International Journal of Modern Physics D, Vol. 3, No. 1 (1994) 81–91

CANONICAL TREATMENT OF COSET SPACE SIGMA MODELS

H.-J. MATSCHULL and H. NICOLAI
II. Institut für theoretische Physik
Universität Hamburg, Luruper Chaussee 149
22761 Hamburg, Germany

ABSTRACT

The canonical treatment and quantization of non-linear coset space sigma models are discussed.

1. Introduction

In this contribution we present a canonical treatment of non-linear coset space sigma models which is based on our recent work on matter-coupled supergravity in three space-time dimensions [1, 2]. To bring out the features of the sigma model more clearly, we will restrict our discussion here to the flat space models (for a related discussion, see also [3]). The main result will be that a polynomial represention of the relevant operators can be found inspite of the inherent non-linearity of the underlying model. In this respect, our construction is very similar in spirit to recent approaches to quantum gravity based on Ashtekar's new variables [4].

Our conventions and notation are entirely taken over from [1, 2], except that we will be dealing with flat three dimensional Minkowski space with signature $-+\dots+$, the points of which are denoted by $x, y, \dots$. These coordinates decompose into a time coordinate t and space coordinates $\mathrm{x}, \mathrm{y}, \dots$ (in canonical gravity, the choice of the time coordinate is quite arbitrary); a dot stands for the derivative with respect to time. Three-dimensional indices are designated by $\mu, \nu, \dots = 0, 1, 2$ while spacelike indices are given by $i, j, .. = 1, 2$. As for the sigma model, we denote the coordinate fields on the N dimensional target manifold by $\varphi^m(x)$ with curved indices $m, n, \dots = 1, \dots, N$.

2. General Sigma Models and Notation

The general sigma model considered here describes a scalar field φ that maps the d-dimensional Lorentzian space-time into an arbitrary target manifold $\mathcal{M}$ with metric G_{mn}.

The standard Lagrangian of the sigma model reads

$$\mathcal{L}(\varphi^m(x), \partial_\mu \varphi^m(x)) = -\tfrac{1}{2} G_{mn}(\varphi(x)) \partial_\mu \varphi^m(x) \partial^\mu \varphi^n(x), \tag{2.1}$$

To define the canonical momenta we will also need the Lagrange function $L = \int \mathrm{d}\mathbf{x}\, \mathcal{L}$, where $\mathrm{d}\mathbf{x}$ denotes integration over the $d-1$ dimensional space.

Obviously, the main problem here is posed by the non-linear interactions induced by the geometrical form of this Lagrangian, and this problem also makes its appearance in the canonical formalism. The canonical momenta, which are conjugate to the coordinate fields φ^m, are defined by

$$p_m := \frac{\delta L}{\delta \dot{\varphi}^m} = G_{mn}(\varphi) \dot{\varphi}^n \tag{2.2}$$

The basic Poisson brackets are given by

$$\{\varphi^m(\mathbf{x}), p_n(\mathbf{y})\} = \delta^m_n \delta(\mathbf{x}, \mathbf{y}) \tag{2.3}$$

and the Hamiltonian is

$$H := \int \mathrm{d}\mathbf{x}\, (p_m \dot{\varphi}^m - \mathcal{L}). \tag{2.4}$$

Although the momenta do transform properly under reparametrizations (namely as vectors, i.e. elements of the tangent space $T_\varphi \mathcal{M}$), the coordinate fields φ^m do not. A first step to avoid such difficulties is to remember how the Poisson bracket is originally defined: It maps two functions on phase space to a new function.

Given two functions f, g on phase space, i.e. functionals of $\varphi^m(\mathbf{x})$ and $p_m(\mathbf{x})$, then

$$\{f, g\} = \int \mathrm{d}\mathbf{x} \left(\frac{\delta f}{\delta \varphi^m(\mathbf{x})} \frac{\delta g}{\delta p_m(\mathbf{x})} - \frac{\delta g}{\delta \varphi^m(\mathbf{x})} \frac{\delta f}{\delta p_m(\mathbf{x})} \right). \tag{2.5}$$

This definition still uses coordinates but is obviously independent of them. Now the main problem is to find "good canonical variables", i.e. functions on the phase space which

- parametrize the phase space completely;
- have simple Poisson brackets, thus can be used as basic quantum operators;
- respect the geometric structure of the model, i.e. transform as tensors on $\mathcal{M}$;
- respect additional structures of the target space (e.g. group multiplication, symmetries etc.).

In the next section we will show that this can be achieved by introducing a vielbein on $\mathcal{M}$. In addition, we will see that the canonical formulation is possible without explicit use of coordinates. Rather, we will introduce canonical variables in terms of which all relevant canonical expressions become polynomial. In particular, for group and coset manifolds, the introduction of a matrix representation will be useful to provide canonical variables having all the properties listed above.

3. The Vielbein Formalism

Let $E_A^{\ m}(\varphi)$ be a vielbein on $\mathcal{M}$. The inverse vielbein $E_m^{\ A}(\varphi)$ is then defined by

$$G_{mn} = E_m^{\ A} E_n^{\ B} \eta_{AB}, \qquad E_A^{\ m} E_m^{\ B} = \delta_A^B. \tag{3.1}$$

where η is any constant ('flat') metric. We assume that a global set of such vectors exists, as for the special cases considered later (group manifolds) it always does. As a consequence, the index A is a global index labeling the vector fields, whereas the index m refers to local coordinates.

The vector field $\partial_\mu \varphi^m$ can now be expanded in terms of the vielbein, giving

$$\partial_\mu \varphi^m = P_\mu^A E_A^{\ m} \quad \Leftrightarrow \quad P_\mu^A = E_m^{\ A} \partial_\mu \varphi^m. \tag{3.2}$$

Note that the quantity P_μ^A introduced here is inert under coordinate transformations on $\mathcal{M}$ and the standard Lagrangian takes the simple form

$$\mathcal{L} = -\tfrac{1}{2} P_\mu^A P^{\mu B} \eta_{AB}. \tag{3.3}$$

We will now show that, starting from a Lagrangian $\mathcal{L}(\varphi, P_\mu^A)$ and introducing special momenta and Poisson brackets, we obtain a canonical theory that is completely equivalent to the one defined by (2.3) and (2.4) without making any use of the 'old' canonical variables. The advantage of this procedure is that, for all examples considered below, the Lagrangian depends on the coordinate fields φ only implicitly, but let us nevertheless study the general Lagrangian $\mathcal{L}(P_\mu^A, \varphi)$.

As our new momenta we define the derivatives of L with respect to the time component P_t^A:

$$P_A := \frac{\delta L}{\delta P_t^A} = p_m \, E_A^{\ m}(\varphi). \tag{3.4}$$

The variables P_A, which we now regard as the momenta, evidently correspond to an anholonomic basis in tangent space (and thus in phase space), whereas the p_m's are like a coordinate basis.

The Hamiltonian (2.4) can be obtained directly, i.e. without using the 'old' momenta, by solving (3.4) for P_t^A and computing

$$H(P_A, \varphi) = \int \mathrm{dx}\,(P_A P_t^A - \mathcal{L}), \tag{3.5}$$

If we parametrize phase space in terms of the variables φ^m and the momenta P_A, the Poisson brackets (2.3) are reproduced via

$$\{f, g\} = \int \mathrm{dx} \ \Big(E_A^{\ m}(\varphi) \frac{\delta f}{\delta \varphi^m(\mathrm{x})} \frac{\delta g}{\delta P_A(\mathrm{x})} - E_A^{\ m}(\varphi) \frac{\delta g}{\delta \varphi^m(\mathrm{x})} \frac{\delta f}{\delta P_A(\mathrm{x})}$$
$$+ \Omega_{AB}^{\ \ \ C}(\varphi) P_C(\mathrm{x}) \frac{\delta f}{\delta P_A(\mathrm{x})} \frac{\delta g}{\delta P_B(\mathrm{x})} \Big), \tag{3.6}$$

where f and g are arbitrary functionals of φ and P_A, and

$$\Omega_{AB}^{\ \ \ C} := 2\, E_{[A}^{\ m} E_{B]}^{\ n} \partial_m E_n^{\ C} \tag{3.7}$$

are the coefficients of anholonomy of the vielbein. Here the coordinates still appear, but

$$E_A^m \frac{\delta f}{\delta\varphi^m} = E_A(f) \tag{3.8}$$

is just the action of the vector field E_A on f. If there is no explicit dependence of $\mathcal{L}$ on φ, the Hamiltonian depends on φ only via the "spatial derivatives" P_i^A; it is therefore useful to have the canonical brackets of P_A and P_i^A. A straightforward calculation yields

$$\begin{aligned}
\{P_A(\mathrm{x}), P_B(\mathrm{y})\} &= \Omega_{AB}{}^C(\varphi(\mathrm{x}))P_C(\mathrm{x})\,\delta(\mathrm{x},\mathrm{y}),\\
\{P_A(\mathrm{x}), P_i^B(\mathrm{y})\} &= \left(\delta_A^B\partial_i - \Omega_{AC}{}^B(\varphi(\mathrm{x}))P_i^C(\mathrm{x})\right)\delta(\mathrm{x},\mathrm{y}),\\
\{P_i^A(\mathrm{x}), P_j^B(\mathrm{y})\} &= 0.
\end{aligned} \tag{3.9}$$

Here and in the remainder, spatial derivatives ∂_i will always be understood to act on the *first* argument in the δ-function (i.e. x in (3.9)). The Poisson brackets (3.9) can be regarded as the basic relations from now on, but observe that the quantities P_i^A are not independent (they obey $\partial_{[i}P_{j]}^A = \frac{1}{2}\Omega_{BC}{}^A P_i^B P_j^C$), nor do they span the whole phase space: general phase space functions, such as global charges introduced below, still depend explicitly on φ.

At first sight the advantage of these relations over (2.3) is not entirely obvious, but for group manifolds the coefficients of anholonomy are just the structure constants of the algebra (for a suitably defined vielbein) and thus the coefficients on the right hand side of (3.9) become numerical, and, in addition, we will find better functions than the coordinates φ^m to parametrize the whole phase space, and the basic brackets will become even simpler.

The transition to the quantized theory is implemented by replacing any function of φ alone by the corresponding multiplication operator and

$$P_A(\mathrm{x}) \longrightarrow \widehat{P}_A(\mathrm{x}) = iE_A^m(\varphi(\mathrm{x}))\frac{\delta}{\delta\varphi^m(\mathrm{x})}. \tag{3.10}$$

The quantum operators then act on wave functionals $\Psi[\varphi(\mathrm{x})]$. The ordering prescription implicit in this replacement ensures that the relations (3.9) can be directly replaced by quantum mechanical commutators (modulo factors of i), and the that geometrical structure of (3.9) is thus preserved.

4. Group Manifolds

At this point, not much more can be said if the target space $\mathcal{M}$ is an arbitrary Riemannian manifold. For this reason, we will now make further assumptions on the structure of $\mathcal{M}$. The simplest possibility is to assume that the target space is a group manifold, i.e. $\mathcal{M} = G$ for some semisimple Lie group G. In fact, we can also treat coset space models with $\mathcal{M} = G/H$ for any subgroup H of G in this way, just by changing the explicit form on the Lagrangian and adding suitable gauge degrees of freedom.

For group manifolds, we assume the vielbein (3.1) to be a set of left invariant vector fields; this means that

$$E_A^m(\tilde{\varphi}(\varphi)) = E_A^n(\varphi)\frac{\partial\tilde{\varphi}^m(\varphi)}{\partial\varphi^n}, \tag{4.1}$$

where $\varphi \longrightarrow \tilde{\varphi}(\varphi)$ is a diffeomorphism induced by left multiplication. Then, the commutator of two vielbein vector fields is

$$[E_A, E_B] = f_{AB}{}^C E_C, \tag{4.2}$$

where $f_{AB}{}^C$ are the structure constants of the group, which also define the coefficients of anholonomy and the flat metric via

$$\Omega_{AB}{}^C = -f_{AB}{}^C, \qquad \eta_{AB} = f_{AC}{}^D f_{BD}{}^C. \tag{4.3}$$

The field theoretic model obtained in this way with the standard Lagrangian goes by the the name of "principal chiral model". From the results listed above, we can immediately derive the relevant brackets by substituting the structure constants for the coefficients of anholonomy, but still these P-variables do not parametrize the whole phase space.

The left invariance of the vielbein fields implies the invariance of P_μ^A (and thus $\mathcal{L}$, if there is no explicit φ-dependence) under left multiplication with a constant group element. This symmetry, of course, provides a conserved Noether current and a charge that canonically generates the left multiplication. However, this charge cannot be expressed in terms of the P-variables, as already mentioned in the last section, because, as the left invariant vector fields are the generators of right multiplication, the momenta P_A also generate right multiplication.

To obtain a parametrization of the whole phase space, it is convenient to introduce a (faithful) matrix representation $\mathcal{V}(\varphi)$ of the group G. A basis for the matrix representation of the algebra is then given by

$$Z_A := E_A^m(\varphi)\mathcal{V}^{-1}(\varphi)\partial_m\mathcal{V}(\varphi), \tag{4.4}$$

which is independent of φ by (4.1). The commutation relations of these matrices are

$$[Z_A, Z_B] = f_{AB}{}^C Z_C, \tag{4.5}$$

and they define a new 'flat' metric by the Killing form

$$\bar{\eta}_{AB} := \mathrm{Tr}(Z_A Z_B), \tag{4.6}$$

which may differ from (4.3) by a constant factor for each simple factor of G, depending on the representation $\mathcal{V}(\varphi)$. Using this metric we can invert (4.4) to obtain the vielbein in terms of the matrix representation and the inverse metric $\bar{\eta}^{AB}$

$$E_m^A = \mathrm{Tr}(\mathcal{V}^{-1}\partial_m\mathcal{V}Z_B)\bar{\eta}^{AB}. \tag{4.7}$$

Our new phase space variables are thus $\mathcal{V}$ and P_A, and they provide a complete parametrization (as the representation is faithful). Note that this formalism does not use the entries of the $\mathcal{V}$-matrix as the primary configuration variables, as this would necessitate an analysis of the constraints that restrict the general matrix to be an element of the group G. For exceptional groups, these constraints are not even known. Instead, the entries of the matrix are functions on the phase space. And they are chosen such that every function on phase space can be written as a function of them and the P_A's.

The Poisson brackets of $\mathcal{V}$ and P_A are very simple. One can obtain them by using (3.6) and (4.4):

$$\begin{aligned} \{P_A(\mathbf{x}), P_B(\mathbf{y})\} &= -f_{AB}{}^C P_C\, \delta(\mathbf{x},\mathbf{y}), \\ \{\mathcal{V}(\mathbf{x}), P_A(\mathbf{y})\} &= \mathcal{V} Z_A\, \delta(\mathbf{x},\mathbf{y}), \end{aligned} \tag{4.8}$$

It is essentially the new algebraic structure (matrix multiplication) that appears on the right hand side of the last equation and provides such a simple expression, showing that the momenta generate right multiplication.

If we use these quantities as the basic operators in the quantized theory, the wave functional has to be given as a function $\Psi[\mathcal{V}(\mathbf{x})]$. To obtain the expression for the operators $\widehat{P}_A$, we define the matrix derivative operator

$$\left(\frac{\partial}{\partial \mathcal{V}}\right)_{pq} := \frac{\partial}{\partial \mathcal{V}_{qp}}. \tag{4.9}$$

As it stands it can only act on functions that are defined on open subsets of $\mathbb{R}^{D\times D}$ (D the dimension of the matrix representation) and it has the property

$$\frac{\partial}{\partial \mathcal{V}} \operatorname{Tr}(A\mathcal{V}) = A \tag{4.10}$$

for any matrix A, which makes computations very simple. It is now easy to see that the polynomial operators

$$\widehat{P}_A = i \operatorname{Tr}\Big(\mathcal{V} Z_A \frac{\delta}{\delta \mathcal{V}}\Big) \tag{4.11}$$

provide the correct commutation relation, but are still defined on functions on open subsets of $\mathbb{R}^{D\times D}$ only. If the group is not open in $\mathbb{R}^{D\times D}$, one can always extend the wave functional into a neighbourhood of G. Then the operators are well defined because the result does not depend on the extension chosen: $\mathcal{V} Z_A$ is a vector field tangent to G in $\mathbb{R}^{D\times D}$.

In fact, the quantum theories defined by this representation and by (3.10) are equivalent: Let $\Psi[\mathcal{V}]$ be a wave functional in the matrix representation. It corresponds to the wave functional $\Psi[\mathcal{V}(\varphi)]$ in the original representation. When acting on it with (3.10), we obtain

$$\begin{aligned} \widehat{P}_A \Psi[\mathcal{V}(\varphi)] &= i E_A^m(\varphi) \frac{\delta}{\delta \varphi^m}\ \Psi[\mathcal{V}(\varphi)] \\ &= i E_A^m(\varphi) \operatorname{Tr}\Big(\partial_m \mathcal{V} \frac{\delta}{\delta \mathcal{V}}\Big)\ \Psi[\mathcal{V}] \\ &= i \operatorname{Tr}\Big(\mathcal{V} Z_A \frac{\delta}{\delta \mathcal{V}}\Big)\ \Psi[\mathcal{V}] \end{aligned} \tag{4.12}$$

Finally, we have to discuss the global symmetry under left multiplication with an arbitrary group element for a Lagrangian that depends on P_μ^A only. From (4.11) it is obvious that

$$\widehat{Q}_A = \int \mathrm{dx}\, i\mathrm{Tr}\Big(Z_A \mathcal{V} \frac{\delta}{\delta \mathcal{V}}\Big) \tag{4.13}$$

is the corresponding quantum operator, that generates left multiplication with a constant group element (whereas (4.11) generates right multiplication). The classical quantity is a little more complicated and reads

$$Q_A = \int \mathrm{dx}\, \mathrm{Tr}(\mathcal{V}^{-1} Z_A \mathcal{V} Z_B) \bar{\eta}^{BC} P_C. \tag{4.14}$$

Straightforward calculation yields the Poisson brackets

$$\{Q_A, \mathcal{V}\} = -Z_A \mathcal{V}, \quad \{Q_A, P_i^B\} = 0, \quad \{Q_A, P_B\} = 0. \tag{4.15}$$

As the Hamiltonian depends on $\mathcal{V}$ only via P_i^A, it follows immediately that the charges Q_A are constants of motion.

5. Coset Manifolds

Up to now we have not specified the action for the coset space sigma model in terms of the variables $\mathcal{V}(\varphi)$ on the group manifold. If we choose the standard Lagrangian (3.3), we are dealing with a physical field that takes its values on the whole group manifold. We will now show that we can treat coset space sigma models in the same manner simply by modifying the action.

A coset space is defined as the set of equivalence classes G/H of a group G modulo any subgroup H, where equivalence is defined by right multiplication, i.e. $g_1 \sim g_2$, iff $g_1 = g_2 h$ for some $h \in H$.

The distance of two equivalence classes, and thus the metric on G/H, is the "orthogonal distance" of the two submanifolds in G, which is actually independent of the special points on m the submanifolds where it is measured.

Let us now fix the Lagrangian for the coset space model to be

$$\mathcal{L}(\varphi^m, \partial_\mu \varphi^m) = G_{mn}^{\mathrm{coset}} \partial_\mu \varphi^m \partial^\mu \varphi^n, \tag{5.1}$$

where φ^m are coordinates on the coset and G_{mn}^{coset} denotes the metric on the coset space, which has to be distinguished from the metric on the group.

To treat this theory as described in the last section, we introduce additional coordinates u^r on H such that, together with φ^m, they parametrize the whole group G in the following way: let $g_0(\varphi)$ by any representative of the equivalence class with coordinates φ^m, then

$$g(\varphi, u) := g_0(\varphi) h(u) \tag{5.2}$$

provides a parametrization of G, which thereby becomes a H bundle over G/H with local section $g_0(\varphi)$. For the matrix representation we get the same formula

$$\mathcal{V}(\varphi, u) = \mathcal{V}_0(\varphi) \mathcal{W}(u), \tag{5.3}$$

where $\mathcal{W}(u)$ is a matrix representation for H. To obtain the vielbein, we first define a basis of the matrix algebra, and then use the inverse of (4.4) to get the left invariant vector fields.

Let the matrices X_α be a basis of the H subalgebra, obeying

$$[X_\alpha, X_\beta] = f_{\alpha\beta}{}^\gamma X_\gamma, \quad \mathrm{Tr}(X_\alpha X_\beta) = \bar{\eta}_{\alpha\beta}, \tag{5.4}$$

and denote by Y_a the remaining independent generators of G, which we choose to be orthogonal to the X_α:

$$\mathrm{Tr}(Y_a X_\beta) = 0, \quad \mathrm{Tr}(Y_a Y_b) = \bar{\eta}_{ab}. \tag{5.5}$$

As a consequence, all structure constants with two greek indices vanish and there is a representation of H on the coset generators Y_a defined by

$$[X_\alpha, Y_a] = f_{\alpha a}{}^b Y_b, \tag{5.6}$$

which preserves the Cartan Killing metric $\eta_{ab} = f_{ac}{}^d f_{bd}{}^c + f_{ac}{}^\gamma f_{b\gamma}{}^c + f_{a\gamma}{}^c f_{bc}{}^\gamma$.

The corresponding (inverse) vielbein on G can be obtained from (4.7). It has a triangular form and reads

$$\begin{aligned}
E_m^{\ a} &= = \mathrm{Tr}\big((\mathcal{V}_0^{-1}\partial_m \mathcal{V}_0)(\mathcal{W} Y_b \mathcal{W}^{-1})\big)\bar{\eta}^{ab}, \\
E_m^{\ \alpha} &= \mathrm{Tr}\big((\mathcal{V}_0^{-1}\partial_m \mathcal{V}_0)(\mathcal{W} X_\beta \mathcal{W}^{-1})\big)\bar{\eta}^{\alpha\beta}, \\
E_r^{\ a} &= \mathrm{Tr}\big(\mathcal{W}^{-1}\partial_r \mathcal{W} Y_b\big)\eta^{ab} = 0, \\
E_r^{\ \alpha} &= \mathrm{Tr}\big(\mathcal{W}^{-1}\partial_r \mathcal{W} X_\beta\big)\eta^{\alpha\beta}.
\end{aligned} \tag{5.7}$$

Here we see that $E_r^{\ \alpha}$ depends on u only and provides a vielbein on H, whereas $E_m^{\ a}$ and $E_m^{\ \alpha}$ still depend on φ and u. In fact, $E_m^{\ \alpha}$ transforms as a gauge field on the H principal bundle $G \to G/H$.

Since $E_m^{\ a}$ is u-dependent, it cannot be used as a vielbein on G/H; but it depends on u only via the "rotation" $Y_a \mapsto \mathcal{W} Y_a \mathcal{W}^{-1}$, which is the integrated form of (5.6). Thus the metric

$$G_{mn}^{\mathrm{coset}}(\varphi) = E_m^{\ a}(\varphi, u) E_n^{\ b}(\varphi, u)\eta_{ab} \tag{5.8}$$

is independent of u. It is exactly the metric defined at the beginning of this section, because it measures the length of a vector projected onto the plane spanned by the vielbein vectors E_a, which are by definition orthogonal to the subgroup H.

At this point one should remember that the 'flat' indices a, b and α, β globally label the vector fields E_a and E_α, whereas the 'curved' indices m, n and r, s only refer to local coordinates on the group manifold. It is therefore essential that we are dealing with a group manifold, because this ensures that global non-vanishing (left invariant) vector fields exist. It is thus the extra u-dependence in (5.8) that makes it possible to write the metric on a coset space (which generally does not admit a global vielbein) as a 'square' of a vielbein.

With these tools at hand, we can now treat the Lagrangian (5.1) as a function of the fields $\varphi^m(x)$ and $u^r(x)$, which is actually independent of u (the u^r are thus trivial gauge parameters), but to which we can apply the methods of the last section, as it is a sigma model on a group manifold. We only have to express the action in terms of the derivatives P_μ^A, which now split into P_μ^a along the coset space and Q_μ^α along the subgroup,

$$P_\mu^a = E_m^{\ a}\,\partial_\mu\varphi^m\,,$$
$$Q_\mu^\alpha = E_m^{\ \alpha}\,\partial_\mu\varphi^m + E_r^{\ \alpha}\,\partial_\mu u^r. \tag{5.9}$$

Obviously, the Lagrangian is just

$$\mathcal{L} = -\tfrac{1}{2}P_\mu^a P^{\mu b}\eta_{ab}. \tag{5.10}$$

It is formally the same as the standard action for the group manifold, but now the sum runs over the coset indices only. As a consequence, the fields Q_μ^α do not appear in the Lagrangian. So we see that it is the Lagrangian that determines which degrees of freedom are physical and which are not; we can convert the principal chiral model into a coset space sigma model simply by omitting those P_μ^A corresponding to a subgroup of H from the sum (3.3). Of course, for a non-compact group G, there is only one choice of the subgroup H for which the Hamiltonian is positive definite. If we compute the canonical momenta

$$P_a = \frac{\delta L}{\delta P_t^a} = \eta_{ab}P_t^b\,, \qquad Q_\alpha = \frac{\delta L}{\delta Q_t^\alpha} = 0 \tag{5.11}$$

the absence of Q_t^α from the Lagrangian immediately implies the constraint $Q_\alpha = 0$; this must be interpreted as a weak equality in accordance with the general theory of constraints [5].

Again we parametrize the phase space by the momenta P_a and Q_α together with the matrix $\mathcal{V}$. The Hamiltonian is then given by

$$H(P_a, Q_\alpha, \mathcal{V}) = \int \mathrm{dx}\, \tfrac{1}{2}\big(P_a P_b \eta^{ab} + P_i^a P_i^b \eta_{ab} + q^\alpha Q_\alpha\big), \tag{5.12}$$

where q^α are arbitrary functions and the composite fields P_i^a can be read off directly from

$$\mathcal{V}^{-1}\partial_i\mathcal{V} = P_i^a Y_a + Q_i^\alpha X_\alpha, \tag{5.13}$$

which combines (5.7) and (5.9).

We repeat that the main difference from the canonical point of view between the principal chiral model and the coset space sigma model characterized by this Hamiltonian is that the momenta Q_α corresponding to the subgroup H have become constraints generating gauge transformations with parameters q^α. Nonetheless, the combined set of momenta P_a and Q_α still obeys the same Poisson brackets as before; consequently, we can read off the result directly from (4.8). So, we get

$$\big\{Q_\alpha(\mathbf{x}), Q_\beta(\mathbf{y})\big\} = -f_{\alpha\beta}{}^\gamma Q_\gamma\,\delta(\mathbf{x},\mathbf{y}),$$
$$\big\{Q_\alpha(\mathbf{x}), P_b(\mathbf{y})\big\} = -f_{\alpha b}{}^c P_c\,\delta(\mathbf{x},\mathbf{y}),$$
$$\big\{P_a(\mathbf{x}), P_b(\mathbf{y})\big\} = -f_{ab}{}^\gamma Q_\gamma\,\delta(\mathbf{x},\mathbf{y}) - f_{ab}{}^c P_c\,\delta(\mathbf{x},\mathbf{y}). \tag{5.14}$$

The first equation shows that the constraints form a first class algebra. Furthermore,

$$\{\mathcal{V}(\mathrm{x}), P_a(\mathrm{y})\} = \mathcal{V}Y_a\, \delta(\mathrm{x},\mathrm{y}),$$
$$\{\mathcal{V}(\mathrm{x}), Q_\alpha(\mathrm{y})\} = \mathcal{V}X_\alpha\, \delta(\mathrm{x},\mathrm{y}) \tag{5.15}$$

which shows that the Q_α generate local H transformations on $\mathcal{V}$, which are realized by right multiplication with an element of H.

To construct an operator representation for P_a and Q_α, we can simply take over formula (3.10). Inserting the properties of the vielbein, one arrives at

$$\widehat{P}_a(\mathrm{x}) = iE_a{}^m(\mathrm{x})\frac{\delta}{\delta\varphi^m(\mathrm{x})} + iE_a{}^r(\mathrm{x})\frac{\delta}{\delta u^r(\mathrm{x})},$$
$$\widehat{Q}_\alpha(\mathrm{x}) = iE_\alpha{}^r(\mathrm{x})\frac{\delta}{\delta u^r(\mathrm{x})}. \tag{5.16}$$

Observe that the operator

$$-iE_m^a(\mathrm{x})\widehat{P}_a(\mathrm{x}) = \frac{\delta}{\delta\varphi^m(\mathrm{x})} + E_m^\alpha(\mathrm{x})\widehat{Q}_\alpha(\mathrm{x}) \tag{5.17}$$

can be viewed as a connection on the principal fiber bundle $G \to G/H$ with base space G/H and fiber H (it defines a "horizontal subspace" of $T_{\{\varphi,u\}}G$ at each point); note, however, that we are dealing with *functional*, not ordinary derivatives here.

We recall that in the quantized theory, any physical wave functional $\Psi[\varphi, u]$ must satisfy $\widehat{Q}_\alpha\Psi = 0$; with the above parametrization, this is simply solved by $\Psi = \Psi[\varphi]$. We emphasize however, that the u-dependence of $\widehat{P}_A$ cannot be dropped since otherwise the constraint algebra (5.14) would not be obeyed.

Obviously, in this representation, i.e. writing the wave function as $\Psi[\varphi, u]$, the constraints are easy to solve, but we would simply end up with a functional $\Psi[\varphi]$ and the whole group structure that has been introduced to simplify the canonical formalism would be lost. Instead, when using the matrix representation, i.e. writing $\Psi[\mathcal{V}]$, the constraints and momentum operators read

$$\widehat{Q}_\alpha = i\mathrm{Tr}\Big(\mathcal{V}X_\alpha\frac{\delta}{\delta\mathcal{V}}\Big), \qquad \widehat{P}_a = i\mathrm{Tr}\Big(\mathcal{V}Y_a\frac{\delta}{\delta\mathcal{V}}\Big), \tag{5.18}$$

which are easier to deal with than the 'geometrical' operators (5.16). The constraints now require that the wave functional is gauge invariant under local transformations $\mathcal{V} \mapsto \mathcal{V}\mathcal{W}$.

As for the group manifolds the Lagrangian in invariant under left multiplication with an arbitrary constant element of G. It follows that G acts as a group of isometry transformations on the target space $\mathcal{M} = G/H$. The associated charges $\mathcal{Q}_a$ and $\mathcal{Q}_\alpha$ are again given by (4.14). Note that the split of indices here has nothing to do with the split into physical and gauge degrees of freedom since we are now dealing with left multiplication whereas the gauge group acts by right multiplication. We may omit the terms proportional to Q_α, as these are constraints and would only generate extra gauge transformations, thus the charges are now

$$Q_a = \int \mathrm{dx}\, \mathrm{Tr}(\mathcal{V}^{-1} Y_a \mathcal{V} Y_b) \bar{\eta}^{bc} P_c,$$

$$Q_\alpha = \int \mathrm{dx}\, \mathrm{Tr}(\mathcal{V}^{-1} X_\alpha \mathcal{V} Y_b) \bar{\eta}^{bc} P_c. \tag{5.19}$$

They constitute the canonical generators of the isometry group and generate the isometry transformations on the fields, as can be verified from the relations (5.15).

The corresponding quantum operators are simply

$$Q_a = \int \mathrm{dx}\, i\mathrm{Tr}\Big(Y_a \mathcal{V} \frac{\delta}{\delta \mathcal{V}}\Big), \quad Q_\alpha = \int \mathrm{dx}\, i\mathrm{Tr}\Big(X_\alpha \mathcal{V} \frac{\delta}{\delta \mathcal{V}}\Big). \tag{5.20}$$

Note, however, that (5.20) differs from (5.19) by terms proportional to constraints. Again the charges commute with all the momenta P_a and derivatives P_i^a, thus they provide constants of motion and, in addition, they also commute with the constraints Q_α. This means that they are observables in the sense of Dirac.

References

[1] H. Nicolai and H.J. Matschull, J. Geom. Phys. **11** (1993) 15

[2] H.J. Matschull and H. Nicolai, *Canonical Quantum Supergravity in Three Dimensions*, preprint DESY 93-073, gr-qc/9306018, to appear in Nucl. Phys. B

[3] M. Forger, J. Laartz and U. Schaper, Comm. Math. Phys. **146** (1992) 397 and Freiburg Preprint THEP-92-24 (1992)

[4] A. Ashtekar, *Lectures on Non-Perturbative Canonical Gravity* World Scientific, Singapore, 1991

[5] P.A.M. Dirac, *Lectures on Quantum Mechanics*, Academic Press, New York, 1965;
A.J. Hanson, T. Regge and C. Teitelboim, *Constrained Hamiltonian Systems*, Accademia Nazionale dei Lincei, Roma, 1976;
M. Henneaux, Phys.Rep.126 (1985) 1