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CAPTURE INTO RESONANCE IN NONLINEAR OSCILLATORY SYSTEMS WITH DECAYING PERTURBATIONS

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We study the influence of oscillatory perturbations on nonlinear nonisochronous oscillatory systems in the plane. We assume that the perturbation amplitude decays and the frequency is unboundedly increasing in time. We study capture into resonance in the case where the amplitude of the system unboundedly increases and the frequency adjusts to the perturbation frequency. We discuss the existence, stability, and asymptotic behavior of resonance solutions at long times. We propose the technique based on averaging method and construction of the Lyapunov functions. The results obtained are applied to the Duffing oscillator with decaying parametric perturbations. Bibliography: 14 *titles. Illustrations*: 3 *figures.*

In this paper, we study resonance phenomena in Hamiltonian systems in the plane under decaying oscillatory perturbations. The problem under consideration is connected with bifurcations in asymptotically autonomous systems discussed, for example, in [1]–[4]. In particular, asymptotically autonomous Hamiltonian systems were considered in [5], where conditions were described under which perturbations do not violate the global behavior of the solution to the limit equations. Bifurcations in almost Hamiltonian systems with decaying oscillatory perturbations were studied in [6, 7], where possible asymptotic regimes were described for solutions in a neighborhood of the equilibrium state in the resonance and nonresonance cases. Moreover, the frequency of oscillatory perturbations was assumed to be asymptotically constant. The asymptotics of solutions at long times for similar linear equations was obtained in [8, 9]. Decaying perturbations with growing frequency and a small parameter were considered in [10], where the solutions were considered in a neighborhood of the equilibrium state on an asymptotically large, but finite time-interval. At the same time, the behavior of trajectories far from the equilibrium state of the limit system has not been studied earlier. In this paper, we discuss this topic and study decaying oscillatory perturbations with growing frequency. The presence of a small parameter

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is not assumed. We study the stability and asymptotic behavior of resonance solutions on a semi-infinite time-interval.

The paper is organized as follows. In Section 1, we state the problem. The main results are formulated in Section 2 and justified in Sections 2–5. In particular, we obtain a necessary condition for the existence of resonance solutions in Section 3. In Section 4, we prove the stability and asymptotics of resonance solutions at long times in a model case. In Section 5, we apply the obtained results to the Duffing oscillator with nonlinear parametric perturbation.

1 Statement of the Problem

We consider the nonautonomous system of ordinary differential equations

$$\begin{aligned} \frac{d\rho}{dt} &= t^{-a} f(\rho, \varphi, S(t)), \\ \frac{d\varphi}{dt} &= \omega(\rho) + t^{-a} g(\rho, \varphi, S(t)), \\ S(t) &= s t^{b+1}, \quad t \geqslant t_0 > 0, \end{aligned} \tag{1.1}$$

where $\omega(\rho)$, $f(\rho, \varphi, S)$, $g(\rho, \varphi, S)$ are smooth functions defined for all $(\rho, \varphi, S) \in \mathbb{R}^3$ and $a, b, s \in \mathbb{R}_+$ are constant parameters, $b \geqslant 1$. We assume that the functions $f(\rho, \varphi, S)$ and $g(\rho, \varphi, S)$ are 2π-periodic in φ and S, and the following asymptotic expansions hold:

$$\begin{aligned} f(\rho, \varphi, S) &= \rho^{\beta+1} \sum_{j=0}^{\infty} \rho^{-j} f_j(\varphi, S), \\ g(\rho, \varphi, S) &= \rho^{\beta} \sum_{j=0}^{\infty} \rho^{-j} g_j(\varphi, S), \\ \omega(\rho) &= \rho^{h} \sum_{j=0}^{\infty} \rho^{-j} \omega_j \end{aligned} \tag{1.2}$$

as $\rho \to \infty$ with periodic coefficients $f_j(\varphi, S)$, $g_j(\varphi, S)$, constant coefficients ω_j ($\omega_0 > 0$), and integer parameters $\beta, h \in \mathbb{Z}$, $h \geqslant 1$.

The system (1.1) is nonlinear and oscillating far from the equilibrium state under perturbations decaying in time. The unknown functions $\rho(t)$ and $\varphi(t)$ are interpreted as the amplitude and phase oscillations. We study capture into resonance in the case where $\rho(t)$ unboundedly increases and $\varphi(t)$ adjusts to the perturbation phase $S(t)$.

For an example we consider the equation

$$\frac{d^2 x}{dt^2} + U'(x) = t^{-a} Q(x) \cos S(t), \tag{1.3}$$

where

$$U(x) = \frac{x^{2h+2}}{2h+2}(1 + \mathscr{O}(x^{-1})), \quad Q(x) = x^p (q + \mathscr{O}(x^{-1})), \quad x \to \infty,$$

with parameters $h, p \in \mathbb{Z}$, $q \in \mathbb{R}$. The corresponding unperturbed equation

$$\frac{d^2 x}{dt^2} + U'(x) = 0$$

written in the variables x, $y \equiv dx/dt$ takes the form of an autonomous Hamiltonian system with $H(x, y) = y^2/2 + U(x)$. It is easy to verify that there exists $\rho_0 > 0$ such that the level lines $\{(x, y) \in \mathbb{R}^2 : H(x, y) = \rho^{2h+2}\}$ for every $\rho \geqslant \rho_0$ determine closed curves in the phase plane (x, y) and correspond to periodic solutions $x_0(t, \rho)$, $y_0(t, \rho)$ with period

$$T(\rho) = \rho^{-h}(T_0 + \mathcal{O}(\rho^{-1})), \quad \rho \to \infty,$$

where

$$T_0 = (2h+2)^{\frac{1}{2h+2}} \int\limits_{-1}^{1} \frac{\sqrt{2}d\zeta}{\sqrt{1-\zeta^{2h+2}}}.$$

We note that the functions

$$X(\varphi, \rho) = x_0\Big(\frac{\varphi}{\omega(\rho)}, \rho\Big), \quad Y(\varphi, \rho) = y_0\Big(\frac{\varphi}{\omega(\rho)}, \rho\Big) \tag{1.4}$$

with $\omega(\rho) \equiv 2\pi/T(\rho)$ are 2π-periodic in φ and can be used to write Equation (1.3) in the variables (ρ, φ). From the identity

$$H(X(\varphi, \rho), Y(\varphi, \rho)) \equiv \rho^{2h+2}$$

it follows that

$$\begin{vmatrix} \partial_\varphi X & \partial_\rho X \\ \partial_\varphi Y & \partial_\rho Y \end{vmatrix} = \frac{(2h+2)\rho^{2h+1}}{\omega(\rho)} \neq 0$$

for $\rho \geqslant \rho_0$ and $\varphi \in \mathbb{R}$. Hence the transformation (1.4) is invertible. Equation (1.3) written in the variables (ρ, φ) has the form (1.1) with

$$\begin{aligned} f(\rho, \varphi, S) &\equiv \frac{Y(\varphi, \rho) Q(X(\varphi, \rho)) \cos S}{(2h+2)\rho^{2h+1}}, \\ g(\rho, \varphi, S) &\equiv -\frac{\omega(\rho)\partial_\rho X(\varphi, \rho) Q(X(\varphi, \rho)) \cos S}{(2h+2)\rho^{2h+1}}. \end{aligned} \tag{1.5}$$

We note that the functions $X(\varphi, \rho)$, $Y(\varphi, \rho)$ satisfy the system of differential equations

$$\omega(\rho)\partial_\varphi X = Y,$$

$$\omega(\rho)\partial_\varphi Y = -U'(X)$$

and have the following asymptotics (cf., for example, [11]):

$$X(\rho, \varphi) = \rho \sum_{j=0}^{\infty} \rho^{-j} X_j(\varphi),$$

$$Y(\rho, \varphi) = \rho^{h+1} \sum_{j=0}^{\infty} \rho^{-j} Y_j(\varphi)$$

as $\rho \to \infty$, where the coefficients $X_j(\varphi)$ and $Y_j(\varphi)$ are 2π-periodic; moreover, $X_0(\varphi)$ and $Y_0(\varphi)$ satisfy the system

$$\omega_0 \partial_\varphi X_0 = Y_0,$$

$$\omega_0 \partial_\varphi Y_0 = -X_0^{2h+1},$$

$$Y_0^2/2 + X_0^{2h+2}/(2h+2) = 1.$$

Hence (1.5) implies the asymptotic expansion (1.2) with $\beta = p - h - 1$ and

$$f_0(\varphi, S) \equiv \frac{qY_0(\varphi)(X_0(\varphi))^p \cos S}{2h+2},$$
$$g_0(\varphi, S) \equiv -\frac{\pi q X_0(\varphi)(X_0(\varphi))^p \cos S}{(h+1)T_0},$$
$$\omega_0 = \frac{2\pi}{T_0}.$$

A numerical analysis of Equation (1.3) in the case $U(x) \equiv x^4/4 - x^2/2$ and $Q(x) \equiv qx^p$ shows that the existence of resonance solutions with growing amplitude $\rho(t) \equiv (H(x(t), y(t)))^{1/4}$ depends on the initial data and perturbation parameters (cf. Figure 1). The goal of this paper is to describe the conditions for the existence and stability of resonance solutions far from the equilibrium state in systems with decaying oscillatory perturbation.

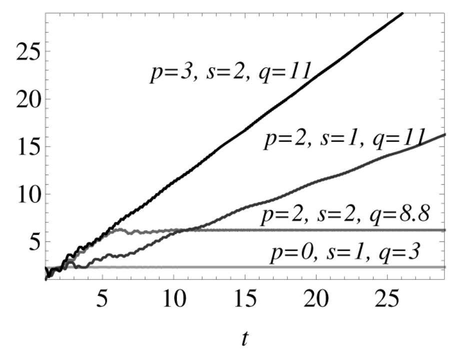

Figure 1. Evolution $\rho(t)$ of solutions to Equation (1.3) in the case $U(x) \equiv x^4/4 - x^2/2$ and $Q(x) \equiv qx^p$ for $a = b = 1$ with different values of parameters p, s, q and the initial data.

2 The Main Results

We note that there exists $\rho_0 > 0$ such that $\omega(\rho) > 0$ and $\omega'(\rho) > 0$ for $\rho \geqslant \rho_0$. Consequently, for any $\varkappa \in \mathbb{Z}_+$ the equation

$$\omega(\rho_\varkappa) = \varkappa^{-1} S'(t)$$

has a solution $\rho_\varkappa(t) > 0$, $t \geqslant t_0$, such that

$$\rho_\varkappa(t) = t^{\frac{b}{h}} \Big(c_\varkappa - \frac{\omega_1}{\omega_0} t^{-\frac{b}{h}} + \mathscr{O}(t^{-\frac{2b}{h}}) \Big), \quad t \to \infty,$$

where

$$c_\varkappa = \Big(\frac{s(b+1)}{\varkappa \omega_0} \Big)^{\frac{1}{h}}.$$

By a *resonance* solution to the system (1.1) we mean a solution $\rho(t)$, $\varphi(t)$ admitting the asymptotics $\rho(t) \sim \rho_\varkappa(t)$ and $\varphi(t) \sim \varkappa^{-1} S(t)$ as $t \to \infty$.

Theorem 2.1. *If the system* (1.1) *has a resonance solution, then*

$$\frac{a-1}{b} \leqslant \frac{\beta}{h} < 1 + \frac{a}{b}. \tag{2.1}$$

We describe conditions for the existence and stability of resonance solutions in the model case where the right-hand sides of the system (1.1) are the leading terms of their asymptotics

$$\begin{aligned} f(r,\varphi,S) &\equiv \rho^{\beta+1} f_0(\varphi,S),\\ g(r,\varphi,S) &\equiv \rho^{\beta} g_0(\varphi,S), \\ \omega(r) &\equiv \rho^h \omega_0 \end{aligned} \tag{2.2}$$

as $\rho\to\infty$. In this case, $\rho_\varkappa(t)\equiv c_\varkappa t^{b/h}$. We set

$$\langle \mathscr{Z}(\xi)\rangle_{\varkappa\xi} := \frac{1}{2\pi\varkappa}\int\limits_0^{2\pi\varkappa} \mathscr{Z}(s)\,ds, \quad \{\mathscr{Z}(\xi)\}_{\varkappa\xi} \equiv \mathscr{Z}(\xi) - \langle \mathscr{Z}(\xi)\rangle_{\varkappa\xi},$$

$$\Gamma_0 := \begin{cases} \dfrac{b}{h}\Big(\dfrac{\varkappa\omega_0}{s(b+1)}\Big)^{\beta/h}, & \beta = \dfrac{h(a-1)}{b} \\ 0, & \beta \neq \dfrac{h(a-1)}{b} \end{cases},$$

$$\Gamma_1 := \begin{cases} \Big(a+\dfrac{b-1}{2}-\dfrac{1}{h}\Big)\Big(\dfrac{\varkappa\omega_0}{s(b+1)}\Big)^{\beta/h}, & \beta = \dfrac{h(a-1)}{b} \\ 0, & \beta \neq \dfrac{h(a-1)}{b} \end{cases}.$$

Theorem 2.2. *Let the conditions* (2.1) *and* (2.2) *hold, and let for any* $\varkappa\in\mathbb{Z}_+$

$$\exists\,\vartheta_0\in\mathbb{R}: \quad \langle f_0(\vartheta_0+\varkappa^{-1}\xi,\xi)\rangle_{\varkappa\xi} = \Gamma_0, \quad \langle \partial_\varphi f_0(\vartheta_0+\varkappa^{-1}\xi,\xi)\rangle_{\varkappa\xi} < 0, \tag{2.3}$$

$$\gamma_\varkappa := \langle \partial_\varphi g_0(\vartheta_0+\varkappa^{-1}\xi,\xi)\rangle_{\varkappa\xi} + \Gamma_1 < 0. \tag{2.4}$$

Then for any $\epsilon>0$ *there exist* $d_\epsilon>0$ *and* $t_\epsilon>0$ *such that the solution* $\rho(t)$, $\varphi(t)$ *to the system* (1.1) *with the initial data such that*

$$|\rho(t_\epsilon)-\rho_\varkappa(t_\epsilon)| + |\varphi(t_\epsilon)-\varkappa^{-1}S(t_\epsilon)-\vartheta_0| \leqslant d_\epsilon$$

satisfies the inequality

$$\sup_{t\geqslant t_\epsilon}(|\rho(t)-\rho_\varkappa(t)| + |\varphi(t)-\varkappa^{-1}S(t)-\vartheta_0|) \leqslant \epsilon.$$

Furthermore, $\rho(t)=\rho_\varkappa(t)(1+o(1))$ *and* $\varphi(t)=\varkappa^{-1}S(t)+\vartheta_0+o(1)$ *as* $t\to\infty$.

The analysis of stability and asymptotics of resonance solutions in a general (not model) case requires special attention and will not be considered in this paper.

3 Proof of Theorem 2.1

By (1.2), there exists $M>0$ such that $|f(\rho,\varphi,S)|\leqslant M\rho^{\beta+1}$ and $|g(\rho,\varphi,S)|\leqslant M\rho^\beta$ for $\rho\geqslant\rho_0$ and all $(\varphi,S)\in\mathbb{R}^2$. Hence the first equation of the system (1.1) implies

$$\begin{cases} \Big|\dfrac{d}{dt}\frac{1}{\rho^\beta(t)}\Big| \leqslant |\beta| M t^{-a}, & \beta\neq 0, \\ \Big|\dfrac{d}{dt}\log\,\rho(t)\Big| \leqslant M t^{-a}, & \beta = 0, \end{cases} \tag{3.1}$$

for $t \geqslant t_0$. It is easy to verify that for the resonance solutions

$$\begin{cases} \Big| \dfrac{d}{dt} \dfrac{1}{\rho_{\varkappa}^{\beta}(t)} \Big| \sim c_{\varkappa}^{-\beta} \dfrac{|\beta| b}{h} t^{-\dfrac{\beta b}{h} - 1}, & \beta \neq 0, \\ \Big| \dfrac{d}{dt} \log \rho_{\varkappa}(t) \Big| \sim \dfrac{b}{h} t^{-1}, & \beta = 0, \end{cases}$$

as $t \to \infty$. Hence (3.1) implies $\beta/h \geqslant (a-1)/b$.

From the second equation of the system (1.1) it follows that

$$\Big| \frac{\varphi(t)}{S(t)} - \varkappa^{-1} \Big| \sim t^{\frac{\beta b}{h} - (b+a)} G(t),$$

where $G(t) = \mathscr{O}(1)$ as $t \to \infty$ for $\rho(t) \sim \rho_{\varkappa}(t)$. Since for the resonance solutions the left-hand side of the last expression converges to zero as $t \to \infty$, we have $\beta/h < 1 + a/b$.

4 Proof of Theorem 2.2

The proof consists of two steps. At the first step, we look for an invertible change of variables, which allows us to simplify the system in leading terms of the asymptotics. After that we construct the Lyapunov functions for the transformed system, which allows us to justify the stability of resonance solutions.

4.1. Change of variables. Substituting

$$\rho(t) = \rho_{\varkappa}(t)(1 + t^{-A} r(\tau)), \quad \varphi(t) = \varkappa^{-1} S(t) + \theta(\tau), \quad \tau = \frac{t^B}{B}, \tag{4.1}$$

where

$$A = \frac{b}{2}\Big(1 + \frac{a}{b} - \frac{\beta}{h}\Big) > 0, \quad B = b + 1 > 0,$$

into (1.1), we obtain the system of equations for $r(\tau)$, $\theta(\tau)$

$$\begin{aligned} \frac{dr}{d\tau} &= F(r, \theta, \xi(\tau), \tau), \\ \frac{d\theta}{d\tau} &= G(r, \theta, \xi(\tau), \tau), \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} F(r, \theta, \xi, \tau) &\equiv c_{\varkappa}^{\beta} (B\tau)^{-\frac{A}{B}} \Big(1 + r(B\tau)^{-\frac{A}{B}}\Big)^{\beta+1} f_0(\theta + \varkappa^{-1}\xi, \xi) \\ &\quad - \frac{b}{h}(B\tau)^{-\frac{B-A}{B}} + (B\tau)^{-1}\Big(A - \frac{b}{h}\Big) r, \\ G(r, \theta, \xi, \tau) &\equiv \omega_0 c_{\varkappa}^{h} \Big(\Big(1 + r(B\tau)^{-\frac{A}{B}}\Big)^{h} - 1\Big) \\ &\quad + c_{\varkappa}^{\beta} (B\tau)^{-\frac{2A}{B}} \Big(1 + r(B\tau)^{-\frac{A}{B}}\Big)^{\beta} g_0(\theta + \varkappa^{-1}\xi, \xi), \\ \xi(\tau) &\equiv S\Big((B\tau)^{\frac{1}{B}}\Big) = \nu\tau, \end{aligned}$$

with the number parameter $\nu = sB > 0$. By the condition (2.1), $B \geqslant 2A$. Consequently, the right-hand sides of (4.2) admit the asymptotics

$$
\begin{aligned}
F(r,\theta,\xi,\tau) &\equiv \tau^{-\frac{A}{B}}\Big(F_A(\theta,\xi) + \tau^{-\frac{B-2A}{B}} F_{B-A}\Big) \\
&\qquad + \tau^{-\frac{2A}{B}}\Big(F_{2A}(r,\theta,\xi) + \tau^{-\frac{B-2A}{B}} F_B(r)\Big) + \mathscr{O}(\tau^{-\frac{3A}{B}}), \\
G(r,\theta,\xi,\tau) &\equiv \tau^{-\frac{A}{B}} G_A(r) + \tau^{-\frac{2A}{B}} G_{2A}(r,\theta,\xi) + \mathscr{O}(\tau^{-\frac{3A}{B}})
\end{aligned}
$$

as $\tau \to \infty$ with coefficients

$$
\begin{aligned}
&F_A(\theta,\xi) \equiv c_\varkappa^\beta B^{-\frac{A}{B}} f_0(\theta + \varkappa^{-1}\xi, \xi), \\
&F_{B-A} \equiv -\frac{b}{h} B^{-1+\frac{A}{B}}, \\
&F_{2A}(r,\theta,\xi) \equiv \lambda_{2A} F_A(\theta,\xi) r, \\
&F_B(r) \equiv \Big(\frac{A}{B} - \frac{1}{h}\Big) r, \\
&G_A(r) \equiv \mu_A r, \\
&G_{2A}(r,\theta,\xi) \equiv B^{-\frac{2A}{B}} c_\varkappa^\beta g_0(\theta + \varkappa^{-1}\xi, \xi) + (h-1) B^{-\frac{A}{B}} G_A(r)\frac{r}{2},
\end{aligned}
$$

where $\lambda_{2A} = (\beta+1)B^{-A/B} > 0$ and $\mu_A = h\omega_0 c_\varkappa^h B^{-A/B} > 0$. Thus, the system (4.2) is asymptotically autonomous; moreover, the corresponding limit system

$$
\frac{dr}{d\tau} = 0, \qquad \frac{d\theta}{d\tau} = 0
$$

as $\tau \to \infty$ is trivial. The right-hand sides $F(r,\theta,\xi,\tau)$ and $G(r,\theta,\xi,\tau)$ are 2π-periodic in θ and $2\pi\varkappa$-periodic in ξ. Since $d\xi/d\tau = \nu > 0$, we see that $\xi(\tau)$ varies faster than $r(\tau)$ and $\theta(\tau)$ as $\tau \to \infty$ and can play the role of the fast variable at long times. The further simplification of the system can be done by using the averaging procedure [12, 13] for the leading terms of the equations with respect to $\xi(\tau)$.

We consider the transformation of the system (4.2) which is close to the identity one:

$$
\begin{aligned}
R(r,\theta,\tau) &= r + \tau^{-\frac{A}{B}}\Big(R_A(r,\theta,\xi(\tau)) + \tau^{-\frac{B-2A}{B}} R_{B-2A}(r,\theta,\xi(\tau))\Big) \\
&\qquad + \tau^{-\frac{2A}{B}}\Big(R_{2A}(r,\theta,\xi(\tau)) + \tau^{-\frac{B-2A}{B}} R_B(r,\theta,\xi(\tau))\Big), \\
\Theta(r,\theta,\tau) &= \theta + \tau^{-\frac{A}{B}}\Theta_A(r,\theta,\xi(\tau)) + \tau^{-\frac{2A}{B}}\Big(\Theta_{2A}(r,\theta,\xi(\tau)) + \tau^{-\frac{B-2A}{B}}\Theta_B(r,\theta,\xi(\tau))\Big),
\end{aligned}
\tag{4.3}
$$

where the functions $R_k(r,\theta,\xi)$ and $\Theta_k(r,\theta,\xi)$ are periodic, have zero means in ξ, and are chosen in such a way that the system (4.2) written in the new variables

$$
\varrho(\tau) = R(r(\tau),\theta(\tau),\tau), \qquad \vartheta(\tau) = \Theta(r(\tau),\theta(\tau),\tau)
$$

is independent of $\xi(\tau)$ in the leading terms of the asymptotics

$$\begin{aligned}\frac{d\varrho}{d\tau} &= \tau^{-\frac{A}{B}}\Big(\Lambda_A(\varrho,\vartheta)+\tau^{-\frac{B-2A}{B}}\Lambda_{B-A}(\varrho,\vartheta)\Big)\\ &\quad+\tau^{-\frac{2A}{B}}\Big(\Lambda_{2A}(\varrho,\vartheta)+\tau^{-\frac{B-2A}{B}}\Lambda_B(\varrho,\vartheta)\Big)+\mathscr{O}(\tau^{-\frac{3A}{B}}),\\ \frac{d\vartheta}{d\tau} &= \tau^{-\frac{A}{B}}\Omega_A(\varrho,\vartheta)+\tau^{-\frac{2A}{B}}\Big(\Omega_{2A}(\varrho,\vartheta)+\tau^{-\frac{B-2A}{B}}\Omega_B(\varrho,\vartheta)\Big)+\mathscr{O}(\tau^{-\frac{3A}{B}})\end{aligned} \tag{4.4}$$

as $\tau\to\infty$, where $|\varrho|<\text{const}$ uniformly for all $\vartheta\in\mathbb{R}$. Substituting (4.3) into the system (4.2) and comparing the result with (4.4), we obtain the following chain of differential equations for the coefficients $R_k(r,\theta,\xi)$ and $\Theta_k(r,\theta,\xi)$:

$$\begin{aligned}\nu\partial_\xi R_A &= \Lambda_A(r,\theta)-F_A(\theta,\xi),\\ \nu\partial_\xi\Theta_A &= \Omega_A(r,\theta)-G_A(r),\\ \nu\partial_\xi R_{B-A} &= \Lambda_{A-B}(r,\theta)-F_{B-A},\\ \nu\partial_\xi R_{2A} &= \Lambda_{2A}(r,\theta)-F_{2A}(r,\theta,\xi)-\mathscr{F}_{2A}(r,\theta,\xi),\\ \nu\partial_\xi\Theta_{2A} &= \Omega_{2A}(r,\theta)-G_{2A}(r,\theta,\xi)-\mathscr{G}_{2A}(r,\theta,\xi),\\ \nu\partial_\xi R_B &= \Lambda_B(r,\theta)-F_B(r)-\mathscr{F}_B(r,\theta,\xi),\\ \nu\partial_\xi\Theta_B &= \Omega_B(r,\theta)-\mathscr{G}_B(r,\theta,\xi),\end{aligned}$$

where

$$\begin{aligned}\mathscr{F}_{2A}(r,\theta,\xi) &\equiv (F_A(\theta,\xi)\partial_r+G_A(r)\partial_\theta)R_A(r,\theta,\xi)-(R_A(r,\theta,\xi)\partial_r+\Theta_A(r,\theta,\xi)\partial_\theta)\Lambda_A(r,\theta),\\ \mathscr{G}_{2A}(r,\theta,\xi) &\equiv (F_A(\theta,\xi)\partial_r+G_A(r)\partial_\theta)\Theta_A(r,\theta,\xi)-(R_A(r,\theta,\xi)\partial_r+\Theta_A(r,\theta,\xi)\partial_\theta)\Omega_A(r,\theta),\\ \mathscr{F}_B(r,\theta,\xi) &\equiv (F_A(\theta,\xi)\partial_r+G_A(r)\partial_\theta)R_{B-A}(r,\theta,\xi)-(R_A(r,\theta,\xi)\partial_r\\ &\quad+\Theta_A(r,\theta,\xi)\partial_\theta)\Lambda_{B-A}(r,\theta)+F_{B-A}\partial_r R_A(r,\theta,\xi)-R_{B-A}(r,\theta,\xi)\partial_r\Lambda_A(r,\theta),\\ \mathscr{G}_B(r,\theta,\xi) &\equiv F_{B-A}\partial_r\Theta_A(r,\theta,\xi)-R_{B-A}(r,\theta,\xi)\partial_r\Omega_A(r,\theta),\end{aligned}$$

The periodicity condition on the coefficients $R_k(r,\theta,\xi)$ and $\Theta_k(r,\theta,\xi)$ in ξ leads to the following definition of functions $\Lambda_k(r,\theta)$ and $\Omega_k(r,\theta)$:

$$\begin{aligned}\Lambda_A(r,\theta) &\equiv \langle F_A(\theta,\xi)\rangle_{\varkappa\xi},\\ \Omega_A(r,\theta) &\equiv G_A(r),\\ \Lambda_{B-A}(r,\theta) &\equiv F_{B-A},\\ \Lambda_{2A}(r,\theta) &\equiv \langle F_{2A}(r,\theta,\xi)+\mathscr{F}_{2A}(r,\theta,\xi)\rangle_{\varkappa\xi},\\ \Omega_{2A}(r,\theta) &\equiv \langle G_{2A}(r,\theta,\xi)+\mathscr{G}_{2A}(r,\theta,\xi)\rangle_{\varkappa\xi},\\ \Lambda_B(r,\theta) &\equiv F_B(r)+\langle\mathscr{F}_B(r,\theta,\xi)\rangle_{\varkappa\xi},\\ \Omega_B(r,\theta) &\equiv \langle\mathscr{G}_B(r,\theta,\xi)\rangle_{\varkappa\xi}.\end{aligned}$$

In this case, we have

$$
\begin{aligned}
&R_A(r,\theta,\xi) \equiv -\frac{1}{\nu}\left\{ \int\limits_0^{\xi} \{F_A(\theta,\sigma)\}_\sigma \, d\sigma \right\}_{\varkappa\xi}, \\
&\Theta_A(r,\theta,\xi) \equiv 0, \quad R_{B-A}(r,\theta,\xi) \equiv 0, \\
&\mathscr{F}_{2A}(r,\theta,\xi) \equiv G_A(r)\partial_\theta R_A(r,\theta,\xi), \quad \mathscr{G}_{2A}(r,\theta,\xi) \equiv -G'_A(r)R_A(r,\theta,\xi), \\
&\mathscr{F}_B(r,\theta,\xi) \equiv 0, \quad \mathscr{G}_B(r,\theta,\xi) \equiv 0.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\Lambda_{2A}(r,\theta) \equiv \langle F_{2A}(r,\theta,\xi)\rangle_{\varkappa\xi}, \quad \Omega_{2A}(r,\theta) \equiv \langle G_{2A}(r,\theta,\xi)\rangle_{\varkappa\xi}, \\
&\Lambda_B(r,\theta) \equiv F_B(r), \quad \Omega_B(r,\theta) \equiv 0.
\end{aligned}
$$

Using the change of variables (4.3), we transform the system (4.2) to the form (4.4). Furthermore, by the structure of (4.3), for any $r_0 > 0$ and $\varepsilon \in (0, r_0)$ there is $\tau_0 > 0$ such that

$$
\begin{array}{lll}
|R(r,\theta,\tau) - r| \leqslant \varepsilon, & |\partial_r R(r,\theta,\tau) - 1| \leqslant \varepsilon, & |\partial_\theta R(r,\theta,\tau)| \leqslant \varepsilon, \\
|\Theta(r,\theta,\tau) - \theta| \leqslant \varepsilon, & |\partial_r \Theta(r,\theta,\tau)| \leqslant \varepsilon, & |\partial_\theta \Theta(r,\theta,\tau) - 1| \leqslant \varepsilon
\end{array}
$$

for all $|r| \leqslant r_0$, $\theta \in \mathbb{R}$, and $\tau \geqslant \tau_0$. Consequently, the transformation $(r,\theta,\tau) \mapsto (\varrho,\vartheta,\tau)$ is invertible for all $|\varrho| \leqslant \varrho_0$, $\vartheta \in \mathbb{R}$, and $\tau \geqslant \tau_0$ with $\varrho_0 = r_0 - \varepsilon > 0$.

4.2. The Lyapunov functions. We begin with the case $B = 2A$. In this case, the system (4.4) has the form

$$
\begin{aligned}
&\frac{d\varrho}{dt} = \tau^{-\frac{1}{2}}\widetilde{\Lambda}_A(\vartheta) + \tau^{-1}\widetilde{\Lambda}_{2A}(\varrho,\vartheta) + \mathscr{O}(\tau^{-\frac{3}{2}}), \\
&\frac{d\vartheta}{dt} = \tau^{-\frac{1}{2}}\mu_A\varrho + \tau^{-1}\Omega_{2A}(\varrho,\vartheta) + \mathscr{O}(\tau^{-\frac{3}{2}})
\end{aligned}
\tag{4.5}
$$

as $\tau \to \infty$ uniformly for all $|\varrho| \leqslant \varrho_0$ and $\vartheta \in \mathbb{R}$, where

$$
\begin{aligned}
&\widetilde{\Lambda}_A(\vartheta) \equiv \langle F_A(\vartheta,\xi)\rangle_{\varkappa\xi} + F_{B-A}, \\
&\widetilde{\Lambda}_{2A}(\varrho,\vartheta) \equiv \lambda_{2A}\varrho\langle F_A(\vartheta,\xi)\rangle_{\varkappa\xi} + F_B(\varrho).
\end{aligned}
\tag{4.6}
$$

Lemma 4.1. *We assume that there exists $\vartheta_0 \in \mathbb{R}$ such that $\widetilde{\Lambda}_A(\vartheta_0) = 0$, $\widetilde{\Lambda}'_A(\vartheta_0) < 0$, and $\widetilde{\gamma}_0 := \partial_\varrho\widetilde{\Lambda}_{2A}(0,\vartheta_0) + \partial_\vartheta\Omega_{2A}(0,\vartheta_0) < 0$. Then for any $\epsilon \in (0,\varrho_0)$ there exist $\delta_\epsilon > 0$ and $\tau_\epsilon \geqslant \tau_0$ such that the solution $\varrho(\tau)$, $\vartheta(\tau)$ to the system (4.5) with initial data $|\varrho(\tau_\epsilon)| + |\vartheta(\tau_\epsilon) - \vartheta_0| \leqslant \delta_\epsilon$ satisfies the inequality*

$$
\sup_{\tau \geqslant \tau_\epsilon} \left(|\varrho(\tau)| + |\vartheta(\tau) - \vartheta_0|\right) \leqslant \epsilon.
$$

Furthermore, $|\varrho(\tau)| + |\vartheta(\tau) - \vartheta_0| \to 0$ as $\tau \to \infty$.

Proof. It is easy to verify that the substitution

$$
\varrho(\tau) = \tau^{-\frac{1}{2}}\varrho_1 + \widehat{\varrho}(\tau), \quad \vartheta(\tau) = \vartheta_0 + \tau^{-\frac{1}{2}}\vartheta_1 + \widehat{\vartheta}(\tau)
$$

with $\varrho_1 = -\Omega_{2A}(0,\vartheta_0)/\mu_A$ and $\vartheta_1 = -\widetilde\Lambda_{2A}(0,\vartheta_0)/\widetilde\Lambda'_A(\vartheta_0)$ reduces the system (4.5) to the form

$$
\begin{aligned}
\frac{d\widehat\varrho}{d\tau} &= -\tau^{-\frac12}\partial_{\widehat\vartheta}\widetilde{\mathscr H}(\widehat\varrho,\widehat\vartheta,\tau) + \mathscr O(\tau^{-\frac32}),\\
\frac{d\widehat\vartheta}{d\tau} &= \tau^{-\frac12}\partial_{\widehat\varrho}\widetilde{\mathscr H}(\widehat\varrho,\widehat\vartheta,\tau) + \tau^{-1}\widetilde{\mathscr Z}(\widehat\varrho,\widehat\vartheta) + \mathscr O(\tau^{-\frac32})
\end{aligned}
\tag{4.7}
$$

as $\tau\to\infty$ and $|\widehat\varrho| + |\widehat\vartheta| < \infty$, where

$$
\begin{aligned}
\widetilde{\mathscr H}(\widehat\varrho,\widehat\vartheta,\tau) \equiv{}& \mu_A\frac{\widehat\varrho^2}{2} - \int\limits_0^{\widehat\vartheta}\Big(\widetilde\Lambda_A(\vartheta_0+\tau^{-\frac12}\vartheta_1+\theta) - \widetilde\Lambda_A(\vartheta_0) - \tau^{-\frac12}\vartheta_1\widetilde\Lambda'_A(\vartheta_0)\Big)\,d\theta\\
&- \tau^{-\frac12}\int\limits_0^{\widehat\vartheta}(\widetilde\Lambda_{2A}(\widehat\varrho,\vartheta_0+\theta) - \widetilde\Lambda_{2A}(0,\vartheta_0))\,d\theta + \tau^{-\frac12}\int\limits_0^{\widehat\varrho}(\Omega_{2A}(r,\vartheta_0) - \Omega_{2A}(0,\vartheta_0)\,)\,dr,
\end{aligned}
$$

$$
\widetilde{\mathscr Z}(\widehat\varrho,\widehat\vartheta) \equiv \Omega_{2A}(\widehat\varrho,\vartheta_0+\widehat\vartheta) - \Omega_{2A}(\widehat\varrho,\vartheta_0) + \int\limits_0^{\widehat\vartheta}\partial_\varrho\widetilde\Lambda_{2A}(\widehat\varrho,\vartheta_0+\theta)\,d\theta.
$$

We note that

$$
\begin{aligned}
\widetilde{\mathscr H}(\widehat\varrho,\widehat\vartheta,\tau) &= \frac{\Delta^2}{2}(1+\mathscr O(\tau^{-\frac12}) + \mathscr O(\Delta)),\\
\widetilde{\mathscr Z}(\widehat\varrho,\widehat\vartheta) &= \widetilde\gamma_0\widehat\vartheta + \mathscr O(\Delta)
\end{aligned}
\tag{4.8}
$$

as $\tau\to\infty$ and $\Delta \equiv \sqrt{\mu_A\widehat\varrho^2 + \lambda_A\widehat\vartheta^2}\to 0$, where $\lambda_A := |\widetilde\Lambda'_A(\vartheta_0)| > 0$. Hereinafter, we assume that the asymptotic estimates are uniform for all $(\widehat\varrho,\widehat\vartheta,\tau)\in\mathbb R^3$ such that $\Delta\leqslant\Delta_*$ and $\tau\geqslant\tau_*$ with some constants $\Delta_*>0$ and $\tau_*>0$.

As a candidate for the Lyapunov function of the system (4.7) we consider the function

$$
\widetilde{\mathscr V}(\widehat\varrho,\widehat\vartheta,\tau) = \widetilde{\mathscr H}(\widehat\varrho,\widehat\vartheta,\tau) + \tau^{-\frac12}\frac{\widetilde\gamma_0}{2}\widehat\varrho\widehat\vartheta.
$$

By (4.8), the total derivative of this function along the trajectories of the system (4.7) admits the asymptotics

$$
\frac{d\widetilde{\mathscr V}}{d\tau}\Big|_{(4.7)} = \tau^{-1}\widetilde\gamma_0\frac{\Delta^2}{2}(1+\mathscr O(\Delta)) + \mathscr O(\Delta)\mathscr O(\tau^{-\frac32}),\quad \tau\to\infty,\quad \Delta\to 0.
$$

Consequently, for any $\sigma\in(0,1)$ there are $\Delta_0>0$ and $\widehat\tau_0\geqslant\tau_0$ such that

$$
\begin{aligned}
&(1-\sigma)\frac{\Delta^2}{2} \leqslant \widetilde{\mathscr V}(\widehat\varrho,\widehat\vartheta,\tau) \leqslant (1+\sigma)\frac{\Delta^2}{2},\\
&\frac{d\widetilde{\mathscr V}}{d\tau}\Big|_{(4.7)} \leqslant -\tau^{-1}(1-\sigma)|\widetilde\gamma_0|\frac{\Delta^2}{2} + M\tau^{-\frac32}\Delta
\end{aligned}
\tag{4.9}
$$

for all $(\widehat\varrho,\widehat\vartheta,\tau)\in\mathbb R^3$ such that $\Delta\leqslant\Delta_0$ and $\tau\geqslant\widehat\tau_0$ with some constant $M>0$. We fix $\epsilon\in(0,\Delta_0)$. Then

$$
\frac{d\widetilde{\mathscr V}}{d\tau}\Big|_{(4.7)} \leqslant -\tau^{-1}\Big((1-\sigma)|\widetilde\gamma_0| - \frac{2M}{\delta_\epsilon}\tau_\epsilon^{-\frac12}\Big)\frac{\Delta^2}{2} \leqslant 0
$$

for all $(\widehat{\varrho}, \widehat{\vartheta}, \tau) \in \mathbb{R}^3$ such that $\delta_\epsilon \leqslant \Delta \leqslant \epsilon$ and $\tau \geqslant \tau_\epsilon$, where

$$\delta_\epsilon = \frac{4M}{(1-\sigma)|\widetilde{\gamma}_0|}\tau_\epsilon^{-\frac{1}{2}}, \quad \tau_\epsilon = \max\Big\{\frac{16M^2(1+\sigma)}{(1-\sigma)^3|\widetilde{\gamma}_0|^2\epsilon^2}, \widehat{\tau}_0\Big\}.$$

We note that

$$\delta_\epsilon < \epsilon\sqrt{(1-\sigma)/(1+\sigma)},$$

$$\sup_{\Delta\leqslant\delta_\epsilon} \widetilde{\mathscr{V}}(\widehat{\varrho}, \widehat{\vartheta}, \tau) \leqslant \inf_{\Delta=\epsilon} \widetilde{\mathscr{V}}(\widehat{\varrho}, \widehat{\vartheta}, \tau), \quad \tau \geqslant \tau_\epsilon.$$

Therefore, since the total derivative of $\widetilde{\mathscr{V}}(\widehat{\varrho}, \widehat{\vartheta}, \tau)$ is nonnegative, the trajectories of the system (4.7) outgoing from the domain $\{(\widehat{\varrho}, \widehat{\vartheta}) : \Delta \leqslant \delta_\epsilon\}$ for $\tau = \tau_s \geqslant \tau_\epsilon$ cannot leave the neighborhood $\{(\widehat{\varrho}, \widehat{\vartheta}) : \Delta \leqslant \epsilon\}$ for $\tau > \tau_s$. Furthermore, (4.9) implies

$$\frac{d\widetilde{\mathscr{V}}}{d\tau}\Big|_{(4.7)} \leqslant -\tau^{-1}\left(\frac{1-\sigma}{1+\sigma}\right)|\widetilde{\gamma}_0|\widetilde{\mathscr{V}}(\widehat{\varrho}, \widehat{\vartheta}, \tau) + M\tau^{-\frac{3}{2}}\Delta_0$$

for all $(\widehat{\varrho}, \widehat{\vartheta}, \tau) \in \mathbb{R}^3$ such that $\Delta \leqslant \Delta_0$ and $\tau \geqslant \tau_0$. Integrating the last inequality with respect to τ, we find

$$\widetilde{\mathscr{V}}(\widehat{\varrho}(\tau), \widehat{\vartheta}(\tau), \tau) = \mathscr{O}(\tau^{-\frac{1}{2}}\log\tau) + \mathscr{O}(\tau^{-|\widetilde{\gamma}_0|(1-\sigma)/(1+\sigma)}), \quad \tau \to \infty.$$

By properties of Lyapunov functions, $|\widehat{\varrho}(\tau)| + |\widehat{\vartheta}(\tau)| \to 0$ as $\tau \to \infty$. □

We consider the case $B > 2A$ where the system (4.4) takes the form

$$\begin{aligned} \frac{d\varrho}{dt} &= \tau^{-\frac{A}{B}}\Big(\widehat{\Lambda}_A(\vartheta) + \tau^{-\frac{B-2A}{B}}F_{B-A}\Big) + \tau^{-\frac{2A}{B}}\widehat{\Lambda}_{2A}(\varrho, \vartheta) + \mathscr{O}(\tau^{-\frac{C}{B}}), \\ \frac{d\vartheta}{dt} &= \tau^{-\frac{A}{B}}\mu_A\varrho + \tau^{-\frac{2A}{B}}\Omega_{2A}(\varrho, \vartheta) + \mathscr{O}(\tau^{-\frac{3A}{B}}) \end{aligned} \tag{4.10}$$

as $\tau \to \infty$ uniformly for all $|\varrho| \leqslant \varrho_0$ and $\vartheta \in \mathbb{R}$, where

$$\widehat{\Lambda}_A(\vartheta) \equiv \langle F_A(\vartheta, \xi)\rangle_\xi, \quad \widehat{\Lambda}_{2A}(\varrho, \vartheta) \equiv \lambda_{2A}\varrho\widehat{\Lambda}_A(\vartheta), \quad C = \min\{B, 3A\}. \tag{4.11}$$

Lemma 4.2. *We assume that there exists $\vartheta_0 \in \mathbb{R}$ such that $\widehat{\Lambda}_A(\vartheta_0) = 0$, $\widehat{\Lambda}'_A(\vartheta_0) < 0$, and $\widehat{\gamma}_0 := \partial_\vartheta\Omega_{2A}(0, \vartheta_0) < 0$. Then for any $\epsilon \in (0, \varrho_0)$ there are $\delta_\epsilon > 0$ and $\tau_\epsilon \geqslant \tau_0$ such that the solution $\varrho(\tau)$, $\vartheta(\tau)$ to the system* (4.10) *with the initial data $|\varrho(\tau_\epsilon)| + |\vartheta(\tau_\epsilon) - \vartheta_0| \leqslant \delta_\epsilon$ satisfies the inequality*

$$\sup_{\tau\geqslant\tau_\epsilon}(|\varrho(\tau)| + |\vartheta(\tau) - \vartheta_0|) \leqslant \epsilon.$$

Furthermore, $|\varrho(\tau)| + |\vartheta(\tau) - \vartheta_0| \to 0$ as $\tau \to \infty$.

Proof. We consider the change of variables

$$\varrho(\tau) = \tau^{-\frac{A}{B}}\varrho_1 + \widehat{\varrho}(\tau), \quad \vartheta(\tau) = \vartheta_0 + \tau^{-\frac{B-2A}{B}}\widetilde{\vartheta}_{B-2A}(\tau) + \widehat{\vartheta}(\tau)$$

with $\varrho_1 = -\Omega_{2A}(0, \vartheta_0)/\mu_A$ and $\widetilde{\vartheta}_{B-2A}(\tau)$ such that

$$\widehat{\Lambda}_A(\vartheta_0 + \tau^{-\frac{B-2A}{B}}\widetilde{\vartheta}_{B-2A}(\tau)) + \tau^{-\frac{B-2A}{B}}F_{B-A} \equiv 0.$$

It is easy to verify that

$$\widetilde{\vartheta}_{B-2A}(\tau) = -F_{B-A}/\widehat{\Lambda}'_A(\vartheta_0) + \mathscr{O}(\tau^{-(B-2A)/B}), \quad \tau \to \infty.$$

The system (4.10) is written in the new variables in the form

$$\begin{aligned} \frac{d\widehat{\varrho}}{d\tau} &= -\tau^{-\frac{A}{B}} \partial_{\widehat{\vartheta}} \widehat{\mathscr{H}}(\widehat{\varrho}, \widehat{\vartheta}, \tau) + \mathscr{O}(\tau^{-\frac{C}{B}}), \\ \frac{d\widehat{\vartheta}}{d\tau} &= \tau^{-\frac{A}{B}} \partial_{\widehat{\varrho}} \widehat{\mathscr{H}}(\widehat{\varrho}, \widehat{\vartheta}, \tau) + \tau^{-\frac{2A}{B}} \widehat{\mathscr{Z}}(\widehat{\varrho}, \widehat{\vartheta}) + \mathscr{O}(\tau^{-\frac{C}{B}}) \end{aligned} \tag{4.12}$$

as $\tau \to \infty$ and $|\widehat{\varrho}| + |\widehat{\vartheta}| < \infty$, where

$$\widehat{\mathscr{H}}(\widehat{\varrho}, \widehat{\vartheta}, \tau) \equiv \mu_A \frac{\widehat{\varrho}^2}{2} - \int\limits_0^{\widehat{\vartheta}} \Big(\widehat{\Lambda}_A(\vartheta_0 + \tau^{-\frac{B-2A}{B}} \widetilde{\vartheta}_{B-2A}(\tau) + \theta) - \widehat{\Lambda}_A(\vartheta_0 + \tau^{-\frac{B-2A}{B}} \widetilde{\vartheta}_{B-2A}(\tau)) \Big) \, d\theta$$

$$- \tau^{-\frac{A}{B}} \lambda_{2A} \widehat{\varrho} \int\limits_0^{\widehat{\vartheta}} (\widehat{\Lambda}_A(\vartheta_0 + \theta) - \widehat{\Lambda}_A(\vartheta_0)) \, d\theta + \tau^{-\frac{A}{B}} \int\limits_0^{\widehat{\varrho}} (\Omega_{2A}(r, \vartheta_0) - \Omega_{2A}(0, \vartheta_0)) \, dr,$$

$$\widehat{\mathscr{Z}}(\widehat{\varrho}, \widehat{\vartheta}) \equiv \Omega_{2A}(\widehat{\varrho}, \vartheta_0 + \widehat{\vartheta}) - \Omega_{2A}(\widehat{\varrho}, \vartheta_0) + \lambda_{2A} \int\limits_0^{\widehat{\vartheta}} (\widehat{\Lambda}_A(\vartheta_0 + \theta) - \widehat{\Lambda}_A(\vartheta_0)) \, d\theta.$$

The functions $\widehat{\mathscr{H}}(\widehat{\varrho}, \widehat{\vartheta}, \tau)$ and $\widehat{\mathscr{Z}}(\widehat{\varrho}, \widehat{\vartheta})$ admit the following asymptotics:

$$\begin{aligned} \widehat{\mathscr{H}}(\widehat{\varrho}, \widehat{\vartheta}, \tau) &= \frac{\Delta^2}{2} (1 + \mathscr{O}(\tau^{-\frac{1}{2}}) + \mathscr{O}(\Delta)), \\ \widehat{\mathscr{Z}}(\widehat{\varrho}, \widehat{\vartheta}) &= \widehat{\gamma}_0 \widehat{\vartheta} + \mathscr{O}(\Delta) \end{aligned}$$

as $\tau \to \infty$ and

$$\Delta \equiv \sqrt{\mu_A \widehat{\varrho}^2 + \lambda_A \widehat{\vartheta}^2} \to 0,$$

where $\lambda_A := |\widehat{\Lambda}_A(\vartheta_0)| > 0$. For the system (4.12) the Lyapunov function is constructed in the form

$$\widehat{\mathscr{V}}(\widehat{\varrho}, \widehat{\vartheta}, \tau) = \widehat{\mathscr{H}}(\widehat{\varrho}, \widehat{\vartheta}, \tau) + \tau^{-\frac{A}{B}} \frac{\widehat{\gamma}_0}{2} \widehat{\varrho} \widehat{\vartheta}.$$

Computing the total derivative of this function, we find the estimate

$$\frac{d\widehat{\mathscr{V}}}{d\tau}\Big|_{(4.12)} = -\tau^{-\frac{2A}{B}} |\widehat{\gamma}_0| \frac{\Delta^2}{2} (1 + \mathscr{O}(\Delta)) + \mathscr{O}(\Delta) \mathscr{O}(\tau^{-\frac{C}{B}}), \quad \tau \to \infty, \quad \Delta \to 0.$$

Consequently, for any $\sigma \in (0,1)$ there are $\Delta_0 > 0$ and $\widehat{\tau}_0 \geqslant \tau_0$ such that

$$\begin{aligned} (1-\sigma) \frac{\Delta^2}{2} &\leqslant \widehat{\mathscr{V}}(\widehat{\varrho}, \widehat{\vartheta}, \tau) \leqslant (1+\sigma) \frac{\Delta^2}{2}, \\ \frac{d\widehat{\mathscr{V}}}{d\tau}\Big|_{(4.12)} &\leqslant -\tau^{-\frac{2A}{B}} (1-\sigma) |\widehat{\gamma}_0| \frac{\Delta^2}{2} + M \tau^{-\frac{C}{B}} \Delta \end{aligned}$$

for all $(\widehat{\varrho}, \widehat{\vartheta}, \tau) \in \mathbb{R}^3$ such that $\Delta \leqslant \Delta_0$ and $\tau \geqslant \widehat{\tau}_0$ with some constant $M > 0$. Thus, the constructed Lyapunov function satisfies estimates similar to (4.9). Repeating the proof of Lemma 4.1, we obtain the required assertion. □

Combining Lemmas 4.1 and 4.2 with formulas (4.1) and (4.3) of change of variables, and the expressions (4.6) and (4.11) for the coefficients, we complete the proof of Theorem 2.2.

5 Examples

We consider the Duffing equation with decaying perturbation

$$\frac{d^2x}{dt^2} - x + x^3 = qt^{-a}x^p \cos\left(st^{b+1}\right), \tag{5.1}$$

where $a, b, s \in \mathbb{R}_+$, $p \in \mathbb{Z}$, $q \in \mathbb{R}$, $b \geqslant 1$. Equation (5.1) is a particular case of (1.3) with $h = 1$, $U(x) \equiv x^4/4 - x^2/2$, and $Q(x) \equiv qx^p$. The corresponding change of variables (1.4) reduces Equation (5.1) to the form (1.1) with $\beta = p - 2$. Moreover,

$$T_0 = 2\sqrt{2}K\left(\frac{1}{2}\right), \quad X_0(\varphi) = \sqrt{2}\,\mathrm{cn}\left(\frac{T_0\varphi}{\pi\sqrt{2}}; \frac{1}{2}\right), \quad Y_0(\varphi) = \frac{2\pi}{T_0}\partial_\varphi X_0(\varphi),$$

where $K(k)$ is a complete elliptic integral of the first kind and $\mathrm{cn}(t, k)$ is the Jacobi elliptic function. Furthermore, the 2π-periodic function $X_0(\varphi)$ is expanded into the Fourier series [14]

$$X_0(\varphi) = \sum_{j=1}^{\infty} x_j \cos((2j-1)\varphi),$$

where

$$x_j = \frac{4\pi\sqrt{2}}{T_0}\,\mathrm{sech}\left((2j-1)\frac{\pi}{2}\right).$$

We note that Theorem 2.1 provides a necessary condition for the existence of resonance solutions to Equation (5.1):

$$2 + \frac{a-1}{b} \leqslant p < 3 + \frac{a}{b}.$$

To obtain the conditions for the stability of resonance solutions, we consider the equation in the corresponding model case (2.2), where

$$\begin{aligned} f_0(\varphi, S) &\equiv \frac{q\pi}{2T_0(p+1)}\partial_\varphi (X_0(\varphi))^{p+1}\cos S, \\ g_0(\varphi, S) &\equiv -\frac{q\pi}{2T_0}(X_0(\varphi))^{p+1}\cos S, \\ \omega_0 &= \frac{2\pi}{T_0}. \end{aligned}$$

We assume that $p = 2$ and $a = b = 1$. Then $\beta = 0$, $\Gamma_0 = 1$, and $\Gamma_1 = 0$. We verify the assumptions of Theorem 2.2 with $\varkappa = 1$. We note that for any 2π-periodic function $Z(\varphi)$

$$\langle Z(\vartheta+\xi)\cos\xi\rangle_\xi \equiv \langle Z(\xi)\cos(\xi-\vartheta)\rangle_\xi \equiv \langle Z(\xi)\cos\xi\rangle_\xi \cos\vartheta + \langle Z(\xi)\sin\xi\rangle_\xi \sin\vartheta.$$

Hence

$$\langle f_0(\vartheta+\xi,\xi)\rangle_\xi \equiv -\frac{q\pi C_3}{6T_0}\sin\vartheta,$$

$$\langle \partial_\varphi f_0(\vartheta+\xi,\xi)\rangle_\xi \equiv -\frac{q\pi C_3}{6T_0}\cos\vartheta,$$

$$\langle \partial_\varphi g_0(\vartheta+\xi,\xi)\rangle_\xi \equiv \frac{q\pi C_3}{2T_0}\sin\vartheta,$$

where $C_3 = \langle X_0^3(\xi)\cos\xi\rangle_\xi \approx 0.954$. Consequently, if $|q| > q_*$, $q_* = 6T_0/(\pi C_3)$, then there is ϑ_0 satisfying the condition (2.3):

$$\vartheta_0 = \begin{cases} -\arcsin\left(\frac{6T_0}{q\pi C_3}\right) + 2\pi k, & q > 0, \\ \pi - \arcsin\left(\frac{6T_0}{|q|\pi C_3}\right) + 2\pi k, & q < 0, \end{cases} \quad k \in \mathbb{Z}.$$

It is easy to verify that, in this case, the condition (2.4) is satisfied with $\gamma_\varkappa = -3 < 0$. Thus, if $|q| > q_*$, then Theorem 2.2 implies the stability of resonance solutions to the model system and asymptotic estimates

$$\rho(t) \sim c_\varkappa t, \quad \varphi(t) - S(t) \sim \vartheta_0, \quad t \to \infty, \quad c_\varkappa = \frac{sT_0}{\pi}.$$

A numerical analysis shows that the resonance solutions to Equation (5.1) are adequately described by the solutions to the model system (cf. Figure 2).

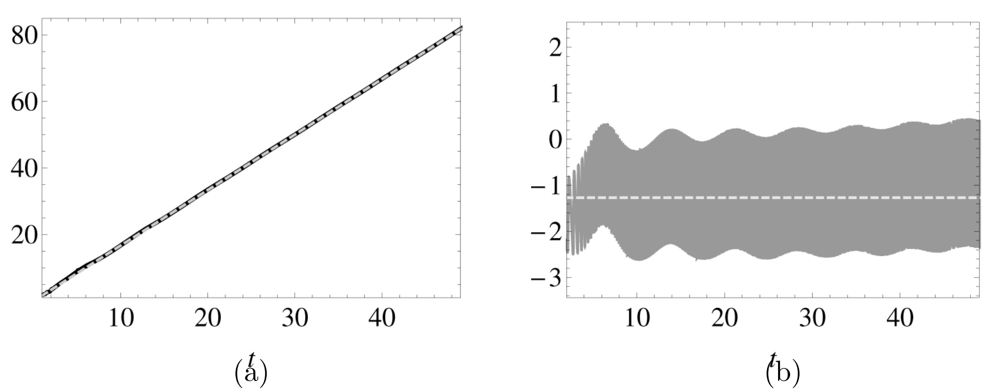

Figure 2. Evolution of the amplitude $\rho(t) \equiv (H(x(t), \dot{x}(t)))^{1/4}$ (a) and the phase difference $\widetilde{\theta}(t) \equiv \widetilde{\varphi}(t) - S(t)$, $\tan\widetilde{\varphi}(t) = -\dot{x}(t)/x(t)$ (b) for the solutions to Equation (5.1) with $p = 2$, $q = 11$, $a = b = s = 1$ ($q_* \approx 10.49$). The dashed lines correspond to $c_\varkappa t$ ($c_\varkappa \approx 1.67$) (a) and $\vartheta_0 \approx -1.27$ (b).

We assume that $p = 3$, $a = 2$, and $b = 1$. Then $\beta = 1$ and $\Gamma_0 = \Gamma_1 = \varkappa\pi/(sT_0)$. We verify the assumptions of Theorem 2.2 for $\varkappa = 2$. We note that

$$\langle f_0(\vartheta + \frac{\xi}{2}, \xi)\rangle_{2\xi} \equiv -\frac{q\pi C_4}{4T_0}\sin 2\vartheta,$$

$$\langle \partial_\varphi f_0(\vartheta + \frac{\xi}{2}, \xi)\rangle_{2\xi} \equiv -\frac{q\pi C_4}{2T_0}\cos 2\vartheta,$$

$$\langle \partial_\varphi g_0(\vartheta + \frac{\xi}{2}, \xi)\rangle_{2\xi} \equiv \frac{q\pi C_4}{T_0} \sin 2\vartheta,$$

where $C_4 = \langle X_0^4(\xi)\cos 2\xi\rangle_\xi \approx 0.911$. Consequently, if $|q| > q_*$, $q_* = 8/(sC_4)$, then there exists ϑ_0 satisfying the condition (2.3):

$$\vartheta_0 = \begin{cases} -\frac{1}{2}\arcsin\left(\frac{8}{sqC_4}\right) + \pi k, & q > 0, \\ \frac{\pi}{2} - \frac{1}{2}\arcsin\left(\frac{8}{s|q|C_4}\right) + \pi k, & q < 0, \end{cases} \quad k \in \mathbb{Z}.$$

It is easy to verify that, in this case, the condition (2.4) is satisfied with $\gamma_\varkappa = -6\pi/(sT_0) < 0$. Thus, in the case $|q| > q_*$, Theorem 2.2 implies the stability of resonance solutions in the model system (cf. Figure 3). Moreover,

$$\rho(t) \sim c_\varkappa t, \quad \varphi(t) - \frac{S(t)}{2} \sim \vartheta_0, \quad t \to \infty, \quad c_\varkappa = \frac{sT_0}{2\pi}.$$

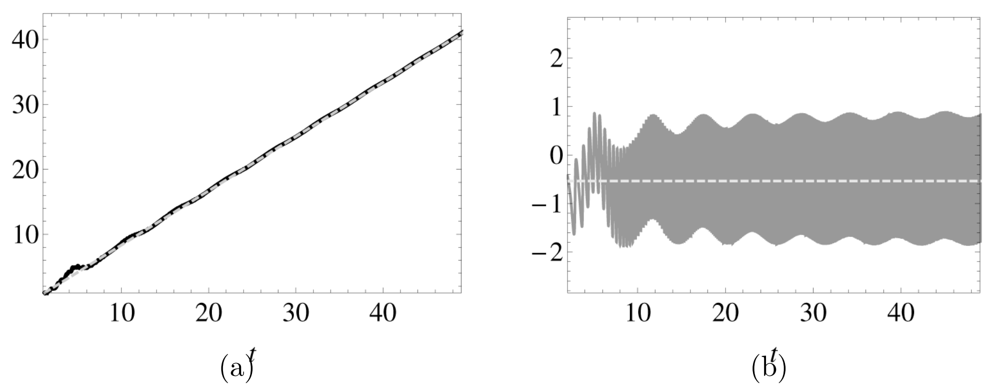

Figure 3. Evolution of the amplitude $\rho(t) \equiv (H(x(t), \dot{x}(t)))^{1/4}$ (a) and the phase difference $\widetilde{\theta}(t) \equiv \widetilde{\varphi}(t) - S(t)/2$, $\tan\widetilde{\varphi}(t) = -\dot{x}(t)/x(t)$ (b) for the solution to Equation (5.1) with $p = 3$, $q = 10$, $a = 2$, $b = s = 1$ ($q_* \approx 8.78$). The dashed lines correspond to $c_\varkappa t$ ($c_\varkappa \approx 0.84$) (a) and $\vartheta_0 \approx -0.54$ (b).

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