Czechoslovak Mathematical Journal

Josef Šlapal

Cardinal arithmetic of general relational systems

Czechoslovak Mathematical Journal, Vol. 43 (1993), No. 1, 125–139

Persistent URL: http://dml.cz/dmlcz/128380

CARDINAL ARITHMETIC OF GENERAL RELATIONAL SYSTEMS

JOSEF ŠLAPAL, Brno

(Received June 24, 1991)

Dedicated to Professor M. Novotný on the occasion of his 70th birthday.

General relations, i.e. the relations whose domains are arbitrary sets, have been investigated in [7]. To complete this investigation, in the present paper we introduce and study three cardinal operations of addition, multiplication and exponentiation for general relational systems that generalize the three Birkhoff's cardinal operations for ordered sets discussed in [1] and [2]. The results attained also generalize those of [3], [4] and [5] where the three operations have been studied for sets with reflexive binary relations, for n-ary relational systems and for general relational systems with the same domains, respectively.

1. Preliminaries

Let F, I be non-empty sets. Then a set of mappings $R \subseteq F^I$ is called a *relation* on F and the ordered pair $\boldsymbol{F} = (F, R)$ is said to be a *relational system*. The set F is called the *carrier of* $\boldsymbol{F}$ and the set I the *domain of* $\boldsymbol{F}$. The relation R of $\boldsymbol{F}$ (i.e. on F) will be sometimes denoted by $\mathscr{R}(\boldsymbol{F})$. Let $\boldsymbol{F}$ and $\boldsymbol{G}$ be relational systems with domains I and J, respectively. Then $\boldsymbol{F}$ and $\boldsymbol{G}$ are said to be *of the same type* if there exists a bijection of I onto J.

Besides the usual conventions, such as the associativity of the cartesian product, we accept the following one: A nonempty set I and the set $\{(x,x) \mid x \in I\}$ called the identity mapping (briefly the identity) of I are considered as the same domains of relational systems. More precisely, if $\boldsymbol{F}$ and $\boldsymbol{G}$ are relational systems with domains I and $\{(x,x) \mid x \in I\}$, respectively, and with the same carrier, and if the following condition holds: $g \in \mathscr{R}(\boldsymbol{G}) \Leftrightarrow$ there exists $f \in \mathscr{R}(\boldsymbol{F})$ with $g(x,x) = f(x)$ for all $x \in I$, then $\boldsymbol{F}$ and $\boldsymbol{G}$ are identified.

1.1. Definition. Let $\boldsymbol{F} = (F, R)$ and $\boldsymbol{G} = (G, S)$ be two relational systems with the same domain I. We say that $\boldsymbol{F}$ is a *subsystem of* $\boldsymbol{G}$ and write $\boldsymbol{F} \subseteq \boldsymbol{G}$ iff $F \subseteq G$ and $R = S \cap F^I$.

1.2. Definition. Let $\boldsymbol{F} = (F, R)$ with domain I and $\boldsymbol{G} = (G, S)$ with domain J be two relational systems of the same type. Let $\alpha: I \longrightarrow J$ be a bijection and let $\varphi: F \longrightarrow G$ be a mapping. If the implication $f \in R \Rightarrow \varphi \circ f \circ \alpha^{-1} \in S$ holds, then φ is called a *homomorphism of* $\boldsymbol{F}$ *into* $\boldsymbol{G}$ *with regard to* α. By $\operatorname{Hom}_\alpha(\boldsymbol{F}, \boldsymbol{G})$ we denote the set of all homomorphisms of $\boldsymbol{F}$ into $\boldsymbol{G}$ with regard to α. A bijective homomorphism φ of $\boldsymbol{F}$ onto $\boldsymbol{G}$ with regard to α such that φ^{-1} is a homomorphism of $\boldsymbol{G}$ onto $\boldsymbol{F}$ with regard to α^{-1} is called an *isomorphism of* $\boldsymbol{F}$ *onto* $\boldsymbol{G}$ *with regard to* α. We write $\boldsymbol{F} \overset{\alpha}{\sim} \boldsymbol{G}$ and say that $\boldsymbol{F}$ and $\boldsymbol{G}$ are *isomorphic with regard to* α if there exists an isomorphism of $\boldsymbol{F}$ onto $\boldsymbol{G}$ with regard to α. If $\boldsymbol{F} \overset{\alpha}{\sim} \boldsymbol{H}$ holds for some subsystem $\boldsymbol{H} \subseteq \boldsymbol{G}$, then we write $\boldsymbol{F} \overset{\alpha}{\prec} \boldsymbol{G}$. If $I = J$ and α is the identity of I, then $\operatorname{Hom}(\boldsymbol{F}, \boldsymbol{G})$ will be written briefly instead of $\operatorname{Hom}_\alpha(\boldsymbol{F}, \boldsymbol{G})$, $\boldsymbol{F} \sim \boldsymbol{G}$ instead of $\boldsymbol{F} \overset{\alpha}{\sim} \boldsymbol{G}$ and $\boldsymbol{F} \prec \boldsymbol{G}$ instead of $\boldsymbol{F} \overset{\alpha}{\prec} \boldsymbol{G}$.

1.3. Example. Consider the teaching process (regarding a certain time table) in a school. Let F, G, I, J be the sets of teachers, subjects, classes and class-rooms, respectively, and let $\operatorname{card} I = \operatorname{card} J$. For $x \in I$ or $x \in J$ we denote by $F(x)$ the set of all teachers that teach the class x or that teach in the class-room x, respectively. Next, for $t \in F$ we denote by $G(t)$ the set of all subjects that are taught by the teacher t. Let $\alpha: I \longrightarrow J$ be a bijection such that the implication $t \in F(x) \Rightarrow t \in F(\alpha(x))$ is valid for each class $x \in I$. (This is fulfilled, for example, if each class $x \in I$ always occupies the same single class-room $\alpha(x)$). Let $R \subseteq F^I$ be the relation defined by $f \in R \Leftrightarrow f(x) \in F(x)$ for each $x \in I$ and let $S \subseteq G^J$ be the relation defined by $g \in S \Leftrightarrow$ for each $y \in J$ there exists $t \in F(y)$ such that $g(y) \in G(t)$ is valid. Let $\varphi: F \longrightarrow G$ be an arbitrary mapping with $\varphi(t) \in G(t)$ for every $t \in F$. Then φ is a homomorphism of (F, R) into (G, S) with regard to α.

1.4. Remark. a) The homomorphism of relational systems with the same domain I with regard to the identity of I coincides with the homomorphism defined in [5]. In particular, if $I = \{1, 2, \ldots, n\}$, then we get the well-known homomorphism of sets with n-ary relations. By the antihomomorphism of sets with n-ary relations we usually understand the homomorphism with regard to the permutation α of $I = \{1, 2, \ldots, n\}$ defined by $\alpha(x) = n - x + 1$ for each $x \in I$.

b) The identity of the carrier of a relational system $\boldsymbol{F}$ is clearly an isomorphism of $\boldsymbol{F}$ onto itself with regard to the identity of the domain of $\boldsymbol{F}$. Further, if φ is a homomorphism of a relational system $\boldsymbol{F}$ into another one, $\boldsymbol{G}$, with regard to α and ψ is a homomorphism of $\boldsymbol{G}$ into a relational system $\boldsymbol{H}$ with regard to β, then $\psi \circ \varphi$ is evidently a homomorphism of $\boldsymbol{F}$ into $\boldsymbol{H}$ with regard to $\beta \circ \alpha$. For relational systems

$\boldsymbol{F}$ and $\boldsymbol{G}$ of the same type, by a morphism from $\boldsymbol{F}$ into $\boldsymbol{G}$ let us understand any homomorphism of $\boldsymbol{F}$ into $\boldsymbol{G}$ with regard to some bijection of the domain of $\boldsymbol{F}$ onto the domain of $\boldsymbol{G}$. Consequently, the class of all relational systems of the same type together with these morphisms forms a category. The presented results attained on the level of the theory of sets are more detailed than those which can be attained on the level of the theory of categories (see [6]).

c) From $\boldsymbol{F} \overset{\alpha}{\prec} \boldsymbol{G}$ and $\boldsymbol{G} \overset{\alpha^{-1}}{\prec} \boldsymbol{F}$ it does not follow that $\boldsymbol{F} \overset{\alpha}{\sim} \boldsymbol{G}$ (not even if α is the identity—see [2]).

Similarly to the papers [3], [4] and [5], the present one is intended as a generalization of Birkhoff's arithmetic of ordered sets ([1], [2]). We shall define and study three cardinal operations of addition, multiplication and exponentiation for relational systems of the same type. For relational systems with the same domain these operations coincide with those investigated in [5] and if, moreover, this domain is finite, then we obtain the direct operations introduced in [4]. For ordered sets we get the cardinal operations discussed in [1] and [2].

2. Cardinal addition

2.1. Definition. Let $\boldsymbol{F} = (F, R)$ with domain I and $\boldsymbol{G} = (G, S)$ with domain J be two relational systems of the same type. Let $\alpha: I \to J$ be a bijection and let $F \cap G = \emptyset$. The *cardinal sum* $\boldsymbol{F} \overset{\alpha}{+} \boldsymbol{G}$ *of* $\boldsymbol{F}$ *and* $\boldsymbol{G}$ *with regard to* α is the relational system $\boldsymbol{H} = (H, T)$ with domain α where $H = F \cup G$ and T is defined as follows: $h \in H^{\alpha}$, $h \in T \Leftrightarrow$ there exists $f \in R$ such that $h(x, y) = f(x)$ for all $(x, y) \in \alpha$ or there exists $g \in S$ such that $h(x, y) = g(y)$ for all $(x, y) \in \alpha$.

If $I = J$ and α is the identity of I, then we write briefly $\boldsymbol{F} + \boldsymbol{G}$ instead of $\boldsymbol{F} \overset{\alpha}{+} \boldsymbol{G}$.

Let $\boldsymbol{F} = (F, R)$ and $\boldsymbol{G} = (F, S)$ be two relational systems with the same domain and the same carrier. Put $\boldsymbol{F} \leqslant \boldsymbol{G}$ iff $R \subseteq S$. Clearly, $\leqslant$ is an order on the set of all relational systems with the same given domain and with the same given carrier.

2.2. Proposition. *Let* $\boldsymbol{F} = (F, R)$ *with domain* I *and* $\boldsymbol{G} = (G, S)$ *with domain* J *be two relational systems of the same type. Let* $\alpha: I \to J$ *be a bijection and let* $F \cap G = \emptyset$. *Let* $\boldsymbol{H} = (H, T) = \boldsymbol{F} \overset{\alpha}{+} \boldsymbol{G}$. *Then* $\boldsymbol{H}$ *is the least element (with respect to* $\leqslant$*) in the set of all relational systems* $\mathbf{L}$ *with the same domain* α *and the same carrier* H *for which the following two conditions are true:*

(1) The identity of F *is a homomorphism of* $\boldsymbol{F}$ *into* $\mathbf{L}$ *with regard to the bijection* $\beta: I \to \alpha$ *defined by* $\beta(x) = (x, \alpha(x))$ *for all* $x \in I$.

(2) The identity of G *is a homomorphism of* $\boldsymbol{G}$ *into* $\mathbf{L}$ *with regard to the bijection* $\gamma: J \to \alpha$ *defined by* $\gamma(y) = (\alpha^{-1}(y), y)$ *for all* $y \in J$.

Proof. By id_F denote the identity of F and by id_G the identity of G. Clearly, $\mathrm{id}_F \in \mathrm{Hom}_\beta(\boldsymbol{F}, \boldsymbol{H})$ and $\mathrm{id}_G \in \mathrm{Hom}_\gamma(\boldsymbol{G}, \boldsymbol{H})$. Let $\boldsymbol{L} = (H, U)$ be a relational system with domain α fulfilling both the conditions (1) and (2). Let $h \in T$ be a mapping. Then (i) there exists $f \in R$ such that $h(x,y) = f(x)$ for all $(x,y) \in \alpha$ or (ii) there exists $g \in S$ such that $h(x,y) = g(y)$ for all $(x,y) \in \alpha$. Let the condition (i) be true. Then $\mathrm{id}_F \circ f \circ \beta^{-1} = f \circ \beta^{-1} \in U$. Since $f(x) = h(x,y) = h(x, \alpha(x)) = h(\beta(x))$ for any $x \in I$, we have $f = h \circ \beta$. Therefore $f \circ \beta^{-1} = h \circ \beta \circ \beta^{-1} = h$. Thus $h \in U$. Similarly we can show that $h \in U$ if the condition (ii) is true. Hence $T \subseteq U$, i.e. $\boldsymbol{H} \leqslant \boldsymbol{L}$. This proves the statement. □

2.3. Lemma. *Let $\boldsymbol{F}_1 = (F_1, R_1)$ with domain I an $\boldsymbol{G}_1 = (G_1, S_1)$ with domain J be relational systems of the same type. Let $\boldsymbol{F}_2 = (F_2, R_2)$ with domain I and $\boldsymbol{G}_2 = (G_2, S_2)$ with domain J be relational systems (of the same type) as well. Let $\alpha : I \to J$ be a bijection. Then*

(1) if $F_1 \cap F_2 = G_1 \cap G_2 = \emptyset$, then $f \in \mathrm{Hom}_\alpha(\boldsymbol{F}_1, \boldsymbol{G}_1)$ and $g \in \mathrm{Hom}_\alpha(\boldsymbol{F}_2, \boldsymbol{G}_2)$ imply $f \cup g \in \mathrm{Hom}_\alpha(\boldsymbol{F}_1 + \boldsymbol{F}_2, \boldsymbol{G}_1 + \boldsymbol{G}_2)$;

(2) if $F_1 \cap G_1 = F_2 \cap G_2 = \emptyset$, then $f \in \mathrm{Hom}(\boldsymbol{F}_1, \boldsymbol{F}_2)$ and $g \in \mathrm{Hom}(\boldsymbol{G}_1, \boldsymbol{G}_2)$ imply $f \cup g \in \mathrm{Hom}(\boldsymbol{F}_1 \overset{\alpha}{+} \boldsymbol{G}_1, \boldsymbol{F}_2 \overset{\alpha}{+} \boldsymbol{G}_2)$.

Proof. (1) Let $F_1 \cap F_2 = G_1 \cap G_2 = \emptyset$ and let $f \in \mathrm{Hom}_\alpha(\boldsymbol{F}_1, \boldsymbol{G}_1)$, $g \in \mathrm{Hom}_\alpha(\boldsymbol{F}_2, \boldsymbol{G}_2)$. Put $h = f \cup g$. Let $p \in \mathscr{R}(\boldsymbol{F}_1 + \boldsymbol{F}_2) = R_1 \cup R_2$. Suppose $p \in R_1$. Then $f \circ p \circ \alpha^{-1} \in S_1$ and since $f \circ p \circ \alpha^{-1} = h \circ p \circ \alpha^{-1}$, we have $g \circ p \circ \alpha^{-1} \in S_1$. Similarly, supposing $p \in R_2$ we get $h \circ p \circ \alpha^{-1} \in S_2$. Hence $p \in R_1 \cup R_2 \Rightarrow h \circ p \circ \alpha^{-1} \in S_1 \cup S_2$. Therefore $h \in \mathrm{Hom}_\alpha(\boldsymbol{F}_1 + \boldsymbol{F}_2, \boldsymbol{G}_1 + \boldsymbol{G}_2)$.

(2) Let $F_1 \cap G_1 = F_2 \cap G_2 = \emptyset$ and let $f \in \mathrm{Hom}(\boldsymbol{F}_1, \boldsymbol{F}_2)$, $g \in \mathrm{Hom}(\boldsymbol{G}_1, \boldsymbol{G}_2)$. Again, put $h = f \cup g$. Let $p \in \mathscr{R}(\boldsymbol{F}_1 \overset{\alpha}{+} \boldsymbol{G}_1)$. Then (i) there exists $q_1 \in R_1$ such that $p(x,y) = q_1(x)$ for all $(x,y) \in \alpha$ or (ii) there exists $q_2 \in S_1$ such that $p(x,y) = q_2(y)$ for all $(x,y) \in \alpha$. Let the condition (i) be true. Then $f \circ q_1 \in R_2$. Put $q(x,y) = f(q_1(x))$ for all $(x,y) \in \alpha$. We have $h \circ p = q$ and $q \in \mathscr{R}(\boldsymbol{F}_2 \overset{\alpha}{+} \boldsymbol{G}_2)$. Similarly, if the condition (ii) is true, then $g \circ q_2 \in S_2$ and putting $q(x,y) = g(q_2(y))$ for all $(x,y) \in \alpha$ we get $h \circ p = q$ and $q \in \mathscr{R}(\boldsymbol{F}_2 \overset{\alpha}{+} \boldsymbol{G}_2)$. Consequently, $h \in \mathrm{Hom}(\boldsymbol{F}_1 \overset{\alpha}{+} \boldsymbol{G}_1, \boldsymbol{F}_2 \overset{\alpha}{+} \boldsymbol{G}_2)$. The proof is complete. □

By virtue of the lemma we obtain

2.4. Theorem. *Let $\boldsymbol{F}_1 = (F_1, R_1)$ with domain I and $\boldsymbol{G}_1 = (G_1, S_1)$ with domain J be relational systems of the same type. Let $\boldsymbol{F}_2 = (F_2, R_2)$ with domain I and $\boldsymbol{G}_2 = (G_2, S_2)$ with domain J be relational systems (of the same type) as well. Let $\alpha : I \to J$ be a bijection. Then*

(1) if $F_1 \cap F_2 = G_1 \cap G_2 = \emptyset$, then $\boldsymbol{F}_1 \overset{\alpha}{\sim} \boldsymbol{G}_1$ and $\boldsymbol{F}_2 \overset{\alpha}{\sim} \boldsymbol{G}_2$ imply $\boldsymbol{F}_1 + \boldsymbol{F}_2 \overset{\alpha}{\sim} \boldsymbol{G}_1 + \boldsymbol{G}_2$;

(2) if $F_1 \cap G_1 = F_2 \cap G_2 = \emptyset$, then $\boldsymbol{F}_1 \sim \boldsymbol{F}_2$ and $\boldsymbol{G}_1 \sim \boldsymbol{G}_2$ imply $\boldsymbol{F}_1 \overset{\alpha}{+} \boldsymbol{G}_1 \sim \boldsymbol{F}_2 \overset{\alpha}{+} \boldsymbol{G}_2$.

2.5. Remark. If the assumptions of Theorem 2.4. are fulfilled, then it can be easily seen that the implication $\boldsymbol{F}_1 \subseteq \boldsymbol{F}_2$ and $\boldsymbol{G}_1 \subseteq \boldsymbol{G}_2 \Rightarrow \boldsymbol{F}_1 \overset{\alpha}{+} \boldsymbol{G}_1 \subseteq \boldsymbol{F}_2 \overset{\alpha}{+} \boldsymbol{G}_2$ is true whenever $F_2 \cap G_2 = \emptyset$. Consequently, in Theorem 2.4 the symbols $\overset{\alpha}{\sim}$ and $\sim$ can be replaced by the symbols $\overset{\alpha}{\prec}$ and $\prec$, respectively.

Clearly, we have

2.6. Theorem. *Let $\boldsymbol{F} = (F, R)$ with domain I, $\boldsymbol{G} = (G, S)$ with domain J and $\boldsymbol{H} = (H, T)$ with domain K be relational systems of the same type. Let $\alpha: I \to J$ and $\beta: J \to K$ be bijections and let $F \cap G = G \cap H = F \cap H = \emptyset$. Let $\gamma: \alpha \to K$ and $\delta: I \to \beta$ be the bijections defined by $\gamma(x, y) = \beta(y)$ for all $(x, y) \in \alpha$ and $\delta(x) = (\alpha(x), \beta(\alpha(x)))$ for all $x \in I$. Finally, let $\theta: \alpha \to \alpha^{-1}$ be the bijection defined by $\theta(x, y) = (y, x)$ for all $(x, y) \in \alpha$. Then*

$$(\boldsymbol{F} \overset{\alpha}{+} \boldsymbol{G}) \overset{\gamma}{+} \boldsymbol{H} = \boldsymbol{F} \overset{\delta}{+} (\boldsymbol{G} \overset{\beta}{+} \boldsymbol{H}), \tag{1}$$

$$\boldsymbol{F} \overset{\alpha}{+} \boldsymbol{G} \overset{\theta}{\sim} \boldsymbol{G} \overset{\alpha^{-1}}{+} \boldsymbol{F}. \tag{2}$$

2.7. Remark. By virtue of (1) of the previous theorem we can write both the sum $(\boldsymbol{F} \overset{\alpha}{+} \boldsymbol{G}) \overset{\gamma}{+} \boldsymbol{H}$ and $\boldsymbol{F} \overset{\delta}{+} (\boldsymbol{G} \overset{\beta}{+} \boldsymbol{H})$ by the same symbol $\boldsymbol{F} \overset{\alpha}{+} \boldsymbol{G} \overset{\beta}{+} \boldsymbol{H}$. More generally, let n be a positive integer and $\{\boldsymbol{F}_i \mid i = 0, 1, \ldots, n\}$ a family of relational systems of the same type and with pairwise disjoint carriers. Let α_i be a bijection of the domain of $\boldsymbol{F}_{i-1}$ onto the domain of $\boldsymbol{F}_i$ for every $i \in \{1, \ldots, n\}$. Then we can define the sum $\boldsymbol{F}_0 \overset{\alpha_1}{+} \boldsymbol{F}_1 \overset{\alpha_2}{+} \ldots \overset{\alpha_n}{+} \boldsymbol{F}_n$ as any one obtained by inserting parentheses and replacing the bijections $\alpha_1, \ldots, \alpha_n$ by the corresponding ones.

3. Cardinal multiplication

3.1. Definition. Let $\boldsymbol{F} = (F, R)$ with domain I and $\boldsymbol{G} = (G, S)$ with domain J be two relational systems of the same type. Let $\alpha: I \to J$ be a bijection. The *cardinal product* $\boldsymbol{F} \overset{\alpha}{\cdot} \boldsymbol{G}$ *of* $\boldsymbol{F}$ *and* $\boldsymbol{G}$ *with regard to* α is the relational system $\boldsymbol{H} = (H, T)$ with domain α where $H = F \times G$ and $T \subseteq H^\alpha$ is defined as follows: $h \in H^\alpha$, $h \in T \Leftrightarrow$ there exist $f \in R$ and $g \in S$ such that $h(x, y) = (f(x), g(y))$ for all $(x, y) \in \alpha$.

If $I = J$ and α is the identity of I, then we write briefly $\boldsymbol{F} \cdot \boldsymbol{G}$ instead of $\boldsymbol{F} \overset{\alpha}{\cdot} \boldsymbol{G}$.

3.2. Proposition. *Let $\boldsymbol{F} = (F, R)$ with domain I and $\boldsymbol{G} = (G, S)$ with domain J be two relational systems of the same type. Let $\alpha: I \to J$ be a bijection and let*

$\boldsymbol{H} = (H,T) = \boldsymbol{F} \overset{\alpha}{\cdot} \boldsymbol{G}$. *Then* $\boldsymbol{H}$ *is the greatest element (with respect to* $\leqslant$*) in the set of all relational systems* $\boldsymbol{L}$ *with the same domain* α *and the same carrier* H *for which the following two conditions are true:*

(1) *The projection of* H *onto* F *is a homomorphism of* $\boldsymbol{L}$ *onto* $\boldsymbol{F}$ *with regard to the bijection* $\beta\colon \alpha \to I$ *defined by* $\beta(x,y) = x$ *for all* $(x,y) \in \alpha$.

(2) *The projection of* H *onto* G *is a homomorphism of* $\boldsymbol{L}$ *onto* $\boldsymbol{G}$ *with regard to the bijection* $\gamma\colon \alpha \to J$ *defined by* $\gamma(x,y) = y$ *for all* $(x,y) \in \alpha$.

Proof. By pr_F we denote the projection of H onto F and by pr_G the projection of H onto G. Clearly, $\mathrm{pr}_F \in \mathrm{Hom}_\beta(\boldsymbol{H},\boldsymbol{F})$ and $\mathrm{pr}_G \in \mathrm{Hom}_\gamma(\boldsymbol{H},\boldsymbol{G})$. Let $\boldsymbol{L} = (H,U)$ be a relational system with domain α fulfilling both the conditions (1) and (2). Let $h \in U$ be a mapping. Then putting $f = \mathrm{pr}_F \circ h \circ \beta^{-1}$ and $g = \mathrm{pr}_G \circ h \circ \gamma^{-1}$ we get $f \in R$ and $g \in S$. We have $\mathrm{pr}_F\,(h(x,y)) = f(\beta(x,y)) = f(x)$ and $\mathrm{pr}_G\,(h(x,y)) = g(\gamma(x,y)) = g(y)$ for all $(x,y) \in \alpha$. Thus $h(x,y) = (f(x),g(y))$ for all $(x,y) \in \alpha$. This yields $h \in T$. Hence $U \subseteq T$, i.e. $\boldsymbol{L} \leqslant \boldsymbol{H}$. The proof is complete. □

3.3. Lemma. *Let* $\boldsymbol{F}_1 = (F_1,R_1)$ *with domain* I *and* $\boldsymbol{G}_1 = (G_1,S_1)$ *with domain* J *be relational systems of the same type. Let* $\boldsymbol{F}_2 = (F_2,R_2)$ *with domain* I *and* $\boldsymbol{G}_2 = (G_2,S_2)$ *with domain* J *be relational systems (of the same type) as well. Let* $\alpha: I \to J$ *be a bijection. Then*

(1) *if* $f \in \mathrm{Hom}_\alpha(\boldsymbol{F}_1,\boldsymbol{G}_1)$ *and* $g \in \mathrm{Hom}_\alpha(\boldsymbol{F}_2,\boldsymbol{G}_2)$, *then* $f * g \in \mathrm{Hom}_\alpha(\boldsymbol{F}_1 \cdot \boldsymbol{F}_2, \boldsymbol{G}_1 \cdot \boldsymbol{G}_2)$;

(2) *if* $f \in \mathrm{Hom}(\boldsymbol{F}_1,\boldsymbol{F}_2)$ *and* $g \in \mathrm{Hom}(\boldsymbol{G}_1,\boldsymbol{G}_2)$, *then* $f * g \in \mathrm{Hom}(\boldsymbol{F}_1 \overset{\alpha}{\cdot} \boldsymbol{G}_1, \boldsymbol{F}_2 \overset{\alpha}{\cdot} \boldsymbol{G}_2)$. *Here,* $f * g$ *means the direct product of the mappings* f *and* g, *i.e.* $f * g(x_1,x_2) = (f(x_1),g(x_2))$.

Proof. Let $f \in \mathrm{Hom}_\alpha(\boldsymbol{F}_1,\boldsymbol{G}_1)$ and $g \in \mathrm{Hom}_\alpha(\boldsymbol{F}_2,\boldsymbol{G}_2)$ and put $h = f * g$. Let $p \in \mathscr{R}(\boldsymbol{F}_1 \cdot \boldsymbol{F}_2)$. Then there exist $q_1 \in R_1$ and $q_2 \in R_2$ such that $p(x) = (q_1(x),q_2(x))$ for all $x \in I$. Further, $f \circ q_1 \circ \alpha^{-1} \in S_1$ and $g \circ q_2 \circ \alpha^{-1} \in S_2$. Put $q(y) = (f(q_1(\alpha^{-1}(y))),g(q_2(\alpha^{-1}(y))))$ for all $y \in J$. Then $q \in \mathscr{R}(\boldsymbol{G}_1 \cdot \boldsymbol{G}_2)$ and for any $y \in J$ we have $h(p(\alpha^{-1}(y))) = h(q_1(\alpha^{-1}(y)),q_2(\alpha^{-1}(y))) = (f(q_1(\alpha^{-1}(y))),g(q_2(\alpha^{-1}(y)))) = q(y)$. Thus $h \circ p \circ \alpha^{-1} = q$ and hence $h \circ p \circ \alpha^{-1} \in \mathscr{R}(\boldsymbol{G}_1 \cdot \boldsymbol{G}_2)$. Consequently, $h \in \mathrm{Hom}(\boldsymbol{F}_1 \cdot \boldsymbol{F}_2, \boldsymbol{G}_1 \cdot \boldsymbol{G}_2)$.

(2) Let $f \in \mathrm{Hom}(\boldsymbol{F}_1,\boldsymbol{F}_2)$, $g \in \mathrm{Hom}(\boldsymbol{G}_1,\boldsymbol{G}_2)$ and put $h = f * g$. Let $p \in \mathscr{R}(\boldsymbol{F}_1 \overset{\alpha}{\cdot} \boldsymbol{G}_1)$. Then there exist $q_1 \in R_1$ and $q_2 \in S_1$ such that $p(x,y) = (q_1(x),q_2(y))$ for all $(x,y) \in \alpha$. Further, $f \circ q_1 \in R_2$ and $g \circ q_2 \in S_2$. Put $q(x,y) = (f(q_1(x)),g(q_2(y)))$ for all $(x,y) \in \alpha$. Then $q \in \mathscr{R}(\boldsymbol{F}_2 \overset{\alpha}{\cdot} \boldsymbol{G}_2)$ and for every $(x,y) \in \alpha$ we have $h(p(x,y)) = h(q_1(x),q_2(y)) = (f(q_1(x)),g(q_2(y))) = q(x,y)$. Thus $h \circ p = q$ and hence $h \circ p \in \mathscr{R}(\boldsymbol{F}_2 \overset{\alpha}{\cdot} \boldsymbol{G}_2)$. Therefore $h \in \mathrm{Hom}(\boldsymbol{F}_1 \overset{\alpha}{\cdot} \boldsymbol{G}_1, \boldsymbol{F}_2 \overset{\alpha}{\cdot} \boldsymbol{G}_2)$. The statement is proved. □

As a consequence of the lemma we get

3.4. Theorem. *Let $\boldsymbol{F}_1$ with domain I and $\boldsymbol{G}_1$ with domain J be relational systems of the same type. Let also $\boldsymbol{F}_2$ with domain I and $\boldsymbol{G}_2$ with domain J be relational systems (of the same type). Let $\alpha\colon I \to J$ be a bijection. Then*
(1) *if $\boldsymbol{F}_1 \overset{\alpha}{\sim} \boldsymbol{G}_1$ and $\boldsymbol{F}_2 \overset{\alpha}{\sim} \boldsymbol{G}_2$, then $\boldsymbol{F}_1 \cdot \boldsymbol{F}_2 \overset{\alpha}{\sim} \boldsymbol{G}_1 \cdot \boldsymbol{G}_2$;*
(2) *if $\boldsymbol{F}_1 \sim \boldsymbol{F}_2$ and $\boldsymbol{G}_1 \sim \boldsymbol{G}_2$, then $\boldsymbol{F}_1 \overset{\alpha}{\cdot} \boldsymbol{G}_1 \sim \boldsymbol{F}_2 \overset{\alpha}{\cdot} \boldsymbol{G}_2$.*

3.5. Remark. The reader can easily verify that if the assumptions of Theorem 3.4 are fulfilled, then the implication $\boldsymbol{F}_1 \subseteq \boldsymbol{F}_2$ and $\boldsymbol{G}_1 \subseteq \boldsymbol{G}_2 \Rightarrow \boldsymbol{F}_1 \overset{\alpha}{\cdot} \boldsymbol{G}_1 \subseteq \boldsymbol{F}_2 \overset{\alpha}{\cdot} \boldsymbol{G}_2$ is true. Consequently, in Theorem 3.4 we can replace the symbols $\overset{\alpha}{\sim}$ and $\sim$ by the symbols $\overset{\alpha}{\prec}$ and $\prec$, respectively.

The following two statements are evident:

3.6. Theorem. *Let $\boldsymbol{F}$ with domain I and $\boldsymbol{G} = (G, S)$ with domain J be relational systems of the same type. Let G be a singleton and $S \neq \emptyset$. Let $\alpha\colon I \to J$ be a bijection and let $\beta\colon \alpha \to I$ be the bijection defined by $\beta(x,y) = x$ whenever $(x,y) \in \alpha$. Then*

$$\boldsymbol{F} \overset{\alpha}{\cdot} \boldsymbol{G} \overset{\beta}{\sim} \boldsymbol{F}.$$

3.7. Theorem. *Let $\boldsymbol{F} = (F, R)$ with domain I, $\boldsymbol{G} = (G, S)$ with domain J and $\boldsymbol{H} = (H, T)$ with domain K be relational systems of the same type. Let $\alpha\colon I \to J$ and $\beta\colon J \to K$ be bijections. Let $\gamma\colon \alpha \to K$ and $\delta\colon I \to \beta$ be the bijections defined by $\gamma(x,y) = \beta(y)$ for all $(x,y) \in \alpha$ and $\delta(x) = \big(\alpha(x), \beta(\alpha(x))\big)$ for all $x \in I$. Let $\theta\colon \alpha \to \alpha^{-1}$ be the bijection defined by $\theta(x,y) = (y,x)$ for all $(x,y) \in \alpha$. Then*

$$(1) \qquad (\boldsymbol{F} \overset{\alpha}{\cdot} \boldsymbol{G}) \overset{\gamma}{\cdot} \boldsymbol{H} = \boldsymbol{F} \overset{\delta}{\cdot} (\boldsymbol{G} \overset{\beta}{\cdot} \boldsymbol{H}),$$

$$(2) \qquad \boldsymbol{F} \overset{\alpha}{\cdot} \boldsymbol{G} \overset{\theta}{\sim} \boldsymbol{G} \overset{\alpha^{-1}}{\cdot} \boldsymbol{F}.$$

3.8. Remark. By virtue of (1) of the previous theorem, an analogue of Remark 2.7 is valid for cardinal multiplication of relational systems (of course, now the assumption of pairwise disjoin carriers of the systems $\boldsymbol{F}_i$ ($i = 0, 1, \ldots, n$) can be omitted).

3.9. Theorem. *Let the assumptions of Theorem 3.7 be fulfilled. Moreover, let $\lambda\colon \beta \circ \alpha \to \beta$ and $\mu\colon \alpha \to \beta \circ \alpha$ be the bijections defined by $\lambda(x,z) = \big(\alpha(x), z\big)$ for all $(x,z) \in \beta \circ \alpha$ and $\mu(x,y) = \big(x, \beta(y)\big)$ for all $(x,y) \in \alpha$. Further, let $\varrho\colon \gamma \to \alpha$ and*

$\sigma : \delta \to \mu$ be the bijections defined by $\varrho(x, y, z) = (x, z, y, z)$ for all $(x, y, z) \in \gamma$ and $\sigma(x, y, z) = (x, y, x, z)$ for all $(x, y, z) \in \delta$. Then

(1) *if $F \cap G = \emptyset$, then $(\boldsymbol{F} \overset{\alpha}{+} \boldsymbol{G}) \overset{\gamma}{\cdot} \boldsymbol{H} \overset{\varrho}{\simeq} (\boldsymbol{F} \overset{\beta\circ\alpha}{\cdot} \boldsymbol{H}) \overset{\lambda}{+} (\boldsymbol{G} \overset{\beta}{\cdot} \boldsymbol{H})$;*

(2) *if $G \cap H = \emptyset$, then $\boldsymbol{F} \overset{\delta}{\cdot} (\boldsymbol{G} \overset{\beta}{+} \boldsymbol{H}) \overset{\sigma}{\simeq} (\boldsymbol{F} \overset{\alpha}{\cdot} \boldsymbol{G}) \overset{\mu}{+} (\boldsymbol{F} \overset{\beta\circ\alpha}{\cdot} \boldsymbol{H})$.*

Proof. (1) We shall prove that the identity of $(F \cup G) \times H$ is an isomorphism of $(\boldsymbol{F} \overset{\alpha}{+} \boldsymbol{G}) \overset{\gamma}{\cdot} \boldsymbol{H}$ onto $(\boldsymbol{F} \overset{\beta\circ\alpha}{\cdot} \boldsymbol{H}) \overset{\lambda}{+} (\boldsymbol{G} \overset{\beta}{\cdot} \boldsymbol{H})$ with regard to ϱ. To this end, let $h \in \mathscr{R}((\boldsymbol{F} \overset{\alpha}{+} \boldsymbol{G}) \overset{\gamma}{\cdot} \boldsymbol{H})$ be a mapping. Then there exist $f \in \mathscr{R}(\boldsymbol{F} \overset{\alpha}{+} \boldsymbol{G})$ and $g \in T$ such that $h(x, y, z) = \big(f(x, y), g(z)\big)$ for all $(x, y, z) \in \gamma$. Next, (i) there exists $f_1 \in R$ such that $f(x, y) = f_1(x)$ for all $(x, y) \in \alpha$ or (ii) there exists $f_2 \in S$ such that $f(x, y) = f_2(y)$ for all $(x, y) \in \alpha$. Let the condition (i) be fulfilled. Then for any $(x, z, y, z) \in \lambda$ we have $h\big(\varrho^{-1}(x, z, y, z)\big) = h(x, y, z) = \big(f(x, y), g(z)\big) = \big(f_1(x), g(z)\big)$. Put $q_1(x, z) = \big(f_1(x), g(z)\big)$ for all $(x, z) \in \beta \circ \alpha$. Then $q_1 \in \mathscr{R}(\boldsymbol{F} \overset{\beta\circ\alpha}{\cdot} \boldsymbol{H})$ and $h\big(\varrho^{-1}(x, z, y, z)\big) = q_1(x, z)$ for all $(x, z, y, z) \in \lambda$. Therefore $h \circ \varrho^{-1} \in \mathscr{R}\big((\boldsymbol{F} \overset{\beta\circ\alpha}{\cdot} \boldsymbol{H}) \overset{\lambda}{+} (\boldsymbol{G} \overset{\beta}{\cdot} \boldsymbol{H})\big)$. Similarly, if the condition (ii) is fulfilled, then putting $q_2(y, z) = \big(f_2(y), g(z)\big)$ for any $(y, z) \in \beta$ we get $q_2 \in \mathscr{R}(\boldsymbol{G} \overset{\beta}{\cdot} \boldsymbol{H})$ and $h\big(\varrho^{-1}(x, z, y, z)\big) = q_2(y, z)$ for all $(x, z, y, z) \in \lambda$. Again, $h \circ \varrho^{-1} \in \mathscr{R}\big((\boldsymbol{F} \overset{\beta\circ\alpha}{\cdot} \boldsymbol{H}) \overset{\lambda}{+} (\boldsymbol{G} \overset{\beta}{\cdot} \boldsymbol{H})\big)$. Conversely, having $h \in \big((F \cup G) \times H\big)^{\gamma}$ with $h \circ \varrho^{-1} \in \mathscr{R}\big((\boldsymbol{F} \overset{\beta\circ\alpha}{\cdot} \boldsymbol{H}) \overset{\lambda}{+} (\boldsymbol{G} \overset{\beta}{\cdot} \boldsymbol{H})\big)$, reversing the considerations we can easily show that $h \in \mathscr{R}\big((\boldsymbol{F} \overset{\alpha}{+} \boldsymbol{G}) \overset{\gamma}{\cdot} \boldsymbol{H}\big)$. The assertion (1) is proved. As for (2), the proof is similar. □

4. Cardinal exponentiation

4.1. Definition. Let $\boldsymbol{F} = (F, R)$ with domain I and $\boldsymbol{G} = (G, S)$ with domain J be relational systems of the same type. Let $\alpha : I \to J$ be a bijection. The *cardinal power $\boldsymbol{F} \overset{\alpha}{\triangle} \boldsymbol{G}$ of $\boldsymbol{F}$ and $\boldsymbol{G}$ with regard to α* is the relational system $\boldsymbol{H} = (H, T)$ with domain α where $H = \mathrm{Hom}_{\alpha^{-1}}(\boldsymbol{G}, \boldsymbol{F})$ and $T \subseteq H^{\alpha}$ is defined as follows: $h \in H^{\alpha}$, $h \in T \Leftrightarrow {}^{t}h \in R$ for all $t \in G$.

Here, for any $t \in G$ and $h \in H^{\alpha}$, ${}^{t}h$ is the mapping ${}^{t}h : I \to F$ defined by ${}^{t}h(x) = h\big(x, \alpha(x)\big)(t)$ whenever $x \in I$. (We should write more precisely ${}^{t}h_{\alpha}$ instead of ${}^{t}h$. Since it will be always clear which bijection α is considered, we will omit the index α.)

If $I = J$ and α is the identity of I, then we write $\boldsymbol{F}^{\mathbf{G}}$ instead of $\boldsymbol{F} \overset{\alpha}{\triangle} \boldsymbol{G}$.

4.2. Theorem. *Let $\boldsymbol{F}_1 = (F_1, R_1)$ with domain I and $\boldsymbol{G}_1 = (G_1, S_1)$ with domain J be relational systems of the same type. Let also $\boldsymbol{F}_2 = (F_2, R_2)$ with domain I and*

$G_2 = (G_2, S_2)$ *with domain* J *be relational systems (of the same type). Let* $\alpha : I \to J$ *be a bijection. Then*

(1) *if* $F_1 \overset{\alpha}{\sim} G_1$ *and* $F_2 \overset{\alpha}{\sim} G_2$, *then* $F_1^{\mathbf{F}_2} \overset{\alpha}{\sim} G_1^{\mathbf{G}_2}$;

(2) *if* $F_1 \sim F_2$ *and* $G_1 \sim G_2$, *then* $F_1 \overset{\alpha}{\triangle} G_1 \sim F_2 \overset{\alpha}{\triangle} G_2$.

Proof. (1) Let $f_1 : F_1 \to G_1$ be an isomorphism of F_1 onto G_1 with regard to α and let $f_2 : F_2 \to G_2$ be an isomorphism of F_2 onto G_2 with regard to α. For any $f \in \mathrm{Hom}(F_2, F_1)$ put $\varphi(f) = f_1 \circ f \circ f_2^{-1}$. We shall prove that φ is a bijection of $\mathrm{Hom}(F_2, F_1)$ onto $\mathrm{Hom}(G_2, G_1)$. Clearly, φ is injective. Let $f \in \mathrm{Hom}(F_2, F_1)$ and let $g \in S_2$ be a mapping. Then $f_2^{-1} \circ g \circ \alpha \in R_2$. Hence $f \circ f_2^{-1} \circ g \circ \alpha \in R_1$ and thus $f_1 \circ f \circ f_2^{-1} \circ g \circ \alpha \circ \alpha^{-1} = \varphi(f) \circ g \in S_1$. We have proved the implication $f \in \mathrm{Hom}(F_2, F_1) \Rightarrow \varphi(f) \in \mathrm{Hom}(G_2, G_1)$. Similarly we can prove that $f \in \mathrm{Hom}(G_2, G_1) \Rightarrow f_1^{-1} \circ f \circ f_2 = \varphi^{-1}(f) \in \mathrm{Hom}(F_2, F_1)$. Therefore φ is a bijection of $\mathrm{Hom}(F_2, F_1)$ onto $\mathrm{Hom}(G_2, G_1)$. Let $h \in \mathscr{R}(F_1^{\mathbf{F}_2})$. Then ${}^t h \in R_1$ for every $t \in F_2$. Thus, for any $t \in F_2$ we have $f_1 \circ {}^t h \circ \alpha^{-1} \in S_1$. Let $u \in G_2$ be an element. Then ${}^u(\varphi \circ h \circ \alpha^{-1})(y) = \varphi(h(\alpha^{-1}(y)))(u) = f_1(h(\alpha^{-1}(y))(f_2^{-1}(y))) = f_1({}^{f_2^{-1}(u)}f(\alpha^{-1}(y)))$ holds for every $y \in J$. Thus ${}^u(\varphi \circ h \circ \alpha^{-1}) \in S_1$. Consequently, $\varphi \circ h \circ \alpha^{-1} \in \mathscr{R}(G_1^{\mathbf{G}_2})$. Therefore φ is a homomorphism of $F_1^{\mathbf{F}_2}$ onto $G_1^{\mathbf{G}_2}$ with regard to α. Now, reversing the considerations we can show that $h \in \mathscr{R}(F_1^{\mathbf{F}_2})$ whenever $h \in (\mathrm{Hom}(F_2, F_1))^I$ and $\varphi \circ h \circ \alpha^{-1} \in \mathscr{R}(G_1^{\mathbf{G}_2})$. Therefore φ is an isomorphism of $F_1^{\mathbf{F}_2}$ onto $G_1^{\mathbf{G}_2}$ with regard to α. The proof of (1) is complete. The assertion (2) can be proved similarly. □

4.3. Remark. It can be easily shown that if the assumptions of Theorem 4.2 are fulfilled, then the implication $F_1 \subseteq F_2$ and $G_1 = G_2 \Rightarrow F_1 \overset{\alpha}{\triangle} G_1 \subseteq F_2 \overset{\alpha}{\triangle} G_2$ is true. Consequently, in Theorem 4.2 the assertions (1) and (2) can be replaced by the following ones:

(1) If $F_1 \overset{\alpha}{\prec} G_1$ and $F_2 \overset{\alpha}{\sim} G_2$, then $F_1^{\mathbf{F}_2} \overset{\alpha}{\prec} G_1^{\mathbf{G}_2}$.

(2) If $F_1 \prec F_2$ and $G_1 \sim G_2$, then $F_1 \overset{\alpha}{\triangle} G_1 \prec F_2 \overset{\alpha}{\triangle} G_2$.

The following result is evident.

4.4 Theorem. *Let* F *with domain* I *and* $G = (G, S)$ *with domain* J *be relational systems of the same type. Let* G *be a singleton and* $S \neq \emptyset$. *Let* $\alpha : I \to J$ *be a bijection. Let* $\beta : \alpha \to I$ *and* $\gamma : \alpha^{-1} \to J$ *be the bijections defined by* $\beta(x, y) = x$ *and* $\gamma(y, x) = y$ *for all* $(x, y) \in \alpha$. *Then*

(1)
$$F \overset{\alpha}{\triangle} G \overset{\beta}{\sim} F,$$

(2)
$$G \overset{\alpha^{-1}}{\triangle} F \overset{\gamma}{\sim} G.$$

4.5. Theorem. *Let $\boldsymbol{F} = (F, R)$ with domain I, $\boldsymbol{G} = (G, S)$ with domain J and $\boldsymbol{H} = (H, T)$ with domain K be relational systems of the same type. Let $\alpha\colon I \to J$ and $\beta\colon J \to K$ be bijections. Let $\gamma\colon \alpha \to K$ and $\delta\colon \beta\circ\alpha \to \beta$ be the bijections defined by $\gamma(x, y) = \beta(y)$ for all $(x, y) \in \alpha$ and $\delta(x, z) = (\alpha(x), z)$ for all $(x, z) \in \beta\circ\alpha$. Finally, let $\lambda\colon \gamma \to \delta$ be the bijection defined by $\lambda(x, y, z) = (x, z, y, z)$ for all $(x, y, z) \in \gamma$. Then*

$$(\boldsymbol{F} \overset{\alpha}{\cdot} \boldsymbol{G}) \overset{\gamma}{\triangle} \boldsymbol{H} \overset{\lambda}{\sim} (\boldsymbol{F} \overset{\beta\circ\alpha}{\triangle} \boldsymbol{H}) \overset{\delta}{\cdot} (\boldsymbol{G} \overset{\beta}{\triangle} \boldsymbol{H}).$$

Proof. Let $\mathrm{pr}_F : F \times G \to F$ and $\mathrm{pr}_G : F \times G \to G$ be the projections. For any $h \in \mathrm{Hom}_{\gamma^{-1}}(\boldsymbol{H}, \boldsymbol{F} \overset{\alpha}{\cdot} \boldsymbol{G})$ put $h_F = \mathrm{pr}_F \circ h$ and $h_G = \mathrm{pr}_G \circ h$. Clearly, $\mathrm{pr}_F \in \mathrm{Hom}_\varrho(\boldsymbol{F} \overset{\alpha}{\cdot} \boldsymbol{G}, \boldsymbol{F})$ and $\mathrm{pr}_G \in \mathrm{Hom}_\sigma(\boldsymbol{F} \overset{\alpha}{\cdot} \boldsymbol{G}, \boldsymbol{G})$ where $\varrho\colon \alpha \to I$ and $\sigma\colon \alpha \to J$ are the bijections defined by $\varrho(x, y) = x$ and $\sigma(x, y) = y$ whenever $(x, y) \in \alpha$. Since $\varrho\circ\gamma^{-1} = (\beta\circ\alpha)^{-1}$ and $\sigma\circ\gamma^{-1} = \beta^{-1}$, by 1.4.b) we have $h_F \in \mathrm{Hom}_{(\beta\circ\alpha)^{-1}}(\boldsymbol{H}, \boldsymbol{F})$ and $h_G \in \mathrm{Hom}_{\beta^{-1}}(\boldsymbol{H}, \boldsymbol{G})$. Further, let $h_1 \in \mathrm{Hom}_{(\beta\circ\alpha)^{-1}}(\boldsymbol{H}, \boldsymbol{F})$ and $h_2 \in \mathrm{Hom}_{\beta^{-1}}(\boldsymbol{H}, \boldsymbol{G})$ and put $h(t) = \big(h_1(t), h_2(t)\big)$ for all $t \in H$. Let $f \in T$. Then $h\big(f\big(\gamma(x, y)\big)\big) = \Big(h_1\big(f\big(\gamma(x, y)\big)\big), h_2\big(f\big(\gamma(x, y)\big)\big)\Big) = \Big(h_1\Big(f\big(\beta\big(\alpha(x)\big)\big)\Big), h_2\big(f\big(\beta(y)\big)\big)\Big)$ for all $(x, y) \in \alpha$. Since $h_1 \circ f \circ \beta \circ \alpha \in R$ and $h_2 \circ f \circ \beta \in S_1$ we have $h \circ f \circ \gamma \in \mathscr{R}(\boldsymbol{F} \overset{\alpha}{\cdot} \boldsymbol{G})$. Therefore $h \in \mathrm{Hom}_{\gamma^{-1}}(\boldsymbol{H}, \boldsymbol{F} \overset{\alpha}{\cdot} \boldsymbol{G})$ and clearly $h_1 = h_F$, $h_2 = h_G$. Now, let $\varphi\colon \mathrm{Hom}_{\gamma^{-1}}(\boldsymbol{H}, \boldsymbol{F} \overset{\alpha}{\cdot} \boldsymbol{G}) \to \mathrm{Hom}_{(\beta\circ\alpha)^{-1}}(\boldsymbol{H}, \boldsymbol{F}) \times \mathrm{Hom}_{\beta^{-1}}(\boldsymbol{H}, \boldsymbol{G})$ be the mapping defined by $\varphi(h) = (h_F, h_G)$ whenever $h \in \mathrm{Hom}_{\gamma^{-1}}(\boldsymbol{H}, \boldsymbol{F} \overset{\alpha}{\cdot} \boldsymbol{G})$. We have proved that φ is surjective. But φ is clearly injective and hence it is a bijection. Let $g \in \mathscr{R}((\boldsymbol{F} \overset{\alpha}{\cdot} \boldsymbol{G}) \overset{\gamma}{\triangle} \boldsymbol{H})$. Then ${}^t g \in \mathscr{R}(\boldsymbol{F} \overset{\alpha}{\cdot} \boldsymbol{G})$ for all $t \in H$. Thus, there exist $p \in R$ and $q \in S$ such that ${}^t g(x, y) = \big(p(x), q(y)\big)$ for all $(x, y) \in \alpha$. For any $(x, z, y, z) \in \delta$ we have $\varphi\big(g\big(\lambda^{-1}(x, z, y, z)\big)\big) = \varphi\big(g(x, y, z)\big) = \Big(\big(g(x, y, z)\big)_F, \big(g(x, y, z)\big)_G\Big)$. Put $r(x, z) = \big(g\big(x, \alpha(x), z\big)\big)_F$ for every $(x, z) \in \beta\circ\alpha$ and $s(y, z) = \big(g\big(\alpha^{-1}(y), y, z\big)\big)_G$ for every $(y, z) \in \beta$. Then $\varphi\big(g\big(\lambda^{-1}(x, z, y, z)\big)\big) = \big(r(x, z), s(y, z)\big)$ for all $(x, z, y, z) \in \delta$ and $r \in \big(\mathrm{Hom}_{(\beta\circ\alpha)^{-1}}(\boldsymbol{H}, \boldsymbol{F})\big)^{\beta\circ\alpha}$, $s \in \big(\mathrm{Hom}_{\beta^{-1}}(\boldsymbol{H}, \boldsymbol{G})\big)^{\beta}$. Now we have

$$\begin{aligned}
{}^t r(x) &= r\big(x, \beta\big(\alpha(x)\big)\big)(t) = \Big(g\Big(x, \alpha(x), \beta\big(\alpha(x)\big)\Big)\Big)_F(t) \\
&= \mathrm{pr}_F\Big(g\Big(x, \alpha(x), \beta\big(\alpha(x)\big)\Big)(t)\Big) = \mathrm{pr}_F\Big({}^t g\big(x, \alpha(x)\big)\Big) \\
&= \mathrm{pr}_F\Big(p(x), q\big(\alpha(x)\big)\Big) = p(x)
\end{aligned}$$

for any $t \in H$ and $x \in I$. Hence ${}^t r = p$ for all $t \in H$. Similarly,

$$\begin{aligned}
{}^t s(y) &= s\big(y, \beta(y)\big)(t) = \Big(g\big(\alpha^{-1}(y), y, \beta(y)\big)\Big)_G(t) \\
&= \mathrm{pr}_G\Big(g\big(\alpha^{-1}(y), y, \beta(y)\big)(t)\Big) = \mathrm{pr}_G\Big({}^t g\big(\alpha^{-1}(y), y\big)\Big) \\
&= \mathrm{pr}_G\Big(p\big(\alpha^{-1}(y)\big), q(y)\Big) = q(y)
\end{aligned}$$

for any $t \in H$ and $y \in J$. Thus ${}^t s = q$ for all $t \in H$. Therefore ${}^t r \in R$ and ${}^t s \in S$ for all $t \in H$. Consequently, $r \in \mathscr{R}(\boldsymbol{F} \overset{\beta\circ\alpha}{\triangle} \boldsymbol{H})$ and $s \in \mathscr{R}(\boldsymbol{G} \overset{\beta}{\triangle} \boldsymbol{H})$. This results in $\varphi \circ g \circ \lambda^{-1} \in \mathscr{R}\big((\boldsymbol{F} \overset{\beta\circ\alpha}{\triangle} \boldsymbol{H}) \overset{\delta}{\cdot} (\boldsymbol{G} \overset{\beta}{\triangle} \boldsymbol{H})\big)$ and we have proved that φ is a homomorphism of $(\boldsymbol{F} \overset{\alpha}{\cdot} \boldsymbol{G}) \overset{\gamma}{\triangle} \boldsymbol{H}$ onto $(\boldsymbol{F} \overset{\beta\circ\alpha}{\triangle} \boldsymbol{H}) \overset{\delta}{\cdot} (\boldsymbol{G} \overset{\beta}{\triangle} \boldsymbol{H})$ with regard to λ. Reversing the argument we can easily show that $\varphi \circ g \circ \lambda^{-1} \in \mathscr{R}\big((\boldsymbol{F} \overset{\beta\circ\alpha}{\triangle} \boldsymbol{H}) \overset{\delta}{\cdot} (\boldsymbol{G} \overset{\beta}{\triangle} \boldsymbol{H})\big)$ implies $g \in \mathscr{R}\big((\boldsymbol{F} \overset{\alpha}{\cdot} \boldsymbol{G}) \overset{\gamma}{\triangle} \boldsymbol{H}\big)$ whenever $g \in \big(\operatorname{Hom}_{\gamma^{-1}}(\boldsymbol{H}, \boldsymbol{F} \overset{\alpha}{\cdot} \boldsymbol{G})\big)^{\gamma}$. Therefore φ is an isomorphism with regard to λ and the proof is complete. $\square$

4.6. Theorem. *Let $\boldsymbol{F} = (F, R)$ with domain I, $\boldsymbol{G} = (G, S)$ with domain J and $\boldsymbol{H} = (H, T)$ with domain K be relational systems of the same type. Let $G \cap H = \emptyset$ and let $\alpha : I \to J$ and $\beta : J \to K$ be bijections. Let $\gamma : I \to \beta$ and $\delta : \alpha \to \beta\circ\alpha$ be the bijections defined by $\gamma(x) = \big(\alpha(x), \beta(\alpha(x))\big)$ for all $x \in I$ and $\delta(x, y) = \big(x, \beta(y)\big)$ for all $(x, y) \in \alpha$. Finally, let $\lambda : \gamma \to \delta$ be the bijection defined by $\lambda(x, y, z) = (x, y, x, z)$ for all $(x, y, z) \in \gamma$. Then*

$$\boldsymbol{F} \overset{\gamma}{\triangle} (\boldsymbol{G} \overset{\beta}{+} \boldsymbol{H}) \overset{\lambda}{\sim} (\boldsymbol{F} \overset{\alpha}{\triangle} \boldsymbol{G}) \overset{\delta}{\cdot} (\boldsymbol{F} \overset{\beta\circ\alpha}{\triangle} \boldsymbol{H}).$$

Proof. For any $h \in \operatorname{Hom}_{\gamma^{-1}}(\boldsymbol{G} \overset{\beta}{+} \boldsymbol{H}, \boldsymbol{F})$ let h_G denote the restriction $h|G$ and h_H the restriction $h|H$, i.e. let $h_G = h \circ \mathrm{id}_G$ and $h_H = h \circ \mathrm{id}_H$. Clearly, $\mathrm{id}_G \in \operatorname{Hom}_{\varrho}(\boldsymbol{G}, \boldsymbol{G} \overset{\beta}{+} \boldsymbol{H})$ and $\mathrm{id}_H \in \operatorname{Hom}_{\sigma}(\boldsymbol{H}, \boldsymbol{G} \overset{\beta}{+} \boldsymbol{H})$ where $\varrho : J \to \beta$ and $\sigma : K \to \beta$ are the bijections defined by $\varrho(y) = \big(y, \beta(y)\big)$ for all $y \in J$ and $\sigma(z) = \big(\beta^{-1}(z), z\big)$ for all $z \in K$. Since $\gamma^{-1} \circ \varrho = \alpha^{-1}$ and $\gamma^{-1} \circ \sigma = (\beta \circ \alpha)^{-1}$, by 1.4.b) we have $h_G \in \operatorname{Hom}_{\alpha^{-1}}(\boldsymbol{G}, \boldsymbol{F})$ and $h_H \in \operatorname{Hom}_{(\beta\circ\alpha)^{-1}}(\boldsymbol{H}, \boldsymbol{F})$. Further, let $h_1 \in \operatorname{Hom}_{\alpha^{-1}}(\boldsymbol{G}, \boldsymbol{F})$, $h_2 \in \operatorname{Hom}_{(\beta\circ\alpha)^{-1}}(\boldsymbol{H}, \boldsymbol{F})$ and put $h = h_1 \cup h_2$. Let $f \in \mathscr{R}(\boldsymbol{G} \overset{\beta}{+} \boldsymbol{H})$. Then (i) there exists $p \in S$ such that $f(y, z) = p(y)$ for all $(y, z) \in \beta$, or (ii) there exists $q \in T$ such that $f(y, z) = q(z)$ for all $(y, z) \in \beta$. Let the condition (i) be fulfilled. Then $h\big(f(\gamma(x))\big) = h\big(f\big(\alpha(x), \beta(\alpha(x))\big)\big) = h_2\big(q\big(\beta(\alpha(x))\big)\big)$ for any $x \in I$. Hence $h \circ f \circ \gamma = h_1 \circ p \circ \alpha \in R$. Similarly, if the condition (ii) is fulfilled, then we obtain $h\big(f(\gamma(x))\big) = h\big(f\big(\alpha(x), \beta(\alpha(x))\big)\big) = h_2\big(q\big(\beta(\alpha(x))\big)\big)$ for all $x \in I$, i.e. $h \circ f \circ \gamma = h_2 \circ q \circ \beta \circ \alpha \in R$. Therefore $h \in \operatorname{Hom}_{\gamma^{-1}}(\boldsymbol{G} \overset{\beta}{+} \boldsymbol{H}, \boldsymbol{F})$ and clearly $h_1 = h_G$ and $h_2 = h_G$. Now, let $\varphi : \operatorname{Hom}_{\gamma^{-1}}(\boldsymbol{G} \overset{\beta}{+} \boldsymbol{H}, \boldsymbol{F}) \to \operatorname{Hom}_{\alpha^{-1}}(\boldsymbol{G}, \boldsymbol{F}) \times \operatorname{Hom}_{(\beta\circ\alpha)^{-1}}(\boldsymbol{H}, \boldsymbol{F})$ be the mapping defined by $\varphi(h) = (h_G, h_H)$ whenever $h \in \operatorname{Hom}_{\gamma^{-1}}(\boldsymbol{G} \overset{\beta}{+} \boldsymbol{H}, \boldsymbol{F})$. We have shown that φ is surjective. Since φ is obviously injective, it is a bijection. Let $g \in \mathscr{R}\big(\boldsymbol{F} \overset{\gamma}{\triangle} (\boldsymbol{G} \overset{\beta}{+} \boldsymbol{H})\big)$. Then ${}^t g \in R$ for every $t \in G \cup H$. For any $(x, y, x, z) \in \delta$ we have $\varphi\big(g\big(\lambda^{-1}(x, y, x, z)\big)\big) = \varphi\big(g(x, y, z)\big) =$

$\left((g(x,y,z))_G, (g(x,y,z))_H\right)$. Put $r(x,y) = \left(g(x,y,\beta(y))\right)_G$ for every $(x,y) \in \alpha$ and $s(x,z) = \left(g(x,\alpha(x),z)\right)_H$ for every $(x,z) \in \beta \circ \alpha$. Then $\varphi(g(\lambda^{-1}(x,y,x,z))) = (r(x,y), s(x,z))$ and $r \in (\operatorname{Hom}_{\alpha^{-1}}(\boldsymbol{G}, \boldsymbol{F}))^{\alpha}$, $s \in (\operatorname{Hom}_{(\beta\circ\alpha)^{-1}}(\boldsymbol{H}, \boldsymbol{F}))^{\beta\circ\alpha}$. We have ${}^t r(x) = r(x,\alpha(x))(t) = \left(g(x,\alpha(x),\beta(\alpha(x)))\right)_G(t) = g(x,\alpha(x),\beta(\alpha(x)))(t) = {}^t g(x)$ for every $t \in G$ and $x \in I$. Similarly,

$$\begin{aligned} {}^t s(x) &= s(x,\beta(\alpha(x)))(t) = \left(g(x,\alpha(x),\beta(\alpha(x)))\right)_H(t) \\ &= g(x,\alpha(x),\beta(\alpha(x)))(t) = {}^t g(x) \end{aligned}$$

for every $t \in H$ and $x \in I$. Thus ${}^t r = {}^t g \in R$ for every $t \in G$ and ${}^t s = {}^t g \in R$ for every $t \in H$. This yields $r \in \mathscr{R}(\boldsymbol{F} \overset{\alpha}{\triangle} \boldsymbol{G})$ and $s \in \mathscr{R}(\boldsymbol{F} \overset{\beta\circ\alpha}{\triangle} \boldsymbol{H})$. Consequently, $\varphi \circ g \circ \lambda^{-1} \in \mathscr{R}((\boldsymbol{F} \overset{\alpha}{\triangle} \boldsymbol{G}) \overset{\delta}{\cdot} (\boldsymbol{F} \overset{\beta\circ\alpha}{\triangle} \boldsymbol{H}))$. Therefore φ is a homomorphism of $\boldsymbol{F} \overset{\gamma}{\triangle} (\boldsymbol{G} \overset{\beta}{+} \boldsymbol{H})$ onto $(\boldsymbol{F} \overset{\alpha}{\triangle} \boldsymbol{G}) \overset{\delta}{\cdot} (\boldsymbol{F} \overset{\beta\circ\alpha}{\triangle} \boldsymbol{H})$ with regard to λ. By the reverse considerations we can show that $\varphi \circ g \circ \lambda^{-1} \in \mathscr{R}((\boldsymbol{F} \overset{\alpha}{\triangle} \boldsymbol{G}) \overset{\delta}{\cdot} (\boldsymbol{F} \overset{\beta\circ\alpha}{\triangle} \boldsymbol{H}))$ implies $g \in \mathscr{R}(\boldsymbol{F} \overset{\gamma}{\triangle} (\boldsymbol{G} \overset{\beta}{+} \boldsymbol{H}))$ whenever $g \in (\operatorname{Hom}_{\gamma^{-1}}(\boldsymbol{G} \overset{\beta}{+} \boldsymbol{H}, \boldsymbol{F}))^{\gamma}$. Therefore φ is an isomorphism and the statement is proved. □

However, the law $\boldsymbol{F} \overset{\gamma}{\triangle} (\boldsymbol{G} \overset{\beta}{\cdot} \boldsymbol{H}) \sim (\boldsymbol{F} \overset{\alpha}{\triangle} \boldsymbol{G}) \overset{\delta}{\triangle} \boldsymbol{H}$ does not hold in general for relational systems $\boldsymbol{F}$, $\boldsymbol{G}$, $\boldsymbol{H}$ of the same type and for the corresponding bijections α, β, γ, δ. Now we are aiming at giving some sufficient conditions for the validity of this law.

Let $\boldsymbol{F} = (F, R)$ be a relational system with domain I. The system $\boldsymbol{F}$ is called

(1) *discrete* iff $R = \{f \in F^I \mid \exists t \in F : f(x) = t \quad \text{for all } x \in I\}$.

(2) *reflexive* iff the discrete relational system $\boldsymbol{G}$ with domain I and with carrier $\boldsymbol{F}$ satisfies $\boldsymbol{G} \leqslant \boldsymbol{F}$,

(3) *complete* iff $\boldsymbol{R} = F^I$.

4.7. Theorem. *Let $\boldsymbol{F} = (F, R)$ with domain I, $\boldsymbol{G} = (G, S)$ with domain J and $\boldsymbol{H} = (H, T)$ with domain K be relational systems of the same type. Let $\alpha : I \to J$ and $\beta : J \to K$ be bijections. Let $\gamma : I \to \beta$ and $\delta : \alpha \to K$ be the bijections defined by $\gamma(x) = (\alpha(x), \beta(\alpha(x)))$ for all $x \in I$ and $\delta(x, y) = \beta(y)$ for all $(x, y) \in \alpha$. Let $\boldsymbol{G}$ and $\boldsymbol{H}$ be reflexive. Then*

$$\boldsymbol{F} \overset{\gamma}{\triangle} (\boldsymbol{G} \overset{\beta}{\cdot} \boldsymbol{H}) \prec (\boldsymbol{F} \overset{\alpha}{\triangle} \boldsymbol{G}) \overset{\delta}{\triangle} \boldsymbol{H}.$$

Proof. First, note that $\gamma = \delta$ is valid. Let $f \in \operatorname{Hom}(\boldsymbol{G} \overset{\beta}{\cdot} \boldsymbol{H}, \boldsymbol{F})$ and $v \in H$. By $f_v : G \to F$ we denote the mapping defined by $f_v(u) = f(u, v)$ whenever $u \in G$. Let

$g \in S$. Putting $g^*(y,z) = (g(y), v)$ for all $(y,z) \in \beta$ we get $g^* \in \mathscr{R}(\boldsymbol{G} \overset{\beta}{\cdot} \boldsymbol{H})$ since $\boldsymbol{H}$ is reflexive. Hence $f \circ g^* \circ \gamma \in R$. However, $f(g^*(\gamma(x))) = f\left(g^*(\alpha(x), \beta(\alpha(x)))\right) = f(g(\alpha(x)), v) = f_v(g(\alpha(x)))$ for any $x \in I$. Therefore $f \circ g^* \circ \gamma = f_v \circ g \circ \alpha$ which yields $f_v \circ g \circ \alpha \in R$. Consequently, $f_v \in \mathrm{Hom}_{\alpha^{-1}}(\boldsymbol{G}, \boldsymbol{F})$. Let $u \in G$, $h \in T$ and put $\bar{h}(y,z) = (u, h(z))$ for all $(y,z) \in \beta$. Then $\bar{h} \in \mathscr{R}(\boldsymbol{G} \overset{\beta}{\cdot} \boldsymbol{H})$ because $\boldsymbol{G}$ is reflexive. Thus $f \circ \bar{h} \circ \gamma \in R$. Let $f' \colon H \to \mathrm{Hom}_{\alpha^{-1}}(\boldsymbol{G}, \boldsymbol{F})$ be the mapping defined by $f'(v) = f_v$ for every $v \in H$. Then ${}^u(f' \circ h \circ \delta)(x) = f'\left(h(\delta(x, \alpha(x)))\right)(u) = f'\left(h(\beta(\alpha(x)))\right)(u) = f_{h(\beta(\alpha(x)))}(u) = f\left(u, h(\beta(\alpha(x)))\right) = f\left(\bar{h}(\alpha(x), \beta(\alpha(x)))\right) = f(\bar{h}(\gamma(x)))$ for all $u \in G$ and $x \in I$. So ${}^u(f' \circ h \circ \delta) = f \circ \bar{h} \circ \gamma$ for all $u \in G$ and this implies ${}^u(f' \circ h \circ \delta) \in R$ for all $u \in G$. Hence $f' \circ h \circ \delta \in \mathscr{R}(\boldsymbol{F} \overset{\alpha}{\triangle} \boldsymbol{G})$ and $f' \in \mathrm{Hom}_{\delta^{-1}}(\boldsymbol{H}, \boldsymbol{F} \overset{\alpha}{\triangle} \boldsymbol{G})$. Now we can define a mapping $\varphi \colon \mathrm{Hom}_{\gamma^{-1}}(\boldsymbol{G} \overset{\beta}{\cdot} \boldsymbol{H}, \boldsymbol{F}) \to \mathrm{Hom}_{\delta^{-1}}(\boldsymbol{H}, \boldsymbol{F} \overset{\alpha}{\triangle} \boldsymbol{G})$ by $\varphi(f) = f'$ for every $f \in \mathrm{Hom}_{\gamma^{-1}}(\boldsymbol{G} \overset{\beta}{\cdot} \boldsymbol{H}, \boldsymbol{F})$. It is easy to see that φ is an injection. Let $p \in \mathscr{R}(\boldsymbol{F} \overset{\gamma}{\triangle} (\boldsymbol{G} \overset{\beta}{\cdot} \boldsymbol{H}))$. Then ${}^{(u,v)}p \in R$ for all $(u,v) \in G \times H$. We are to show that $\varphi \circ p \in \mathscr{R}((\boldsymbol{F} \overset{\alpha}{\triangle} \boldsymbol{G}) \overset{\delta}{\triangle} \boldsymbol{H})$, i.e. ${}^v(\varphi \circ p) \in \mathscr{R}(\boldsymbol{F} \overset{\alpha}{\triangle} \boldsymbol{G})$ for every $v \in H$, but this is equivalent to ${}^u({}^v(\varphi \circ p)) \in R$ for all $(u,v) \in G \times H$. For any $x \in I$ and $(u,v) \in G \times H$ we have ${}^u({}^v(\varphi \circ p))(x) = {}^v(\varphi \circ p)(x, \alpha(x))(u) = \varphi\left(p(x, \alpha(x), \beta(\alpha(x)))\right)(v)(u) = \left(p(x, \alpha(x), \beta(\alpha(x)))\right)'(v)(u) = \left(p(x, \alpha(x), \beta(\alpha(x)))\right)_v(u) = p(x, \alpha(x), \beta(\alpha(x)))(u,v) = {}^{(u,v)}p(x)$. So ${}^u({}^v(\varphi \circ p)) = {}^{(u,v)}p$ for all $(u,v) \in G \times H$ and hence ${}^u({}^v(\varphi \circ p)) \in R$ for all $(u,v) \in G \times H$. Thus $\varphi \circ p \in \mathscr{R}((\boldsymbol{F} \overset{\alpha}{\triangle} \boldsymbol{G}) \overset{\delta}{\triangle} \boldsymbol{H})$. Reversing the previous considerations we can easily show that $\varphi \circ p \in \mathscr{R}((\boldsymbol{F} \overset{\alpha}{\triangle} \boldsymbol{G}) \overset{\delta}{\triangle} \boldsymbol{H})$ implies $p \in \mathscr{R}(\boldsymbol{F} \overset{\gamma}{\triangle} (\boldsymbol{G} \overset{\beta}{\cdot} \boldsymbol{H}))$ whenever $p \in \left(\mathrm{Hom}_{\gamma^{-1}}((\boldsymbol{G} \overset{\beta}{\cdot} \boldsymbol{H}) \overset{\gamma}{\triangle} \boldsymbol{F})\right)^\gamma$. Thus φ is an isomorphism of $\boldsymbol{F} \overset{\gamma}{\triangle} (\boldsymbol{G} \overset{\beta}{\cdot} \boldsymbol{H})$ onto the subsystem of $(\boldsymbol{F} \overset{\alpha}{\triangle} \boldsymbol{G}) \overset{\delta}{\triangle} \boldsymbol{H}$, whose carrier is $\varphi(\mathrm{Hom}_{\gamma^{-1}}(\boldsymbol{G} \overset{\beta}{\cdot} \boldsymbol{H}, \boldsymbol{F}))$, with regard to the identity $\mathrm{id} : \gamma \to \delta$. Therefore $\boldsymbol{F} \overset{\gamma}{\triangle} (\boldsymbol{G} \overset{\beta}{\cdot} \boldsymbol{H}) \prec (\boldsymbol{F} \overset{\alpha}{\triangle} \boldsymbol{G}) \overset{\delta}{\triangle} \boldsymbol{H}$ and the proof is complete. $\square$

4.8. Theorem. *Let the assumptions of Theorem 4.7 be fulfilled. If, moreover,* $\boldsymbol{F}$ *is reflexive and both* $\boldsymbol{G}$ *and* $\boldsymbol{H}$ *are discrete, then*

$$\boldsymbol{F} \overset{\gamma}{\triangle} (\boldsymbol{G} \overset{\beta}{\cdot} \boldsymbol{H}) \sim (\boldsymbol{F} \overset{\alpha}{\triangle} \boldsymbol{G}) \overset{\delta}{\triangle} \boldsymbol{H}.$$

Proof. If $\boldsymbol{F}$ is reflexive and both $\boldsymbol{G}$ and $\boldsymbol{H}$ are discrete, then clearly $\boldsymbol{G} \overset{\beta}{\cdot} \boldsymbol{H}$ is discrete and $\boldsymbol{F} \overset{\alpha}{\triangle} \boldsymbol{G}$ is reflexive. Therefore $\mathrm{Hom}_{\gamma^{-1}}(\boldsymbol{G} \overset{\beta}{\cdot} \boldsymbol{H}, \boldsymbol{F}) = F^{G \times H}$ and $\mathrm{Hom}_{\delta^{-1}}(\boldsymbol{H}, \boldsymbol{F} \overset{\alpha}{\triangle} \boldsymbol{G}) = (F^G)^H$. The mapping φ defined in the proof of Theorem 4.7 is obviously a bijection of $F^{G \times H}$ onto $(F^G)^H$. This fact implies the statement. $\square$

Let $\boldsymbol{F} = (F, R)$ be a relational system with domain I. Let J, K be sets equipotent with I and let $\alpha\colon I \to J$, $\beta\colon I \to K$ be bijections. The system $\boldsymbol{F}$ is called *diagonal with regard to the pair* (α, β) iff the following holds:

Let $\{f_j \mid j \in J\}$ be a family where $f_j \in R$ for all $j \in J$. Let $\{g_k \mid k \in K\}$ be the family of elements of F^I defined by $g_k(i) = f_{\alpha(i)}(\beta^{-1}(k))$ for every $i \in I$ and $k \in K$. If $g_k \in R$ for all $k \in K$, then putting $h(i) = f_{\alpha(i)}(i)$ whenever $i \in I$ we get $h \in R$.

It can be easily seen that $\boldsymbol{F}$ is diagonal with regard to (α, β) iff it is diagonal with regard to (β, α).

If $I = J = K$ and both α and β are identities, then the diagonality of $\boldsymbol{F}$ with regard to (α, β) coincides with the diagonality of $\boldsymbol{F}$ introduced in [5]. If, moreover, I is finite, then $\boldsymbol{F}$ is diagonal with regard to (α, β) iff R satisfies the diagonal property defined in [4]. In particular, if card $I = 2$, i.e. if R is a binary relation on F, then $\boldsymbol{F}$ is diagonal with regard to (α, β) iff R is transitive.

4.9. Theorem. *Let the assumptions of Theorem 4.7 be fulfilled. If, moreover, $\boldsymbol{F}$ is diagonal with regard to $(\alpha, \beta \circ \alpha)$, then*

$$\boldsymbol{F} \overset{\gamma}{\triangle} (\boldsymbol{G} \overset{\beta}{\cdot} \boldsymbol{H}) \sim (\boldsymbol{F} \overset{\alpha}{\triangle} \boldsymbol{G}) \overset{\delta}{\triangle} \boldsymbol{H}.$$

P r o o f. Let $g \in \mathrm{Hom}_{\delta^{-1}}(\boldsymbol{H}, \boldsymbol{F} \overset{\alpha}{\triangle} \boldsymbol{G})$ and put $f(u, v) = g(v)(u)$ for any $u \in G$ and $v \in H$. Let $h \in \mathscr{R}(\boldsymbol{G} \overset{\beta}{\cdot} \boldsymbol{H})$. Then there exist $h_1 \in S$ and $h_2 \in T$ such that $h(y, z) = \big(h_1(y), h_2(z)\big)$ for all $(y, z) \in \beta$. As $g \circ h_2 \circ \delta \in \mathscr{R}(\boldsymbol{F} \overset{\alpha}{\triangle} \boldsymbol{G})$, we have ${}^{u}(g \circ h_2 \circ \delta) \in R$ for every $u \in G$. Thus, putting $f_j = {}^{h_1(j)}(g \circ h_2 \circ \delta)$ whenever $j \in J$ we get $f_j \in R$. Next, as $g(v) \in \mathrm{Hom}_{\alpha^{-1}}(\boldsymbol{G}, \boldsymbol{F})$ for all $v \in H$, we have $g\big(h_2(k)\big) \in \mathrm{Hom}_{\alpha^{-1}}(\boldsymbol{G}, \boldsymbol{F})$ for all $k \in K$. Therefore, putting $g_k = g\big(h_2(k)\big) \circ h_1 \circ \alpha$ we get $g_k \in R$ whenever $k \in K$. Further,

$$\begin{aligned}
f_{\alpha(i)}\big((\beta \circ \alpha)^{-1}(k)\big) &= {}^{h_1(\alpha(i))}(g \circ h_2 \circ \delta)\big((\beta \circ \alpha)^{-1}(k)\big) \\
&= g\left(h_2\big(\delta\big((\beta \circ \alpha)^{-1}(k), (\alpha \circ (\beta \circ \alpha)^{-1})(k)\big)\big)\right)\left(h_1\big(\alpha(i)\big)\right) \\
&= g\left(h_2\big((\beta \circ \alpha \circ (\beta \circ \alpha)^{-1})(k)\big)\right)\left(h_1\big(\alpha(i)\big)\right) \\
&= f\big(h_1\big(\alpha(i)\big), h_2(k)\big)
\end{aligned}$$

and

$$g_k(i) = g\big(h_2(k)\big)\big(h_1\big(\alpha(i)\big)\big) = f\big(h_1\big(\alpha(i)\big), h_2(k)\big)$$

for all $i \in I$ and $k \in K$. Hence $g_k(i) = f_{\alpha(i)}\big((\beta \circ \alpha)^{-1}(k)\big)$ for every $i \in I$ and $k \in K$. Since $\boldsymbol{F}$ is diagonal with regard to $(\alpha, \beta \circ \alpha)$ and since

$$f_{\alpha(i)}(i) = f\left(h_1\big(\alpha(i)\big), h_2\big(\beta\big(\alpha(i)\big)\big)\right) = f\left(h\big(\alpha(i), \beta\big(\alpha(i)\big)\big)\right) = f\big(h\big(\gamma(i)\big)\big)$$

holds for every $i \in I$, we have $f \circ h \circ \gamma \in R$. Consequently, $f \in \mathrm{Hom}_{\gamma^{-1}}(\boldsymbol{G} \overset{\beta}{\cdot} \boldsymbol{H}, \boldsymbol{F})$. Now, if φ is the mapping defined in th proof of Theorem 4.7, then $g = \varphi(f)$ and therefore φ is a surjection. This yields the statement. $\square$

Let us conclude with the following evident assertion:

4.10. Proposition. *Let $\boldsymbol{F}$ with domain I, $\boldsymbol{G}$ with domain J and $\boldsymbol{H}$ with domain K be relational systems of the same type. Let α, β, γ, δ be the bijections defined in the same way as in Theorem 4.7. If $\boldsymbol{F}$ is complete, then*

$$\boldsymbol{F} \overset{\gamma}{\triangle} (\boldsymbol{G} \overset{\beta}{\cdot} \boldsymbol{H}) \sim (\boldsymbol{F} \overset{\alpha}{\triangle} \boldsymbol{G}) \overset{\delta}{\triangle} \boldsymbol{H}.$$

References

[1] *G. Birkhoff*: An extended arithmetic, Duke Math. J. *3* (1937), 311–316.
[2] *G. Birkhoff*: Generalized arithmetic, Duke Math. J. *9* (1942), 283–302.
[3] *M. M. Day*: Arithmetic of ordered systems, Trans. Am. Math. Soc. *58* (1945), 1–43.
[4] *V. Novák*: On a power of relational structures, Czech. Math. J. *35* (1985), 167–172.
[5] *J. Šlapal*: Direct arithmetic of relational systems, Publ. Math. Debrecen *38* (1991), 39–48.
[6] *J. Šlapal*: Cartesian closedness in categories of relational systems, Arch. Math. (Basel) *52* (1989), 603–606.
[7] *J. Šlapal*: On relations, Czech. Math. J. *39* (1989), 198–214.

Author's address: 616 69 Brno, Technická 2, Czech Republic (katedra matematiky FS VUT).