Capacity and Optimal Power Allocation for Fading Broadcast Channels with Minimum Rates

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Abstract—**We derive the capacity and optimal power allocation scheme for a multi-user fading broadcast channel in which minimum rates must be maintained for each user in all fading states, assuming perfect channel state information at the transmitter and at all receivers. We show that superposition coding can achieve the capacity of such channels and explicitly characterize the boundary of the capacity region. The optimal power allocation scheme is a two-step process: We first allocate the minimum power required to achieve the minimum rates in all fading states, and we then optimally allocate the excess power to maximize the ergodic rates averaged over all fading states in excess of the minimum rate requirements. The optimal allocation of the excess power is a multi-level water-filling relative to effective noise that incorporates the minimum rate constraints. Numerical results are provided for different fading broadcast channel models.**

I. Introduction

The ergodic capacity of fading broadcast channels determines the maximum average rates achievable in the downlink of a single cell. Ergodic capacity is achieved via multi-level water-filling of power over time and users using superposition coding [1,2]. An unfortunate consequence of the optimal power allocation scheme is that users with poor channels may receive no data for large periods of time, depending on the duration of channel fades. Such a situation may be unacceptable in delay-constrained applications such as video transmission.

With the above motivation in mind, this paper finds the ergodic capacity of a broadcast channel subject to an average power constraint and with minimum rate requirements for all users. We will show that the optimal power allocation scheme with minimum rates reduces to first allocating the minimum power required to meet the minimum rates and then allocating the excess power according to a multi-level water-filling scheme relative to effective noise. Minimum-rate capacity is essentially a combination of outage capacity [3] and ergodic capacity: some power is used to maintain the minimum rates in all fading states, similar to outage capacity with zero-outage probability, while the remaining power is used to maximize the average rates *in excess* of the minimum rates, similar to ergodic capacity. Minimum-rate capacity with a peak power constraint and with outage are extensions of our derivations which are solved in [6].

This paper is organized as follows. Section II describes the system model. Section III considers a single-user fading channel with a minimum rate constraint. In Section IV we show that superposition coding is optimal for the fading two-user broadcast channel and find the optimal power allocation scheme using both Lagrangian and greedy techniques. Numerical results are presented in Section V, followed by our conclusions.

II. System Model

For notation, we use boldface to denote vectors and $\mathbb{E}_n$ to denote expectation over the random variable n.

We consider a flat-fading Gaussian broadcast channel with a single transmitter communicating independent information to M users over bandwidth B. The signal intended for user j at time i is denoted by $X_j(i)$ and has power $P_j(i)$. Each receiver has additive white Gaussian noise (AWGN) with noise density v_j. The time-varying channel gain of user j is denoted by $\sqrt{g_j(i)}$. By incorporating the channel gain into the noise term as in [2], we define an effective noise density $n_j(i) = v_j/g_j(i)$ and get an equivalent form for the received signal:

$$Y_j(i) = \sum_{k=1}^{M} X_k(i) + z_j(i) \qquad (1)$$

where $z_j(i) \sim N(0, n_j(i)B)$. We assume that the noise density vector $\mathbf{n}(i) = (n_1(i), \ldots, n_M(i))$ is known to the transmitter and all M receivers at time instant i. As the noise density vector incorporates the effects of the channel gain, we will alternatively refer to $\mathbf{n}$ as the *fading state*.

The transmitter can vary power $P_j(i)$ (and therefore the rate) for each user relative to the noise density vector, subject to average power constraint $\overline{P}$. Using superposition coding with successive decoding, the rate of user j for fading state $\mathbf{n}$ is given by $R_j(\mathbf{n}) = B\log(1 + \frac{P_j(\mathbf{n})}{n_j B + \sum_{i=1}^{M} P_i(\mathbf{n})\mathbf{1}[n_j > n_i]})$ where $\mathbf{1}[.]$ is the indicator function. For simplicity, we assume $B = 1$. In this paper we impose minimum rate constraints $\mathbf{R}^* = (R_1^*, \ldots, R_M^*)$ which must be maintained in all fading states, or $R_j(\mathbf{n}) \geq R_j^*, j = 1, \ldots, M$, for all $\mathbf{n}$. Clearly, $\mathbf{R}^*$ must be in the zero-outage capacity region [3] of the channel in order for the minimum rates to be achievable in all fading states.

III. Single User Fading Channel

Before analyzing the broadcast channel, we first find the capacity-achieving scheme for a single user fading channel subject to minimum rate constraints. We must find the optimal power allocation scheme $P(n)$ (n is a scalar for the single user scenario), or equivalently:

$$\max_{P(n)} \mathbb{E}_n \left[\log(1 + \frac{P(n)}{n})\right] \qquad (2)$$

$$\text{subject to:} \quad \mathbb{E}_n[P(n)] \leq \overline{P}, \quad R(n) \geq R^* \quad \forall\, n.$$

We denote the minimum power required to achieve the minimum rate as $P^*(n) \triangleq n(e^{R^*}-1)$ and define $\hat{P}(n)$ as the power allocated to fading state n in excess of $P^*(n)$. The total power allocated to fading state n then is $P(n) = P^*(n) + \hat{P}(n)$. Using standard Lagrangian techniques, the optimal allocation of excess power is modified water-filling:

$$\hat{P}(n) = \begin{cases} \frac{1}{\lambda} - (n + P^*(n)) & n + P^*(n) \leq \frac{1}{\lambda} \\ 0 & n + P^*(n) > \frac{1}{\lambda} \end{cases} \tag{3}$$

where $\frac{1}{\lambda}$ is the water-filling level satisfying the reduced power constraint $P^* = \overline{P} - \mathbf{E_n}[P^*(n)]$.

The special structure of the capacity formula for Gaussian channels allows us to treat power allocated to a channel as an additional source of noise. Therefore, the interpretation of this scheme is very simple: First allocate minimum power $P^*(n)$ to each state. Then, treating $P^*(n)$ as additional noise, use the standard water-filling algorithm with effective noise $n + P^*(n)$ rather than n and with power constraint P^* instead of $\overline{P}$. We will soon seen that the multi-user broadcast channel can be interpreted in a similar manner.

IV. Two-User Fading Broadcast Channel

Now consider a multi-user fading broadcast channel as described in Section II with $M = 2$. We explicitly characterize the minimum rate capacity for the two-user case, but these results can be extended to an arbitrary M-user broadcast channel. For simplicity, we assume $n_2 > n_1$ in all states. Later we show how our results extend to the general case where the ordering of noises differs from state to state.

Before we find the optimal power allocation scheme, let us first prove that superposition coding achieves capacity for the broadcast channel with minimum rates. Consider a two-user, constant broadcast channel with minimum rates. Our goal then is to find all possible rate pairs (R_1, R_2) satisfying minimum rates R_1^* and R_2^*, or equivalently all rate pairs such that $R_1 \geq R_1^*$ and $R_2 \geq R_2^*$. Without minimum rates, all possible rate pairs are achievable by superposition coding. The rate pairs satisfying the minimum rates are simply a subset of all rate pairs without minimum rates, and are therefore achievable by superposition coding. Because the fading broadcast channel can be equivalently viewed as a set of parallel constant broadcast channels [2], one for each fading state, superposition coding can be used in each of these parallel channels to achieve capacity. Using the time-sharing argument of [2], we also can show that the capacity region is convex.

Due to the convexity of the region, the boundary of the capacity region can be found by the following maximization:

$$\max_{\mathbf{P}(\mathbf{n})} \mathbb{E}_n [\mu_1 R_1(\mathbf{n}) + \mu_2 R_2(\mathbf{n})] \tag{4}$$
$$\text{subject to:} \quad \mathbb{E}_n [P_1(\mathbf{n}) + P_2(\mathbf{n})] \leq \overline{P}$$
$$R_1(\mathbf{n}) \geq R_1^*, \quad R_2(\mathbf{n}) \geq R_2^* \quad \forall \mathbf{n}$$

over $0 \leq \mu_1 \leq 1$ and $\mu_2 = 1 - \mu_1$.

Let us now introduce notation similar to that used in Section III. Defining minimum powers $P_1^*(\mathbf{n}) \triangleq n_1(e^{R_1^*}-1)$ and $P_2^*(\mathbf{n}) \triangleq (P_1^*(\mathbf{n}) + n_2)(e^{R_2^*} - 1)$, (4) simplifies to:

$$\max_{\hat{\mathbf{P}}(\mathbf{n})} \mathbb{E}_n \left[\mu_1 \log\left(1 + \frac{\hat{P}_1(\mathbf{n}) + P_1^*(\mathbf{n})}{n_1}\right) \right. \tag{5}$$
$$\left. + \mu_2 \log\left(1 + \frac{\hat{P}_2(\mathbf{n}) + P_2^*(\mathbf{n})}{\hat{P}_1(\mathbf{n}) + P_1^*(\mathbf{n}) + n_2}\right) \right]$$
$$\text{subject to:} \quad \mathbb{E}_n [\hat{P}(\mathbf{n})] \leq P^*, \quad 0 \leq \hat{P}_1(\mathbf{n}) \leq \hat{P}(\mathbf{n}) e^{-R_2^*}$$

where $P^* \triangleq \overline{P} - \mathbb{E}_n[P_1^*(\mathbf{n}) + P_2^*(\mathbf{n})]$ is the excess power constraint and $\hat{P}(\mathbf{n}) \triangleq \hat{P}_1(\mathbf{n}) + \hat{P}_2(\mathbf{n})$ is the excess power allocated to fading state $\mathbf{n}$. As before, $\hat{P}_1(\mathbf{n})$ and $\hat{P}_2(\mathbf{n})$ represent excess power.

Though the minimum rate capacity of the single-user channel was found rather easily, the broadcast channel problem is considerably more difficult because stronger users interfere with weaker users. Whenever excess power $\hat{P}_1(\mathbf{n})$ is allocated to user 1, the interference seen by user 2 increases and R_2 decreases. Therefore, whenever user 1 is allocated excess power some excess power must also be allocated to user 2 to overcome the additional interference. As a result, user 1 cannot be allocated all of the excess power in a state and $\hat{P}_1(\mathbf{n})$ is constrained to $0 \leq \hat{P}_1(\mathbf{n}) \leq \hat{P}(\mathbf{n}) e^{-R_2^*}$.

In order to solve (5), we decompose the maximization into two steps:

1. Given $\hat{P}(\mathbf{n})$ for all $\mathbf{n}$, we must optimally distribute the excess power between the users in each state:

$$F_n(\hat{P}(\mathbf{n})) \triangleq \max_{\hat{P}_1(\mathbf{n})} \mu_1 \log\left(1 + \frac{\hat{P}_1(\mathbf{n}) + P_1^*(\mathbf{n})}{n_1}\right) \tag{6}$$
$$+ \mu_2 \log\left(1 + \frac{\hat{P}(\mathbf{n}) - \hat{P}_1(\mathbf{n}) + P_2^*(\mathbf{n})}{\hat{P}_1(\mathbf{n}) + P_1^*(\mathbf{n}) + n_2}\right)$$
$$\text{subject to:} \quad 0 \leq \hat{P}_1(\mathbf{n}) \leq \hat{P}(\mathbf{n}) e^{-R_2^*}.$$

2. After we find $F_n(\hat{P}(\mathbf{n}))$ for each $\mathbf{n}$, we must optimally allocate excess power $\hat{P}(\mathbf{n})$ over all fading states:

$$\max_{\hat{P}(\mathbf{n})} \mathbb{E}_n [F_n(\hat{P}(\mathbf{n}))] \quad \text{subject to:} \quad \mathbb{E}_n [\hat{P}(\mathbf{n})] \leq P^*. \tag{7}$$

Equation (6) is a one-dimensional optimization over $\hat{P}_1(\mathbf{n})$ and is therefore easily solved. $F_n(\hat{P}(\mathbf{n}))$ is achieved by the following power distribution:

1) If $\frac{\mu_2}{\mu_1} \leq 1$, then $\hat{P}_1(\mathbf{n}) = \hat{P}(\mathbf{n}) e^{-R_2^*}$ and $\hat{P}_2(\mathbf{n}) = \hat{P}(\mathbf{n})(1 - e^{-R_2^*})$.

2) If $\frac{\mu_2}{\mu_1} \geq \frac{n_2 + P_1^*}{n_1 + P_1^*}$, then $\hat{P}_2(\mathbf{n}) = \hat{P}(\mathbf{n})$ and $\hat{P}_1(\mathbf{n}) = 0$.

3a) If $1 < \frac{\mu_2}{\mu_1} < \frac{n_2 + P_1^*}{n_1 + P_1^*}$ and $\hat{P}(\mathbf{n}) \leq P_{thresh}$, then $\hat{P}_1(\mathbf{n}) = \hat{P}(\mathbf{n}) e^{-R_2^*}$ and $\hat{P}_2(\mathbf{n}) = \hat{P}(\mathbf{n})(1 - e^{-R_2^*})$.

3b) If $1 < \frac{\mu_2}{\mu_1} < \frac{n_2 + P_1^*}{n_1 + P_1^*}$ and $\hat{P}(\mathbf{n}) > P_{thresh}$, then $\hat{P}_1(\mathbf{n}) =$

$P_{thresh}e^{-R_2^*}$ and $\hat{P}_2(\mathbf{n}) = \hat{P}(\mathbf{n}) - \hat{P}_1(\mathbf{n})$ where $P_{thresh} = (\frac{\mu_1 n_2 - \mu_2 n_1}{\mu_2 - \mu_1} - P_1^*)e^{R_2^*}$.

Power allocated to user 1 increases R_1, but also leads to a decrease in R_2 due to interference. On the other hand, power allocated to user 2 will not lead to as large of an increase in R_2 because $n_2 > n_1$, but it does not affect R_1. In case 1, because $\mu_1 \geq \mu_2$, it is optimal to give user 1 as much power as allowable while ensuring that user 2 can still maintain R_2^*. On the other extreme, in case 2, when $\mu_2 \gg \mu_1$, user 2 is given all available power. When $1 < \frac{\mu_2}{\mu_1} < \frac{n_2+P_1^*}{n_1+P_1^*}$ as in case 3 , the weights are such that increasing R_1 is optimal up to the point where $\hat{P}_1 = P_{thresh}e^{-R_2^*}$ (case 3a). Past this point, it is optimal to use the power above P_{thresh} to increase R_2 (case 3b).

Now that $F_n(\hat{P}(\mathbf{n}))$ is known, we must solve (7). We introduce a Lagrangian multiplier λ to get:

$$\max_{\hat{P}(\mathbf{n})} \ \mathbb{E}_n[F_n(\hat{P}(\mathbf{n}))] - \lambda(\mathbb{E}_n[\hat{P}(\mathbf{n})] - P^*). \tag{8}$$

The standard solution to such a problem satisfies $F'(\hat{P}(\mathbf{n})) = \lambda$, similar to single user water-filling. The optimal power allocation scheme is a two-level water-filling scheme where $\frac{1}{\lambda}$ is chosen to satisfy excess power constraint P^*:

1) If $\mu_2 \leq \mu_1$, then

$$\begin{cases} \hat{P}_1(\mathbf{n}) = \frac{\mu_1}{\lambda}e^{-R_2^*} - (P_1^*(\mathbf{n}) + n_1) \\ \hat{P}_2(\mathbf{n}) = \frac{\mu_1}{\lambda}(1 - e^{-R_2^*}) - (P_1^*(\mathbf{n}) + n_1)(e^{R_2^*} - 1). \end{cases} \tag{9}$$

2) If $\frac{\mu_2}{\mu_1} \geq \frac{n_2+P_1^*}{n_1+P_1^*}$, then

$$\begin{cases} \hat{P}_1(\mathbf{n}) = 0 \\ \hat{P}_2(\mathbf{n}) = \frac{\mu_2}{\lambda} - (P_1^*(\mathbf{n}) + n_2)e^{R_2^*}. \end{cases} \tag{10}$$

3a) If $1 \leq \frac{\mu_2}{\mu_1} \leq \frac{n_2+P_1^*}{n_1+P_1^*}$ and $\lambda \leq (\frac{\mu_2-\mu_1}{n_2-n_1})e^{-R_2^*}$ then

$$\begin{cases} \hat{P}_1(\mathbf{n}) = \frac{\mu_1}{\lambda}e^{-R_2^*} - (P_1^*(\mathbf{n}) + n_1) \\ \hat{P}_2(\mathbf{n}) = \frac{\mu_1}{\lambda}(1 - e^{-R_2^*}) - (P_1^*(\mathbf{n}) + n_1)(e^{R_2^*} - 1). \end{cases} \tag{11}$$

3b) If $1 \leq \frac{\mu_2}{\mu_1} \leq \frac{n_2+P_1^*}{n_1+P_1^*}$ and $\lambda > (\frac{\mu_2-\mu_1}{n_2-n_1})e^{-R_2^*}$ then

$$\begin{cases} \hat{P}_1(\mathbf{n}) = P_{thresh}e^{-R_2^*} \\ \hat{P}_2(\mathbf{n}) = \frac{\mu_2}{\lambda} - (P_1^* + n_2)e^{R_2^*} - P_{thresh}e^{-R_2^*}. \end{cases} \tag{12}$$

After some algebraic manipulation, the state-by-state excess power allocation simplifies to:

$$\hat{P}(\mathbf{n}) = \max\left(\frac{\mu_1}{\lambda} - (P_1^*(\mathbf{n}) + n_1)e^{R_2^*}, \frac{\mu_2}{\lambda} - (P_1^*(\mathbf{n}) + n_2)e^{R_2^*}, 0\right) \tag{13}$$

and $\hat{P}_1(\mathbf{n})$ and $\hat{P}_2(\mathbf{n})$ are specified by $F_n(\hat{P}(\mathbf{n}))$ Extending this scheme to allow states in which $n_1 > n_2$ only requires reversing all subscripts in (13).

To simplify this solution, let us first define effective noises n_1' and n_2':

$$\begin{cases} n_i' = (P_1^*(\mathbf{n}) + n_i)e^{R_2^*} & n_1 < n_2 \\ n_i' = (P_2^*(\mathbf{n}) + n_i)e^{R_1^*} & n_1 \geq n_2 \end{cases} \tag{14}$$

Let us examine the $n_1 < n_2$ equations more carefully. In Section III we saw that the effective noise in a single-user channel is the sum of the actual noise and power already used in the channel. If we expand n_2', we find that $n_2' = n_2 + P_1^*(\mathbf{n}) + P_2^*(\mathbf{n})$. In this expression the first two terms represent noise and interference, and the last term $P_2^*(\mathbf{n})$, represents power already allocated to user 2. It appears that user 1 should only have an effective noise term of $n_1 + P_1^*(\mathbf{n})$, but the factor of $e^{R_2^*}$ in n_1' is due to the fact that user 1 can only use a fraction (specifically $e^{-R_2^*}$) of the power allocated to it due to the interference user 1 causes on user 2.

Using the effective noises, the optimal scheme is water-filling with two water-levels scaled by weights μ_1 and μ_2:

$$\hat{P}(\mathbf{n}) = \max\left(\frac{\mu_1}{\lambda} - n_1', \frac{\mu_2}{\lambda} - n_2', 0\right). \tag{15}$$

The allocation of excess power to each fading state is identical to the optimal power allocation [1, 2] used to achieve ergodic capacity of the broadcast channel with effective noises n_1' and n_2' and power P^*. The allocation of the excess power between users, however, is not necessarily the same as under ergodic capacity maximization because under minimum rate constraints all excess power in a fading state cannot be allocated to the stronger user due to the interference it causes on the weaker user. Nonetheless, the state-by-state rates achieved by each user, and therefore the average rates, are equal to the ergodic capacity maximization rates. The minimum rate capacity is therefore equal to the ergodic capacity of the broadcast channel with effective noises n_1' and n_2' and power P^* plus the minimum rates.

We see that water-level $\frac{\mu_1}{\lambda}$ is used for channels that are filled on n_1' and $\frac{\mu_2}{\lambda}$ is used for n_2' channels. Fig. 1 illustrates a four fading state example where $\mu_2 > \mu_1$. Note that (15) does not specify the distribution of excess power between the two users. If water-filling is done up to $\frac{\mu_1}{\lambda}$, then some power is given to user 1, but power may also be allocated to user 2 according to $F_n(\hat{P}(\mathbf{n}))$ Both users are allocated some power in all fading states, due to the minimum power allocation, and therefore superposition coding is necessary in *every* fading state.

In recent work on the broadcast channel [2, 4], a greedy interpretation of the optimal power allocation scheme is derived. We now use the same approach to find an alternative derivation of the minimum rate capacity. To use the greedy approach, we must reformulate our problem slightly. We again assume $n_2 > n_1$ for simplicity and generalize our results later. Instead of allocating power to users 1 and 2, we distribute power between two policies in each fading state:

Policy 1: Increase $R_1(\mathbf{n})$ while ensuring $R_2(\mathbf{n}) \geq R_2^*$. Denote the power allocated to this policy as $P_{R_1}(\mathbf{n})$. To main-

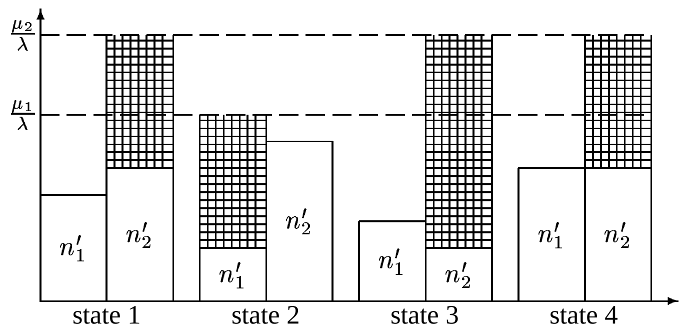

Fig. 1. Water-filling diagram for two-user channel with min rates

tain $R_2(\mathbf{n})$ above the minimum rate, user 2 receives a fraction of $P_{R_1}(\mathbf{n})$: $\hat{P}_2(\mathbf{n}) = P_{R_1}(\mathbf{n})(1 - e^{-R_2^*})$ and $\hat{P}_1(\mathbf{n}) = P_{R_1}(\mathbf{n})e^{-R_2^*}$.

Policy 2: Increase $R_2(\mathbf{n})$ while keeping $R_1(\mathbf{n})$ constant. Denote the power allocated to this policy as $P_{R_2}(\mathbf{n})$. Because user 2 does not interfere with user 1, user 2 is given all the power: $\hat{P}_2(\mathbf{n}) = P_{R_2}(\mathbf{n})$ and $\hat{P}_1(\mathbf{n}) = 0$.

We now perform the optimization in (5) over $P_{R_1}(\mathbf{n})$ and $P_{R_2}(\mathbf{n})$. By adding the constant $(-\mu_1 R_1^* - \mu_2 R_2^*)$ to the function to be maximized and introducing a Lagrangian, it can be shown that the following is an equivalent maximization:

$$\max_{P_{R_1}(\mathbf{n}), P_{R_2}(\mathbf{n})} \mathbb{E}_n[\mu_1 x_1 + \mu_2 x_2] - \lambda \mathbb{E}_n[P_{R_1}(\mathbf{n}) + P_{R_2}(\mathbf{n})] \quad (16)$$

where $x_1(P_{R_1}, P_{R_2}) = \log(1 + \frac{P_{R_1}(\mathbf{n})e^{-R_2^*}}{n_1 + P_1^*})$ and $x_2(P_{R_1}, P_{R_2}) = \log(1 + \frac{P_{R_2}(\mathbf{n})}{P_{R_1}(\mathbf{n}) + P_1^*(\mathbf{n}) + P_2^*(\mathbf{n}) + n_2})$.

Due to the convexity of the problem, we know there exists a $\lambda > 0$ such that the solution of the above problem solves the original optimization problem (4). As is done in the standard broadcast channel [2, 4], we decompose (16) into a set of independent optimization problems, one for each fading state:

$$\max_{P_{R_1}(\mathbf{n}), P_{R_2}(\mathbf{n})} \mu_1 x_1 + \mu_2 x_2 - \lambda(P_{R_1}(\mathbf{n}) + P_{R_2}(\mathbf{n})) \quad (17)$$

By Lemma 3.1 of [4], a power allocation scheme is optimal if and only if it is the solution to (17) for every fading states. We can find the solution to (17) using the greedy method of [2, 4]. For $i = 1, 2$, define utility functions u_i as:

$$u_i(z) = \frac{\mu_i}{z + (P_1^*(\mathbf{n}) + n_i)e^{R_2^*}} - \lambda, \quad i = 1, 2. \quad (18)$$

The solution to (17) then is:

$$\int_0^\infty [\max_{i=1,2} u_i(z)]^+ dz. \quad (19)$$

At each interference level z, power dp is allocated to the policy with the larger utility function $u_i(z)$ up to the point where both utility functions are negative. The utility functions are the derivatives of x_1 and x_2 minus λ. In each fading state, power is allocated to the better of the two policies at each interference level, or to the policy leading to the larger marginal increase of the objective. At the optimal point, the marginal increase of the objective function in each fading state will equal λ. The optimal power allocation scheme then is:

1) If $\mu_2 \leq \mu_1$, then $u_1(z) > u_2(z)$ for $z \geq 0$ so policy 1 receives all power. Therefore, $P_{R_1} = \frac{\mu_1}{\lambda} - (P_1^* + n_1)e^{R_2^*}$.

2) If $\frac{\mu_2}{\mu_1} \geq \frac{n_2 + P_1^*}{n_1 + P_1^*}$, then $u_2(z) > u_1(z)$ for $z \geq 0$ so policy 2 receives all power and $P_{R_2} = \frac{\mu_2}{\lambda} - (P_1^* + n_2)e^{R_2^*}$.

3) If $1 \leq \frac{\mu_2}{\mu_1} \leq \frac{n_2 + P_1^*}{n_1 + P_1^*}$, then $u_1(z)$ and $u_2(z)$ intersect at the point $z = (\frac{\mu_1 n_2 - \mu_2 n_1}{\mu_2 - \mu_1} - P_1^*)e^{R_2^*}$ and $u_1(0) \geq u_2(0)$. If $\lambda \leq (\frac{\mu_2 - \mu_1}{n_2 - n_1})e^{-R_2^*}$, then the utility functions are negative at the intersection point and only policy 1 receives power according to $P_{R_1} = \frac{\mu_1}{\lambda} - (P_1^* + n_1)e^{R_2^*}$. If $\lambda > (\frac{\mu_2 - \mu_1}{n_2 - n_1})e^{-R_2^*}$, then the functions intersect where the utility functions are positive and both policies receive power: $P_{R_1} = (\frac{\mu_1 n_2 - \mu_2 n_1}{\mu_2 - \mu_1} - P_1^*)e^{R_2^*}$ and $P_{R_2} = \frac{\mu_2}{\lambda} - (P_1^* + n_2)e^{R_2^*} - P_{R_1}$.

These three cases correspond exactly to (9)-(12) and the scheme can be extended to any ordering of noises by reversing all subscripts. The greedy approach gives valuable insight on how power is allocated within each fading state (by choosing the policy with the larger marginal increase at every interference level) and also across fading states (via water-filling level $\frac{1}{\lambda}$).

V. Numerical Results

In this section we present numerical results on two-user broadcast channels with symmetric minimum rates and symmetric fading distributions. In all plots, the power constraint is 10 mW and the bandwidth is 100 kHz.

In Fig. 2 the capacity region of a two-user channel with very different noise levels is plotted. In one fading state, n_1 is 40 dB less than n_2, and vice versa in the second fading state. Without minimum rates, capacity is achieved by allocating most of the power to the better of the two users in each channel state. When minimum rate constraints are applied, much of the power must be allocated to the weaker user in every fading state to meet the minimum rates, leading to a large capacity reduction.

As the difference in the noise levels of the two users decreases, the difference between the minimum-rate capacity region and the ergodic capacity region decreases. The capacity region of a channel where n_1 and n_2 differ by 20 dB in each fading state is plotted in Fig. 3. Minimum rate constraints force power to be allocated to both users in every state, but because the poorer channel is only 20 dB weaker than the stronger channel, as opposed to the 40 dB in the first example, allocating power to the weaker user is not quite as sub-optimal. Therefore, the difference between the ergodic and minimum rate capacity regions is not as large as in the first example.

In the final two plots, results for more realistic channel models are presented. Independent fading is assumed for both receivers. In Fig. 4, Rician fading with $K = 1$ is modeled. This is not as severe as Rayleigh fading, but the power of the

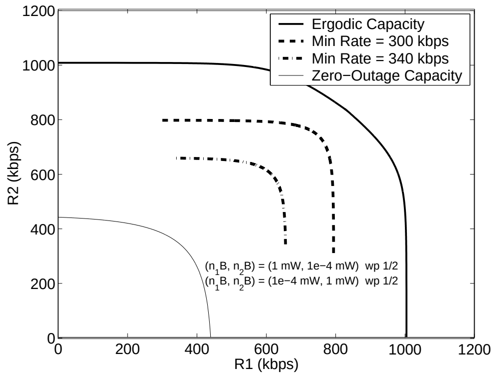

Fig. 2. Capacity of symmetric channel with 40 dB difference in SNR

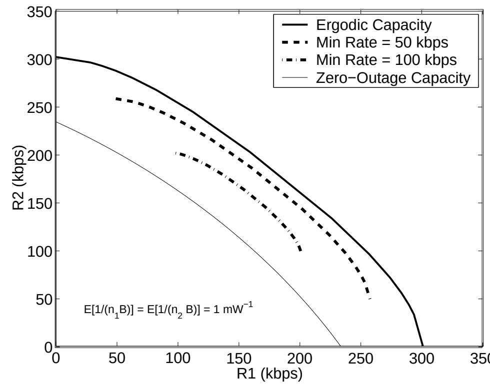

Fig. 4. Rician fading with $K = 1$, Average SNR = 10 dB

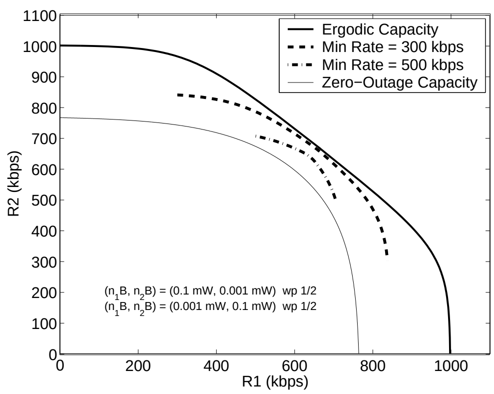

Fig. 3. Capacity of symmetric channel with 20 dB difference in SNR

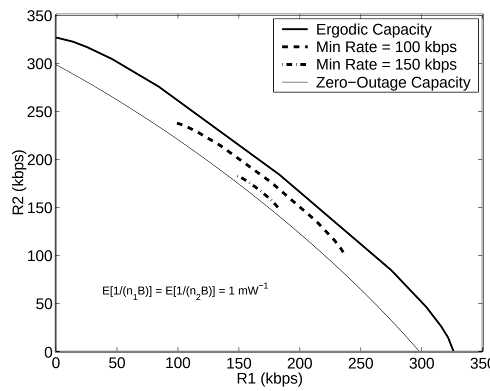

Fig. 5. Rician fading with $K = 5$, Average SNR = 10 dB

multipath component equals the power of the line of sight component. The channels of the two users often vary quite significantly, and, as expected by our earlier results, minimum rates reduce capacity significantly. In Fig. 5, Rician fading with $K = 5$ is modeled. Because the line of sight component is quite large, both users generally have strong channels. As a result, minimum rates do not reduce capacity significantly.

In each of the plots, the zero-outage capacity region is also shown. Notice that the difference between the minimum-rate and ergodic capacity regions is roughly proportional to the difference between the zero-outage and ergodic capacity regions. This relationship is due to the fact that minimum-rate capacity is a combination of zero-outage and ergodic capacity.

VI. Conclusion

We have obtained the capacity region of a multi-user fading broadcast channel with minimum rates. We found that the minimum-rate capacity region is achievable by superposition coding with successive decoding and we derived the optimal power allocation scheme. By using minimum power and effective noise terms, we saw that the minimum rate problem decomposes into two independent problems: a zero-outage capacity problem (i.e. minimizing the power needed to achieve the minimum rates), and an ergodic capacity problem (i.e. maximizing the ergodic capacity of the broadcast channel with the effective noise terms and the excess power constraint). We also derived a greedy power allocation scheme to give additional insight into the optimal power allocation scheme. Finally, by analyzing our numerical results, we determined that severely fading channels, i.e channels with wide-ranging noise levels, incur a large capacity reduction due to minimum rate constraints, while benign fading environments are able to support large minimum rates with little capacity reduction.

References

[1] D. Hughes-Hartog, “The capacity of the degraded spectral Gaussian broadcast channel”, Ph.D. Dissertation, Stanford University, 1975.

[2] L. Li and A. Goldsmith, “Capacity and optimal resource allocation for fading broadcast channels: Part I: Ergodic capacity”, IEEE Trans. Inform. Theory, pp. 1083-1102, March 2001.

[3] L. Li and A. Goldsmith, “Capacity and optimal resource allocation for fading broadcast channels: Part II: Outage capacity”, IEEE Trans. Inform. Theory, pp. 1103-1127, March 2001.

[4] D.N. Tse, “Optimal power allocation over parallel Gaussian broadcast channels”, Proceedings of ISIT, Ulm, Germany, June 1997, p. 27.

[5] A. Goldsmith and M. Effros, “The capacity region of broadcast channels with intersymbol interference and colored Gaussian noise”, IEEE Trans. Inform. Theory, vol. 47, pp. 219-240, Jan. 2001.

[6] N. Jindal and A. Goldsmith, “Capacity and optimal power allocation for fading broadcast channels with minimum rates”, in preparation for journal submission.