# Canonical transformations theory for presymplectic systems

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We develop a theory of canonical transformations for presymplectic systems, reducing this concept to that of canonical transformations for regular coisotropic canonical systems. In this way we can also link these with the usual canonical transformations for the symplectic reduced phase space. Furthermore, the concept of a generating function arises in a natural way as well as that of gauge group.

## I. INTRODUCTION

Since the well-known Dirac's pioneer work[1] on constrained Hamiltonian systems, the interest in such theory has been growing because it provides an appropriate framework to deal with many physical theories either for finite-dimensional systems as time-dependent [or more generally ($n$-parameter)-dependent] systems, mechanical systems defined by singular Lagrangians, etc., or infinite-dimensional systems exhibiting gauge invariance. A good test of the relevance of these systems is the amount of papers trying to develop the mathematical framework for these systems, which has been shown to be that of presymplectic geometry, which has been possible thanks to the papers by Gotay,[2] Lichnerowicz,[3] Sniatycki,[4] and others, to whom we apologize for omitting their names. For a recent review see, e.g., Ref. 5.

The essential characteristic of these systems is the existence of contraint functions, limiting the possible values of the dynamical variables that Dirac classified in first and second class according to the vanishing or not of their mutual Poisson brackets. This classification was motivated because the second-class constraints may be eliminated from the theory up to a redefinition of Poisson brackets becoming now the so-called Dirac's brackets, and they may be considered as corresponding to spurious degrees of freedom. This aspect is really clarified when using appropriate coordinates as indicated by Shanmugadhasan.[6]

On the other hand, the invariance of the Poincaré–Cartan integral has also been proved to be a sound principle for the study of nondegenerate systems and it has motivated a recent paper[7,8] devoted to the study of the Hamilton–Jacobi method for degenerate systems. Our experience with regular systems suggests for us to look for a concept generalizing that of canonical transformation, and it has been carried out[9] for regular canonical systems by making use of a generalization of the Hwa-Chung theorem.[10] We aim in this paper to give a general concept of canonical transformation for any presymplectic system, as well as attempt to go deep in the analysis of this concept in order to characterize such transformations, studying the group structure of such a set of canonical transformations, some remarkable subgroups (in particular, the subgroup of gauge transformations), and the theory of the corresponding generating functions, which follows the track of Weinstein's theory for symplectic systems.[11]

The paper is organized as follows: Section II is devoted to analyzing the structure of locally Hamiltonian presymplectic systems, and the main result of this section, given in Theorem 3, is that the study of the locally Hamiltonian presymplectic systems can be done by means of its local structure coisotropic germ. The concept of canonical transformation for presymplectic systems is given in Sec. III and after a deep analysis it is shown that it is enough to consider the case of canonical regular systems because any other can be reduced to it. Section IV contains a study of the group structure of the set of canonical transformations. When the process of reduction of the presymplectic system is carried out, the canonical transformations pass to the quotient and it singularizes the subgroup of canonical transformations, inducing the identity in the quotient, called the gauge group. The concept of a generating function is introduced in Sec. V and Sec. VI is devoted to showing some interesting properties of the generating functions, which will be of interest to manage with in local coordinates.

## II. THE STRUCTURE OF PRESYMPLECTIC SYSTEMS

The mathematical framework for a geometrical description of the Dirac's theory of constrained systems[1,12,13] has been shown to be that of presymplectic dynamical systems.[2–4,14,15] In this section we will analyze the local structure of such systems and it will be shown how it is possible to imbed a presymplectic manifold as a coisotropic submanifold of a symplectic manifold in which a family of locally Hamiltonian vector fields extending the dynamics of the original system can be constructed. This result is based on some theorems by Sniatycki[4] and Gotay,[16] which will be restated in order to make this paper more self-contained.

*Definition 1:* A presymplectic manifold is a pair $(M,\omega)$ where $\omega$ is a closed two-form of constant rank on the differentiable manifold $M$. If $\alpha$ is a closed one-form on $M$, the

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triplet $(M,\omega,\alpha)$ is said to be a locally Hamiltonian presymplectic dynamical system.

The Dirac–Bergmann theory of constrained systems corresponds to taking $M = D_{\mathscr{L}}(T^*Q)$ and $\omega$, the pullback to $M$, of the canonical two-form $\omega_0$ on $T^*Q$. Here $D_{\mathscr{L}}$ denotes the Legendre map $D_{\mathscr{L}}: TQ \to T^*Q$, with $Q$ the configuration space and $\mathscr{L}$ the Lagrangian function which is assumed to be singular; that is, $D_{\mathscr{L}}$ is not a local diffeomorphism. Alternatively, we can consider in this case another presymplectic manifold $(M = TQ, \omega_{\mathscr{L}} = D^*_{\mathscr{L}}\omega_0)$.

There are a lot of other relevant presymplectic manifolds arising in physics. For instance, we can mention parameter-dependent systems where the mainfold $M$ is $P \times \Lambda$ with $(P,\Omega)$ a symplectic manifold and $\Lambda$ the parameter space. The closed two-form $\tilde{\omega}$ is given by $\tilde{\omega} = \pi^*\omega$, where $\pi$ denotes the natural projection on the first factor $\pi: P \times \Lambda \to P$. This is the case of the usual way of dealing with time-dependent systems.[17]

Given a locally Hamiltonian presymplectic dynamical system, the constraint algorithm, developed by Gotay *et al.*,[2,14–16,18] provides a method for obtaining a maximal submanifold $C$, called the final constraint submanifold, for which the equation

$$\iota(\Gamma)\omega_{|C} = \alpha_{|C} \tag{2.1}$$

is meaningful, and it is possible to endow $\alpha$ with a dynamical sense. The vector field $\Gamma$ is not uniquely defined and this ambiguity corresponds to what is usually called gauge freedom.[2,15]

When dealing with Dirac's constrained Hamiltonian systems, the functions locally defining this final constraint submanifold are both the primary constraints defining $M \subset T^*Q$ and the secondary constraints. But Dirac gave a new classification of constraints in first and second class depending on the possibility of eliminating the ambiguity in the corresponding multiplier in the expression of the total Hamiltonian. In the general case constraints of both classes can appear, but Sniatycki proved[4] that it is possible to imbed coisotropically the final constraint submanifold $C$ in a symplectic manifold, in the very general case of $C$ defining a regular canonical system, and the second-class constraints are eliminated.

*Definition 2:* Let $(P,\Omega)$ be a symplectic manifold and $j: C \hookrightarrow P$ a submanifold of $P$. Then $(P,C,\Omega)$ is said to be a regular canonical system if $\ker j^*\Omega \cap TC$ is a subbundle of the tangent bundle $TC$.

**Theorem 1[4]:** Let $(P,C,\Omega)$ be a regular canonical system. If $C$ is a closed submanifold of $P$, there exist a symplectic submanifold of $(P,\Omega)$, $k: (\tilde{P},\tilde{\Omega}) \hookrightarrow (P,\Omega)$, and a coisotropic imbedding of $C$ into $(\tilde{P},\tilde{\Omega})$, $l: (C, j^*\Omega) \hookrightarrow (\tilde{P},\tilde{\Omega})$, such that $k \circ l = j$.

The existence of a symplectic manifold $(P,\Omega)$ containing $C$ may be forgotten for the presymplectic case if we make use of the coisotropic imbedding theorem recently given by Gotay.[16]

**Theorem 2:** Let $(M,\omega)$ be a presymplectic manifold. Then, we have the following.

(i) There exists a symplectic form $\Omega$ on a tubular neighborhood of the zero section of the dual bundle $E^*$ of the characteristic bundle $E$ of $(M,\omega)$, where $M$ can be coisotropically imbedded.

(ii) Any two coisotropic imbeddings of $(M,\omega)$ are locally equivalent: if $j_i$ $(M,\omega) \mapsto (P_i,\Omega_i)$, $i = 1,2$, are two of such coisotropic imbeddings, there exist two neighborhoods $U_i = 1,2$, of $j_i(M)$ in $P_i$ and a symplectomorphism $\phi: U_1 \to U_2$ such that $\phi^*\Omega_2 = \Omega_1$ and $\phi \circ j_1 = j_2$.

We introduce next some definitions and notations we are going to use concerning functions and one-forms defined on a symplectic manifold $(P,\Omega)$.

*Definition 3:* Let $C$ be a submanifold of $(P,\Omega)$. A function $f \in \mathscr{C}^\infty(P)$ is said to be a constraint function for $C$ if $f_{|C}$ is constant, and the set of such functions will be denoted $C(P,C)$. A function $g \in \mathscr{C}(P,C)$ is called a first-class function if $\{f,g\}_{|C=0}\ \forall f \in \mathscr{C}(P,C)$, and we will write $\mathscr{B}(P,C)$ for the set of all first-class functions. Finally, the first-class constraint functions are those of $\mathscr{B}(P,C) \cap \mathscr{C}(P,C)$, and the corresponding set will be denoted $\mathscr{A}(P,C)$.

Here $\{\ ,\ \}$ will denote the Poisson bracket defined on the set $\Lambda^1(P)$ of one-forms by the form $\Omega$ as follows: $\{\alpha,\beta\} = \hat{\Omega}[\hat{\Omega}^{-1}(\alpha), \hat{\Omega}^{-1}(\beta)]$ for any pair of one-forms $\alpha,\beta \in \Lambda^1(P)$. The map $\hat{\Omega}: \mathfrak{X}(P) \to \Lambda^1(P)$ is defined by contraction with $\Omega, \hat{\Omega}(X) = \iota(X)\Omega$. When $f$ and $g$ are functions, the Poisson bracket is defined by $\{f,g\} = \Omega(\hat{\Omega}^{-1}(df), \hat{\Omega}^{-1}(dg))$. The concepts of constraint and first-class functions can be generalized for one-forms on $P$ as follows.

*Definition 4:* A one-form $\alpha \in \Lambda^1(P)$ is a constraint one-form for $C$ if $j^*\alpha = 0$, $j$ being the immersion $j: C \hookrightarrow P$. The set of these one-forms will be denoted $C^1(P,C)$. The one-form $\beta$ is a first-class one-form if $j^*\{\alpha,\beta\} = 0$, $\forall \alpha \in C^1(P,C)$, and the set of all such one-forms will be written $B^1(P,C)$. Finally, by $A^1(P,C)$ we will denote the set $A^1(P,C) = B^1(P,C) \cap C^1(P,C)$ of the first-class constraint one-forms.

*Proposition 1:* With the above notations, we have the following.

(i) $d\mathscr{A}(P,C) \subset A^1(P,C) \cap Z^1(P)$,

$d\mathscr{B}(P,C) \subset B^1(P,C) \cap Z^1(P)$,

$d\mathscr{C}(P,C) \subset C^1(P,C) \cap Z^1(P)$.

(ii) If $(C, j^*\Omega) \hookrightarrow (P,\Omega)$ is a coisotropic imbedding, then $C^1(P,C) \cap Z^1(P) \subset B^1(P,C) \cap Z^1(P)$ and therefore $A^1(P,C) \cap Z^1(P) = C^1(P,C) \cap Z^1(P)$.

*Proof:* (i) If $f \in \mathscr{C}(P,C)$ then $j^*\,df = d\,(j^*f) = 0$ and therefore $df \in C^1(P,C) \cap Z^1(P)$. Moreover, if $g \in \mathscr{B}(P,C)$, then $j^*\,d\,\{f,g\} = 0$ and therefore $j^*\,\{df,dg\} = 0$. But it implies that $j^*\{\alpha,dg\} = 0$ for any $\alpha \in C^1(P,C) \cap Z^1(P)$ because of the local existence of a neighborhood and a function $f \in \mathscr{C}(P,C)$ such that $\alpha = df$ according to the relative Poincaré lemma.[9]

(ii) If $\alpha \in C^1(P,C) \cap Z^1(P)$, the lemma of Poincaré shows that there is a function $f$ (only locally defined) such that $df = \alpha$ and then $j^*\,\alpha = 0$ implies that $j^*f$ is constant on the neighborhood $\mathscr{V}$ where $f$ was defined. Now, if $\beta \in C^1(P,C) \cap Z^1(P)$ and $g$ is a function (locally defined) such that $\beta = dg$, and $g \in \mathscr{C}(P,C)$, we see that $j^*\,\{\alpha,\beta\} = j^*\{df,dg\} = j^*\,d\,\{f,g\} = dj^*\,\{f,g\}$. If $C$ is coisotropically imbedded in $P, \mathscr{C}(P,C) \subset \mathscr{B}(P,C)$ and therefore $j^*\,\{f,g\} = 0$, which implies $j^*\,\{\alpha,\beta\} = 0$. In order to prove that $j^*\,\{\alpha,\beta\} = 0$, $\forall \beta \in C^1(P,C)$, we remark that $C^1(P,C) \cap Z^1(P)$ generates locally $C^1(P,C)$ as a $C^\infty(P)$ module and for every $\beta \in C^1(P,C)$ there exist $b_i \in C^\infty(P)$ and $f^i \in \mathscr{C}(P,C)$ such that $\beta$ can be written as $\beta = \Sigma b_i\, df^i$. Then, using the identity $\{\alpha,h\gamma\} = X_\alpha(h)\gamma + h\,\{\alpha,\gamma\}$,

$\forall h \in C^\infty(P)$, $\alpha,\gamma \in A^1(P)$, with $X_\alpha = \hat{\Omega}^{-1}(\alpha)$, we find that for every $\alpha \in C^1(P,C)\cap Z^1(P)$ and $\beta \in C^1(P,C)$,

$$\{\alpha,\beta\} = \left\{\alpha, \sum b_i\, df^i\right\} = \sum X_\alpha(b_i) df^i + \sum b_i \{\alpha, df^i\},$$

and therefore

$$j^*\{\alpha,\beta\} = \sum (X_\alpha(b_i)\circ j)\, d(j^* f^i) + \sum (b_i \circ j) j^*\{\alpha, df^i\} = 0.$$

The main goal of this section is the following theorem.

**Theorem 3:** Let $(M,\omega,\alpha)$ be a locally Hamiltonian presymplectic system and $i: C \hookrightarrow M$ the final constraint submanifold. There exist a symplectic manifold $(P,\Omega)$ and a coisotropic imbedding $j: C \hookrightarrow P$ such that $j^*\Omega = i^*\omega$ and we have the following.

(i) For each vector field $\Gamma$ on $M$, tangent to $C$, satisfying $\iota(\Gamma)\omega_{|C} = \alpha_{|C}$, there is a locally Hamiltonian vector field $\Gamma_\xi$ on $P$, tangent to $C$, such that $\Gamma_{|C} = \Gamma_{\xi|C}$.

(ii) The vector fields $\Gamma_\xi$ associated to the dynamical system $\Gamma$ satisfying the above conditions are given by

$$\Gamma_\xi = \hat{\Omega}^{-1}(\alpha_P + \xi), \tag{2.2}$$

where $\alpha_P$ is a closed one-form on $P$ such that $j^*\alpha_P = i^*\alpha$, and $\xi$ any closed first-class constraint one-form on $P$ for $C$, $\xi \in A^1(P,C)\cap Z^1(P)$.

(iii) (local uniqueness) The coisotropic imbedding and the family

$$D(P,C) = \{\hat{\Omega}^{-1}(\alpha_P + \xi) \mid \xi \in A^1(P,C)\cap Z^1(P)\}$$

are locally unique.

Here local uniqueness means that if $j': C \hookrightarrow P'$ is another coisotropic imbedding, there will exist a family of locally Hamiltonian vector fields

$$D(P',C) = \{\hat{\Omega}'^{-1}(\alpha_{P'} + \xi) \mid \xi \in A^1(P',C)\cap Z^1(P'),\ \alpha_{P'} \in Z^1(P')\},$$

and a local symplectomorphism $\phi$ from a neighborhood of $j(C)$ in $P$ in a neighborhood of $j'(C)$ in $P'$ such that $j' \circ \phi = \phi \circ j$ and maps $D(P,C)$ on $D(P',C)$.

*Proof:* According to Theorem 2, there is a symplectic manifold $(\tilde{P},\tilde{\Omega})$ and a coisotropic imbedding $l:(M,\omega) \hookrightarrow (\tilde{P},\tilde{\Omega})$. On the other hand, if $(C,i^*\omega)$ is the presymplectic manifold which is obtained from application of the constraint algorithm, Theorem 2 furnishes a new symplectic manifold where $(C,i^*\omega)$ is coisotropically imbedded. Let $j_2$ denote such an imbedding $j_2: C \to P_2$. The relation between both symplectic structures is given by Theorem 1. We can also see $C$ as a submanifold $J: C \to P_1$ with $J = l \circ i$ and then Theorem 1 asserts the existence of a symplectic submanifold $k:(P_3,\Omega_3) \to (\tilde{P},\tilde{\Omega})$ and a coisotropic imbedding $j_3:(C,i^*\omega) \to (P_3,\Omega_3)$ such that $k \circ j_3 = J$. The local uniqueness part of Theorem 2 leads to the existence of a symplectomorphism $\phi$ of a neighborhood of $j_3(C)$ in $P_3$ on a neighborhood of $j_2(C)$ in $P_2$. If $(P,\Omega)$ is any of such neighborhoods and $j$ the corresponding immersion of $C$ in $P$, we have a coisotropic imbedding of $C$ in $(P,\Omega)$. (See Fig. 1.)

In order to prove the points concerning the dynamics, we remark that both $(\tilde{P},\tilde{\Omega})$ and $(P,\Omega)$ are neighborhoods of the zero sections of vector bundles over $M$ and $C$, respectively. Let $\pi_k, \pi_j$, and $\pi_l$ be the corresponding projections

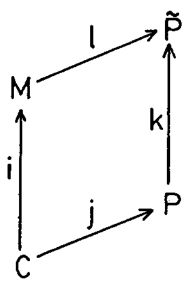



FIG. 1. Diagram of the coisotropic imbedding.

$\pi_k: \tilde{P} \to P$, $\pi_j: P \to C$, $\pi_l: \tilde{P} \to M$, verifying $\pi_k \circ k = \mathrm{id}_P$, $\pi_l \circ l = \mathrm{id}_M$, and $\pi_j \circ j = \mathrm{id}_C$. Let $\Gamma_\circ$ be a vector field in $M$ tangent to $C$ verifying the dynamical condition (2.1), i.e., $\iota(\Gamma_\circ)\omega_{|C} = \alpha_{|C}$. From the relation $k \circ j = l \circ i$ we see that the images of the manifold $C$ under $l \circ i$ and $k \circ j$ are contained in $l(M)$ and $k(P)$, respectively, and then $\mathrm{C} = l \circ i(C) \subset l(M) \cap k(P) \equiv W$. We define a vector field in $l(M)$ by $l_* \Gamma_\circ$ and take its restriction to $W$, that it is not necessarily tangent to $W$ but it will be tangent to $\mathrm{C}$ because the tangency of $\Gamma_\circ$ to $C$ implies that there exists a vector field $\Gamma'_\circ$ in $C$ such that $i_* \Gamma'_\circ = \Gamma_\circ$ and therefore $l_* \Gamma_\circ = (l \circ i)_* \Gamma'_\circ$ is tangent to $\mathrm{C}$. The map $\pi_k: k(P) \to P$ is a diffeomorphism, so that it is meaningful to take the restriction of $\pi_k$ to $W = l(M) \cap k(P)$ and define the vector field $\pi_{k*}(l_* \Gamma_\circ)$ (the respective restrictions of $l_*$ and $\pi_k$ to $W$ are understood). We remark that $\pi_{k*} l_* \Gamma_\circ$ is tangent to $C$ in $P$ because if we take the vector field $\Gamma'_\circ$ in $C$ as above and compute $\pi_{k*} l_* \Gamma_\circ$ we find that $\pi_{k*} l_* \Gamma_\circ = j_* \Gamma'_\circ$. It is now easy to see that the vector field $\pi_{k*} l_* \Gamma_\circ$ defined in a submanifold of $P$ satisfies on it the equation $\iota(\pi_{k*} l_* \Gamma_\circ)\Omega = \alpha_P$ with $\alpha_P = k^* \pi_l^* \alpha$. In fact, the following computation shows that we can associate the vector field $X_{\alpha_P} = \hat{\Omega}^{-1}(\alpha_P)$ with $\Gamma_\circ$ because

$$\begin{aligned}\iota(\pi_{k*} l_* \Gamma_\circ)\Omega(Y) &= k^* \Omega_1(\pi_{k*} l_* \Gamma_\circ, Y) \\ &= (\pi_l \circ k)^* \omega((\pi_k \circ l)_* \Gamma_\circ, Y) \\ &= \omega(\Gamma_\circ, (\pi_l \circ k)_* Y) \\ &= \iota(\Gamma_\circ)\omega(\pi_l \circ k)_* Y = \alpha_P(Y).\end{aligned}$$

The vector field $X_\xi = \hat{\Omega}^{-1}(\xi)$ corresponding to an element of $A^1(P,C)\cap Z^1(P)$ is tangent to $C$ and is such that $X_{\xi|C} \in \ker i^*\omega$, and consequently the vector field $X_{\alpha_P} + X_\xi$ is a solution of the dynamical equation, too. Therefore, by addition of vector fields $X_\xi$ with $\xi \in A^1(P,C)\cap Z^1(P)$ to the vector field $X_{\alpha_P}$ we obtain vector fields in $P$ tangent to $C$. Noteworthy is that if $\Gamma_1$ is another vector field in $M$ satisfying the dynamical equation, then the difference $(\Gamma_1 - \Gamma_\circ)_{|C}$ lies in $\ker i^*\omega$ and therefore $\pi_{k*} j_{1*} \Gamma_1 = \pi_{k*} j_{1*} \Gamma_\circ + X_\xi$, with $\xi \in A^1(P,C)\cap Z^1(P)$. Actually $\ker i^*\omega = \pi_{j*} \hat{\Omega}^{-1}(A^1(P,C) Z^1(P))$, because the closed first-class constraint one-forms generate via $\hat{\Omega}^{-1}$ the submodule $\Gamma(TC^\perp)$ of $\mathfrak{X}(P,C) = \{X \in \mathfrak{X}(P) \mid X_{|C} \in \Gamma(TC)\}$.

As far as the local uniqueness is concerned we must prove that given two coisotropic imbeddings $j_1, j_2$ into two symplectic manifolds $(P_i,\Omega_i)$, $i = 1,2$, there will be a local symplectomorphism $\phi: P_1 \to P_2$ mapping locally Hamiltonian vector fields on $P_1$ in locally Hamiltonian vector fields on $P_2$ and $\phi \circ j_1 = j_2$. Now, let $(P,\Omega)$ be a symplectic manifold where the final constraint manifold is coisotropically imbedded, obtained from any symplectic manifold $(S,\sigma)$ in which $(M,\omega)$ is imbedded.

The local uniqueness part of the statement of Theorem 2 says that any two coisotropic imbeddings are locally equivalent and consequently the two symplectic manifolds we will obtain, either from $(M,\omega)$ using $(S,\sigma)$ or from $(C,i^*\omega)$ using the coisotropic imbedding theorem, have to be locally symplectomorphic. The second assertion of the statement follows from the fact that any symplectomorphism preserves the locally Hamiltonian character of the vector fields, and from the condition $\phi \circ j_1 = j_2$, which says that $\phi$ transforms constraint one-forms into constraint one-forms; and, as we have shown that $C^1(P,C)\cap Z^1(P,C) = A^1(P,C)\cap Z^1(P,C)$, the proof ends.

*Definition 5:* If $(P,C,\Omega)$ is a regular canonical system such that the immersion $j$ is a coisotropic imbedding we will say that $(P,C,\Omega)$ is a regular canonical coisotropic system.

The preceding results can also be presented in a different language using the concept of local manifold pair, as Weinstein does,[19] or that of a germ of a manifold as a submanifold of another one; that is, if $C$ is a submanifold of $M$ and $(M,C)$ a pair of manifolds, we will say that $(M',C)$ is equivalent to $(M,C)$ if there is another pair $(M'',C)$ such that $M''$ is an open submanifold of both $M$ and $M'$. An equivalence class of pairs of manifolds is called a local manifold pair or germ of $C$ in $M$ and will be denoted $[M,C]$. A map between two germs is defined by an equivalence class of maps. This equivalence is defined as follows: two maps $f_i:(M_i,C)\mapsto(M'_i,C')$, $i = 1,2$ are said to be equivalent if there exists a map $g:(M_3,C)\mapsto(M'_3,C')$ such that $M_3$ is an open submanifold of both $M_1$ and $M_2$, and $M'_3$ is an open submanifold of $M'_1$ and $M'_2$ with $f_{1|M_3} = f_{2|M_3} = g$.

A germ $[P,C]$ is said to be coisotropic if $(P,C)$ is a pair where $C$ is a coisotropic submanifold of the symplectic manifold $(P,\Omega)$. We can consider the category with objects the germs $[P,C]$ and morphisms the symplectic maps between germs $[\phi]:[P,C]\to[P',C']$. We will say that a germ $[P,C]$ is the local structure germ for a presymplectic germ if it verifies the universal property of being an initial object in this category, i.e., for every $[P',C]$ there is a morphism $[\phi]:[P,C]\to[P',C']$ such that $\phi_{|C} = id_C$. With this language Theorem 3 can be restated as follows: For every locally Hamiltonian presymplectic system $(M,\omega,\alpha)$, there exists a local structure germ $[P,C]$, with $C$ the final constraint manifold for $(M,\omega,\alpha)$. It is uniquely defined and there is on it a family of locally Hamiltonian vector fields furnishing a dynamical description of the system.

## III. CANONICAL TRANSFORMATIONS FOR PRESYMPLECTIC SYSTEMS

The traditional concept of canonical transformations for Hamiltonian dynamical systems as symplectomorphisms has recently been generalized[9] for application to regular canonical systems $(P,S,\Omega)$. The definition of canonical transformation depends on the choice of a particular kind of vector field, called locally weakly Hamiltonian fields relative to $(P,S,\Omega)$, and therefore depends on the immersion of $S$ in the ambient manifold $P$. We aim to find a generalization of the concept of canonical transformation for a presymplectic system with no reference to an ambient symplectic manifold containing it, that it will reduce to that proposed in Ref. 9 in the case of a regular canonical system. Moreover, we will prove, by making use of the results of the preceding section, that the general problem of studying the canonical transformations of a presymplectic system can be reduced to that of the canonical transformations of a regular canonical system $(P,C,\Omega)$.

*Definition 6:* Let $(M,\omega,\alpha)$ be a locally Hamiltonian presymplectic system and let $i_C:C\hookrightarrow M$ be the final constraint submanifold. A vector field $X\in\mathfrak{X}(M)$ is said to be a locally Hamiltonian vector field relative to $C$ if (i) $X$ is tangent to $C$, $X_{|C}\in\Gamma(TC)$ and (ii) there exists a closed one-form $\beta\in Z^1(M)$ such that

$$i_C^*(\iota(X)\omega - \beta) = 0. \quad (3.1)$$

The set of such vector fields will be denoted $\mathfrak{X}_{\rm LH}(M,C)$. It is to be remarked that the condition (ii) is weaker than $\iota(X)\omega_{|C} = \beta_{|C}$ and any vector field $X$ satisfying this equation will satisfy (3.1), too. As an example, the dynamical vector fields provided by the Constraint algorithm are locally Hamiltonian vector fields relative to $C$. On the other hand condition (ii) is equivalent to $i_C^* L_X\omega = 0$.

As a corollary of the theorems of Ref. 9 we can write down the generalization of the Hwa-Chung theorem for presymplectic systems.

**Theorem 4:** Let $(M,\omega,\alpha)$ be a locally Hamiltonian presymplectic system with final constraint submanifold $i_C:C\hookrightarrow M$ and rank $(i_C^*\omega) = 2r$. If $\beta\in\Lambda^p(M)$ is such that $i_C^*(L_X\beta) = 0$, $\forall X\in\mathfrak{X}_{\rm LH}(M,C)$, then we have the following.

(i) $i_C^*\beta = 0$ if $p > 2r$ or $p = 2l+1$ with $l < r$.

(ii) If $p = 2l$, $l\leqslant r$, there exists a function $f\in C^\infty(M)$ such that $i_C^*(\beta - f\omega^{\wedge l}) = 0$ and $i_C^* f$ is constant on each connected component of $C$.

In this context the concept of canonical transformation generalizing that of Ref. 9 is the following one.

*Definition 7:* Let $(M_k,\omega_k,\alpha_k)$, $k = 1,2$, be two locally Hamiltonian presymplectic systems and $i_k:C_k\to M_k$ the corresponding final constraint submanifolds. A pair $(\Phi,\phi)$ of diffeomorphisms $\Phi:M_1\to M_2$ and $\phi:C_1\to C_2$ is said to be a canonical transformation between $(M_1,\omega_1,\alpha_1)$ and $(M_2,\omega_2,\alpha_2)$ if (i) $\Phi\circ i_1 = i_2\circ\phi$ and (ii) $\Phi_*(\mathfrak{X}_{\rm LH}(M_1,C_1))\subset\mathfrak{X}_{\rm LH}(M_2,C_2)$.

A characterization of a canonical transformation for such systems, which is a straightforward consequence of the former theorem, is given by the following.

**Theorem 5:** A pair $(\Phi,\phi)$ of diffeomorphisms $\Phi:M_1\to M_2$ and $\phi:C_1\to C_2$, such that $\Phi\circ i_1 = i_2\circ\phi$, is a canonical transformation if and only if there is a real number $c$ such that $i_1^*(\Phi^*\omega_2 - c\omega_1) = 0$.

Only the particular case $c = 1$ will be considered in the following. It corresponds to the restricted canonical transformations for Hamiltonian systems in the terminology of the book by Saletan and Cromer,[20] but we will omit the word restricted.

A convenient characterization of the locally Hamiltonian vector fields which is also an immediate consequence of Theorem 4 is given next.

**Theorem 6:** Let $(M,\omega,\alpha)$ be a locally Hamiltonian presymplectic system and $i_C:C\to M$ the final constraint submanifold. A vector field $X$ in $M$ tangent to $C$ is locally Hamiltonian relative to $C$ if and only if the flow of $X$ is a family of canonical transformations of $(M,\omega,\alpha)$.

The fundamental result of this section concerns the reduction for a general presymplectic system to the case of a canonical system which is given by the structure theorem of the precedent section. In fact, the next theorem asserts that the set of canonical transformations of a presymplectic system can be seen as the set of canonical transformations of a regular canonical system coisotropically imbedded.

**Theorem 7:** With the same notations as in Definition 7, for each canonical transformation $(\Phi,\phi)$ between $(M_k,\omega_k,\alpha_k)$, $k=1,2$, if $(P_k,C_k,\Omega_k)$ are their corresponding regular canonical coisotropic systems given by Theorem 2, there exists a symplectomorphism $\Psi$ between them such that $\Psi\circ j_1 = j_2\circ\phi$, with $j_k$ being the injections $j_k:C_k\to P_k$.

*Proof:* Let $l_2:M_2\hookrightarrow(\tilde{P}_2,\tilde{\Omega}_2)$ be the coisotropic imbedding to $M_2$ in $(\tilde{P}_2,\tilde{\Omega}_2)$, $k_2:(P_2,\Omega_2)\to(\tilde{P}_2,\tilde{\Omega}_2)$ the symplectic submanifold, and $j_2:C_2\hookrightarrow(P_2,\Omega_2)$ the coisotropic imbedding given by Gotay's and Sniatycki's theorems verifying $k_2\circ j_2 = l_2\circ i_2$ as in Theorem 3. (see Fig. 2.)

Now, the point is that the composite map $j_2\circ\phi:C_1\hookrightarrow(P_2,\Omega_2)$ is a coisotropic imbedding satisfying $(j_2\circ\phi)^*\Omega_2 = i_1^*\,\omega_1$. In fact, a little computation gives $(j_2\circ\phi)^*\Omega_2 = \phi^*\,i_2^*\,\omega_2 = (\Phi\circ i_1)^*\,\omega_2 = i_1^*\,\omega_1$. In order to prove that $j_2\circ\phi$ is coisotropic, we must show that $TC_1^{\perp\Omega_2}\subset \mathbf{TC}_1$, where $\mathbf{TC}_1$ denotes the set of tangent vectors to $C_1$ through $j_2\circ\phi$, that is, $\mathbf{TC}_1 = (j_2\circ\phi)_*(TC_1)$. Let $u\in TC_1^{\perp\Omega_2}|_p$, where $p = j_2\circ\phi(m_1)$, i.e., $\Omega_2(p)(u,v)=0$, $\forall v\in\mathbf{TC}_1|_p$. If $v\in\mathbf{TC}_1|_p$, there exists a tangent vector $v'\in TC_1|_{m_1}$ such that $(j_2\circ\phi)_*(m_1)v' = v$, so that $\Omega_2(m_2)(u,j_{2*}\phi_*(m_1)v')=0$, where $m_2=\phi(m_1)$, $\forall v'\in T_{m_1}C_1$, or in the same way $\Omega_2(p)(u,j_{2*}(m_2)v'')=0$, $\forall v''\in T_{m_2}C_2$, because $\phi$ is a diffeomorphism. Then, $u\in TC_2^{\perp\Omega_2}$, and from the coisotropy of $C_2$ we have that $u\in\mathbf{TC}_2$; but $\mathbf{TC}_2=\mathbf{TC}_1$ and $j_2\circ\phi:C_1\hookrightarrow(P_2,\Omega_2)$ is a coisotropic imbedding.

In this point the local uniqueness of Theorem 2 shows that there exists a symplectomorphism $\Psi$ from $(P_1,\Omega_1)$, the initial symplectic manifold where $C_1$ is coisotropically imbedded, into $(P_2,\Omega_2)$, such that $\Psi\circ j_1 = j_2\circ\phi$, and the proof ends.

*Remarks:* (i) The function $\Psi$ is defined only locally on a neighborhood of $j_1(C)$ in $P$, but taking this neighborhood as the whole manifold the result still holds.

(ii) This theorem shows the possibility of studying canonical transformations for presymplectic systems using only their local structures as in Theorem 3. This simplification permits development of the study of the group of canonical transformations and its subgroups, so in the following sections we will use both points of view to deal with canonical transformations for a presymplectic system. That is, given a canonical transformation $(\Phi,\phi)$ between $(M_1,\omega_1,\alpha_1)$ and $(M_2,\omega_2,\alpha_2)$, we use without mention of it the associated canonical transformation $(\Psi,\phi)$ between the associated coisotropic regular canonical systems $(P_1,C_1,\Omega_1)$ and $(P_2,C_2,\Omega_2)$.

(iii) It is also to be remarked that there are canonical transformations between canonical regular systems that are not symplectomorphisms. In fact, it is possible to consider canonical transformations between two canonical regular systems associated to presymplectic systems $(M_k,\omega_k,\alpha_k)$ that are not symplectic transformations.

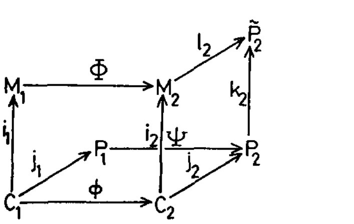



FIG. 2. Diagram displaying the maps of Theorem 7.

## IV. THE GROUP OF CANONICAL TRANSFORMATIONS FOR PRESYMPLECTIC SYSTEMS

Instead of dealing with presymplectic systems as indicated in the preceding sections, there is an alternative way which is called the reduction of the phase space.[17,19] The kernel of the presymplectic form $\omega_C = i_C^*\,\omega$ defines an involutive distribution and therefore it is integrable because of the well-known Frobenius theorem. The maximal connected integral submanifolds are the leaves of a foliation that gives rise to an equivalence relation in $C$. Suppose we discard the points of $C$, where $\omega_C$ fails to be of constant rank, and denote $\pi_C:C\to\hat{C}$ the natural projection of $C$ onto the quotient space. Then, if $\pi_C$ is a submersion, there exists a symplectic form $\hat{\omega}$ defined on $\hat{C}$ such that $\pi_C^*\,\hat{\omega} = \omega_C = i_C^*\,\omega$. It is defined by means of $\hat{\omega}_{\hat{m}}(\hat{X},\hat{Y}) = \omega_m(X,Y)$, where $m\in C$, $\pi_C(m)=\hat{m}$ and $X,Y\in T_mC$, $\hat{X},\hat{Y}\in T_{\hat{m}}\hat{C}$ are related by $\pi_{C_*m}(X)=\hat{X}$, $\pi_{C_*m}(Y)=\hat{Y}$. The pair $(\hat{C},\hat{\omega})$ is called the reduced phase space.

This is the usual approach to the study of dynamical systems with gauge degrees of freedom, as Yang–Mills fields[21,23] and gravitational fields.[23] In this scheme the canonical transformations are but symplectomorphisms of the reduced structure. In this section both alternative definitions will be related; we will prove that there is a canonical epimorphism of the group of generalized canonical transformations we have defined onto the group $\mathrm{Sp}(\hat{C},\hat{\omega})$ of symplectomorphisms of $(\hat{C},\hat{\omega})$.

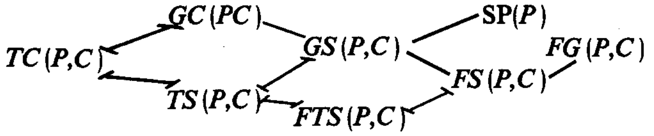
In order to explain this deep relation we need some notations referring to the group of (generalized) canonical transformations and its more relevant subgroups, which we present next.

We will denote $GC(P,C)$ the set of canonical transformations for the coisotropic canonical system $(P,C,\Omega)$ which can be endowed with a group structure in the natural way.

There are a lot of important subgroups of this with physical and mathematical meaning. For instance, $GS(P,C) = GC(P,C)\cap\mathrm{Sp}(P)$, which is not a normal subgroup in the general case. Now, if $\pi:P\to C$ denotes the above-mentioned projection, a very important subgroup of $GS(P,C)$ is made up by the elements that commute with $\pi$ and leave invariant the symplectic form $\Omega$. The set of such fibered symplectomorphisms is a subgroup to be denoted $FS(P,C)$, and it has been studied for time-dependent systems in Ref. 24. In a similar way we can define $FG(P,C)$ as made up from all fibered canonical transformations.

We will denote $TC(P,C)$ the set of canonical transformations that are trivial on $C$. This set is a normal subgroup of $GC(P,C)$ and has a subgroup to $TS(P,C) = TC(P,C)\cap GS(P,C)$.

The lattice of these subgroups as well as the relationship between them are shown in the diagram below. The symbol ⊢—⊣ means that the lower is normal in the upper one, and a subgroup in the link of two means that it is the intersection of both groups on the opposite edges




The group of the equivalence classes, $GC(P,C)/TC(P,C)$, will be denoted Can $C$ and it is obvious that each class $[(\Phi,\phi)]\in$ Can $C$ is uniquely defined by $\phi\in \mathrm{Diff}\, C$, hence Can $C$ is isomorphic to the group of those diffeomorphisms of $C$ preserving the presymplectic structure $\Omega_C = j^*\Omega$. Another related matter is to know whether it is possible to choose a symplectic transformation of $(P,\Omega)$ in any class or not. All this and related questions will be dealt with in next section.

The main theorem in this section is based on the following proposition.

*Proposition 2:* Any canonical transformation $(\Phi,\phi)$ for $(P,C,\Omega)$ leaves invariant the distribution defined by $\Omega_C$.

*Proof:* If $v\in T_m C$ is in $\ker\Omega_C(m)$, $X$ is a vector field defined in a neighborhood of $m$ in $P$ such that $X_m = v$ and $X_{|C}\in\Gamma(\ker\Omega_C)$, and we take into account that $C$ is coisotropic in $P$, we can conclude that $\Gamma(\ker\Omega_C)=\Gamma(TC^\perp)$ and consequently $\Gamma(\ker\Omega_C)$ is generated by constraint first-class functions; namely, $f\in\mathscr{A}(P,C)$ will exist such that $X_f=\hat\Omega^{-1}(df)=X$. A canonical transformation maps the set of locally Hamiltonian vector fields tangent to $C$ onto itself and the subset of those corresponding to constraint first-class functions on itself and therefore $\phi_*\ker\Omega_C=\ker\Omega_C$.

**Theorem 8:** With the same notations as above, the map $\hat\phi:\hat C\to\hat C$, defined by $\hat\phi\circ\pi_C=\pi_C\circ\phi$, is a symplectic map in $(\hat C,\hat\Omega)$.

*Proof:* The map is well defined because the foliation defined by $\ker\Omega_C$ is invariant under $\phi$. Moreover, if we compute $\pi_C^*\hat\phi^*\hat\Omega$ we find the chain of identities $\pi_C^*\hat\phi^*\hat\Omega = (\phi\circ\pi_C)^*\hat\Omega=\phi^*\Omega_C=\Omega_C=\pi^*\hat\Omega$. Now, $\pi$ being a submersion, we can conclude that $\phi^*\hat\Omega=\hat\Omega$.

*Corollary 1:* There is a canonical homomorphism $p$ between Can $C$ and $\mathrm{Sp}(\hat C,\hat\Omega)$ given by $p(\phi)=\hat\phi$.

*Definition 8:* The kernel of the homomorphism $p$ will be called the group $\mathscr{G}$ of $(P,C,\Omega)$ and is made up by the canonical transformations preserving every leaf of the foliation defined by $\ker\Omega_C$.

If $A(P,C)$ is the set of Hamiltonian constraint first-class vector fields in $P$ over $C$, according to Gotay's notation,[2] namely, $A(P,C)=\{X_f=\hat\Omega^{-1}(df)|f\in\mathscr{A}(P,C)\}$, we can write an exact sequence of Lie algebras as indicated by the following theorem.

**Theorem 9:** With the above notations, the sequence

$$0\to A(P,C)\overset{i_*}{\rightarrowtail}\mathfrak{X}_{\mathrm{LH}}(P,C)\overset{\pi_{C*}}{\to}\mathfrak{X}_{\mathrm{LH}}(\hat C)\to 0$$

is exact. Here $i_*$ is the natural injection of $A(P,C)$ in $\mathfrak{X}(C)$.

*Proof:* Notice that the vector fields in $\mathfrak{X}_{\mathrm{LH}}(C)$ are $\pi_C$ projectable and therefore $\pi_{C*}$ is well defined. The Hamiltonian constraint first-class vector fields generate $\ker\Omega_C=\Gamma(TC^\perp)$ and they are mapped by $\pi_{C*}$ on the zero vector field. Conversely, if a vector field $X\in\mathfrak{X}_{\mathrm{LH}}(C)$ is mapped by $\pi_{C*}$ on the zero vector field, each integral curve is contained in a leaf of the foliation defined by $\ker\Omega_C$, so $X$ is in $\Gamma(TC^\perp)$ and it belongs to $A(P,C)$.

Corresponding to this exact Lie algebra sequence we have another sequence of Lie groups

$$1\to\mathscr{G}\to\text{Can } C\to\mathrm{Sp}(\hat C,\hat\Omega)\to 1.$$

It is noteworthy that in the case of Yang–Mills fields, the gauge group $\mathscr{G}$ is a Lie Hilbert group and $A(P,C)$ is actually the Lie algebra of this group.[21,25]

## V. GENERATING FUNCTIONS FOR GENERALIZED CANONICAL TRANSFORMATIONS

The generating functions for canonical transformations of Hamiltonian systems arise as associated to the Lagrangian manifolds corresponding to the graph of the transformation in a symplectic product space.[11,17] If $(P_1,\Omega_1)$ and $(P_2,\Omega_2)$ are symplectic manifolds, a symplectic structure $\Omega_{12}$ is defined on the product manifold $P_1\times P_2$ by $\Omega_{12}=\pi_1^*\Omega_1-\pi_2^*\Omega_2$, where $\pi_i:P_1\times P_2\to P_i$ $(i=1,2)$ are the canonical projections. Then $\phi:(P_1,\Omega_1)\to(P_2\circ\Omega_2)$ is a symplectomorphism if and only if its graph is a Lagrangian submanifold of $(P_1\times P_2,\Omega_{12})$.[11,17,26] Before trying to generalize the concept of generating function we establish a similar property characterizing the canonical transformations for presymplectic systems.

**Theorem 10:** Let $(P_i,S_i,\Omega_i)$, $i=1,2$, be two canonical regular systems. A pair of diffeomorphisms $(\Phi,\phi)$, $\Phi:P_1\to P_2$, $\phi:S_1\to S_2$ is a canonical transformation if and only if (i) $\Phi\circ j_1=j_2\circ\phi$, where $j_i:S_i\to P_i$ are the imbeddings of the submanifolds into the symplectic manifolds $(P_i,\Omega_i)$; and (ii) graph $\Phi$ is an isotropic submanifold of $(P_1\times P_2,\Omega_{12})$.

*Proof:* Let $\bar i$ denote the canonical injection $\bar i$: graph $\phi\to S_1\times S_2$ and $i$ the canonical injection $i$:graph $\Phi\to P_1\times P_2$. The map $j$:graph $\phi\to$graph $\Phi$ defined by $j(x,\phi(x))=(j_1(x),\ \Phi(j_1(x)),\ \forall x\in S_1$ is such that $(j_1\times j_2)\circ\bar i=i\circ j$. The map $i\circ j$:graph $\phi\to P_1\times P_2$ is an imbedding and

$$(i\circ j)^*\Omega_{12}=j_1^*(\Omega_1-\Phi^*\Omega_2).$$

Consequently $(i\circ j)^*\Omega_{12}=0$ if and only if $j_1^*\times(\Omega_1-\Phi^*\Omega_2)=0$.

We recall that if $k:I\to P$ is an isotropic submanifold of the symplectic manifold $(P,\Omega)$, then $k^*\Omega=0$, and if $\theta$ is a locally defined one-form such that $\Omega=d\theta$, the one-form $k^*\theta$ is closed and there will be a locally defined function $S$ on $I$ with $dS=k^*\theta$. Such a function $S$ is called a generalized generating function for the isotropic submanifold $I$. The important point to be remarked is that the generating function for Lagrangian submanifolds describes the local structure of these,[11] whereas the generalized generating functions for isotropic submanifolds only partially describe such submanifolds. We can, however, define generalized generating functions for canonical transformations of presymplectic systems in a similar way as in the classical case of canonical transformations for Hamiltonian systems.

With the same notations as in Theorem 10, if $\mathscr{U}_1$ and $\mathscr{U}_2$ are two neighborhoods in $P_1$ and $P_2$, respectively, in which one-forms $\theta_i$, $i = 1,2$ are defined such that $d\theta_i = \Omega_i$, the one-form $\Theta_{12} = \pi_1^* \theta_1 - \pi_2^* \theta_2$ defined in $\mathscr{U}_{12} = \mathscr{U}_1 \times \mathscr{U}_2$ satisfies $\Omega_{12} = d\Theta_{12}$. The relation between $\theta_1$ and $\Phi^*\theta_2$ for a canonical transformation of $(P,S,\Omega)$ is given by the following theorem.

**Theorem 11:** Let $\Phi: P_1 \to P_2$ be a map such that there exists $\phi: S_1 \to S_2$ with $\Phi \circ j_1 = j_2 \circ \phi$. Then, we have the following.

(i) $(\Phi,\phi)$ is a canonical transformation if and only if there is a function $G$ locally defined on graph $\phi$ such that $(i \circ j)^*\Theta_{12} = dG$.

(ii) $(\Phi,\phi)$ is a canonical transformation if and only if there is a function $F$ locally defined on $S_1$ such that $j_1^* \times (\theta_1 - \Phi^*\theta_2) = -dF$.

(iii) In the case of $(\Phi,\phi)$ being a canonical transformation, there exist connected neighborhoods $\mathscr{V}$ in $S_1$ and $\mathscr{U}$ in graph $\phi$ such that $G \circ \rho - F$ is constant, where $\rho$ is the inverse of the restriction of $\pi_1$ to $\mathscr{U}$.

*Proof:* (i) The submanifold graph $\Phi$ of $P_1 \times P_2$ is isotropic if and only if $0 = (i \circ j)^*\Omega_{12} = d(i \circ j)^*\Theta_{12}$ and therefore iff there exists a function $G$ locally defined on a neighborhood $\mathscr{U}$ in graph $\phi$ for every point in graph $\phi$ with $(i \circ j)^*\Theta_{12|\mathscr{U}} = dG$.

(ii) The canonicity condition $j_1^*(\Phi^*\Omega_2 - \Omega_1) = 0$, when restricted to $\mathscr{U}_1 \times \mathscr{U}_2$, becomes the closedness of $j_1^* \times (\Phi^*\theta_2 - \theta_1)$ on a neighborhood $\mathscr{V}$ in $S_1$ such that $j_1(\mathscr{V}) \subset \mathscr{U}_1 \cap \Phi^{-1}(\mathscr{U}_2)$. It is equivalent to the local existence of a function $F$ on $\mathscr{V}$ with

$$j_1^*(\Phi^*\theta_2 - \theta_1) = -dF.$$

(iii) Let $\rho$ be the inverse map for the restriction of $\pi_1$ to graph $\phi$.

Then, $\rho^* dG = j_1^*(\theta_1 - \Phi^*\theta_2) = dF$ and therefore $G \circ \rho - F$ is constant.

*Definition 9:* The functions $F$ and $G$ locally defined as above on $S_1$ and graph $\phi$, respectively, will be called Poincaré and Weinstein generating functions for the canonical transformation $(\Phi,\phi)$.

These functions are but generalizations of the corresponding concepts for Hamiltonian systems[17,24] and they admit extensions to open neighborhoods in $P_1$ and graph $\Phi$, respectively, as shown in the following proposition.

*Proposition 3:* With the notations of Theorem 10, if $\{\xi^i\}_{i=1}^k$ is a set of independent functions defining $\mathscr{V}$ in $P_1$ and $\{\zeta^i\}_{i=1}^k$ is another defining $\phi(\mathscr{V})$ in $P_2$, then we have the following.

(i) The neighborhood $\mathscr{U} = \rho(\mathscr{V})$ in graph $\phi$ can be defined in graph $\Phi$ by the set $\{\eta^i\}_{i=1}^k$ of independent functions given by $\eta^i = \pi_1^*\xi^i + \pi_2^*\zeta^i$.

(ii) There exists a function $\tilde{G}$ defined on $j \circ \rho(\mathscr{V}) \subset$ graph $\Phi$ such that $i^*\Theta_{12|_{\text{graph}\,\phi}} = d\tilde{G}_{|\text{graph}\,\phi}$ and $j^*\tilde{G} = G$.

(iii) There exists a function $\tilde{F}$ defined on a neighborhood $\tilde{\mathscr{V}}$ of $P_1$ such that $\tilde{\mathscr{V}} \cap C_1 = \mathscr{V}$, $j_1^*\tilde{F} = F$, $\tilde{G} \circ \rho - \tilde{F}$ is constant, and $(\Phi^*\theta_2 - \theta_1)|_{C_1} = dF_{|C_1}$.

*Proof:* (i) Let $(y,\Phi(y))$ be an element of graph $\Phi$. Then, from the identity

$$\eta(y,\Phi(y)) = (\pi_1^*\xi^i + \pi_2^*\zeta^i)(y,\Phi(y)) = \xi^i(y) + \zeta^i(\Phi(y)),$$

it follows that $\eta^i(y,\Phi(y)) = 0$ $(i = 1,...,k)$ is equivalent to $y \in \mathscr{V}$ and $\Phi(y) \in \Phi(\mathscr{V})$.

(ii) Let $G_e$ be an arbitrary but fixed extension of $G$ to $j(\rho(\mathscr{V}))$. Every extension $\tilde{G}$ can be written as $\tilde{G} = G_e + \Sigma_{i=1}^k f_i \eta^i$ and therefore

$$j^*\tilde{G} = j^*G_e + \sum_{i=1}^{k} (f_i \circ j)(\eta_i \circ j) = j^*G_e = G.$$

Finally, since $j^*(i^*\Theta_{12}) = j^*d\tilde{G}$ and using Lagrange's multiplier theorem, we can conclude that $i^*\Theta_{12|\mathscr{U}} = d\tilde{G}_{|\mathscr{U}}$.

(iii) Let $F_e$ be defined as $F_e = G_e \circ \rho$, and defining $\tilde{F}$ on a neighborhood $\tilde{\mathscr{V}}$ of $S_1$ in $P_1$ by means of $\tilde{F} = F_e + \Sigma_{i=1}^k (f_i \circ \rho)\xi^i$, such a function $F$ is such that $j_1^*\tilde{F} = F$, and furthermore,

$$d\tilde{F} = dF_e + \sum_{i=1}^{k} d(f_i \circ \rho)\xi^i + \sum_{i=1}^{k} (f_i \circ \rho) d\xi^i = d(\tilde{G} \circ \rho),$$

where the functions $\xi^i$ and $\eta^i$ defining $\mathscr{V}$ and $\phi(\mathscr{V})$ have been assumed to be chosen as $\xi^i = \eta^i \circ \rho$. Finally, since $j_1^*(\theta_1 - \Phi^*\theta_2) = dF = j_1^* d\tilde{F}$ we obtain $(\theta_1 - \Phi^*\theta_2)_{|\mathscr{V}} = d\tilde{F}_{|\mathscr{V}}$.

Before ending this section we want to remark that even if $\tilde{F}$ and $\tilde{G}$ seem to play the same role as the classical Poincaré and Weinstein generating functions, they only define locally a symplectic transformation. The point is that in some cases they define a global symplectomorphism $\Phi: P_1 \to P_2$. This case was the one considered in Ref. 24 but it is not the general case in which we are only capable of relating the coordinates of the points in $S_1$ with those of $S_2$. The next section is devoted to explaining how to get the explicit form of $\phi: S_1 \to S_2$ from the generating function $\tilde{G}$ (or $G$) as well as to presenting some remarkable results concerning the generating functions $F$ and $G$.

## VI. LOCAL PROPERTIES OF GENERATING FUNCTIONS

In this section we will analyze the local reconstruction of a generalized canonical transformation $\phi \in \text{Can}\, C$ starting from its Weinstein generating function, as well as its relation with the corresponding generating function in the reduced phase space. Let $(P_i,C_i,\Omega_i)$, with $i = 1,2$, be two coisotropic regular systems. Then, it is to be remarked that if $(\Phi,\phi)$ is a generalized canonical transformation between $(P_1,C_1,\Omega_1)$ and $(P_2,C_2,\Omega_2)$, then graph $\phi$ is an isotropic submanifold of $(P_1 \times P_2,\Omega_{12})$ while $C_1 \times C_2$ is a coisotropic submanifold. The canonical projection of $C_i$ on the corresponding reduced space will be denoted $\eta_i$ instead of the more cumbersome notation $\pi_{C_i}$. The reduced phase space $\widehat{C_1 \times C_2}$ is but $\hat{C}_1 \times \hat{C}_2$ and the projection on the reduced phase space $\widehat{C_1 \times C_2}$ is denoted $\Pi: C_1 \times C_2 \to \widehat{C_1 \times C_2}$, which coincides with $\eta_1 \times \eta_2$. We also recall that if $\hat{\phi}$ is a symplectomorphism between $(\hat{C}_1,\hat{\Omega}_1)$ and $(\hat{C}_2,\hat{\Omega}_2)$, the set graph $\hat{\phi}$ is a Lagrangian submanifold of $(\hat{C}_1 \times \hat{C}_2,\hat{\Omega}_{12})$ with $\hat{\Omega}_{12}$ defined usually as $\hat{\Omega}_{12} = \hat{\pi}_1^* \hat{\Omega}_1 - \hat{\pi}_2^* \hat{\Omega}_2$, where $\hat{\pi}_i: \hat{C}_1 \times \hat{C}_2 \to \hat{C}_1$ $(i = 1,2)$ is the canonical projection. With these notations, we can state the following proposition.

*Proposition 4:* Let $G$ be a locally defined Weinstein function for the canonical transformation $(\Phi,\phi)$. Then, there exists a Weinstein generating function $\hat{G}$ for the reduced symplectomorphism $\hat{\phi}: \hat{C}_1 \to \hat{C}_2$ such that $G = \Pi^*\hat{G}$.

*Proof:* If $\hat{\theta}_i$ and $\theta_i$ are locally defined one-forms such that $d\hat{\theta}_i = \hat{\Omega}_i$ and $d\theta_i = \Omega_i$, the identity $\eta_i^* \hat{\Omega}_i = j_i^* \Omega_i$ implies that there are locally defined functions $f_i$ on neighborhoods of the $C_i$'s with $\eta_i^* \hat{\theta}_i = j_i^* \theta_i + df_i$. If $\theta'_i$ is defined as $\theta'_i = \theta_i + d\pi_{ij}^* f_j$, where $\pi_{ij}: P_i \to C_j$ denotes the projection along the fiber structure of $P_i$ over $C_j$, then $j_i^* \theta'_i = \eta_i^* \hat{\theta}_i$. Let $G$ be the Weinstein generating function defined in Sec. V using the one-form $\Theta'_{12} = \pi_1^* \theta'_1 - \pi_2^* \theta'_2$. The following relation holds locally: $\bar{i}^*(j_1 \times j_2)^* \Theta'_{12} = dG$. If $\hat{i}$ is the natural inclusion of graph $\hat{\phi}$ in $\hat{C}_1 \times \hat{C}_2$, we have $\hat{i} \circ \Pi = \Pi \circ \bar{i}$ and $\hat{\pi}_i \circ \Pi \circ \bar{i} = \eta_i \circ \pi_i$ $(i = 1,2)$. Consequently, the one-form $\hat{\Theta}_{12}$ defined by $\hat{\Theta}_{12} = \hat{\pi}_1^* \hat{\theta}_1 - \hat{\pi}_2^* \hat{\theta}_2$ defines a generating function $\hat{G}$ such that $\Pi^* d\hat{G} = dG$ because

$$\Pi^* d\hat{G} = \Pi^* \hat{i}^* \hat{\Theta}_{12} = (\eta_1 \circ \pi_1)^* \hat{\theta}_1 - (\eta_2 \circ \pi_2) \hat{\theta}_2$$
$$= \bar{i}^* \circ (j_1 \times j_2)^* \Theta'_{12} = dG.$$

As a straightforward consequence we can state the following corollaries.

*Corollary 2:* In the same conditions as the above proposition, we can find Poincaré generalized generating functions for $\phi$ and $\hat{\phi}$, respectively, that are related by $F = \eta_1^* \hat{F}$.

*Corollary 3:* Let $\mathscr{U}_1$ be a coordinate neighborhood of a point $x_1 \in C_1$ in $P_1$ and $(q^1,...,q^n, p_1,...,p_n)$ be Darboux coordinates such that the equations $p_1 = \cdots = p_k = 0$ locally define $C_1 \cap \mathscr{U}_1$. Then, there is a Poincaré generating function such that $\partial F / \partial q^i = 0$, $i = 1,...,k$, for each canonical transformation.

*Proof:* It is an obvious consequence of the form $F = \eta_1^* \hat{F}$ because of the tangency of the vector fields $\{\partial / \partial q^i\}_{i=1}^k$ to the kernel of $j_1^* \Omega_1$ in $C_1$.

This fact is worthy of note: the generating functions $F$ do not depend on the gauge variables.

Before studying mixed generating functions for generalized canonical transformations we introduce some notations. The neighborhoods of $P_i$ in which $\theta_i$ is locally defined will be denoted by $\mathscr{U}_i$ $(1 = 1,2)$. By $x_1 = (q^1,...,q^n, p_1,...,p_n)$ we mean a set of canonical coordinates for $\mathscr{U}_1$ such that the set $\mathscr{V}_1 = \mathscr{U}_1 \cap C_1$ is defined by the vanishing of the first $k$ $p$'s.

*Lemma 1:* Let $(\mathscr{U}_1, x_1)$ be a canonical neighborhood of $m_1 \in C_1$ as defined above. For every canonical transformation $(\Phi,\phi)$ from $(P_1,C_1,\Omega_1)$ to $(P_2,C_2,\Omega_2)$, there exists a canonical neighborhood $(\mathscr{U}_2, x_2)$ of $\phi(m_1)$ such that $\phi(\mathscr{V}_1) \subset \mathscr{U}_2 \subset \Phi(\mathscr{U}_1)$ and if $x_2 = (Q^1,...,Q^n, P_1,...,P_n)$, we have $Q^i \circ \phi = q^i$, $i = 1,...,n$ and $P_{k+i} \circ \phi = p_i$, $i = 1,...,n-k$.

*Proof:* The point is that as $\Phi$ is not a symplectomorphism, $\Phi(\mathscr{U}_1)$ is not a canonical neighborhood. We remark that $\phi(\mathscr{V}_1) \subset C_2$ because $\phi(C_1) \subset C_2$. There exists a canonical neighborhood $\mathscr{U}'$ of $\phi(m_1)$ in $P_2$ such that $\mathscr{U}' = \Phi(\mathscr{U}_1)$, but what we need is that $\phi(\mathscr{V}_1) \subset \mathscr{U}' \subset \Phi(\mathscr{U}_1)$, and it can be found as follows: $\phi(\mathscr{V}_1)$ is a coisotropic submanifold of $(P_2,\Omega_2)$ and we know that there is a tubular neighborhood $\mathscr{W}$ of $\phi(\mathscr{V}_1)$ in $(P_2,\Omega_2)$ symplectomorphic to a tubular neighborhood of the canonical coisotropic imbedding of $\phi(\mathscr{V}_1)$. Then, we can choose $\mathscr{U}_2 = \Phi(\mathscr{U}_1) \cap \mathscr{W}$ and the coordinate functions given by those of the coisotropic imbedding using the identification by the local symplectomorphism, and on $\phi(\mathscr{V}_1)$ the set of coordinates given by $Q^i = q^i \circ \phi^{-1}$, $P_i = p_i \circ \phi^{-1}$. This is a canonical set satisfying the required conditions.

Instead of using the projection of graph $\phi \cap (\mathscr{U}_1 \times \mathscr{U}_2)$ on $\mathscr{V}_1$ we can also project on other sets and in this way we can define generating functions that are not of type I. The neighborhood $\mathscr{U}_{12} = \mathscr{U}_1 \times \mathscr{U}_2$ is identified with an open set of $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ via the map $x_1 \times x_2$. If we think of $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ as the product $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, each factor being labeled by a number $\alpha = 1,...,4$, the map that projects canonically in the $i \times j$ factor will be called $\pi^{ij}$. So we can construct the following six functions from $\mathscr{U}_{12}$ to $\mathbb{R}^n \times \mathbb{R}^n$, $\{\tau^{ij} = \pi^{ij} \circ (x_1 \times x_2)\}$. This family of functions defines a family of functions parametrized with different sets of variables associated by the Weinstein function defined on graph $\phi$ that we will denote by $F_{ij} = G(\tau^{ij})^{-1}$ when $(\tau^{ij})^{-1}$ exists. The first function $F_{12}$ is the coordinate representation of the usual Poincaré generating function for $(\Phi,\phi)$, and the last one, $F_{34}$, the Poincaré generating function for $(\Phi^{-1},\phi^{-1})$. The other ones are the generalized mixed generating functions of type $(ij)$ for $(\Phi,\phi)$ and their main properties will be described in the theorem below. There is an important point to be remarked here, on the definition of the mixed generating functions $F_{ij}$. We have pointed out that it is necessary that there exists $(\tau^{ij})^{-1}$ and this is equivalent to the fact that the submanifold graph $\phi$ in $P_1 \times P_2$ is transverse to the function $\tau^{ij}$; that is, denoting by $T^{ij}(p)$ the set of points which are mapped in $p \in \mathbb{R}^n \times \mathbb{R}^n$ by $\tau^{ij}$, $T^{ij}(p) = \tau^{ij-1}(p)$, $p \in \tau^{ij}(\mathscr{U}_{12})$, graph $\phi$ will be transverse in the point $(x,\phi(x)) \in$ graph $\phi$ to $\tau^{ij}$ if

$$T_{(x,\phi(x))} \text{graph } \Phi \oplus T_{(x,\phi(x))} T^{ij}(\tau^{ij}(x,\phi(x))) = T_{(x,\phi(x))}(P_1 \times P_2).$$

If it occurs we will be able to parametrize locally the submanifold graph $\phi$ (or graph $\Phi$) by means of the function $\tau^{ij}$ and then there will exist $\tau^{ij-1}$. Using these conditions in the following we can state Theorem 12.

**Theorem 12:** With the notation defined above, if $(\Phi,\phi)$ is a canonical transformation from $(P_1,C_1,\Omega_1)$ to $(P_2,C_2,\Omega_2)$, locally we have

$$p_i = \frac{\partial F_{13}}{\partial q^i}, \quad i = k+1,...,n,$$

$$P_i = -\frac{\partial F_{13}}{\partial Q^i}, \quad i = k+1,...,n,$$

$F_{13}$ being the mixed generating function of type (1.3) for the transformation.

*Proof:* The proof is a simple matter of computing the coordinate expression of the Weinstein generating function. That is, since

$$\Theta_{12} = \pi_1^* \theta_1 - \pi_2^* \theta_2 = \sum_{i=1}^n p_i \, dq^i - \sum_{i=1}^n P_i \, dQ^i,$$

then

$$dF_{12} = (\tau^{13})^{-1*} dG$$
$$= (\tau^{13})^{-1*} \Big( \sum (p_i \, dq^i - P_i \, dQ^i) \Big)$$

and then

$$\frac{\partial F_{13}}{\partial q^i} = p_i, \quad i = k+1,...n$$

and

$$-\frac{\partial F_{13}}{\partial Q^i} = P_i, \quad i = k+1,...n.$$

Finally, by the construction of the canonical neighborhood $\mathscr{U}_2$ in $P_2$ we have that $P_i \circ \phi = p_i$, $i = k+1,...n, Q^i \circ \phi = q^i, i = 1,...,n$.

There exists a similar theorem for each mixed function $F_{14}, F_{23}$, and $F_{24}$, and it is very interesting to notice that they define locally the canonical transformation only for $(n-k)\times(n-k)$ variables on the submanifold $C_1$. In the particular case of the mixed function of type (1,4), the equations before become

$$\frac{\partial F_{14}}{\partial q^i} = p_i, \quad i = k+1,...,n,$$

$$-\frac{\partial F_{14}}{\partial P_i} = Q_i, \quad i = k+1,...,n,$$

showing that the gauge ambiguity does not permit the complete reconstruction of the transformation on the manifold $C_1$ from the generating functions.

The extended generating functions $\tilde{G}, \tilde{F}$, defined in Sec. V, give locally a symplectomorphism $(\tilde{\Phi}, \phi)$ such that it coincides with $\phi$ in $\mathscr{V}_1$, but in general, as pointed out in Sec. IV, it will not be possible to extend such a symplectomorphism to a global one, and it will not be possible to construct smoothly a set of such functions such that their graphs overlap correctly.

## VII. CONCLUSIONS

We have introduced the concept of canonical transformation that generalizes the concept introduced for regular systems (see, e.g., Ref. 11), time-dependent systems,[24] and canonical systems.[9] The generalization is based on Theorem 3 where the possibility of finding a symplectic manifold $P$ in an essentially unique way is shown, such that the final constraint submanifold $C$ is coisotropically imbedded in $P$ and for any dynamical vector field $\Gamma$ compatible with $C$ there is a (no uniquely defined) vector field on $P$ with the same restriction on $C$ (up to identification of $C$ with its image). Furthermore, the result of Theorem 7 shows the possibility of studying the canonical transformations using only their local structure and the crucial point is that every canonical transformation defines a symplectic transformation in the (symplectic) reduced space and it is possible to define canonical transformations of the presymplectic space that are trivial on the quotient space; they will be called gauge transformations. In fact, if we start with a gauge theory as is usually meant it will be a presymplectic system and the group of gauge transformations as defined above coincides with the gauge group of the theory.

It is remarkable that the equations of motion can now be considered as a one-parameter family of canonical transformations. Moreover, the equation for the determination of the generating function is but the generalized Hamilton–Jacobi equation. These and other applications will be given in a subsequent paper.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