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Caputo-type modification of the Hadamard fractional derivatives

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Abstract

Generalization of fractional differential operators was subjected to an intense debate in the last few years in order to contribute to a deep understanding of the behavior of complex systems with memory effect. In this article, a Caputo-type modification of Hadamard fractional derivatives is introduced. The properties of the modified derivatives are studied.

Keywords: Hadamard fractional derivatives; Caputo

1 Introduction

The fractional calculus which is as old as the usual calculus is the generalization of differentiation and integration of integer order to arbitrary ones.

In the last 30 years or so, there has been an important development of the field of fractional calculus as it has been applied to many fields of science. It was pointed out that fractional derivatives and integrals are more convenient for describing real materials, and some physical problems were treated by using derivatives of non-integer orders [1, 2, 5–9, 12].

Several authors found interesting results when they used the fractional calculus in control theory [7, 10, 11].

Maybe the most well-known fractional integral is the one developed by Riemann and Liouville and based on the generalization of the usual Riemann integral $\int_a^x f(t)\, dt$.

The left Riemann-Liouville fractional integral and the right Riemann-Liouville fractional integral are defined respectively by

$$_aI^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} f(\tau)\, d\tau, \tag{1}$$

$$I_b^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\tau-x)^{\alpha-1} f(\tau)\, d\tau, \tag{2}$$

where $\alpha > 0$, $n-1 < \alpha < n$. Here and in the following, $\Gamma(\alpha)$ represents the Gamma function.

The left Riemann-Liouville fractional derivative is defined by

$$_aD^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-\tau)^{n-\alpha-1} f(\tau)\, d\tau. \tag{3}$$

The right Riemann-Liouville fractional derivative is defined by

$$D_b^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dx}\right)^n \int_x^b (\tau - x)^{n-\alpha-1} f(\tau)\, d\tau. \tag{4}$$

The fractional derivative of a constant takes the form

$$_aD^\alpha C = C\, \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)} \tag{5}$$

and the fractional derivative of a power of t has the following form:

$$_aD^\alpha (x-a)^\beta = \frac{\Gamma(\alpha+1)(x-a)^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)}, \tag{6}$$

for $\beta > -1$, $\alpha \geq 0$.

Although the definition of the Riemann-Liouville type (3) played a significant role in the development of the theory of fractional calculus, real world problems require fractional derivatives which contain physically interpretable initial conditions $f(a), f'(a), \ldots$ [4, 6]. To overcome such problems, Caputo proposed the following definitions of the left and right fractional derivatives respectively:

$$_a^C D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-\tau)^{n-\alpha-1} \left(\frac{d}{d\tau}\right)^n f(\tau)\, d\tau, \tag{7}$$

and

$$^C D_b^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_x^b (\tau - x)^{n-\alpha-1} \left(-\frac{d}{d\tau}\right)^n f(\tau)\, d\tau, \tag{8}$$

where α represents the order of the derivative such that $n-1 < \alpha < n$. By definition the Caputo fractional derivative of a constant is zero.

The Riemann-Liouville fractional derivatives and Caputo fractional derivatives are connected with each other by the following relations:

$$_a^C D^\alpha f(x) = {}_aD^\alpha \left(f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)} (x-a)^{k-\alpha} \right), \tag{9}$$

$$^C D_b^\alpha f(x) = D_b^\alpha \left(f(x) - \sum_{k=0}^{n-1} \frac{(-1)^k f^{(k)}(b)}{\Gamma(k-\alpha+1)} (b-x)^{k-\alpha} \right). \tag{10}$$

Riemann-Liouville fractional derivative is formally a fractional power $(\frac{d}{dx})^\alpha$ of the differentiation $\frac{d}{dx}$ and is invariant with respect to translation on the whole axis [2]. Hadamard [3] suggested a fractional power of the form $(x\frac{d}{dx})^\alpha$. This fractional derivative is invariant with respect to dilation on the whole axis. The Hadamard approach to the fractional integral was based on the generalization of the nth integral

$$_a\mathcal{J}^n(x) = \int_a^x \frac{dt_1}{t_1} \int_a^{t_1} \frac{dt_2}{t_2} \cdots \int_a^{t_{n-1}} f(t_n)\frac{dt_n}{t_n} = \frac{1}{(n-1)!} \int_a^t \left(\log\frac{x}{t}\right)^{n-1} f(t)\frac{dt}{t}. \tag{11}$$

The left and right Hadamard fractional integrals of order $\alpha \in \mathbb{C}$, $\mathcal{R}(\alpha) > 0$ are respectively defined by

$$_a\mathcal{J}^\alpha y(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log \frac{x}{t}\right)^{\alpha-1} y(t) \frac{dt}{t}, \tag{12}$$

$$\mathcal{J}_b^\alpha y(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\log \frac{t}{x}\right)^{\alpha-1} y(t) \frac{dt}{t}. \tag{13}$$

The corresponding left and right sided Hadamard fractional derivatives are

$$_a\mathcal{D}^\alpha y(x) = \delta^n \left(_a\mathcal{J}^{n-\alpha} y\right)(x) = \left(x\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_a^x \left(\log \frac{x}{t}\right)^{n-\alpha-1} y(t) \frac{dt}{t}, \tag{14}$$

and

$$\mathcal{D}_b^\alpha y(x) = (-\delta)^n \left(\mathcal{J}_b^{n-\alpha} y\right)(x) = \left(-x\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_t^b \left(\log \frac{t}{x}\right)^{n-\alpha-1} y(t) \frac{dt}{t}, \tag{15}$$

where $\delta = x\frac{d}{dx}$, $\delta^0 y(x) = y(x)$, $\alpha \in \mathbb{C}$, $\mathcal{R}(\alpha) \geq 0$.

In [13–17], the authors contributed to the development of the theory of Hadamard-type fractional calculus. However, this calculus is still studied less than that of Riemann-Liouville.

The question arises here is whether we can modify the Hadamard fractional derivatives so that the derivatives of a constant is 0 and these derivatives contain physically interpretable initial conditions similar to the ones in Caputo fractional derivatives.

The main goal of this article is to define the Caputo-type modification of the Hadamard fractional derivatives and present properties of such derivatives.

2 Caputo-type Hadamard fractional derivatives

Let $\mathcal{R}(\alpha) \geq 0$ and $n = [\mathcal{R}(\alpha)] + 1$. If $y(x) \in AC_\delta^n[a,b]$, where $0 < a < b < \infty$ and $AC_\delta^n[a,b] = \{g : [a,b] \to \mathbb{C} : \delta^{n-1}g(x) \in AC[a,b], \delta = x\frac{d}{dx}\}$. We define the Caputo-type modification of left- and right-sided Hadamard fractional derivatives respectively as follows:

$$_a^C\mathcal{D}^\alpha y(x) = _a\mathcal{D}^\alpha \left[y(t) - \sum_{k=0}^{n-1} \frac{\delta^k y(a)}{k!} \left(\log \frac{t}{a}\right)^k\right](x), \tag{16}$$

and

$$^C\mathcal{D}_b^\alpha y(x) = \mathcal{D}_b^\alpha \left[y(t) - \sum_{k=0}^{n-1} \frac{(-1)^k \delta^k y(b)}{k!} \left(\log \frac{b}{t}\right)^k\right](x). \tag{17}$$

In particular, if $0 < \mathcal{R}(\alpha) < 1$, we have

$$_a^C\mathcal{D}^\alpha y(x) = _a\mathcal{D}^\alpha \left[y(t) - y(a)\right](x), \tag{18}$$

and

$$^C\mathcal{D}_b^\alpha y(x) = \mathcal{D}_b^\alpha \left[y(t) - y(b)\right](x). \tag{19}$$

Theorem 2.1 *Let $\mathcal{R}(\alpha) \geq 0$ and $n = [\mathcal{R}(\alpha)] + 1$. If $y(x) \in AC_\delta^n[a,b]$, where $0 < a < b < \infty$. Then ${}_a^C\mathcal{D}^\alpha y(x)$ and ${}^C\mathcal{D}_b^\alpha y(x)$ exist everywhere on $[a,b]$ and*

(i) if $\alpha \notin \mathbb{N}_0$, ${}_a^C\mathcal{D}^\alpha y(x)$ can be represented by

$$
{}_a^C\mathcal{D}^\alpha y(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \left(\log \frac{x}{t}\right)^{n-\alpha-1} \delta^n y(t) \frac{dt}{t} = {}_a\mathcal{J}^{n-\alpha}\delta^n y(x), \tag{20}
$$

and ${}^C\mathcal{D}_b^\alpha y(x)$ can be represented by

$$
{}^C\mathcal{D}_b^\alpha y(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \left(\log \frac{t}{x}\right)^{n-\alpha-1} \delta^n y(t) \frac{dt}{t} = (-1)^n \mathcal{J}_b^{n-\alpha}\delta^n y(x), \tag{21}
$$

(ii) if $\alpha \in \mathbb{N}$, then

$$
{}_a^C\mathcal{D}^\alpha y(x) = \delta^n y(x), \qquad {}^C\mathcal{D}_b^\alpha y(x) = (-1)^n \delta^n y(x). \tag{22}
$$

In particular,

$$
{}_a^C\mathcal{D}^0 y(x) = {}^C\mathcal{D}_b^0 y(x) = y(x). \tag{23}
$$

Proof Let $\alpha \notin \mathbb{N}_0$. Using the definition of left Hadamard fractional derivative (14) and applying integrating by parts formula $u = y(t) - \sum_{k=0}^{n-1} \frac{\delta^k y(a)}{k!}(\log \frac{t}{a})^k$ and $dv = (\log \frac{x}{t})^{n-\alpha-1}\frac{dt}{t}$ in (16), one gets

$$
\begin{aligned}
{}_a^C\mathcal{D}^\alpha y(x) &= \left(x\frac{d}{dx}\right)^n \left\{\left[-\frac{1}{n-\alpha}\left(\log \frac{x}{t}\right)^{n-\alpha}\left(y(t) - \sum_{k=0}^{n-1}\frac{\delta^k y(a)}{k!}\left(\log \frac{t}{a}\right)^k\right)\right]\Bigg|_a^x\right.\\
&\quad \left. + \frac{1}{n-\alpha}\int_a^x \left(\log \frac{x}{t}\right)^{n-\alpha}\left[\delta y(t) - \sum_{k=1}^{n-1}\frac{\delta^k y(a)}{k!}\left(\log \frac{t}{a}\right)^k\right]\frac{dt}{t}\right\}\\
&= \left(x\frac{d}{dx}\right)^{n-1}\int_a^x \left(\log \frac{x}{t}\right)^{n-\alpha}\left[\delta y(t) - \sum_{k=1}^{n-1}\frac{\delta^k y(a)}{k!}\left(\log \frac{t}{a}\right)^k\right]\frac{dt}{t}\\
&= \cdots = x\frac{d}{dx}\frac{1}{\Gamma(n-\alpha)}\int_a^x \left(\log \frac{x}{t}\right)^{n-\alpha-1}\left(\delta^{n-1}y(t) - \delta^{n-1}y(a)\right)\frac{dt}{t}.
\end{aligned}
$$

Performing once more the integration by parts with the same choice of dv, one gets equation (20). (21) is proved in a similar way. Now, when $\alpha = n$ we have

$$
{}_a^C\mathcal{D}^n y(x) = {}_a\mathcal{D}^n\left[y(t) - \sum_{k=0}^{n-1}\frac{\delta^k y(a)}{k!}\left(\log \frac{t}{a}\right)^k\right](x).
$$

That is

$$
\begin{aligned}
y(x) &= {}_a\mathcal{J}^n{}_a^C\mathcal{D}^n y(x) + \sum_{k=0}^{n-1}\frac{\delta^k y(a)}{k!}\left(\log \frac{x}{a}\right)^k\\
&= \frac{1}{(n-1)!}\int_a^x \left(\log \frac{x}{t}\right)^{n-1}{}_a^C\mathcal{D}^n y(t)\frac{dt}{t} + \sum_{k=0}^{n-1}\frac{\delta^k y(a)}{k!}\left(\log \frac{x}{a}\right)^k.
\end{aligned}
$$

From Lemma 1.2 in [1], one derives $_a^C\mathcal{D}^\alpha y(x) = \delta^n y(x)$. The second formula in (22) can be proved likewise. □

Theorem 2.2 *Let $\mathcal{R}(\alpha) \geq 0$ and $n = [\mathcal{R}(\alpha)] + 1$. If $y(x) \in C_\delta^n[a, b]$, where $0 < a < b < \infty$. Then $_a^C\mathcal{D}^\alpha y(x)$ and $^C\mathcal{D}_b^\alpha y(x)$ are continuous on $[a, b]$ and*

(i) *if $\alpha \notin \mathbb{N}_0$, $_a^C\mathcal{D}^\alpha y(x)$ and $^C\mathcal{D}_b^\alpha y(x)$ can be represented by (20) and (21) respectively and*

$$_a^C\mathcal{D}^\alpha y(a) = 0, \qquad ^C\mathcal{D}_b^\alpha y(b) = 0, \tag{24}$$

(ii) *if $\alpha \in \mathbb{N}$, then the formulas in (22) hold.*

Proof The representations (21) and (22) are proved like in Theorem 2.1. The continuity of $_a^C\mathcal{D}^\alpha y(x)$ and $^C\mathcal{D}_b^\alpha y(x)$ results from their representations and Lemma 2.36 in [1] with $f(t) = \delta^n y(t)$, $\gamma = 0$ and α replaced by $n - \alpha$. The identities in (24) hold since

$$\left| _a^C\mathcal{D}^\alpha y(x) \right| \leq \frac{\|\delta^n y\|_C}{|\Gamma(n - \alpha)|(n - \mathcal{R}(\alpha))} \left(\log \frac{x}{a} \right)^{n - \mathcal{R}(\alpha)}, \tag{25}$$

$$\left| ^C\mathcal{D}_b^\alpha y(x) \right| \leq \frac{\|\delta^n y\|_C}{|\Gamma(n - \alpha)|(n - \mathcal{R}(\alpha))} \left(\log \frac{b}{x} \right)^{n - \mathcal{R}(\alpha)}. \tag{26}$$

When $\alpha \in \mathbb{N}$, the first formula in (22) holds as a result of Lemma 1.4 in [1]. The second formula can be proved likewise. □

Corollary 2.3 *Let $\mathcal{R}(\alpha) \geq 0$ and $n = [\mathcal{R}(\alpha)] + 1$.*

(i) *If $\alpha \notin \mathbb{N}_0$, then $_a^C\mathcal{D}^\alpha$ is bounded from the space $C_\delta^n[a, b]$ to the space $C_a[a, b] = \{f \in C[a, b] \text{ s.t } f(a) = 0\}$ and $^C\mathcal{D}_b^\alpha$ is bounded from the space $C_\delta^n[a, b]$ to the space $C_b[a, b] = \{f \in C[a, b] \text{ s.t } f(b) = 0\}$ and*

$$\left\| _a^C\mathcal{D}^\alpha y \right\|_{C_a} \leq \frac{(\log \frac{b}{a})^{n - \mathcal{R}(\alpha)}}{|\Gamma(n - \alpha)|(n - \mathcal{R}(\alpha))} \|y\|_{C_\delta^n}, \tag{27}$$

$$\left\| ^C\mathcal{D}_b^\alpha y \right\|_{C_b} \leq \frac{(\log \frac{b}{a})^{n - \mathcal{R}(\alpha)}}{|\Gamma(n - \alpha)|(n - \mathcal{R}(\alpha))} \|y\|_{C_\delta^n}. \tag{28}$$

(ii) *If $\alpha = n \in \mathbb{N}_0$, then $_a^C\mathcal{D}^n$ and $^C\mathcal{D}_b^n$ are bounded from $C_\delta^n[a, b]$ to $C[a, b]$ and*

$$\left\| _a^C\mathcal{D}^n y \right\|_C \leq \|y\|_{C_\delta^n}, \qquad \left\| ^C\mathcal{D}_b^n y \right\|_C \leq \|y\|_{C_\delta^n}. \tag{29}$$

Proof (27) and (28) follow from the estimates (25) and (26) keeping in mind that $\|\delta^n y\|_C \leq \|y\|_{C_\delta^n}$ (see formula 1.1.28 in [1] when $\gamma = 0$). The estimates in (29) are straightforward. □

$_a^C\mathcal{D}^\alpha$ and $^C\mathcal{D}_b^\alpha$ provide operations inverse to $_a\mathcal{J}^\alpha$ and $\mathcal{J}_b^\alpha$ respectively for $\mathcal{R}(\alpha) \neq 0$ or $\alpha \in \mathbb{N}$. But it is not the case for $\mathcal{R}(\alpha) \in \mathbb{N}$ and $\mathcal{R}(\alpha) \neq 0$.

Lemma 2.4 *Let $\mathcal{R}(\alpha) > 0$, $n = [\mathcal{R}(\alpha)] + 1$ and $y \in C[a, b]$.*

(i) *If $\mathcal{R}(\alpha) \neq 0$ or $\alpha \in \mathbb{N}$, then*

$$_a^C\mathcal{D}^\alpha \left(_a\mathcal{J}^\alpha y \right)(x) = y(x), \qquad ^C\mathcal{D}_b^\alpha \left(\mathcal{J}_b^\alpha y \right)(x) = y(x). \tag{30}$$

(ii) If $\mathcal{R}(\alpha) \in \mathbb{N}$ and $\mathcal{R}(\alpha) \neq 0$ then

$$
{}^{C}_{a}\mathcal{D}^{\alpha}\big({}_{a}\mathcal{J}^{\alpha}y\big)(x) = y(x) - \frac{{}_{a}\mathcal{J}^{\alpha+1-n}y(a)}{\Gamma(n-\alpha)}\left(\log\frac{x}{a}\right)^{n-\alpha}, \tag{31}
$$

$$
{}^{C}\mathcal{D}^{\alpha}_{b}\big(\mathcal{J}^{\alpha}_{b}y\big)(x) = y(x) - \frac{\mathcal{J}^{\alpha+1-n}_{b}y(b)}{\Gamma(n-\alpha)}\left(\log\frac{b}{x}\right)^{n-\alpha}. \tag{32}
$$

Proof From (16), we have

$$
{}^{C}_{a}\mathcal{D}^{\alpha}\big({}_{a}\mathcal{J}^{\alpha}y\big)(x) = {}_{a}\mathcal{D}^{\alpha}\big({}_{a}\mathcal{J}^{\alpha}y\big)(x) - \sum_{k=0}^{n-1}\frac{\delta^{k}({}_{a}\mathcal{J}^{\alpha}y)(a)}{k!}\left(\log\frac{x}{a}\right)^{k}. \tag{33}
$$

From properties 2.28 and 2.27 in [1], we have ${}_{a}\mathcal{D}^{\alpha}({}_{a}\mathcal{J}^{\alpha}y)(x) = y(x)$ and $\delta^{k}({}_{a}\mathcal{J}^{\alpha}y)(x) = {}_{a}\mathcal{J}^{\alpha-k}y(x)$. It is easy to verify that

$$
\big|{}_{a}\mathcal{J}^{\alpha-k}y(x)\big| \leq \frac{\|y\|_{C}}{|\Gamma(\alpha-k)|(\mathcal{R}(\alpha)-k)}\left(\log\frac{x}{a}\right)^{\mathcal{R}(\alpha)-k}, \quad k = 0,1,\dots,n-1, \tag{34}
$$

from which we conclude that ${}_{a}\mathcal{J}^{\alpha-k}y(a) = 0$, and thus the first identity in (30) holds. The second identity is proved analogously.

If $\alpha = m + i\beta$, $m \in \mathbb{N}$, $\beta \neq 0$, then $n = m + 1 \geq 2$ and $\delta^{k}({}_{a}\mathcal{J}^{\alpha}y)(x) = {}_{a}\mathcal{J}^{\alpha-k}y(x)$, $k = 0,1,\dots,m-1$ and similar to (34) we have

$$
\big|{}_{a}\mathcal{J}^{\alpha-k}y(x)\big| \leq \frac{\|y\|_{C}}{|\Gamma(\alpha-k)|(m-k)}\left(\log\frac{x}{a}\right)^{m-k}, \quad k = 0,1,\dots,m-1. \tag{35}
$$

Thus we have ${}_{a}\mathcal{J}^{\alpha-k}y(a) = 0$, $k = 0,1,\dots,n-2$ and hence (31) holds. (32) can be proved similarly. □

Lemma 2.5 *Let $y \in AC^{n}_{\delta}[a,b]$ or $C^{n}_{\delta}[a,b]$ and $\alpha \in \mathbb{C}$. Then*

$$
{}_{a}\mathcal{J}^{\alpha}\big({}^{C}_{a}\mathcal{D}^{\alpha}\big)y(x) = y(x) - \sum_{k=0}^{n-1}\frac{\delta^{k}y(a)}{k!}\left(\log\frac{x}{a}\right)^{k}, \tag{36}
$$

$$
\mathcal{J}^{\alpha}_{b}\big({}^{C}\mathcal{D}^{\alpha}_{b}\big)y(x) = y(x) - \sum_{k=0}^{n-1}\frac{\delta^{k}y(b)}{k!}\left(\log\frac{b}{x}\right)^{k}. \tag{37}
$$

Proof (36) and (37) follow from the identities ${}^{C}_{a}\mathcal{D}^{\alpha}y(x) = {}_{a}\mathcal{J}^{n-\alpha}y(x)$ and ${}^{C}\mathcal{D}^{\alpha}_{b}y(x) = \mathcal{J}^{n-\alpha}_{b}y(x)$ respectively. □

The Caputo modifications of the left and right Hadamard fractional derivatives have the same properties 2.7.16 and 2.7.18. in [1] but they are different from the ones in 2.7.19.

Property 2.6 *Let $\mathcal{R}(\alpha) \geq 0$, $n = [\mathcal{R}(\alpha)] + 1$ and $\mathcal{R}(\beta) > 0$. Then*

$$
{}^{C}_{a}\mathcal{D}^{\alpha}\left(\log\frac{x}{a}\right)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\log\frac{x}{a}\right)^{\beta-\alpha-1}, \quad \mathcal{R}(\beta) > n, \tag{38}
$$

$$
{}^C\mathcal{D}_b^\alpha \left(\log \frac{b}{x}\right)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\log \frac{b}{x}\right)^{\beta-\alpha-1}, \quad \mathcal{R}(\beta) > n, \tag{39}
$$

$$
{}_a^C\mathcal{D}^\alpha \left(\log \frac{x}{a}\right)^k = 0, \qquad {}^C\mathcal{D}_b^\alpha \left(\log \frac{b}{x}\right)^k = 0, \quad k = 0,1,\ldots,n-1. \tag{40}
$$

In particular, one has

$$
{}_a^C\mathcal{D}^\alpha 1 = 0, \qquad {}^C\mathcal{D}_b^\alpha 1 = 0. \tag{41}
$$

Caputo-Hadamard fractional derivatives can also be defined on the positive half axis $\mathbb{R}^+$ by replacing a by 0 in formula (20) and b by ∞ in formula (21) provided that $y(x) \in AC_\delta^n(\mathbb{R}^+)$ (or $C_\delta^n(\mathbb{R}^+)$). Thus one has

$$
{}^C\mathcal{D}_{0^+}^\alpha y(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \left(\log \frac{x}{t}\right)^{n-\alpha-1} \delta^n y(t)\frac{dt}{t}, \tag{42}
$$

$$
{}^C\mathcal{D}_-^\alpha y(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^\infty \left(\log \frac{t}{x}\right)^{n-\alpha-1} \delta^n y(t)\frac{dt}{t}. \tag{43}
$$

The Mellin transforms of the Caputo modifications of the left and right Hadamard fractional derivatives are the same as Mellin transforms of the left and right Hadamard fractional derivatives 2.7.72 and 2.7.74 in [1].

Lemma 2.7 *Let $\mathcal{R}(\alpha) > 0$ and $y(x)$ be such that the Mellin transforms $(\mathcal{M}y)(s)$, $(\mathcal{M}\delta^n y)(s)$ exist for $s \in \mathbb{C}$.*

(i) If $\mathcal{R}(s) < 0$, then

$$
\left(\mathcal{M}{}^C\mathcal{D}_{0^+}^\alpha y\right)(s) = (-s)^\alpha (\mathcal{M}y)(s). \tag{44}
$$

(ii) If $\mathcal{R}(s) > 0$, then

$$
\left(\mathcal{M}{}^C\mathcal{D}_-^\alpha y\right)(s) = s^\alpha (\mathcal{M}y)(s). \tag{45}
$$

Proof (44) follows from Lemma 2.38 and formula 1.4.35 in [1]

$$
\left(\mathcal{M}{}^C\mathcal{D}_{0^+}^\alpha y\right)(s) = \left(\mathcal{M}\mathcal{J}_{0^+}^{n-\alpha}\delta^n y\right)(s) = (-s)^{\alpha-n}\left(\mathcal{M}\delta^n y\right) = (-s)^\alpha (\mathcal{M}y). \tag{46}
$$

(45) can be proved similarly. □

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