if it is of class C^2, and if there exists a function*

$$\Psi = \Psi(x^j, \dot{x}^j, P_j, R_j), \tag{3.2}$$

also of class C^2, which has the property that

$$d\Psi(x^j, \dot{x}^j, P_j, R_j) \equiv \bar{P}_i d\bar{x}^i + \bar{R}_i d\dot{\bar{x}}^i - P_i dx^i - R_i d\dot{x}^i, \tag{3.3}$$

whenever $\bar{x}^i$, $\dot{\bar{x}}^i$, $\bar{P}_i$, $\bar{R}_i$ on the right-hand side are evaluated according to (3.1).

In order to find a set of necessary and sufficient conditions for a transformation (3.1) to be of this type, we expand both sides of the identity (3.3). A comparison of the coefficients of dx^i, $d\dot{x}^i$, dP_i, dR_i on each side yields the following identities

$$\frac{\partial \Psi}{\partial x^i} = \bar{P}_k \frac{\partial \bar{x}^k}{\partial x^i} + \bar{R}_k \frac{\partial \dot{\bar{x}}^k}{\partial x^i} - P_i, \tag{3.4a}$$

$$\frac{\partial \Psi}{\partial \dot{x}^i} = \bar{P}_k \frac{\partial \bar{x}^k}{\partial \dot{x}^i} + \bar{R}_k \frac{\partial \dot{\bar{x}}^k}{\partial \dot{x}^i} - R_i, \tag{3.4b}$$

$$\frac{\partial \Psi}{\partial P_i} = \bar{P}_k \frac{\partial \bar{x}^k}{\partial P_i} + \bar{R}_k \frac{\partial \dot{\bar{x}}^k}{\partial P_i}, \quad \frac{\partial \Psi}{\partial R_i} = \bar{P}_k \frac{\partial \bar{x}^k}{\partial R_i} + \bar{R}_k \frac{\partial \dot{\bar{x}}^k}{\partial R_i}. \tag{3.4c, d}$$

Conversely, if a function Ψ satisfies these identities, this implies (3.3).

The obvious integrability conditions of (3 4) can be written as follows [3] in LAGRANGE bracket form (where we now take the definition (2.12) to refer to $\bar{x}^i$, $\bar{\dot{x}}^i$, $\bar{P}_i$, $\bar{R}_i$):

$$\begin{gathered}[x^i, x^j]=0,\ [x^i, \dot{x}^j]=0,\ [\dot{x}^i, \dot{x}^j]=0,\\ [x^i, P_j]=\delta^j_i,\ [x^i, R_j]=0,\ [\dot{x}^i, P_j]=0,\ [\dot{x}^i, R_j]=\delta^j_i,\\ [P_i, P_j]=0,\ [P_i, R_j]=0,\ [R_i, R_j]=0.\end{gathered} \tag{3.5}$$

Thus we may formulate

THEOREM 1. - *The Lagrange bracket relations* (3.5) *represent a necessary and sufficient condition in order that the transformation* (3 1) (*of class* C^2) *is a generalized canonical transformation.*

REMARK. - When $R_i \equiv 0$, the LAGRANGE bracket (2.12) reduces to the form associated with the first order case. In general, the above formalism does not, however, reduce to the formalism of the simple case (which is characterized by the condition $R_i \equiv 0$), this being due to the presence of the $\dot{x}^i$ in our theory, whereas these quantities are eliminated from the canonical formalism of the first order theory.

4. - The existence of the inverse transformation.

The LAGRANGE bracket relations (3.5) enable us to prove that any canonical transformation must necessarily always possess a non-vanishing Jacobian J, and therefore an inverse. In fact, if we denote by the indices h, i, j, k the h-th, $(n+i)$-th, $(2n+j)$-th and $(3n+k)$-th *columns* of J, and by h', i', j', k' the corresponding *rows* of J, then one can show, by elementary operations, that

$$J^2 \equiv \det \begin{vmatrix} [x^{h'}, P_h], & [x^{h'}, R_i], & [x^j, x^{h'}], & [\dot{x}^k, x^{h'}] \\ [\dot{x}^{i'}, P_h], & [\dot{x}^{i'}, R_i], & [x^j, \dot{x}^{i'}], & [\dot{x}^k, \dot{x}^{i'}] \\ [P_{j'}, P_h], & [P_{j'}, R_i], & [x^j, P_{j'}], & [\dot{x}^k, P_{j'}] \\ [R_{k'}, P_h], & [R_{k'}, R_i], & [x^j, R_{k'}], & [\dot{x}^k, R_{k'}] \end{vmatrix}. \tag{4.1}$$

If (3.1) now represents a canonical transformation, we infer from (3.5) that all entries in (4.1) vanish expect those arising from the diagonal

[3] Detailed calculations (also as regards sections 4, 5 and 8 of this article) may be found in [2], chapter 3.

LAGRANGE brackets, which are equal to $\delta^h_{h'}$, $\delta^i_{i'}$, $\delta^{j'}_j$, $\delta^{k'}_k$ respectively, so that

$$J^2 = 1. \tag{4.2}$$

This proves

LEMMA 1. - *The generalized canonical transformation possesses an inverse transformation in some neighbourhood of each set* $(x^i, \dot{x}^i, P_i, R_i)$ *at which it is defined.*

In section 7 we shall improve on (4.2) by showing that, in fact, $J = 1$.

5. - Reciprocity relations and generalized Poisson brackets.

We now consider the inverse of the canonical transformation (3.1), *viz.*

$$\begin{aligned} x^i &= x^i(\bar{x}^j, \dot{\bar{x}}^j, \bar{P}_j, \bar{R}_j), \quad \dot{x}^i = \dot{x}^i(\bar{x}^j, \dot{\bar{x}}^j, \bar{P}_j, \bar{R}_j), \\ P_i &= P_i(\bar{x}^j, \dot{\bar{x}}^j, \bar{P}_j, \bar{R}_j), \quad R_i = R_i(\bar{x}^j, \dot{\bar{x}}^j, \bar{P}_j, \bar{R}_j), \end{aligned} \tag{5.1}$$

and we write

$$\bar{\Psi}(\bar{x}^j, \dot{\bar{x}}^j, \bar{P}_j, \bar{R}_j) \overset{\text{def}}{=} \Psi(x^j, \dot{x}^j, P_j, R_j). \tag{5.2}$$

It is obvious that the inverse (5.1) of the canonical transformation is again canonical, its generating function being $-\bar{\Psi}(\bar{x}^j, \dot{\bar{x}}^j, \bar{P}_j, \bar{R}_j)$. Hence theorem 1 can be expressed also in terms of the «barred» variables. In the latter case we use the notation

$$[\bar{x}^i, \bar{x}^j]' \overset{\text{def}}{=} \frac{\partial x^h}{\partial \bar{x}^i}\frac{\partial P_h}{\partial \bar{x}^j} - \frac{\partial x^h}{\partial \bar{x}^j}\frac{\partial P_h}{\partial \bar{x}^i} + \frac{\partial \dot{x}^h}{\partial \bar{x}^i}\frac{\partial R_h}{\partial \bar{x}^j} - \frac{\partial \dot{x}^h}{\partial \bar{x}^j}\frac{\partial R_h}{\partial \bar{x}^i}. \tag{5.3}$$

The partial derivatives of a generalized canonical transformation (3.1) with respect to its variables are related in a definite way to those of the inverse transformation. Again the situation in our case is in complete analogy with that of the first order problem, for which RUND ([7], p. 92) obtained the so-called reciprocity relations. We shall now indicate how this set of reciprocity relations is obtained.

Let us substitute back from the inverse functions (5.1) into (3.1), thereby obtaining a set of $4n$ *identities* in $\bar{x}^i$, $\dot{\bar{x}}^i$, $\bar{P}_i$, $\bar{R}_i$. These, in turn, are differentiated partially with respect to each of $\bar{x}^j$, $\dot{\bar{x}}^j$, $\bar{P}_j$, $\bar{R}_j$, which gives rise to a set of $16n^2$ identities in these variables. Elementary but fairly lengthy calculations then lead to a further set of $16n^2$ identities, of which two typical elements are the following:

$$\frac{\partial \bar{x}^i}{\partial x^j} = [x^j,\, x^k]\frac{\partial x^k}{\partial \bar{P}_i} + [x^j,\, \dot{x}^k]\frac{\partial \dot{x}^k}{\partial \bar{P}_i} + [x^j,\, P_k]\frac{\partial P_k}{\partial \bar{P}_i} + [x^j,\, R_k]\frac{\partial R_k}{\partial \bar{P}_i}, \tag{5.4a}$$

$$\frac{\partial \bar{P}_i}{\partial x^j} = [x^k,\, x^j]\frac{\partial x^k}{\partial \bar{x}^i} + [\dot{x}^k,\, x^j]\frac{\partial \dot{x}^k}{\partial \bar{x}^i} + [P_k,\, x^j]\frac{\partial P_k}{\partial \bar{x}^i} + [R_k,\, x^j]\frac{\partial R_k}{\partial \bar{x}^i}. \tag{5.4b}$$

These identities are valid for any transformation of type (3.1) which is of class C^1 and possesses an inverse. For a canonical transformation the LAGRANGE bracket relations (3.5) reduce (5.4a), (5.4b) respectively to

$$\frac{\partial \bar{x}^i}{\partial x^j} = \frac{\partial P_j}{\partial \bar{P}_i}, \quad \frac{\partial \bar{P}_i}{\partial x^j} = -\frac{\partial P_j}{\partial \bar{x}^i}.$$

In this manner one thus obtains the following set of *generalized reciprocity relations* as *necessary* conditions for our canonical transformation:

$$\begin{aligned}
&\frac{\partial \bar{x}^i}{\partial x^j} = \frac{\partial P_j}{\partial \bar{P}_i}, \quad \frac{\partial \bar{x}^i}{\partial \dot{x}^j} = \frac{\partial R_j}{\partial \bar{P}_i}, \quad \frac{\partial \bar{x}^i}{\partial P_j} = -\frac{\partial x^j}{\partial \bar{P}_i}, \quad \frac{\partial \bar{x}^i}{\partial R_j} = -\frac{\partial \dot{x}^j}{\partial \bar{P}_i},\\
&\frac{\partial \dot{\bar{x}}^i}{\partial x^j} = \frac{\partial P_j}{\partial \bar{R}_i}, \quad \frac{\partial \dot{\bar{x}}^i}{\partial \dot{x}^j} = \frac{\partial R_j}{\partial \bar{R}_i}, \quad \frac{\partial \dot{\bar{x}}^i}{\partial P_j} = -\frac{\partial x^j}{\partial \bar{R}_i}, \quad \frac{\partial \dot{\bar{x}}^i}{\partial R_j} = -\frac{\partial \dot{x}^j}{\partial \bar{R}_i},\\
&\frac{\partial \bar{P}_i}{\partial x^j} = -\frac{\partial P_j}{\partial \bar{x}^i}, \quad \frac{\partial \bar{P}_i}{\partial \dot{x}^j} = -\frac{\partial R_j}{\partial \bar{x}^i}, \quad \frac{\partial \bar{P}_i}{\partial P_j} = \frac{\partial x^j}{\partial \bar{x}^i}, \quad \frac{\partial \bar{P}_i}{\partial R_j} = \frac{\partial \dot{x}^j}{\partial \bar{x}^i},\\
&\frac{\partial \bar{R}_i}{\partial x^j} = -\frac{\partial P_j}{\partial \dot{\bar{x}}^i}, \quad \frac{\partial \bar{R}_i}{\partial \dot{x}^j} = -\frac{\partial R_j}{\partial \dot{\bar{x}}^i}, \quad \frac{\partial \bar{R}_i}{\partial P_j} = \frac{\partial x^j}{\partial \dot{\bar{x}}^i}, \quad \frac{\partial \bar{R}_i}{\partial R_j} = \frac{\partial \dot{x}^j}{\partial \dot{\bar{x}}^i}.
\end{aligned} \tag{5.5}$$

That these conditions also are sufficient can easily be seen as follows. If we are given a transformation (3.1) with inverse (5.1) for which the relations (5.5) are valid, we substitute for $\partial x^h/\partial \bar{x}^i$, $\partial P_h/\partial \bar{x}^i$, $\partial \dot{x}^h/\partial \bar{x}^i$, $\partial R_h/\partial \bar{x}^i$ in (5.3), obtaining

$$\begin{aligned}
[\bar{x}^i,\, \bar{x}^j]' &= \frac{\partial \bar{P}_i}{\partial P_h}\frac{\partial P_h}{\partial \bar{x}^j} + \frac{\partial x^h}{\partial \bar{x}^j}\frac{\partial \bar{P}_i}{\partial x^h} + \frac{\partial \bar{P}_i}{\partial R_h}\frac{\partial R_h}{\partial \bar{x}^j} + \frac{\partial \dot{x}^h}{\partial \bar{x}^j}\frac{\partial \bar{P}_i}{\partial \dot{x}^h}\\
&\equiv \frac{\partial \bar{P}_i}{\partial \bar{x}^j} \equiv 0,
\end{aligned}$$

with similar conclusions in regard of all other LAGRANGE bracket relations. Thus we may state

THEOREM 2. – *The validity of the reciprocity relations* (5.5) *is a necessary and sufficient condition for the transformation* (3.1) *to be a canonical transformation.*

These reciprocity relations may now be used for the introduction of the generalized Poisson brackets in a very natural manner. In fact, if we substitute for all members on the right-hand side of (5.3) from (5.5), we immediately find that

$$[\bar{x}^i, \bar{x}^j]' = \frac{\partial \bar{P}_i}{\partial x^h}\frac{\partial \bar{P}_j}{\partial P_h} - \frac{\partial \bar{P}_i}{\partial P_h}\frac{\partial \bar{P}_j}{\partial x^h} + \frac{\partial \bar{P}_i}{\partial \dot{x}^h}\frac{\partial \bar{P}_j}{\partial R_h} - \frac{\partial \bar{P}_i}{\partial R_h}\frac{\partial \bar{P}_j}{\partial \dot{x}^h},$$

with similar expressions for the other LAGRANGE brackets in (3.5), referring to the «barred» variables as in (5.3).

Now let $F = F(x^j, \dot{x}^j, P_j, R_j)$, $G = G(x^j, \dot{x}^j, P_j, R_j)$ be any two functions of class C^1 of the canonical variables. Then the *generalized Poisson bracket* (4) of these functions with respect to the canonical coordinates is defined by

$$(F, G) \overset{\text{def}}{=} \frac{\partial F}{\partial x^i}\frac{\partial G}{\partial P_i} - \frac{\partial F}{\partial P_i}\frac{\partial G}{\partial x^i} + \frac{\partial F}{\partial \dot{x}^i}\frac{\partial G}{\partial R_i} - \frac{\partial F}{\partial R_i}\frac{\partial G}{\partial \dot{x}^i}. \tag{5.6}$$

In terms of this definition the above-mentioned relations for the LAGRANGE brackets may thus be written

$$\begin{gathered}
[\bar{x}^i, \bar{x}^j]' = (\bar{P}_i, \bar{P}_j),\ [\bar{x}^i, \dot{\bar{x}}^j]' = (\bar{P}_i, \bar{R}_j),\ [\dot{\bar{x}}^i, \dot{\bar{x}}^j]' = (\bar{R}_i, \bar{R}_j), \\
[\bar{x}^i, \bar{P}_j]' = (\bar{x}^j, \bar{P}_i),\ [\bar{x}^i, \bar{R}_j]' = (\dot{\bar{x}}^j, \bar{R}_i),\ [\dot{\bar{x}}^i, \bar{P}_j]' = (\dot{\bar{x}}^j, \bar{P}_i),\ [\dot{\bar{x}}^i, \bar{R}_j]' = (\dot{\bar{x}}^j, \bar{R}_i), \\
[\bar{P}_i, \bar{P}_j]' = (\bar{x}^i, \bar{x}^j),\ [\bar{P}_i, \bar{R}_j]' = (\bar{x}^i, \dot{\bar{x}}^j),\ [\bar{R}_i, \bar{R}_j]' = (\dot{\bar{x}}^i, \dot{\bar{x}}^j).
\end{gathered} \tag{5.7}$$

Since the identities (5.7) are valid under the assumption that the variables $(\bar{x}^i, \dot{\bar{x}}^i, \bar{P}_i, \bar{R}_i)$ and $(x^i, \dot{x}^i, P_i, R_i)$ are connected by a canonical transformation, we infer from theorem 1 that for such a transformation the following *Poisson bracket relations* are valid:

$$\begin{gathered}
(\bar{x}^i, \bar{x}^j) = 0,\ (\bar{x}^i, \dot{\bar{x}}^j) = 0,\ (\dot{\bar{x}}^i, \dot{\bar{x}}^j) = 0, \\
(\bar{x}^i, \bar{P}_j) = \delta^i_j,\ (\bar{x}^i, \bar{R}_j) = 0,\ (\dot{\bar{x}}^i, \bar{P}_j) = 0,\ (\dot{\bar{x}}^i, \bar{R}_j) = \delta^i_j, \\
(\bar{P}_i, \bar{P}_j) = 0,\ (\bar{P}_i, \bar{R}_j) = 0,\ (\bar{R}_i, \bar{R}_j) = 0.
\end{gathered} \tag{5.8}$$

(4) The Poisson bracket (5.6) has been used occasionally in the past (apart from a difference in sign). The first definition seems to be due to WESSEL ([8]); see [2] for a more detailed list of references.

Conversely, let us assume that the relations (5.8) hold for some transformation (3.1). One may now show, analogously to section 4, that, for any transformation (3.1) of class C^1, one has quite generally

$$J^2 \equiv \det \begin{vmatrix} (\bar{P}_{h'},\ \bar{x}^h), & (\bar{P}_{h'},\ \dot{\bar{x}}^i), & (\bar{P}_{h'},\ \bar{P}_j), & (\bar{P}_{h'},\ \bar{R}_k) \\ (\bar{R}_{i'},\ \bar{x}^h), & (\bar{R}_{i'},\ \dot{\bar{x}}^i), & (\bar{R}_{i'},\ \bar{P}_j), & (\bar{R}_{i'},\ \bar{R}_k) \\ (\bar{x}^h,\ \bar{x}^{j'}), & (\dot{\bar{x}}^i,\ \bar{x}^{j'}), & (\bar{P}_j,\ \bar{x}^{j'}), & (\bar{R}_k,\ \bar{x}^{j'}) \\ (\bar{x}^h,\ \dot{\bar{x}}^{k'}), & (\dot{\bar{x}}^i,\ \dot{\bar{x}}^{k'}), & (\bar{P}_j,\ \dot{\bar{x}}^{k'}), & (\bar{R}_k,\ \dot{\bar{x}}^{k'}) \end{vmatrix}.$$

We substitute from (5.8) to find that

$$J^2 = (-1)^{4n} = 1,$$

and therefore our transformation possesses an inverse (5.1).

Proceeding as we did for the reciprocity relations, we substitute from (5.1) back into (3.1) and differentiate to get a set of $16n^2$ identities. These then yield a set of identities in POISSON bracket form which are very similar to (5.4). If we then substitute from the POISSON bracket relations (5.8) into these identities, we obtain precisely the reciprocity relations (5.5). In view of theorem 2 we thus have

THEOREM 3. - *A necessary and sufficient condition for the transformation* (3.1) *to be canonical is that the* Poisson *bracket relations* (5.8) *hold.*

We note that we could have therefore used these relations (5.8), or the LAGRANGE bracket relations (3.5), to *define* a cononical transformation, thereby dispensing with the requirement that it be of class C^2. The POISSON bracket (5.6) is anti-symmetric in the sense that for any two functions F, G we have

$$(F,\ G) = -(G,\ F), \tag{5.9}$$

while it also satisfies the JACOBI identity.

It is instructive to note that the relations (5.8) reduce to *identities* when (3.1) is the identity transformation:

$$\begin{gathered} (x^i,\ x^j) = 0,\ (x^i,\ \dot{x}^j) = 0,\ (\dot{x}^i,\ \dot{x}^j) = 0, \\ (x^i,\ P_j) = \delta^i_j,\ (x^i,\ R_j) = 0,\ (\dot{x}^i,\ P_j) = 0,\ (\dot{x}^i,\ R_j) = \delta^i_j, \\ (P_i,\ P_j) = 0,\ (P_i,\ R_j) = 0,\ (R_i,\ R_j) = 0, \end{gathered} \tag{5.10}$$

these identities being of considerable importance in quantum mechanics [5].

[5] As regards the treatment of certain aspects of the theory of elementary particles by means of a variational principle based on a second order problem in the calculus of variations we refer to [3]. This article also contains an extensive bibliography, while some further references may be found in [2] (pp. 44-47).

6. – A further relation between the generalized canonical transformations and Poisson brackets. Some applications.

Exactly as in the case for the ordinary canonical transformation, our transformation leaves invariant the POISSON bracket of two arbitrary functions. This can be proved as follows: Let us substitute from the inverse (5.1) of (3.1) into the arguments of F and G to obtain two new functions $\bar{F}$, $\bar{G}$ of the variables $\bar{x}^i$, $\dot{\bar{x}}^i$, $\bar{P}_i$, $\bar{R}_i$:

$$F(x^i, \dot{x}^i, P_i, R_i) = \bar{F}(\bar{x}^i, \dot{\bar{x}}^i, \bar{P}_i, \bar{R}_i), \quad G(x^i, \dot{x}^i, P_i, R_i) = \bar{G}(\bar{x}^i, \dot{\bar{x}}^i, \bar{P}_i, \bar{R}_i). \tag{6.1}$$

We denote the POISSON bracket with respect to the «barred» variables by a prime. If we now evaluate the POISSON bracket (F, G) in terms of $\bar{F}$, $\bar{G}$, we find the following identity:

$$\begin{aligned}(F, G) = {} & \frac{\partial \bar{F}}{\partial \bar{x}^j} \frac{\partial \bar{G}}{\partial \bar{x}^h} (\bar{x}^j, \bar{x}^h) + \frac{\partial \bar{F}}{\partial \bar{x}^j} \frac{\partial \bar{G}}{\partial \bar{P}_h} (\bar{x}^j, \bar{P}_h) + \frac{\partial \bar{F}}{\partial \bar{x}^j} \frac{\partial \bar{G}}{\partial \dot{\bar{x}}^h} (\bar{x}^j, \dot{\bar{x}}^h) + \\ & + \frac{\partial \bar{F}}{\partial \bar{x}^j} \frac{\partial \bar{G}}{\partial \bar{R}_h} (\bar{x}^j, \bar{R}_h) + \frac{\partial \bar{F}}{\partial \bar{P}_j} \frac{\partial \bar{G}}{\partial \bar{x}^h} (\bar{P}_j, \bar{x}^h) + \dots + \frac{\partial \bar{F}}{\partial \bar{R}_j} \frac{\partial \bar{G}}{\partial \dot{\bar{x}}^h} (\bar{R}_j, \dot{\bar{x}}^h) + \\ & + \frac{\partial \bar{F}}{\partial \bar{R}_j} \frac{\partial \bar{G}}{\partial \bar{R}_h} (\bar{R}_j, \bar{R}_h).\end{aligned}$$

Hence, in view of theorem 3 we find that

$$(F, G) = \frac{\partial \bar{F}}{\partial \bar{x}^j} \frac{\partial \bar{G}}{\partial \bar{P}_j} - \frac{\partial \bar{F}}{\partial \bar{P}_j} \frac{\partial \bar{G}}{\partial \bar{x}^j} + \frac{\partial \bar{F}}{\partial \dot{\bar{x}}^j} \frac{\partial \bar{G}}{\partial \bar{R}_j} - \frac{\partial \bar{F}}{\partial \bar{R}_j} \frac{\partial \bar{G}}{\partial \dot{\bar{x}}^j} \equiv (\bar{F}, \bar{G})', \tag{6.2}$$

which proves the above assertion.

Let us now conversely assume that the transformation (3.1) leaves invariant the POISSON bracket of *any* two functions of the canonical variables. In particular, this holds true for

$$\begin{aligned} & \underset{1}{F}^i \overset{\text{def}}{=} \bar{x}^i(x^j, \dot{x}^j, P_j, R_j), \quad \underset{2}{F}^i \overset{\text{def}}{=} \dot{\bar{x}}^i(x^j, \dot{x}^j, P_j, R_j), \\ & \underset{i}{F}^3 \overset{\text{def}}{=} \bar{P}_i(x^j, \dot{x}^j, P_j, R_j), \quad \underset{i}{F}^4 \overset{\text{def}}{=} \bar{R}_i(x^j, \dot{x}^j, P_j, R_j). \end{aligned} \tag{6.3}$$

Obviously, in our notation (6.1),

$$\underset{1}{\bar{F}}^i(\bar{x}^j, \dot{\bar{x}}^j, \bar{P}_j, \bar{R}_j) \overset{\text{def}}{=} \bar{x}^i(x^j(\bar{x}^k, \dot{\bar{x}}^k, \bar{P}_k, \bar{R}_k), \dots, R_j(\bar{x}^k, \dot{\bar{x}}^k, \bar{P}_k, \bar{R}_k)) \equiv \bar{x}^i,$$

and similarly

$$\bar{F}^i_2 = \dot{\bar{x}}^i,\ \bar{F}^3_i = \bar{P}_i,\ \bar{F}^4_i = \bar{R}_i.$$

Hence, for example, by the analogue of (5.10) for the «barred» variables,

$$(\bar{F}^i_1,\ \bar{F}^3_j)' = \delta^i_j, \tag{6.4}$$

while, by assumption,

$$(\bar{F}^i_1,\ \bar{F}^3_j)' = (F^i_1,\ F^3_j),$$

so that as a result of (6.3) and (6.4)

$$(F^i_1,\ F^3_j) = (\bar{x}^i,\ \bar{P}_j) = \delta^i_j.$$

The remaining members of the POISSON bracket relations (5.8) are also easily found, so that we have proved

THEOREM 4. - *The transformation* (3.1) *is canonical if and only if it leaves invariant the Poisson bracket of two arbitrary functions.*

Theorem 4 immediately allows us to show that a product of two canonical transformations is itself again a canonical transformation. Furthermore, we have seen that the inverse of a canonical transformation is again canonical, and we may therefore state

THEOREM 5. - *The totality of all generalized canonical transformations forms a group.*

The dynamical application of the fact that the canonical transformations associated with the first order problem leave invariant the canonical equations of the latter, is one of the most important properties of these transformations. Whereas we shall *not* attempt any dynamical interpetation of a *second order* variational problem, we may nevertheless investigate some effects of the generalized canonical transformations on the extremals of problems of this kind.

Let us initially consider the *non-parameter-invariant* problem defined by the integral (2.1). In terms of the corresponding Hamiltonian

$$H = H(t,\ x^i,\ \dot{x}^i,\ P_i,\ R_i) \tag{6.5}$$

the *canonical equations* have the form

$$\frac{dx^i}{dt}=\frac{\partial H}{\partial P_i},\ \frac{d\dot{x}^i}{dt}=\frac{\partial H}{\partial R_i},\ \frac{dP_i}{dt}=-\frac{\partial H}{\partial x^i},\ \frac{dR_i}{dt}=-\frac{\partial H}{\partial \dot{x}^i} \tag{6.6}$$

(cf. [9], p. 266).

As in (6.1) we now *define* the function

$$\bar{H}(t,\ \bar{x}^i,\ \dot{\bar{x}}^i,\ \bar{P}_i,\ \bar{R}_i)\overset{\text{def}}{=\!=}H(t,\ x^i,\ \dot{x}^i,\ P_i,\ R_i), \tag{6.7}$$

where the $\bar{x}^i$, $\dot{\bar{x}}^i$, $\bar{P}_i$, $\bar{R}_i$ arise from the set x^i, $\dot{x}^i$, P_i, R_i by means of a canonical transformation. Then

$$\frac{\partial \bar{H}}{\partial \bar{x}^i}=\frac{\partial H}{\partial x^j}\frac{\partial x^j}{\partial \bar{x}^i}+\frac{\partial H}{\partial \dot{x}^j}\frac{\partial \dot{x}^j}{\partial \bar{x}^i}+\frac{\partial H}{\partial P_j}\frac{\partial P_j}{\partial \bar{x}^i}+\frac{\partial H}{\partial R_j}\frac{\partial R_j}{\partial \bar{x}^i}, \tag{6.8}$$

while

$$\frac{d\bar{P}_i}{dt}=\frac{\partial \bar{P}_i}{\partial x^j}\frac{dx^j}{dt}+\frac{\partial \bar{P}_i}{\partial \dot{x}_j}\frac{d\dot{x}^j}{dt}+\frac{\partial \bar{P}_i}{\partial P_j}\frac{dP_j}{dt}+\frac{\partial \bar{P}_i}{\partial R_j}\frac{dR_j}{dt}. \tag{6.9}$$

An application of the reciprocity relations (5.5) to the appropriate terms on the right-hand side of (6.8), followed by the addition of the resulting equation to (6.9), gives rise to the relations

$$\frac{\partial \bar{H}}{\partial \bar{x}^i}+\frac{d\bar{P}_i}{dt}=\frac{\partial \bar{P}_i}{\partial x^j}\left(\frac{dx^j}{dt}-\frac{\partial H}{\partial P_j}\right)+\frac{\partial \bar{P}_i}{\partial \dot{x}^j}\left(\frac{d\dot{x}^j}{dt}-\frac{\partial H}{\partial R_j}\right)+$$
$$+\frac{\partial \bar{P}_i}{\partial P_j}\left(\frac{dP_j}{dt}+\frac{\partial H}{\partial x^j}\right)+\frac{\partial \bar{P}_i}{\partial R_j}\left(\frac{dR_j}{dt}+\frac{\partial H}{\partial \dot{x}^j}\right).$$

Hence, if the functions x^i, $\dot{x}^i$, P_i, R_i satisfy the canonical equations (6.6), it follows that

$$\frac{d\bar{P}_i}{dt}=-\frac{\partial \bar{H}}{\partial \bar{x}^i}.$$

We apply a similar argument to $\partial\bar{H}/\partial\dot{\bar{x}}^i$, $\partial\bar{H}/\partial\bar{P}_i$, $\partial\bar{H}/\partial\bar{R}_i$ in conjunction with $d\bar{R}_i/dt$, $d\bar{x}^i/dt$, $d\dot{\bar{x}}^i/dt$ respectively. In this way the following equations are found:

$$\frac{d\bar{x}^i}{dt}=\frac{\partial \bar{H}}{\partial \bar{P}_i},\ \frac{d\dot{\bar{x}}^i}{dt}=\frac{\partial \bar{H}}{\partial \bar{R}_i},\ \frac{d\bar{P}_i}{dt}=-\frac{\partial \bar{H}}{\partial \bar{x}^i},\ \frac{d\bar{R}_i}{dt}=-\frac{\partial \bar{H}}{\partial \dot{\bar{x}}^i}. \tag{6.10}$$

These are the canonical equations of a problem with Hamiltonian $\bar{H}(t, \bar{x}^i, \dot{\bar{x}}^i, \bar{P}_i, \bar{R}_i)$ which is defined by (6.7). The canonical transformation (3.1) therefore *maps extremals of a second order variational problem with Hamiltonian H into extremals of a problem with Hamilonian $\bar{H}$. In this sense the canonical equations* (6.6) *are invariant under a generalized canonical transformation.* We note that we have implicitly assumed that *the parameter t is not affected iu this process.*

Before elaborating on the last statement, let us, conversely, investigate those transformations (3.1) which leave invariant the equations (6.6) for *arbitrary* $H(t, x^i, \dot{x}^i, P_i, R_i)$, i.e. transformations which are such that equations (6.6) always imply equations (6.10) (subject to (6.7)). But, from (6.8), (6.9), (6.10), we have

$$0 \equiv \frac{\partial \bar{H}}{\partial \bar{x}^i} + \frac{d\bar{P}_i}{dt} = \frac{\partial H}{\partial x^j}\frac{\partial x^j}{\partial \bar{x}^i} + \frac{\partial H}{\partial \dot{x}^j}\frac{\partial \dot{x}^j}{\partial \bar{x}^i} + \frac{\partial H}{\partial P_j}\frac{\partial P_j}{\partial \bar{x}^i} + \frac{\partial H}{\partial R_j}\frac{\partial R_j}{\partial \bar{x}^i} +$$

$$+ \frac{\partial \bar{P}_i}{\partial x^j}\frac{dx^j}{dt} + \frac{\partial \bar{P}_i}{\partial \dot{x}^j}\frac{d\dot{x}^j}{dt} + \frac{\partial \bar{P}_i}{\partial P_j}\frac{dP_j}{dt} + \frac{\partial \bar{P}_i}{\partial R_j}\frac{dR_j}{dt},$$

into the right-hand side of which we substitute from (6.6) to find that

$$0 = \frac{\partial H}{\partial x^j}\left(\frac{\partial x^j}{\partial \bar{x}^i} - \frac{\partial \bar{P}_i}{\partial P_j}\right) + \frac{\partial H}{\partial \dot{x}^j}\left(\frac{\partial \dot{x}^j}{\partial \bar{x}^i} - \frac{\partial \bar{P}_i}{\partial R_j}\right) + \frac{\partial H}{\partial P_j}\left(\frac{\partial P_j}{\partial \bar{x}^i} + \frac{\partial \bar{P}_i}{\partial x^j}\right) + \frac{\partial H}{\partial R_j}\left(\frac{\partial R_j}{\partial \bar{x}^i} + \frac{\partial \bar{P}_i}{\partial \dot{x}^j}\right).$$

Since H is arbitrary it follows that the first column of the set of reciprocity relations (5.5) must hold. The remaining members of these relations are easily obtained in an analogous fashion, so that *the transformations must be canonical.*

As regards the above-mentioned effect of the canonical transformation on the parameter t, it is well-known that in a non-parameter-invariant variational problem t can be considered as an additional x-coordinate (for first order problems cf. [7], pp. 44-48; a similar approach may be made regarding problems of the second order as is indicated in [5], p. 84). In fact, let us consider an integral

$$I = \int L\left(t, x^\alpha, \frac{dx^\alpha}{dt}, \frac{d^2x^\alpha}{dt^2}\right) dt \tag{6.11}$$

in (t, x^α)-space. We put $t = x^n$ and introduce an arbitrary parameter τ, now denoting differentiation with respect to τ by dots: $\dot{x}^i = dx^i/d\tau$, etc. Then

(6.11) becomes

$$I = \int L^*(x^i, \dot{x}^i, \ddot{x}^i)\, d\tau, \tag{6.12}$$

where

$$L^*(x^i, \dot{x}^i, \ddot{x}^i) = L\left(x^n, x^\alpha, \frac{\dot{x}^\alpha}{\dot{x}^n}, \frac{1}{(\dot{x}^n)^2}\left\{\ddot{x}^\alpha - \frac{\dot{x}^\alpha}{\dot{x}^n}\ddot{x}^n\right\}\right)\dot{x}^n$$

([5], p. 84). The canonical transformation (3.1), as applied to the canonical coordinates corresponding to (6.12), may be regarded as a *t-dependent transformation* in terms of the problem (6.11) if the above construction is kept in mind. One can then prove that this transformation leaves invariant the canonical equations of the problem defined by (6.11) in the following sense: There exists a function $\bar{H}(\bar{t}, \bar{x}^\alpha, \dot{\bar{x}}^\alpha, \bar{P}_\alpha, \bar{R}_\alpha)$ such that equations (6.6) (with indices i replaced by α) give rise to the set

$$\frac{d\bar{x}^\alpha}{d\bar{t}} = \frac{\partial \bar{H}}{\partial \bar{P}_\alpha}, \quad \frac{d\dot{\bar{x}}^\alpha}{d\bar{t}} = \frac{\partial \bar{H}}{\partial \bar{R}_\alpha}, \quad \frac{d\bar{P}_\alpha}{d\bar{t}} = -\frac{\partial \bar{H}}{\partial \bar{x}^\alpha}, \quad \frac{d\bar{R}_\alpha}{d\bar{t}} = -\frac{\partial \bar{H}}{\partial \dot{\bar{x}}^\alpha},$$

where $\bar{t} = \bar{x}^n$.

We shall not do so here, however, and instead turn to the *parameter-invariant* problem (where we note that (6.12) is an example of a parameter-invariant integral). The above method does not apply here, since in general no uniquely defined Hamiltonian exists for such problems, and the canonical equations corresponding to (6.6) involve two arbitrary parameters (see [6] for the canonical formalism in this case). The Lagrangians of these problems are characterized by the identities

$$L = P_i \dot{x}^i + R_i \ddot{x}^i, \quad R_i \dot{x}^i = 0,$$

so that the fundamental integral simply reads

$$I = \int_{C\,Q_1}^{Q_2} (P_i dx^i + R_i d\dot{x}^i)\, d\tau.$$

It immediately follows from (3.3) that a canonical transformation changes this integral into

$$\bar{I} = \int_{C\,\bar{Q}_1}^{\bar{Q}_2} (\bar{P}_i d\bar{x}^i + \bar{R}_i d\dot{\bar{x}}^i)\, d\tau \equiv I + \int d\Psi.$$

In other words, the two integrals are *equivalent* and the canonical transformation therefore *maps extremals of the one into extremals of the other.*

If we return to the canonical equations (6.6) of the non-parameter-invariant problem, we immediately note that they can be written as follows in terms of the POISSON brackets (5.6):

$$\frac{dx^i}{dt} = (x^i,\ H),\ \frac{d\dot{x}^i}{dt} = (\dot{x}^i,\ H),\ \frac{dP_i}{dt} = (P_i,\ H),\ \frac{dR_i}{dt} = (R_i,\ H).$$

Furthermore, if an arbitrary function $F = F(t,\ x^j,\ \dot{x}^j,\ P_j,\ R_j)$ is differentiated along an extremal, one finds, that

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + (F,\ H).$$

7. - Properties of the Jacobian J.

We now derive some further properties of the generalized canonical transformation, and, in particular, we introduce the concept of elementary canonical transformations, with the help of which one can show that the canonical transformation (3.1) can also be generated by a function $\Phi(x^i,\ \dot{x}^i,\ \bar{x}^i,\ \dot{\bar{x}}^i)$ instead of (3.2). This leads to the conclusion that (4.2) can be replaced by the statement that $J = +1$.

Let $(\alpha_1,\ \alpha_2,\ \dots,\ \alpha_n)$, $(\beta_1,\ \beta_2,\ \dots,\ \beta_n)$ be two arbitrary permutations of the numbers 1, 2, ..., n. Then a direct evaluation of the POISSON bracket relations (5.8) (in which the identities (5.10) are used) shows that the following two transformations are canonical:

$$\bar{x}^i = x^{\alpha_i},\quad \dot{\bar{x}}^i = \dot{x}^{\alpha_i},\quad \bar{P}_i = P_{\alpha_i},\quad \bar{R}_i = R_{\alpha_i}, \tag{7.1}$$

$$\bar{x}^i = \dot{x}^{\alpha_i},\quad \dot{\bar{x}}^i = x^{\beta_i},\quad \bar{P}_i = R_{\alpha_i},\quad \bar{R}_i = P_{\beta_i}. \tag{7.2}$$

Both transformations represent a mere relabelling of the coordinates, and it is important to note that in (7.2) the roles of the x^i, $\dot{x}^i$ are interchanged, and similarly those of the P_i, R_i. It is also easily seen by direct manipulation that the Jacobian of each of these transformations is equal to $+1$.

A more general transformation (which includes (7.1), (7.2) as special cases) may be defined as follows: The $2n$ quantities x^i, $\dot{x}^i$ are permuted arbitrarily, after which the first n are denoted by $\bar{x}^i$, and the remaining n by $\dot{\bar{x}}^i$. Then the same permutation is applied to the set P_i, R_i, this process yielding $\bar{\bar{P}}_i$, $\bar{R}_i$. We shall denote this transformation by P, i.e.

$$(7.3) \qquad P\begin{pmatrix} x^i; & \dot{x}^j \\ P_i; & R_j \end{pmatrix} = \begin{pmatrix} \bar{x}^i; & \dot{\bar{x}}^j \\ \bar{P}_i; & \bar{R}_j \end{pmatrix}.$$

The latter is again a generalized canonical transformation with Jacobian to $+1$.

We shall also be concerned with a different type of generalized canonical transformation. Let N be some number: $0 \leq N \leq 2n$. Then we have the the following possibilities:

(i) $1 \leq N \leq n$. We let the indices μ, ν run from 1 to N and, if $N < n$, μ', ν' from $N+1$ to n, and consider the transformation

$$(7.4) \qquad \bar{x}^\mu = x^\mu,\ \bar{x}^{\mu'} = -P_{\mu'},\ \dot{\bar{x}}^i = -R_i;\ \bar{P}_\mu = P_\mu,\ \bar{P}_{\mu'} = x^{\mu'},\ \bar{R}_i = \dot{x}^i.$$

Then one immediately finds that

$$(\bar{x}^\mu, \bar{P}_\nu) = (x^\mu, P_\nu),\ (\bar{x}^\mu, \bar{P}_{\nu'}) = (x^\mu, x^{\nu'}),$$

$$(\bar{x}^{\mu'}, \bar{P}_\nu) = (-P_{\mu'}, P_\nu),\ (\bar{x}^{\mu'}, \bar{P}_{\nu'}) = (-P_{\mu'}, x^{\nu'}) = (x^{\nu'}, P_{\mu'}),$$

and it follows from the identities (5.10) that $(\bar{x}^i, \bar{P}_j) = \delta^i_j$. The other members of the POISSON bracket relations (5.8) are derived in a similar fashion, and thus (7.4) indeed constitutes a generalized canonical transformation. By reducing its Jacobian to diagonal form, we also find that the latter has the value $+1$.

(ii) If $N = 0$ the transformation (7.4) is defined as

$$(7.5) \qquad \bar{x}^i = -P_i,\ \dot{\bar{x}}^i = -R_i;\ \bar{P}_i = x^i,\ \bar{R}_i = \dot{x}^i.$$

These two transformations may be formally combined in terms of the notation

$$(7.6) \quad T_1\begin{pmatrix} x^i; & \dot{x}^j \\ P_i; & R_j \end{pmatrix} = \begin{pmatrix} \bar{x}^i; & \dot{\bar{x}}^j \\ \bar{P}_i; & \bar{R}_j \end{pmatrix} \equiv \begin{pmatrix} x^1, \ldots, x^N, & -P_{N+1}, \ldots, -P_n; & -R_j \\ P_1, \ldots, P_N, & x^{N+1}, \ldots, \quad x^n; & \dot{x}^j \end{pmatrix}, \ 0 \leq N \leq n.$$

(iii) If $N \geq n+1$ we define a similar transformation T_2 by

$$T_2\begin{pmatrix} x^i; & \dot{x}^j \\ P_i; & R_j \end{pmatrix} = \begin{pmatrix} \bar{x}^i; & \dot{\bar{x}}^j \\ \bar{P}_i; & \bar{R}_j \end{pmatrix} \equiv \tag{7.7}$$

$$\equiv \begin{pmatrix} x^i; & \dot{x}^1, & \ldots, & \dot{x}^{N-n}, & -R_{N-n+1}, \ldots, & -R_n \\ P_i; & R_1, & \ldots, & R_{N-n}, & \dot{x}^{N-n+1}, \ldots, & \dot{x}^n \end{pmatrix}, \quad n+1 \leq N \leq 2n,$$

which is also a generalized canonical transformation with $J = +1$.

The transformations P, T_1 and T_2 have the following fundamental property. Let K: $(x^i, \dot{x}^j; P_h, R_k)$ be a set of canonical variables, and q be an *arbitrary* member of the subset $(x^{m+1}, \ldots, x^n, \dot{x}^j)$, $0 \leq m \leq n$, of K. Then it follows from the definitions of P and T_1 that a successive application of these transformations enables us to transform K into the set $\bar{K}$: $(\bar{x}^i, \dot{\bar{x}}^j; \bar{P}_h, \bar{R}_k)$, with

$$\bar{x}^1 = x^1, \ \ldots, \ \bar{x}^m = x^m; \ \bar{P}_1 = P_1, \ \ldots, \ \bar{P}_m = P_m; \ \bar{P}_{m+1} = q. \tag{7.8}$$

If q belongs to $(P_{m+1}, \ldots, P_n, R_k)$ we only need to apply the transformation P to ensure that the members of $\bar{K}$ satisfy (7.8).

Similarly, if $m \geq n+1$, a combination of T_2 and P allows us to find a set $\bar{K}$ for which the conditions

$$\begin{gathered} \bar{x}^i = x^i, \ \dot{\bar{x}}^1 = \dot{x}^1, \ \ldots, \ \dot{\bar{x}}^{m-n-1} = \dot{x}^{m-n-1}; \\ \bar{P}_i = P_i, \ \bar{R}_1 = R_1, \ \ldots, \ \bar{R}_{m-n-1} = R_{m-n-1}; \ \bar{R}_{m-n} = q \end{gathered} \tag{7.9}$$

hold if q is some member of the set $(\dot{x}^{m-n}, \ldots, \dot{x}^n, R_{m-n}, \ldots, R_n)$.

Any combination of the transformations P, T_1, T_2 (and their inverses) will be called an *elementary canonical transformation.* For such a transformation the Jacobian $J = +1$, while it may also be proved that the totality of these transformations forms a group.

The elementary canonical transformations now enable us to prove

THEOREM 6. – *Each generalized canonical transformation can be considered as the product of an elementary canonical transformation and one for which*

$$\det \begin{vmatrix} \dfrac{\partial \bar{x}^i}{\partial P_j}, & \dfrac{\partial \dot{\bar{x}}^i}{\partial P_j} \\ \dfrac{\partial \bar{x}^i}{\partial R_j}, & \dfrac{\partial \dot{\bar{x}}^i}{\partial R_j} \end{vmatrix} \neq 0. \tag{7.10}$$

PROOF. - The theorem is proved by induction. Since, by theorem 3, $(\bar{x}^1, \bar{P}_1) = 1$, not all of the quantities $\partial\bar{x}^1/\partial x^i$, $\partial\bar{x}^1/\partial P_i$, $\partial\bar{x}^1/\partial\dot{x}^i$, $\partial\bar{x}^1/\partial R_i$ can vanish simultaneously. One can thus effect an elementary canonical transformation on x^i, $\dot{x}^i$, P_i, R_i such that

$$\frac{\partial\bar{x}^1}{\partial P_1} \neq 0. \tag{7.11}$$

We now assume that for a number m, $1 < m < n$,

$$\frac{\partial(\bar{x}^1, \ldots, \bar{x}^m)}{\partial(P_1, \ldots, P_m)} \neq 0, \tag{7.12}$$

if necessary after suitable elementary canonical transformations have been carried out.

In what follows the indices a, b, c run from 1 to m, and a', b', c' run from $m+1$ to n, the summation convention being applied to these indices as well.

The matrix

$$H_{(m+1)} \stackrel{\text{def}}{=} \left\| \begin{matrix} \dfrac{\partial\bar{x}^a}{\partial P_b}, & \dfrac{\partial\bar{x}^a}{\partial P_{b'}}, & \dfrac{\partial\bar{x}^a}{\partial x^i}, & \dfrac{\partial\bar{x}^a}{\partial\dot{x}^i}, & \dfrac{\partial\bar{x}^a}{\partial R_i} \\[2ex] \dfrac{\partial\bar{x}^{m+1}}{\partial P_b}, & \dfrac{\partial\bar{x}^{m+1}}{\partial P_{b'}}, & \dfrac{\partial\bar{x}^{m+1}}{\partial x^i}, & \dfrac{\partial\bar{x}^{m+1}}{\partial\dot{x}^i}, & \dfrac{\partial\bar{x}^{m+1}}{\partial R_i} \end{matrix} \right\| \tag{7.13}$$

clearly has rank $m+1$, since otherwise

$$J \equiv \frac{\partial(\bar{x}^i, \dot{\bar{x}}^i, \bar{P}_i, \bar{R}_i)}{\partial(x^j, \dot{x}^j, P_j, R_j)} = 0,$$

which would violate (4.2). We now investigate the following set of $4n - 2m$ determinants of order $m+1$, in which a denotes the a-th *row*, and b the b-th *column*, while the last column is determined by the particular value of of b' or i under consideration:

$$D_P(b') \stackrel{\text{def}}{=} \det \begin{vmatrix} \dfrac{\partial\bar{x}^a}{\partial P_b}, & \dfrac{\partial\bar{x}^a}{\partial P_{b'}} \\[2ex] \dfrac{\partial\bar{x}^{m+1}}{\partial P_b}, & \dfrac{\partial\bar{x}^{m+1}}{\partial P_{b'}} \end{vmatrix}, \qquad D_R(i) \stackrel{\text{def}}{=} \det \begin{vmatrix} \dfrac{\partial\bar{x}^a}{\partial P_b}, & \dfrac{\partial\bar{x}^a}{\partial R_i} \\[2ex] \dfrac{\partial\bar{x}^{m+1}}{\partial P_b}, & \dfrac{\partial\bar{x}^{m+1}}{\partial R_i} \end{vmatrix}$$

$$D_x(b') \overset{\text{def}}{=} \det \begin{vmatrix} \dfrac{\partial \bar{x}^a}{\partial P_b}, & \dfrac{\partial \bar{x}^a}{\partial x^{b'}} \\ \dfrac{\partial \bar{x}^{m+1}}{\partial P_b}, & \dfrac{\partial \bar{x}^{m+1}}{\partial x^{b'}} \end{vmatrix}, \quad D_{\dot{x}}(i) \overset{\text{def}}{=} \det \begin{vmatrix} \dfrac{\partial \bar{x}^a}{\partial P_b}, & \dfrac{\partial \bar{x}^a}{\partial \dot{x}^i} \\ \dfrac{\partial \bar{x}^{m+1}}{\partial P_b}, & \dfrac{\partial \bar{x}^{m+1}}{\partial \dot{x}^i} \end{vmatrix}.$$

If each one of these were to vanish, there would, because of (7.12), exists m multipliers λ_c, not all zero, such that the relations

$$\begin{gathered} \frac{\partial \bar{x}^{m+1}}{\partial P_b} = \lambda_c \frac{\partial \bar{x}^c}{\partial P_b}, \quad \frac{\partial \bar{x}^{m+1}}{\partial P_{b'}} = \lambda_c \frac{\partial \bar{x}^c}{\partial P_{b'}}, \quad \frac{\partial \bar{x}^{m+1}}{\partial R_i} = \lambda_c \frac{\partial \bar{x}^c}{\partial R_i}, \\ \frac{\partial \bar{x}^{m+1}}{\partial x^{b'}} = \lambda_c \frac{\partial \bar{x}^c}{\partial x^{b'}}, \quad \frac{\partial \bar{x}^{m+1}}{\partial \dot{x}^i} = \lambda_c \frac{\partial \bar{x}^c}{\partial \dot{x}^i} \end{gathered} \tag{7.14}$$

would hold simultaneously.

An expansion of the POISSON bracket $(\bar{x}^a, \bar{x}^{m+1})$ according to the definition (5.6), followed by a substitution from (7.14) into the expression thus obtained, now shows that

$$(\bar{x}^a, \bar{x}^{m+1}) - \lambda_c(\bar{x}^a, \bar{x}^c) = \frac{\partial \bar{x}^a}{\partial P_b}\left(\frac{\partial \bar{x}^{m+1}}{\partial x^b} - \lambda_c \frac{\partial \bar{x}^c}{\partial x^b}\right). \tag{7.15}$$

Since we are dealing with a generalized canonical transformation we infer from (5.8) that the left-hand side of (7.15) vanishes, so that, as a result of the assumption (7.12),

$$\frac{\partial \bar{x}^{m+1}}{\partial x^b} = \lambda_c \frac{\partial \bar{x}^c}{\partial x^b}. \tag{7.16}$$

But the relations (7.14) and (7.16) together contradict our statement that the matrix $H_{(m+1)}$ has rank $m+1$. Therefore at least one of the determinants $D_P(b')$, $D_R(i)$, $D_x(b')$, $D_{\dot{x}}(i)$ is non-vanishing, say

$$\det \begin{vmatrix} \dfrac{\partial \bar{x}^a}{\partial P_b}, & \dfrac{\partial \bar{x}^a}{\partial q} \\ \dfrac{\partial \bar{x}^{m+1}}{\partial P_b}, & \dfrac{\partial \bar{x}^{m+1}}{\partial q} \end{vmatrix} \neq 0, \tag{7.17}$$

where q is some member of the set $(P_{b'}, R_i, x^{b'}, \dot{x}^i)$. We have seen above that, in this case, we may apply an elementary canonical transformation to the set of canonical variables $(x^i, \dot{x}^i, P_i, R_i)$ to obtain a new set for which relations corresponding to (7.8) hold. The latter equations imply that the P_b remain unchanged, while q becomes P_{m+1} (if we denote the new set again by $(x^i, \dot{x}^i, P_i, R_i)$), so that (7.17) now reads

$$\frac{\partial(\bar{x}^1, \dots, \bar{x}^{m+1})}{\partial(P_1, \dots, P_{m+1})} \neq 0. \tag{7.18}$$

By induction the inequality (7.18) thus holds for $m = 1, 2, \dots, n-1$.

We therefore assume that

$$\frac{\partial(\bar{x}^i, \dot{\bar{x}}^a)}{\partial(P_i, R_b)} \neq 0, \quad a, b = 1, 2, \dots, m, \tag{7.19}$$

for some m, $0 \leq m < n$ (where (7.19) reduces to $\partial(\bar{x}^i)/\partial(P_j)$ if $m = 0$). The matrix (7.13) is replaced by

$$H_{(m+n+1)} \overset{\text{def}}{=} \left\| \begin{matrix} \dfrac{\partial \bar{x}^i}{\partial P_j}, & \dfrac{\partial \bar{x}^i}{\partial R_b}, & \dfrac{\partial \bar{x}^i}{\partial R_{b'}}, & \dfrac{\partial \bar{x}^i}{\partial x^j}, & \dfrac{\partial \bar{x}^i}{\partial \dot{x}^j} \\[2ex] \dfrac{\partial \dot{\bar{x}}^a}{\partial P_j}, & \dfrac{\partial \dot{\bar{x}}^a}{\partial R_b}, & \dfrac{\partial \dot{\bar{x}}^a}{\partial R_{b'}}, & \dfrac{\partial \dot{\bar{x}}^a}{\partial x^j}, & \dfrac{\partial \dot{\bar{x}}^a}{\partial \dot{x}^j} \\[2ex] \dfrac{\partial \dot{\bar{x}}^{m+1}}{\partial P_j}, & \dfrac{\partial \dot{\bar{x}}^{m+1}}{\partial R_b}, & \dfrac{\partial \dot{\bar{x}}^{m+1}}{\partial R_{b'}}, & \dfrac{\partial \dot{\bar{x}}^{m+1}}{\partial x^j}, & \dfrac{\partial \dot{\bar{x}}^{m+1}}{\partial \dot{x}^j} \end{matrix} \right\|,$$

the rank of the latter being $m + n + 1$. A simultaneous vanishing of those determinants $D(q)$ which consist of the first $n + m$ columns of $H_{(m+n+1)}$ together with $\|\partial \bar{x}^i/\partial q, \partial \dot{\bar{x}}^a/\partial q, \partial \dot{\bar{x}}^{m+1}/\partial q\|$ as last column (where q denotes each one of $R_{b'}$, $\dot{x}^{b'}$ in turn), immediately implies the existence of $n + m$ non-trivial constants λ_k, μ_c, such that

$$\begin{aligned} &\frac{\partial \dot{\bar{x}}^{m+1}}{\partial P_j} = \lambda_k \frac{\partial \bar{x}^k}{\partial P_j} + \mu_c \frac{\partial \dot{\bar{x}}^c}{\partial P_j}, \quad \frac{\partial \dot{\bar{x}}^{m+1}}{\partial R_b} = \lambda_k \frac{\partial \bar{x}^k}{\partial R_b} + \mu_c \frac{\partial \dot{\bar{x}}^c}{\partial R_b}, \\ &\frac{\partial \dot{\bar{x}}^{m+1}}{\partial \dot{x}^{b'}} = \lambda_k \frac{\partial \bar{x}^k}{\partial \dot{x}^{b'}} + \mu_c \frac{\partial \dot{\bar{x}}^c}{\partial \dot{x}^{b'}}, \quad \frac{\partial \dot{\bar{x}}^{m+1}}{\partial R_{b'}} = \lambda_k \frac{\partial \bar{x}^k}{\partial R_{b'}} + \mu_c \frac{\partial \dot{\bar{x}}^c}{\partial R_{b'}}. \end{aligned} \tag{7.20}$$

Furthermore, if we evaluate the expressions $(\bar{x}^i, \dot{\bar{x}}^{m+1}) - \lambda_k(\bar{x}^i, \bar{x}^k) - \mu_c(\bar{x}^i, \dot{\bar{x}}^c)$, $(\dot{\bar{x}}^a, \dot{\bar{x}}^{m+1}) - \lambda_k(\dot{\bar{x}}^a, \bar{x}^k) - \mu_c(\dot{\bar{x}}^a, \dot{\bar{x}}^c)$, and then take (5.8) into account, we find that

$$(7.21) \qquad 0 = \frac{\partial \bar{x}^i}{\partial P_j} A_j + \frac{\partial \bar{x}^i}{\partial R_b} B_b, \quad 0 = \frac{\partial \dot{\bar{x}}^a}{\partial P_j} A_j + \frac{\partial \dot{\bar{x}}^a}{\partial R_b} B_b,$$

where

$$A_j = \lambda_k \frac{\partial \bar{x}^k}{\partial x^j} + \mu_c \frac{\partial \dot{\bar{x}}^c}{\partial x^j} - \frac{\partial \dot{\bar{x}}^{m+1}}{\partial x^j}, \quad B_b = \lambda_k \frac{\partial \bar{x}^k}{\partial \dot{x}^b} + \mu_c \frac{\partial \dot{\bar{x}}^c}{\partial \dot{x}^b} - \frac{\partial \dot{\bar{x}}^{m+1}}{\partial \dot{x}^b}.$$

In view of (7.19) we infer from (7.21) that $A_i = 0$, $B_b = 0$, and these equations cannot hold together with (7.20) if the rank of $H_{(m+n+1)}$ is $m+n+1$. This implies that $D(q) \neq 0$ for at least one member q of $(R_{b'}, \dot{x}^{b'})$, so that we only need to apply an elementary canonical transformation which leaves P_i, R_a unchanged and identifies q with R_{m+1} (cf. (7.9)) to prove that

$$(7.22) \qquad \frac{\partial(\bar{x}^i, \dot{\bar{x}}^a, \dot{\bar{x}}^{m+1})}{\partial(P_j, R_b, R_{m+1})} \neq 0.$$

This immediately completes the proof of the theorem.

Let us now, for the time being, restrict our attention to generalized canonical transformations (3.1) for which the inequality (7.10) holds. Then we may solve equations (3.1a) for

$$(7.23) \qquad P_i = \pi_i(x^j, \dot{x}^j, \bar{x}^j, \dot{\bar{x}}^j), \quad R_i = \rho_i(x^j, \dot{x}^j, \bar{x}^j, \dot{\bar{x}}^j).$$

A substitution into (3.1b) yields $\bar{P}_i$, $\bar{R}_i$ also as functions of $x^j, \dot{x}^j, \bar{x}^j, \dot{\bar{x}}^j$:

$$(7.24) \qquad \bar{P}_i = f_i(x^j, \dot{x}^j, \bar{x}^j, \dot{\bar{x}}^j), \quad \bar{R}_i = g_i(x^j, \dot{x}^j, \bar{x}^j, \dot{\bar{x}}^j).$$

The generating function (3.2) then also becomes a function of these variables, namely

$$(7.25) \qquad \Psi(x^j, \dot{x}^j, \pi_j, \rho_j) \overset{\text{def}}{=} \Phi(x^j, \dot{x}^j, \bar{x}^j, \dot{\bar{x}}^j),$$

so that (3.3) now reads

$$(7.26) \qquad d\Phi = f_i d\bar{x}^i + g_i d\dot{\bar{x}}^i - \pi_i dx^i - \rho_i d\dot{x}^i.$$

Here the x^i, $\dot{x}^i$, $\bar{x}^i$, $\dot{\bar{x}}^i$ are to be considered as the independent variables. We may therefore conclude that

$$\frac{\partial\Phi}{\partial\bar{x}^i}=f_i,\quad \frac{\partial\Phi}{\partial\dot{\bar{x}}^i}=g_i,\quad \frac{\partial\Phi}{\partial x^i}=-\pi_i,\quad \frac{\partial\Phi}{\partial\dot{x}^i}=-\rho_i, \tag{7.27}$$

or, keeping (7.23) and (7.24) in mind,

$$\frac{\partial\Phi}{\partial\bar{x}^i}=\bar{P}_i,\quad \frac{\partial\Phi}{\partial\dot{\bar{x}}^i}=\bar{R}_i,\quad \frac{\partial\Phi}{\partial x^i}=-P_i,\quad \frac{\partial\Phi}{\partial\dot{x}^i}=-R_i. \tag{7.28}$$

The above results now enable us to prove the principal result of this section, *viz.*

THEOREM 7. – *The value of the functional determinant of an arbitrary canonical transformation is always unity.*

PROOF. – In view of theorem 6, and the fact that the Jacobian of an elementary canonical transformation has been observed to be equal to one, theorem 7 will be proved if we show its assertion to be valid for all canonical transformations which are subject to (7.10).

The x^i, $\dot{x}^i$, $\bar{x}^i$, $\dot{\bar{x}}^i$ are therefore chosen as independent variables, and we have quite generally that

$$J=\frac{\partial(\bar{x}^i,\ \dot{\bar{x}}^i,\ \bar{P}_i,\ \bar{R}_i)}{\partial(x^j,\ \dot{x}^j,\ P_j,\ R_j)}=\frac{\partial(\bar{x}^i,\ \dot{\bar{x}}^i,\ f_i,\ g_i)}{\partial(x^j,\ \dot{x}^j,\ \bar{x}^j,\ \dot{\bar{x}}^j)}\cdot\left\{\frac{\partial(x^k,\ \dot{x}^k,\ \pi_k,\ \rho_k)}{\partial(x^h,\ \dot{x}^h,\ \bar{x}^h,\ \dot{\bar{x}}^h)}\right\}^{-1}, \tag{7.29}$$

where we have substituted from (7.23), (7.24). But clearly

$$\frac{\partial(\bar{x}^i,\ \dot{\bar{x}}^i,\ f_i,\ g_i)}{\partial(x^j,\ \dot{x}^j,\ \bar{x}^j,\ \dot{\bar{x}}^j)}=\frac{\partial(f_i,\ g_i)}{\partial(x^j,\ \dot{x}^j)}, \tag{7.30}$$

while

$$\frac{\partial(x^k,\ \dot{x}^k,\ \pi_k,\ \rho_k)}{\partial(x^h,\ \dot{x}^h,\ \bar{x}^h,\ \dot{\bar{x}}^h)}=\frac{\partial(\pi_k,\ \rho_k)}{\partial(\bar{x}^h,\ \dot{\bar{x}}^h)}. \tag{7.31}$$

For the determinants on the right-hand sides of (7.30) and (7.31) we now obtain with the help of (7.27)

$$\frac{\partial(f_i,\ g_i)}{\partial(x^j,\ \dot{x}^j)}=\det\begin{vmatrix}\dfrac{\partial^2\Phi}{\partial x^j\,\partial\bar{x}^i}, & \dfrac{\partial^2\Phi}{\partial\dot{x}^j\,\partial\bar{x}^i}\\[2ex] \dfrac{\partial^2\Phi}{\partial x^j\partial\dot{\bar{x}}^i}, & \dfrac{\partial^2\Phi}{\partial\dot{x}^j\,\partial\dot{\bar{x}}^i}\end{vmatrix}\overset{\mathrm{def}}{=}D,$$

together with

$$\frac{\partial(\pi_i,\ \rho_i)}{\partial(\bar{x}^i,\ \dot{\bar{x}}^j)} = \det \begin{vmatrix} -\dfrac{\partial^2\Phi}{\partial \bar{x}^j\,\partial x^i}, & -\dfrac{\partial^2\Phi}{\partial \dot{\bar{x}}^j\,\partial x^i} \\[2ex] -\dfrac{\partial^2\Phi}{\partial \bar{x}^j\,\partial \dot{x}^i}, & -\dfrac{\partial^2\Phi}{\partial \dot{\bar{x}}^j\,\partial \dot{x}^i} \end{vmatrix} = (-1)^{2n} D = D,$$

where we have used the fact that Φ is of class C^2. Hence, substituting from (7.30), (7.31) into (7.29), we find that

$$J = 1,$$

and, in view of the remarks made above, this now holds for *arbitrary* canonical transformations as well.

REMARK. - The above results may also be derived for a generalized canonical transformation which is defined in terms of the POISSON bracket (or LAGRANGE bracket) relations, and where the functions concerned are only required to be of class C^1. We shall not do so here however; the approach is analogous to that of CARATHÉODORY ([1], § 98 *et seq.*) for the ordinary canonical transformation.

8. - t-independent generalized canonical transformations.

We have observed in section 6 that in a non-parameter-invariant problem the parameter t can be considered as an additional x-coordinate. Conversely, in an integral of type (6.12), one of the coordinates (usually x^n) may be assigned the special role of the parameter t (thereby rendering the problem non-parameter-invariant, whether (6.12) is parameter-invariant or not). In this case one may regard the canonical transformation considered above as t-dependent, and it is interesting to investigate the relation of such a transformation to the so-called t-independent canonical transformations which we now proceed to define. (Our approach is a generalization of the method of RUND ([7]. pp. 96-99)).

We find it convenient to write the transformation (3.1) in the following manner, where the indices α, β again assume the values 1, 2, ..., $n-1$:

$$(8.1)\qquad \begin{aligned} &\bar{x}^\alpha = \bar{x}^\alpha(x^j,\ \dot{x}^j,\ P_j,\ R_j), && \bar{x}^n = \bar{x}^n(x^j,\ \dot{x}^j,\ P_j,\ R_j), \\ &\dot{\bar{x}}^\alpha = \dot{\bar{x}}^\alpha(x^j,\ \dot{x}^j,\ P_j,\ R_j), && \dot{\bar{x}}^n = \dot{\bar{x}}^n(x^j,\ \dot{x}^j,\ P_j,\ R_j), \\ &\bar{P}_\alpha = \bar{P}_\alpha(x^j,\ \dot{x}^j,\ P_j,\ R_j), && \bar{P}_n = \bar{P}_n(x^j,\ \dot{x}^j,\ P_j,\ R_j), \\ &\bar{R}_\alpha = \bar{R}_\alpha(x^j,\ \dot{x}^j,\ P_j,\ R_j), && \bar{R}_n = \bar{R}_n(x^j,\ \dot{x}^j,\ P_j,\ R_j). \end{aligned}$$

This canonical transformation is said to be t–independent if $\bar{x}^\alpha$, $\dot{\bar{x}}^\alpha$, $\bar{P}_i$, $\bar{R}_i$ do do not depend explicitly on x^n and $\dot{x}^n$, and if

$$\bar{x}^n = x^n, \ \dot{\bar{x}}^n = \dot{x}^n. \tag{8.2}$$

Such a transformation is thus characterized by (8.2) together with the relations

$$\frac{\partial \bar{x}^\alpha}{\partial x^n} = 0, \ \frac{\partial \bar{x}^\alpha}{\partial \dot{x}^n} = 0, \ \frac{\partial \dot{\bar{x}}^\alpha}{\partial x^n} = 0, \ \frac{\partial \dot{\bar{x}}^\alpha}{\partial \dot{x}^n} = 0, \tag{8.3a, b, c, d}$$

$$\frac{\partial \bar{P}_i}{\partial x^n} = 0, \ \frac{\partial \bar{P}_i}{\partial \dot{x}^n} = 0, \ \frac{\partial \bar{R}_i}{\partial x^n} = 0, \ \frac{\partial \bar{R}_i}{\partial \dot{x}^n} = 0. \tag{8.4a, b, c, d}$$

We deduce from (8.2) that

$$\frac{\partial \bar{x}^n}{\partial \dot{x}^n} = 0, \ \frac{\partial \dot{\bar{x}}^n}{\partial x^n} = 0. \tag{8.5a, b}$$

Our aim is to construct a canonical transformation involving only the $4n-4$ variables x^α, $\dot{x}^\alpha$, P_α, R_α. If the reciprocity relations (5.5) are applied to (8.3), (8.4), and (8.5), the following equations are obtained:

$$\begin{gathered}
0 = \frac{\partial \bar{x}^\alpha}{\partial x^n} = \frac{\partial P_n}{\partial \bar{P}_\alpha}, \quad 0 = \frac{\partial \bar{x}^i}{\partial \dot{x}^n} = \frac{\partial R_n}{\partial \bar{P}_i}, \quad 0 = \frac{\partial \dot{\bar{x}}^i}{\partial x^n} = \frac{\partial P_n}{\partial \bar{R}_i}, \quad 0 = \frac{\partial \dot{\bar{x}}^\alpha}{\partial \dot{x}^n} = \frac{\partial R_n}{\partial \bar{R}_\alpha}, \\
0 = \frac{\partial \bar{P}_i}{\partial x^n} = -\frac{\partial P_n}{\partial \bar{x}^i}, \ 0 = \frac{\partial \bar{P}_i}{\partial \dot{x}^n} = -\frac{\partial R_n}{\partial \bar{x}^i}, \ 0 = \frac{\partial \bar{R}_i}{\partial x^n} = -\frac{\partial P_n}{\partial \dot{\bar{x}}^i}, \ 0 = \frac{\partial \bar{R}_i}{\partial \dot{x}^n} = -\frac{\partial R_n}{\partial \dot{\bar{x}}^i}, \\
1 = \frac{\partial \bar{x}^n}{\partial x^n} = \frac{\partial P_n}{\partial \bar{P}_n}, \quad 1 = \frac{\partial \dot{\bar{x}}^n}{\partial \dot{x}^n} = \frac{\partial R_n}{\partial \bar{R}_n}.
\end{gathered} \tag{8.6}$$

We immediately conclude that

$$\bar{P}_n = P_n + k_1, \quad \bar{R}_n = R_n + k_2, \tag{8.7}$$

where k_1, k_2 are arbitrary constants.

It is obvious from (8.3), (8.4) and (8.5) that if we take the inverse of (8.1) we again find that

$$x^n = \bar{x}^n, \quad \dot{x}^n = \dot{\bar{x}}^n. \tag{8.8}$$

Hence, applying the reciprocity relations (5.5) once more, we see that

$$\frac{\partial \bar{x}^\alpha}{\partial P_n} = -\frac{\partial x^n}{\partial \bar{P}_\alpha} = 0,\ \frac{\partial \dot{\bar{x}}^\alpha}{\partial R_n} = -\frac{\partial \dot{x}^n}{\partial \bar{R}_\alpha} = 0,\ \frac{\partial \bar{x}^\alpha}{\partial R_n} = -\frac{\partial \dot{x}^n}{\partial \bar{P}_\alpha} = 0,\ \frac{\partial \dot{\bar{x}}^\alpha}{\partial P_n} = -\frac{\partial x^n}{\partial \bar{R}_\alpha} = 0,$$

(8.9)

$$\frac{\partial \bar{P}_\alpha}{\partial P_n} = \frac{\partial x^n}{\partial \bar{x}^\alpha} = 0,\ \frac{\partial \bar{P}_\alpha}{\partial R_n} = \frac{\partial \dot{x}^n}{\partial \bar{x}^\alpha} = 0,\ \frac{\partial \bar{R}_\alpha}{\partial P_n} = \frac{\partial x^n}{\partial \dot{\bar{x}}^\alpha} = 0,\ \frac{\partial \bar{R}_\alpha}{\partial R_n} = \frac{\partial \dot{x}^n}{\partial \dot{\bar{x}}^\alpha} = 0.$$

If we now take (8.6), (8.7), (8.9) into consideration we deduce that any canonical transformation (8.1) which is subject to (8.2), (8.3) and (8.4) has the explicit form

$$\begin{aligned} &\bar{x}^\alpha = \bar{x}^\alpha(x^\beta, \dot{x}^\beta, P_\beta, R_\beta), \quad \dot{\bar{x}}^\alpha = \dot{\bar{x}}^\alpha(x^\beta, \dot{x}^\beta, P_\beta, R_\beta), \\ &\bar{P}_\alpha = \bar{P}_\alpha(x^\beta, \dot{x}^\beta, P_\beta, R_\beta), \quad \bar{R}_\alpha = \bar{R}_\alpha(x^\beta, \dot{x}^\beta, P_\beta, R_\beta), \end{aligned} \tag{8.10}$$

$$\bar{x}^n = x^n,\ \dot{\bar{x}}^n = \dot{x}^n,\ \bar{P}_n = P_n + k_1,\ \bar{R}_n = R_n + k_2, \tag{8.11}$$

where the functions in (8.10) are identical with the corresponding ones in (8.1).

In order to see what happens to the generating function (3.2), we evaluate the n-th members of each of the equations (3.4), e.g. (3.4a):

$$\frac{\partial \Psi}{\partial x^n} = -P_n + \bar{P}_h \frac{\partial \bar{x}^h}{\partial x^n} + \bar{R}_h \frac{\partial \dot{\bar{x}}^h}{\partial x^n},$$

which, as a result of (8.2), (8.3a), (8.3c), (8.5b) and (8.7), reduces to

$$\frac{\partial \Psi}{\partial x^n} = k_1.$$

Similarly we find from (3.4b), (8.3b), (8.5a), (8.2), (8.3d) and (8.7) that

$$\frac{\partial \Psi}{\partial \dot{x}^n} = k_2.$$

In a similar fashion the n-th members of equations (3.4c, d) are found to reduce to

$$\frac{\partial \Psi}{\partial P_n} = 0, \quad \frac{\partial \Psi}{\partial R_n} = 0.$$

The last four relations thus show that the generating function (3.2) must have the form

$$\Psi(x^i,\ \dot{x}^i,\ P_i,\ R_i) = \psi(x^\alpha,\ \dot{x}^\alpha,\ P_\alpha,\ R_\alpha) - k_1 x^n - k_2 \dot{x}^n, \tag{8.12}$$

where ψ is some function of class C^2.

But, by (8.12),

$$d\psi = d\Psi + k_1 dx^n + k_2 d\dot{x}^n,$$

so that, in view of (3.3) and (8.11),

$$d\psi = \bar{P}_\alpha d\bar{x}^\alpha + \bar{R}_\alpha d\dot{\bar{x}}^\alpha - P_\alpha dx^\alpha - R_\alpha d\dot{x}^\alpha,$$

which is precisely of the form (3.3) in terms of the $4n-4$ variables $x^\alpha, \dot{x}^\alpha, P_\alpha, R_\alpha$. We have thus proved

THEOREM 8. - *The t-independent canonical transformation* (8.1) *with generating function* $\Psi(x^j,\ \dot{x}^j,\ P_j,\ R_j)$ *may be written explicitly in the form* (8.10), (8.11), *where* k_1, k_2 *are two arbitrary constants. The transformation* (8.10), *which only involves the* $4n-4$ *variables* x^α, $\dot{x}^\alpha$, P_α, R_α, *is a canonical transformation with generating function* $\psi(x^\alpha,\ \dot{x}^\alpha,\ P_\alpha,\ R_\alpha)$ *related to* Ψ *according to* (8.12).

We note that when $k_1 = k_2 = 0$ we are confronted with the trivial case for which the generating functions are identical and (8.11) reduces to the identity transformation.

9. - Extension to canonical transformations associated with problems of arbitrary order.

We merely state some of the main results of the theory associated with an m-th order variational problem whose Lagrangian is

$$L = L(t,\ x^i,\ x^i_{(1)},\ \ldots,\ x^i_{(m)}), \text{ for some } m \geq 1,$$

were we have employed the notation

$$x^i_{(0)} \overset{\text{def}}{=} x^i,\ x^i_{(1)} = \frac{dx^i}{dt},\ \ldots,\ x^i_{(r)} = \frac{d^r x^i}{dt^r},\ r = 1,\ 2,\ \ldots .$$

It is assumed that L is of class C^{2m} and that the curve $C: x^i = x^i(t)$ in X_{n+1} is of class C^m.

The canonical momenta are defined as follows:

$$P_i^{(m-1)} \stackrel{\text{def}}{=} \frac{\partial L}{\partial x^i_{(m)}}. \quad P_i^{(\varepsilon)} = \frac{\partial L}{\partial x^i_{(\varepsilon+1)}} - \frac{dP_i^{(\varepsilon+1)}}{dt}, \ \varepsilon = 0, 1, \ldots, m-2,$$

(cf. [9], p. 266). Thus the canonical coordinates of the theory are the $2mn$ quantities

$$\{x^i_{(0)}, x^i_{(1)}, \ldots, x^i_{(m-1)}; P_i^{(0)}, P_i^{(1)}, \ldots, P_i^{(m-1)}\} \equiv \{x^i_{(\rho)}, P_i^{(\rho)}\},$$

where in this section the bracketed indices ρ, σ run from 0 to $m-1$, while *the summation convention is also applied to indices of this type.*

Consider the transformation

$$\bar{x}^i_{(\rho)} = \bar{x}^i_{(\rho)}(x^j_{(\sigma)}, P_j^{(\sigma)}), \quad \bar{P}_i^{(\rho)} = \bar{P}_i^{(\rho)}(x^j_{(\sigma)}, P_j^{(\sigma)}), \tag{9.1}$$

which is said to constitute a *canonical transformation associated with the m-th order problem in the calculus of variations* if it is of class C^2, and if a function $\Psi_m(x^i_{(\rho)}, P_i^{(\rho)})$ of class C^2 exists such that

$$d\Psi_m \equiv \bar{P}_i^{(\rho)} d\bar{x}^i_{(\rho)} - P_i^{(\rho)} dx^i_{(\rho)},$$

whenever we substitute on the right-hand side from (9.1).

We now define the *Lagrange bracket* of the canonical coordinates $x^i_{(\rho)}(u, v, \ldots)$, $P_i^{(\rho)}(u, v, \ldots)$ with respect to any two parameters on which they depend (being at least of class C^1 in them) by

$$[u, v] \stackrel{\text{def}}{=} \frac{\partial x^i_{(\rho)}}{\partial u} \frac{\partial P_i^{(\rho)}}{\partial v} - \frac{\partial x^i_{(\rho)}}{\partial v} \frac{\partial P_i^{(\rho)}}{\partial u},$$

and the *Poisson bracket* of any two functions

$$F = F(t, x^i_{(\rho)}, P_i^{(\rho)}), \quad G = G(t, x^i_{(\rho)}, P_i^{(\rho)}),$$

of class C^1 in the canonical coordinates, by

$$(F, G) \stackrel{\text{def}}{=} \frac{\partial F}{\partial x^i_{(\rho)}} \frac{\partial G}{\partial P_i^{(\rho)}} - \frac{\partial F}{\partial P_i^{(\rho)}} \frac{\partial G}{\partial x^i_{(\rho)}}.$$

We then have the following three sets of characterizing properties of the canonical transformation (9.1):

(*a*) The LAGRANGE bracket relations (cf. (3.5) and theorem 1)

$$[x^i_{(\rho)},\ x^j_{(\sigma)}] = 0,\ [x^i_{(\rho)},\ P^{(\sigma)}_j] = \delta^j_i\,\delta^\rho_\sigma,\ [P^{(\rho)}_i,\ P^{(\sigma)}_j] = 0,$$

(*b*) the reciprocity relations (cf. (5.5) and theorem 2)

$$\frac{\partial \bar{x}^i_{(\rho)}}{\partial x^j_{(\sigma)}} = \frac{\partial P^{(\sigma)}_j}{\partial \bar{P}^{(\rho)}_i},\quad \frac{\partial \bar{x}^i_{(\rho)}}{\partial P^{(\sigma)}_j} = -\frac{\partial x^j_{(\sigma)}}{\partial \bar{P}^{(\rho)}_i},$$

$$\frac{\partial \bar{P}^{(\rho)}_i}{\partial x^j_{(\sigma)}} = -\frac{\partial P^{(\sigma)}_j}{\partial \bar{x}^i_{(\rho)}},\quad \frac{\partial \bar{P}^{(\rho)}_i}{\partial P^{(\sigma)}_j} = \frac{\partial x^j_{(\sigma)}}{\partial \bar{x}^i_{(\rho)}},$$

(*c*) the POISSON bracket relations (cf. (5.8) and theorem 3)

$$(\bar{x}^i_{(\rho)},\ \bar{x}^j_{(\sigma)}) = 0,\ (\bar{x}^i_{(\rho)},\ \bar{P}^{(\sigma)}_j) = \delta^i_j\,\delta^\sigma_\rho,\ (\bar{P}^{(\rho)}_i,\ \bar{P}^{(\sigma)}_j) = 0.$$

The other theorems of this article may also be extended to hold for this general case.

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