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Capacitary Inequalities of the Brunn-Minkowski Type

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1. Introduction

The main purpose of this paper is to show how so-called $:\alpha:$-concave capacities may be built up from Newton capacity c_n in $\mathbb{R}^n(n \geqq 3)$ in much the same way as $:\alpha:$-concave measures from Lebesgue measure m_n in $\mathbb{R}^n(n \geqq 1)$.

Throughout, $\Omega, \Omega_0, \ldots$ stand for non-empty, open, and convex subsets of $\mathbb{R}^n$ and $\mathscr{K}(\Omega)$ denotes the class of all convex bodies contained in Ω.

Recall that a positive Radon measure μ in Ω is said to be $:\alpha:$-concave if $-\infty \leqq \alpha < +\infty$ and

$$\begin{cases} \mu(\lambda A + (1-\lambda)B) \geqq (\lambda\mu^\alpha(A) + (1-\lambda)\mu^\alpha(B))^{1/\alpha} \\ \text{all} \quad A, B \in \mathscr{K}(\Omega), \quad 0 < \lambda < 1, \end{cases} \tag{1.1}$$

[2, 3, 6]. In a similar way a capacity μ in Ω will be called $:\alpha:$-concave if (1.1) is true.

Below, for any positive function f, we introduce the short-hand notation $f_{\alpha,n} = -f^{-1/n}$, $\alpha = -\infty$; $= (\operatorname{sgn}\alpha) f^{\alpha/(1-\alpha n)}$, $\alpha \neq 0$, $-\infty < \alpha < 1/n$; $= \ln f$, $\alpha = 0$; and $=$ an arbitrary positive constant for $\alpha = 1/n$, respectively. Using this convention, a positive Radon measure μ in Ω with $\operatorname{supp}\mu = \Omega$ is $:\alpha:$-concave precisely when $\alpha \leqq 1/n, \mu \ll m_n$, and a suitable version of the function $(d\mu/dm_n)_{\alpha,n}$ is concave. In a similar way, given a Borel function $f: \Omega \to]0, +\infty[$, the capacity fc_n is $:\alpha:$-concave as soon as the function $f_{\alpha,n-2}$ is concave (Theorem 3.1). However, as is readily seen, this construction does not exhaust the class of all $:\alpha:$-concave capacities in Ω.

To establish the crucical estimate

$$c_n^{1/(n-2)}(A+B) \geqq c_n^{1/(n-2)}(A) + c_n^{1/(n-2)}(B), A, B \in \mathscr{K}(\mathbb{R}^n) \tag{1.2}$$

it is natural to compare the equilibrium potentials in $\mathbb{R}^n$ of the underlying convex bodies. In what follows, suppose $\bar{\Omega}, \bar{\Omega}_0, \ldots \in \mathscr{K}(\mathbb{R}^n)$ if $n \leqq 2$ and let $u_A^\Omega = P$. [Brownian motion hits A before $\Omega^c = \mathbb{R}^n \backslash \Omega$] be the equilibrium potential of $A \in \mathscr{K}(\Omega)$ relative to Ω. Then, if $A_i \in \mathscr{K}(\Omega_i)$, $i = 0, 1$, we use an idéa of Gabriel [7, 8] to prove that

$$u^{(1-\lambda)\Omega_0+\lambda\Omega_1}_{(1-\lambda)A_0+\lambda A_1} \geqq u^{\Omega_0}_{A_0} \wedge u^{\Omega_1}_{A_1}, \quad 0<\lambda<1, \tag{1.3}$$

(Theorem 2.1), which immediately implies (1.2).

Interestingly enough, the inequality (1.3) also enlarges the previously known collection of $:\alpha:$-concave measures, a fact we express as a simple convexity property of the Poisson kernel in a convex polyhedron (Corollary 2.1).

2. A Brunn-Minkowski Version of the Gabriel-Lewis Inequality

Theorem 2.1. *Let* $x_i \in \mathbb{R}^n$, $A_i \in \mathscr{K}(\Omega_i)$, $i=0,1$, *and set* $\xi_\lambda=(1-\lambda)\xi_0+\lambda\xi_1$, $\xi=x, A, \Omega$, $0<\lambda<1$. *Then*

$$u^{\Omega_\lambda}_{A_\lambda}(x_\lambda) \geqq u^{\Omega_0}_{A_0}(x_0) \wedge u^{\Omega_1}_{A_1}(x_1).$$

The special case $A_0=A_1, \Omega_0=\Omega_1$ is due to Gabriel [7, 8] and Lewis [9]. A parabolic extension of the Gabriel-Lewis inequality has recently been settled by the author [5]. Note that Theorem 2.1 may be interpreted as a Brunn-Minkowski inequality of Brownian motion (for other estimates of this type see [6, 4], and a forthcoming paper by Antoine Ehrhard, Strasbourg).

The proof of Theorem 2.1 given below does not contain any new ingredients. Nevertheless, for the sake of completeness we will provide all details in what follows.

Lemma 2.1. *For any* $A \in \mathscr{K}(\Omega)$, $\nabla u^\Omega_A \neq 0$ *in* $\Omega \backslash A$.

Proof (assuming $0 \in A$). If $0<\alpha<1$, then by the maximum principle $u^\Omega_A(\alpha(\cdot)) \geqq u^\Omega_A$ and it follows that the non-zero harmonic map $x \cdot \nabla u^\Omega_A(x)$, $x \in \Omega \backslash A$, must be negative.

Proof of Theorem 2.1 [assuming $\bar{\Omega}_0, \bar{\Omega}_1 \in \mathscr{K}(\mathbb{R}^n)$]. Let, for brevity, $u_\lambda = u^{\Omega_\lambda}_{A_\lambda} | \bar{\Omega}_\lambda$, $0 \leqq \lambda \leqq 1$.

In what follows, suppose $0<\lambda<1$ is fixed and set

$$u^*_\lambda(x) = \sup\{u_0(x_0) \wedge u_1(x_1); x=(1-\lambda)x_0+\lambda x_1, x_i \in \bar{\Omega}_i, i=0,1\}, \quad x \in \bar{\Omega}_\lambda.$$

We shall prove the inequality $u^*_\lambda \leqq u_\lambda$. To this end, we show that the statement $\neg(u^*_\lambda \leqq u_\lambda)$ leads to a contradiction.

In fact, suppose there exists an $(\varepsilon, \hat{x}) \in]0,1[\times \bar{\Omega}_\lambda$ such that $\sup(u^*_\lambda - u^\varepsilon_\lambda) = u^*_\lambda(\hat{x}) - u^\varepsilon_\lambda(\hat{x})$ is strictly positive. Of course, $\hat{x} \in \Omega_\lambda \backslash A_\lambda$ because $u^*_{\lambda|\partial\Omega_\lambda}=0$ and $u_{\lambda|A_\lambda}=1$. Let $\hat{x}=(1-\lambda)\hat{x}_0+\lambda\hat{x}_1$, where the $\hat{x}_i \in \bar{\Omega}_i$ and $u^*_\lambda(\hat{x})=u_0(\hat{x}_0) \wedge u_1(\hat{x}_1)$. Certainly, $(\hat{x}_0, \hat{x}_1) \in (\Omega_0 \times \Omega_1) \backslash (A_0 \times A_1)$. Also, it is easy to see that the relation $\hat{x}_0 \notin A_0$, $\hat{x}_1 \in A_1$ is contradictory. Indeed, $\hat{x}_0 \in \partial u_0^{-1}(]u_0(\hat{x}_0), +\infty[)$ due to harmonicity and if $x_0 \in u_0^{-1}(]u_0(\hat{x}_0), +\infty[)$ is sufficently close to $\hat{x}_0$, then $u_1((\hat{x}-(1-\lambda)x_0)/\lambda) > u^*_\lambda(\hat{x})$ forcing $u^*_\lambda(\hat{x}) > u^*_\lambda(\hat{x})$, which cannot hold. Thus, by symmetry, $(\hat{x}_0, \hat{x}_1) \in (\Omega_0 \backslash A_0) \times (\Omega_1 \backslash A_1)$.

Now suppose $h \in \mathbb{R}^n$ and $h \cdot \nabla u_i(\hat{x}_i) > 0$ ($i=0$ or 1). Then, if $s>0$ is sufficiently small, $u_i(\hat{x}_i+sh) > u_i(\hat{x}_i)$ and, hence, $u^*_\lambda(\hat{x}+s\lambda_i h) \geqq u^*_\lambda(\hat{x})$, where $\lambda_i=(2i-1)\lambda+1-i$, so that $u^\varepsilon_\lambda(\hat{x}+s\lambda_i h) \geqq u^\varepsilon_\lambda(\hat{x})$. Accordingly, $h \cdot \nabla u_\lambda(\hat{x}) \geqq 0$ and it follows that the non-zero vectors $\nabla u_\lambda(\hat{x})$ and $\nabla u_i(\hat{x}_i)$ are parallel. In the following, let $a = |\nabla u_\lambda^\varepsilon(\hat{x})|$, $a_i = |\nabla u_i(\hat{x}_i)|$, and $v = \nabla u_\lambda^\varepsilon(\hat{x})/a$, respectively.

From now on we assume that $u_\lambda^*(\hat{x}) = u_0(\hat{x}_0)$. The case $u_\lambda^*(\hat{x}) = u_1(\hat{x}_1)$ may be treated in a quite similar way.

Let $h \in \mathbb{R}^n$ be such that the scalar $\kappa = h \cdot v \neq 0$. For each real s close to 0 there exists a unique $r = r(s)$, with $|r|$ minimal, satisfying the equation

$$u_0(\hat{x}_0 + sh/a_0) - u_0(\hat{x}_0) = u_1(\hat{x}_1 + rh/a_1) - u_1(\hat{x}_1).$$

Writing $\hat{x}(s) = (1-\lambda)(\hat{x}_0 + sh/a_0) + \lambda(\hat{x}_1 + r(s)h/a_1) = \hat{x} + ((1-\lambda)s/a_0 + \lambda r(s)/a_1)h$ we now have

$$u_0(\hat{x}_0 + sh/a_0) - u_\lambda^\varepsilon(\hat{x}(s)) \leqq u_\lambda^*(\hat{x}(s)) - u_\lambda^\varepsilon(\hat{x}(s)) \leqq u_0(\hat{x}_0) - u_\lambda^\varepsilon(\hat{x}).$$

In particular,

$$D_s^2(u_0(\hat{x}_0 + sh/a_0) - u_\lambda^\varepsilon(\hat{x}(s)))_{|s=0} \leqq 0. \tag{2.1}$$

To analyse (2.1) we introduce the following Taylor expansions

$$u_i(\hat{x}_i + sh/a_i) = u_i(\hat{x}_i) + \kappa s + b_i s^2 + o(s^2) \quad \text{as} \quad s \to 0 \qquad (i = 0, 1)$$

and get

$$r(s) = s + \kappa^{-1}(b_0 - b_1)s^2 + o(s^2) \quad \text{as} \quad s \to 0.$$

Moreover, setting $p = (1-\lambda)/a_0 + \lambda/a_1$ and

$$u_\lambda^\varepsilon(\hat{x} + sh) = u_\lambda^\varepsilon(\hat{x}) + \kappa a s + B s^2 + o(s^2) \quad \text{as} \quad s \to 0,$$

respectively, a straightforward calculation yields

$$u_\lambda^\varepsilon(\hat{x}(s)) = u_\lambda^\varepsilon(\hat{x}) + \kappa a p s + [p^2 B + \lambda(a/a_1)(b_0 - b_1)]s^2 + o(s^2) \quad \text{as} \quad s \to 0.$$

Thus from (2.1), $\xi \cdot (b_0, b_1) - p^2 B \leqq 0$ for a suitable vector $\xi \in \mathbb{R}^2$, which does not depend on h. But then, for any $h \in \mathbb{R}^n$,

$$\sum_{1 \leqq j,k \leqq n} \left[\xi \cdot \left(\frac{1}{a_0^2} D^2_{x_j x_k} u_0(\hat{x}_0), \frac{1}{a_1^2} D^2_{x_j x_k} u_1(\hat{x}_1)\right) - p^2 D^2_{x_j x_k} u_\lambda^\varepsilon(\hat{x})\right] h_j h_k \leqq 0$$

and we have

$$\xi \cdot \left(\frac{1}{a_0^2} \Delta u_0(\hat{x}_0), \frac{1}{a_1^2} \Delta u_1(\hat{x}_1)\right) - p^2 \Delta u_\lambda^\varepsilon(\hat{x}) \leqq 0,$$

or, stated otherwise, $|\nabla u_\lambda(\hat{x})| = 0$. From this contradiction we conclude that $u_\lambda^* \leqq u_\lambda$ and the proof of Theorem 2.1 is completed.

Corollary 2.1. *Assume $\bar{\Omega}$ is a convex polyhedron in $\mathbb{R}^n$ and denote by F any $(n-1)$-dimensional facet of $\bar{\Omega}$. Let κ_x^Ω be the harmonic measure at $x \in \Omega$ relative to Ω. Then an appropriate version of the restricted Poisson kernel*

$$\Omega \times \mathrm{ri}(\bar{\Omega} \cap F) \to]0, +\infty[, (x, z) \rightsquigarrow [d\kappa_x^\Omega / d\sigma_{\mathrm{ri}(\bar{\Omega} \cap F)}](z)$$

raised to the power $-1/(n-1)$ is convex.

Here ri denotes the relative interior operator.

The special case $\Omega=\{x_n>0\}$ $(n\geqq 2)$ with

$$d\kappa_x^\Omega(z)=\frac{\Gamma(n/2)}{\pi^{n/2}}\frac{x_n}{\|x-z\|^n}dz, \quad x_n>0, \quad z_n=0,$$

shows that, generally, there does not exist any greater real number than $-1/(n-1)$ possessing the property stated in Corollary 2.1.

Proof. Let $A, B\subseteqq \mathrm{ri}(\bar{\Omega}\cap F)$ be arbitrary compact and convex sets. Then, by applying Theorem 2.1, we conclude that

$$\begin{cases}\kappa^\Omega_{\lambda x+(1-\lambda)y}(\lambda A+(1-\lambda)B)\geqq\kappa_x^\Omega(A)\wedge\kappa_y^\Omega(B)\\ \text{all} \quad x, y\in\Omega, \quad 0<\lambda<1\end{cases}$$

which is just another way to state what we want to prove [3, Theorems 3.2, and 3.1]. □

3. :α:-Concave Capacities Induced by Newton Capacity

Recall that if μ is a capacity in Ω [11, p. 7.30] and $f:\Omega\to[0,+\infty]$ is a Borel function, then, by definition,

$$(f\mu)(A)=\int_A f d\mu=\int_0^\infty \mu(A\cap\{f\geqq s\})ds, \quad A\in\mathscr{B}(\Omega).$$

Theorem 3.1. *Let $n\geqq 3$ and $-\infty\leqq\alpha\leqq 1/(n-2)$. Moreover, assume $f:\Omega\to]0,+\infty[$ is such that $f_{\alpha,n-2}$ is concave. Then the capacity fc_n is :α:-concave.*

Proof. We first consider the special case $\alpha=1/(n-2)$.

Following the convention that $c_n(B(0;r))=r^{n-2}$,

$$c_n(A)=\lim_{|x|\to+\infty}\lim_{r\to+\infty}|x|^{n-2}u_A^{B(0;r)}(x), \quad A\in\mathscr{K}(\mathbb{R}^n),$$

and, hence, for any fixed $A, B\in\mathscr{K}(\mathbb{R}^n)$,

$$c_n(\lambda A+(1-\lambda)B)\geqq c_n(A)\wedge c_n(B), \quad 0<\lambda<1,$$

in view of Theorem 2.1. In particular,

$$c_n(\lambda A/c_n^{1/(n-2)}(A)+(1-\lambda)B/c_n^{1/(n-2)}(B))\geqq 1, \quad 0<\lambda<1,$$

and (1.2) follows by choosing λ as the solution of the equation

$$\lambda/c_n^{1/(n-2)}(A)=(1-\lambda)/c_n^{1/(n-2)}(B).$$

Next suppose $0<\alpha<1/(n-2)$ and set $1/\beta=\alpha/(1-\alpha(n-2))$. Given $A, B\in\mathscr{K}(\mathbb{R}^n)$ and $0<\lambda<1$, we shall prove that

$$\int_0^\infty c_n((\lambda A+(1-\lambda)B)\cap\{f\geqq s^\beta\})s^{\beta-1}ds$$
$$\geqq\left[\lambda\left(\int_0^\infty c_n(A\cap\{f\geqq s^\beta\})s^{\beta-1}ds\right)^\alpha+(1-\lambda)\left(\int_0^\infty c_n(B\cap\{f\geqq s^\beta\})s^{\beta-1}ds\right)^\alpha\right]^{1/\alpha},$$

which, however, is an immediate consequence of (1.2) and a standard inequality from the theory of $:\alpha:$-concave measures [3, Theorem 3.1 or 6, Theorem 5.1].

Finally, the case $-\infty \leqq \alpha \leqq 0$ may be treated in a similar way and we omit the details here.

This completes the proof of Theorem 3.1. □

4. Some Simple Examples

Example 4.1. The Newtonian capacity of a rectangular box in $\mathbb{R}^3$ is a concave function of the edge lengths. □

Example 4.2. If a body $A \in \mathscr{K}(\mathbb{R}^3)$ carries the total charge $p>0$, then the electrostatic energy $E(A,p)$ equals $p^2/c_3(A)$. Hence, for all $A, B \in \mathscr{K}(\mathbb{R}^3)$ and $p,q>0$,

$$E(A+B, p+q) \leqq E(A,p) + E(B,q)$$

by the Hölder inequality. □

Example 4.3. Consider an $A \in \mathscr{K}(\mathbb{R}^2)$ and suppose $g=(0,0,1)$ and $\varepsilon>0$. Following the convention that $\mathbb{R}^2 = \mathbb{R}^2 \times \{0\} \subseteqq \mathbb{R}^3$, the sets $A \pm \frac{\varepsilon}{2} g$ define a plate condenser $A(\varepsilon) = \left(A - \frac{\varepsilon}{2} g\right) \cup \left(A + \frac{\varepsilon}{2} g\right)$ possessing the capacity $c(A,\varepsilon)$ equal to the total mass of the equilibrium measure μ of $A + \frac{\varepsilon}{2} g$ relative to the half space $\{x_3>0\}$. Thus, using probabilistic normalizations (Port and Stone [10]), it follows that

$$u^H_{A+\frac{\varepsilon}{2}g}(x) = \frac{1}{2\pi} \int \left(\frac{1}{|x-y|} - \frac{1}{|x-y+2y_3 g|} \right) d\mu(y),$$

where the integrand equals

$$\frac{1}{2} \frac{\varepsilon^2}{|x-y|^3} + o\left(\frac{1}{|x-y|^3}\right)$$

uniformly in $y \in A + \frac{\varepsilon}{2} g$ as $|x| \to +\infty$ subject to the restriction $x_3 = \varepsilon/2$. By applying Theorem 2.1 we now have for all $A, B \in \mathscr{K}(\mathbb{R}^3)$ and $\varepsilon, \delta > 0, 0<\lambda<1$,

$$\begin{aligned} &(\lambda\varepsilon + (1-\lambda)\delta)^2 c(\lambda A + (1-\lambda)B, \lambda\varepsilon + (1-\lambda)\delta) \\ &\geqq [\varepsilon^2 c(A,\varepsilon)] \wedge [\delta^2 c(B,\delta)]. \end{aligned}$$

Finally, noting that $c(\cdot,\cdot)$ is positive homogenous of degree one, we get

$$(\varepsilon+\delta)^{2/3} c^{1/3}(A+B, \varepsilon+\delta) \geqq \varepsilon^{2/3} c^{1/3}(A,\varepsilon) + \delta^{2/3} c^{1/3}(B,\delta). \quad \square$$

Example 4.4. Let $\Omega \in \mathscr{K}(\mathbb{R}^n)$ $(n \geqq 3)$ be bounded and suppose $f: \Omega \to]0, +\infty[$ is concave. Then

$$\int_\Omega f^p dc_n = p \int_0^{\|f\|_\infty} s^{p-1} c_n(f \geqq s) ds,$$

where the function $c_n^{1/(n-2)}(f \geqq s)$, $0 < s < \|f\|_\infty$, is decreasing and concave. We may now apply known elementary integral inequalities [1, Lemma 3.1] to deduce that the function

$$\ln\left[\binom{n-2+p}{n-2}\int_\Omega f^p dc_n\right], \quad p>0,$$

is concave. Furthermore, the function

$$\left[\binom{n-2+p}{n-2}\int_\Omega f^p dc_n/c_n(\Omega)\right]^{1/p}, \quad p>0,$$

decreases. □

Example 4.5. Suppose μ is an $:\alpha:$-concave capacity in $\mathbb{R}^n$ and, in addition, suppose μ is symmetric, that is, $\mu(A)=\mu(-A)$, $A \in \mathscr{B}(\mathbb{R}^n)$. Then, for all barrels $A, B \subseteqq \mathbb{R}^n$ and every $x \in \mathbb{R}^n$ and $\theta > 1$,

$$\mu(A \cap (B+x)) \geqq \mu(A \cap (B+\theta x)).$$

Indeed, μ is increasing and the claim above follows at once from the set relation

$$A \cap (B+x) \supseteqq \frac{\theta+1}{2\theta}(A \cap (B+\theta x)) + \frac{\theta-1}{2\theta}(A \cap (B-\theta x)). \quad \square$$

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