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Accepté par la Rédaction le 21. 2. 1980

Cardinal functions on compact F-spaces and on weakly countably complete Boolean algebras *

by

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Abstract. We investigate limitations on the cardinals $\varkappa$ which occur as the value of cardinal functions on infinite compact F-spaces (or on weakly countably complete Boolean algebras). We find limitations of the form $\varkappa^{\omega} = \varkappa$, or else $\mathrm{cf}(\varkappa) = \omega$, or at least "$\varkappa$ is not a strong limit with $\mathrm{cf}(\varkappa) = \omega$", and show that all infinite cardinals $\varkappa$ with $\varkappa^{\omega} = \varkappa$ do occur (for cardinality one needs the additional restriction $\varkappa \geq 2^{\aleph\omega}$, as is well known).

Contents

1. Introduction. This is a paper on the behavior of cardinal functions on compact F-spaces. The Boolean algebras which occur as the algebra of clopen (= closed and open) sets of a zero-dimensional compact F-space are the weakly countably complete Boolean algebras, or WCC algebras for short, see § 6 for the definition. This class includes the class of countably complete Boolean algebras and has the pleasant property of being closed under homomorphisms. (However, it is consistent

* Completed while supported by NSF-Grant MCS78-09484.

that there be a WCC algebra that is not a quotient of any countably complete Boolean algebra, (vDvM].) So via Stone duality this also is a paper on the behavior of cardinal functions on a significant class of Boolean algebras.

Our main results, in their two forms, are the following.

1.1. THEOREM. *Let X be an infinite compact F-space.*

(1) $|X|^{\omega} = |X|$.

(2) *If $\varkappa$ is the character of X at some point, or is the character of X, or is the hereditary Lindelöf degree of X, then* $\mathrm{cf}\varkappa \neq \omega$.

(3) *The spread (= hereditary cellularity) of X is not a strong limit with countable cofinality.*

1.2. THEOREM. *Let B be an infinite* WCC *algebra.*

(1) *If $\varkappa$ is the number of ultrafilters on B, then* $\varkappa^{\omega} = \varkappa$.

(2) *If $\varkappa$ is the least cardinal such that some ultrafilter has $\leqslant\varkappa$ generators, or such that every ultrafilter has $\leqslant\varkappa$ generators, or such that every filter has $\leqslant\varkappa$ generators, then* $\mathrm{cf}\varkappa \neq \omega$.

(3) *If $\varkappa$ is the smallest cardinal such that $|D| \leqslant \varkappa$ whenever D is a disjointed set in some homomorphic image of B, then $\varkappa$ is not a strong limit with countable cofinality.*

An early version of these results was announced in [vD$_1$].

Theorem 1.1 is more general than Theorem 1.2 since a compact F-space need not be zero-dimensional, see Example 14.9. However, we give several proofs in the zero-dimensional case only since this reduces the technical complexity of proofs; the proof of the general case does not require additional ideas.

Three other cardinal functions are considered in §§ 11, 12, and we also obtain information about cardinal functions on compact extremally disconnected spaces (complete Boolean algebras) in §§ 13, 15.

We have tried to write this paper so that it is intelligible for both topologists and Boolean algebraists. Our language is mostly topological, but we do not need much topology, and as a service to Boolean algebraists we have included dictionaries in §§ 4, 6 and 17, and have included the proofs of some known topological facts.

2. Original motivation. A space X is said to *omit* the cardinal $\varkappa$ if $|X| \geqslant \varkappa$ and if no closed subspace of X has cardinality $\varkappa$ (so $|X| > \varkappa$). We say that $\varkappa$ *can be omitted* if it is omitted by some compact Hausdorff space.

Juhász has shown in [Ju$_2$, 2.3] that under GCH no compact Hausdorff space omits both $\varkappa^{+}$ and $\varkappa^{++}$, for every cardinal $\varkappa$, see also [HJ$_2$]. It is well known that $\beta\omega$ (ω is the space of integers) omits every infinite cardinal $< 2^{2^{\omega}}$, [GJ, 9.12]. This shows that GCH is essential, and also that a compact space can omit both $\varkappa$ and $\varkappa^{+}$ under GCH, at least for $\varkappa = \omega$.

It is unknown if under GCH any cardinal of the form $\varkappa^{++}$, with $\varkappa \geqslant \omega$, can be omitted, even in the special case $\varkappa = \omega$. In fact, until now it was unknown if under GCH any cardinal other than ω or ω_1 can be omitted. Since there are arbitrarily large compact F-spaces, and since a closed subspace of a compact F-space again is a compact F-space, by Proposition 6.2a, and since $|X|^{\omega} = |X|$ for each infinite compact F-space, by Theorem 1.1(1), we see that every infinite cardinal $\varkappa$ with $\varkappa^{\omega} \neq \varkappa$, in particular with $\mathrm{cf}(\varkappa) = \omega$, can be omitted. (An early, weaker version of my result is included in [Ju$_2$].) So ω_1 and every cardinal $\varkappa$ with $\mathrm{cf}(\varkappa) = \omega$ can be omitted. It is unknown if under GCH any other cardinals can be omitted.

3. Conventions and definitions.

A. Set theory. Cardinals are initial (von Neumann) ordinals, $\varkappa$ and λ usually represent infinite cardinals. ω is ω_0. A sequence, i.e. a function s with domain ω, is frequently denoted $\langle s_n \rangle_n$. As usual,

$$[\varkappa]^{\lambda} = \{A \subseteq \varkappa\colon\ |A| = \lambda\}, \qquad [\varkappa]^{<\lambda} = \{A \subseteq \varkappa\colon\ |A| < \varkappa\}.$$

B. Topology. All our spaces are regular and in fact completely regular. "Clopen" abbreviates "closed and open", a space is zero-dimensional if its clopen sets are a base.

A sequence $\langle A_n \rangle_n$ of subsets of a space X will be called
disjoint if $A_m \cap A_n = \emptyset$ for distinct $m, n \in \omega$,
relatively discrete if it is disjoint and if each A_n is open in $\bigcup_m A_m$.

A sequence $\langle x_n \rangle_n$ of points is called relatively discrete if the sequence $\langle \{x_n\} \rangle_n$ is relatively discrete; remember that if $\langle x_n \rangle_n$ is relatively discrete then (by regularity) there is a disjoint open sequence $\langle U_n \rangle_n$ with $x_n \in U_n$ $(n \in \omega)$.

$\varkappa$ also denotes the discrete space with $\varkappa$ points.

C. Boolean algebra. We say that a subset P of a Boolean algebra B is
disjointed if $a \wedge b = \emptyset$ for distinct $a, b \in P$,
weakly disjointed if $a \nleqslant F$ for $a \in P$ and finite $F \subseteq P$ with $a \notin F$.
We say that the subsets P and Q of B are
disjointed if $p \wedge q = \emptyset$ for $p \in P$ and $q \in Q$,
can be separated if there is $s \in B$ with $p \leqslant s$ for $p \in P$ and $q \leqslant s'$ if $q \in Q$. (In this case we say that s separates P from Q.)

4. Cardinal functions. In this section we review the definition of some cardinal functions on topological spaces, give the Boolean algebraic translation and mention a useful well known fact.

A. Definitions.

cellularity	$c(X) = \sup\{|\mathscr{U}|\colon\ \mathscr{U}$ a disjoint family of nonempty open sets$\}$,
character of $A \subseteq X$	$\chi(A, X) = \min\{|\mathscr{L}|\colon\ \mathscr{L}$ is a local base at $A\}$,
character of $a \in X$	$\chi(a, X) = \chi(\{a\}, X)$,
character	$\chi(X) = \sup\{\chi(a, X)\colon\ a \in X\}$,
density	$d(X) = \min\{|D|\colon\ D$ is a dense subset of $X\}$,
Lindelöf degree	$L(X) = \min\{\varkappa\colon$ every open cover has a subcover of cardinality $\leqslant\varkappa\}$,
spread	$s(X) = \sup\{|D|\colon\ D$ is a relatively discrete subset of $X\}$,
weight	$w(X) = \min\{|\mathscr{B}|\colon\ \mathscr{B}$ is a base for $X\}$.

Also, if φ is a cardinal function, then one defines

hereditary φ $\qquad h\varphi(X) = \sup\{\varphi(Y)\colon\ Y\subseteq X\}$.

Note that $s = hc$.

We do not need the common convention that cardinal functions take on values $\geqslant\omega$ only.

B. Boolean algebraic translation.

4.1. Lemma. *Let B be a Boolean algebra, and let X be the Stone space of X,*

(a) $|X| = |\{F\subseteq B\colon\ F$ *is an ultrafilter*$\}|$.

(b) $c(X) = \sup\{|S|\colon\ S$ *is a disjointed subset of* $B\}$.

(c) *if F is a filter on B then* $\chi(\bigcap F, X) = \min\{|G|\colon\ G$ *generates* $F\}$.

(d) $\chi(X) = \min\{\varkappa\colon$ *every ultrafilter on B is generated by at most $\varkappa$ elements of* $B\}$.

(e) $d(X) = \min\{\varkappa\colon\ B$ *is the union of $\varkappa$ ultrafilters*$\}$.

(f) $hL(X) = \min\{\varkappa\colon$ *every filter on B is generated by at most $\varkappa$ elements of* $B\}$.

(g) $s(X) = \sup\{|S|\colon\ S$ *is a weakly disjointed subset of* $B\}$.

(h) $w(X) = |B|$ *if X is infinite.*

(i) $hc(X) = \sup\{c(A)\colon\ A$ *is a homomorphic image of* $B\}$.

With the exception of (c), (d), (f) the proofs are routine, hence we omit them; see below for (c), (d), (f).

If B and X are as in the lemma, we will define $\varphi(B) = \varphi(X)$ for all φ considered, with the exception of $|\ |$. [Note that what Boolean algebraists call the *density* of B is not $d(X)$, but $\pi(X)$, the π-weight of X.] If F is a filter on B, we use $\chi(F, B)$ for $\chi(\bigcap F, X)$ conform with 4.1 c.

C. A useful fact.

4.2. Lemma. *If X is compact and if A is a subset of X that is closed but not open, then*

$$\begin{aligned}\chi(A, X) &= L(X-A)\\ &= \min\{|\mathcal{U}|\colon\ \mathcal{U} \text{ is an open family in } X \text{ with } \textstyle\bigcap\mathcal{U} = A\}\\ &= \min\{|\mathcal{F}|\colon\ \mathcal{F} \text{ is a closed family in } X \text{ with } \textstyle\bigcup\mathcal{F} = X-A\}.\end{aligned}$$

Proof. Call the common value of the two minima μ. Note that $\mu\geqslant\omega$.

$\chi(A, X)\leqslant\mu$: Since X is normal, there is an open family $\mathcal{U}$ in X such that $|\mathcal{U}|\leqslant\mu$ and $A = \bigcap\{\overline{U}\colon\ U\in\mathcal{U}\}$. Since $|\mathcal{U}|\geqslant\omega$ (because A is not open) we may assume that $\mathcal{U}$ is closed under finite intersection. Let V be any open set with $V\supseteq A$, then $\{\overline{U}-V\colon\ U\in\mathcal{U}\}$ is a family of closed sets in the compact space X with empty intersection. Hence there must be a $U\in\mathcal{U}$ with $U\subseteq V$. So $\mathcal{U}$ is a local base at A.

$\mu\leqslant\chi(A, X)$: Trivial.

$\mu\leqslant L(X-A)$: Every point of $X-A$ has a compact neighborhood.

$L(X-A)\leqslant\mu$: $X-A$ is the union of μ compact sets, and $\mu\geqslant\omega$. ∎

This is known of course. The proofs of (c), (d) and (f) of Lemma 4.1 now are easy.

5. Combinatorial tools. The next lemma can be proved by an obvious modification of the proof of the corollary, due to Tarski, [T, Thm. 7]; cf. [CH$_2$, Thm. 4.1].

5.1. Lemma. *Let $\langle\varkappa_n\rangle_n$ be a sequence of cardinals which either is strictly increasing or is nondecreasing with $\varkappa_0\geqslant\omega$, and let $\varkappa = \sup_n\varkappa_n$.*

Let $\langle K_n\rangle_n$ be a disjoint sequence of sets with $|K_n|>\varkappa_n$ for $n\in\omega$. Then there is a family $\mathcal{A}\subseteq[\bigcup_n K_n]^\omega$ with $|\mathcal{A}| = \varkappa^\omega$ such that

(1) *$\mathcal{A}$ is almost disjoint, i.e. $|A\cap B|<\omega$ for distinct $A, B\in\mathcal{A}$,*

(2) *$|A\cap K_n| = 1$ for $A\in\mathcal{A}$, $n\in\omega$.*

5.2. Corollary. *If $\varkappa\geqslant\omega$ then there is an almost disjoint $\mathcal{A}\subseteq[\varkappa]^\omega$ with $|\mathcal{A}| = \varkappa^\omega$.*

Our next lemma is Hajnal's Free Set Lemma, [H] (or [CN, 10.14], [Ju$_1$, A3.5]); we emphasize that $\varkappa$ need not be regular in the lemma.

5.3. Lemma. *Let $\varkappa>\lambda\geqslant\omega$. For every $f\colon\ \varkappa\to[\varkappa]^{<\lambda}$ there is an $F\in[\varkappa]^\varkappa$ which is f-free, i.e. $\xi\notin f(\eta)$ whenever $\xi, \eta\in F$ are distinct.*

We also observe that the fact that for all cardinals $\lambda\geqslant\omega$ one has

$$\lambda^\omega = \lambda\cdot\sup\{\varkappa^\omega\colon\ \varkappa\leqslant\lambda \text{ and } \operatorname{cf}\varkappa = \omega\}$$

(proof by transfinite induction) implies

5.4. Lemma. *If $\lambda\geqslant\omega$ then $\lambda^\omega = \lambda$ if (and only if) $\varkappa^\omega\leqslant\lambda$ for every $\varkappa\leqslant\lambda$ with* $\operatorname{cf}\varkappa = \omega$.

6. Compact F-spaces and WCC algebras. Clearly a Boolean algebra is complete iff every two disjointed subsets can be separated, and is countably complete iff every two disjointed subsets, one of which is countable, can be separated. This suggests the following

Definition. A Boolean algebra is called *weakly countably complete*, or a WCC algebra for short, if every two disjointed countable subsets can be separated.

The topological counter part is given by the next definition and lemma.

Definition. The compact space X is called an *F-space* if the following holds:

if F and G are F_σ-subsets of X with $\overline{F}\cap G = F\cap\overline{G} = \emptyset$, then $\overline{F}\cap\overline{G} = \emptyset$.

6.1. Lemma. *Let B be the clopen algebra of a zero-dimensional compact space X. Then the following are equivalent:*

(a) *B is a WCC-algebra;*

(b) *every two disjoint open F_σ-subsets of X have disjoint closures; and*

(c) *X is an F-space.*

Proof. (a) → (b): Let U and V be disjoint open F_σ's in X. Since X is zero-dimensional, there are countable $\mathcal{U}, \mathcal{V}\subseteq B$ with $U = \bigcup\mathcal{U}$ and $V = \bigcup\mathcal{V}$. Clearly $\mathcal{U}$ and $\mathcal{V}$ are disjointed. If S separates $\mathcal{U}$ from $\mathcal{V}$, then $\overline{U}\subseteq S$ and $\overline{V}\subseteq X-S$.

(b) → (c): Let F and G be F_σ-subsets of X with $\overline{F}\cap G = F\cap\overline{G} = \emptyset$. Let $\langle F_n\rangle_n$ and $\langle G_n\rangle_n$ be sequences of closed sets with $F = \bigcup_n F_n$ and $G = \bigcup_n G_n$. With recursion we can construct sequences $\langle U_n\rangle_n$ and $\langle V_n\rangle_n$ of open sets such that

$$F_n\subseteq U_n \quad\text{and}\quad G_n\subseteq V_n;$$

$$\overline{U}_n\cap\overline{V}_n = U_n\cap\overline{G} = V_n\cap\overline{F} = \emptyset;$$

and

$$\overline{U}_n\subseteq U_{n+1} \quad\text{and}\quad \overline{V}_n\subseteq V_{n+1}, \quad\text{for all } n\in\omega.$$

Then $\bigcup_n U_n$ and $\bigcup_n V_n$ are disjoint open F_σ's, hence $\overline{F}\cap\overline{G}\subseteq(\bigcup_n U_n)^-\cap(\bigcup_n V_n)^- = \emptyset$.

(c) → (b): Trivial.

(b) → (a): Let $\mathscr{U}$, $\mathscr{V}$ be disjointed countable subsets of B. Then $\mathscr{U}$ and $\mathscr{V}$ are disjoint open F_σ's, hence $(\bigcup\mathscr{U})^-\cap(\bigcup\mathscr{V})^- = \emptyset$. Since X is zero-dimensional and compact there is an $S\in B$ with $\bigcup\mathscr{U}\subseteq S$ and $\bigcup\mathscr{V}\subseteq X-S$. Clearly S separates $\mathscr{U}$ from $\mathscr{V}$. ■

6.2. PROPOSITION. (a) *A closed subspace of a compact F-space again is compact F-space.*

(b) *A homomorphic image of a* WCC *algebra again is a* WCC *algebra.*

Proof. (a) is obvious, and (b) is the dual of (a) because of 6.1. ■

6.3. Remarks. (a) WCC-algebras were called almost σ-complete by Koppelberg, [K$_1$], who also noted that 6.2b holds, and are called CSP (≡ Countable Separation Property) Boolean algebra in [vDMR].

(b) Our definition of F-space is not the original definition, [GH], and does not tell when a noncompact space is an F-space, but it is the most convenient definition for our purposes.

(c) Our proof that (b) ↔ (c) in 6.1 does not use the fact that X is zero-dimensional.

(d) In contrast with 6.2 it is well known that $\beta\omega$ is extremally disconnected, but that its closed subspace $\beta\omega-\omega$ is not even basically disconnected, [GJ, 6W.3], but of course it is an F-space. In this context we mention, as in the introduction, that it is consistent that there be a compact zero-dimensional F-space which cannot be embedded in a basically disconnected space, [vDvM].

7. Cardinality.

7.1. THEOREM. *If X is an infinite compact F-space, then $|X|^\omega = |X|$, and $|X|\geqslant 2^{2^\omega}$.*

The first part of the theorem is an easy corollary to the following lemma.

7.2. LEMMA. *Let X be a regular space. If $\varkappa\leqslant|X|$ and $\mathrm{cf}\,\varkappa=\omega$, then there is an almost disjoint $\mathscr{A}\subseteq[X]^\omega$ with $|\mathscr{A}| = \varkappa^\omega$ such that $A\cup B$ is relatively discrete for all $A, B\in\mathscr{A}$.*

Proof. For $x\in X$ define the *local cardinality* $\mathrm{lk}(x)$ of x in X by

$$\mathrm{lk}(x) = \min\{|U|: U \text{ is a neighborhood of } x\}.$$

Let $\langle\varkappa_n\rangle_n$ be a (strictly) increasing sequence of cardinals with $\sup_n\varkappa_n = \varkappa$.

Case 1. There is a sequence $\langle x_n\rangle_n$ of distinct points such that $\mathrm{lk}(x_n)\geqslant\varkappa_n$ for $n\in\omega$.

Since the $\varkappa_n$'s are increasing, and since every infinite subset of X has an infinite relatively discrete subset, we may assume that $\langle x_n\rangle_n$ is relatively discrete. Since X is regular we can find a disjoint open sequence $\langle K_n\rangle_n$ in X with $x_n\in K_n$ for $n\in\omega$.

Then $|K_n|\geqslant\mathrm{lk}(x_n)\geqslant\varkappa_n$ for all $n\in\omega$, hence there is $\mathscr{A}$ as in Lemma 5.1. If $A, B\in\mathscr{A}$, then $|K_n\cap(A\cup B)|\leqslant 2$ for all $n\in\omega$, hence $A\cup B$ is relatively discrete since the K_n's are disjoint and open and $A\cup B\subseteq\bigcup_n K_n$.

Case 2. Not Case 1.

Then an attempt to recursively pick x_n's must fail, so there is a finite (possibly empty) $F\subseteq X$ and there is a $\lambda<\varkappa$ (namely some $\varkappa_n$) such that $\mathrm{lk}(x)<\lambda$ for all $x\in X-F$. For each $x\in X-F$ choose a neighborhood U_x of x with $|U_x|<\lambda$. By Hajnal's Free Set Lemma 5.3 there is a $D\subseteq X$ with $|D| = \varkappa$ (even $|D| = |X|$ is possible) such that $x\notin U_y$ for distinct $x, y\in D$. Then obviously D is relatively discrete. By Corollary 5.2 there is an almost disjoint $\mathscr{A}\subseteq[D]^\omega$ with $|\mathscr{A}| = \varkappa^\omega$. Clearly $A\cup B$ is relatively discrete for all $A, B\in\mathscr{A}$. ■

7.3. Proof of Theorem 7.1. Let $\varkappa$ be any cardinal with $\mathrm{cf}\varkappa = \omega$ and $\varkappa\leqslant|X|$. Let $\mathscr{A}$ be as in Lemma 7.2. For $A\subseteq X$ let $A^* = A^- - A$, the set of cluster points outside A. Note that $A^* = (A-F)^*$ if F is finite.

For any two distinct $A, B\in\mathscr{A}$ the set $F = A\cap B$ is finite, hence

$$A^*\cap B^*\subseteq(A-F)^-\cap(B-F)^- = \emptyset$$

since X is an F-space (for $A\cup B$ is relatively discrete, hence $(A-F)^-\cap(B-F) = (A-F)\cap(B-F)^- = \emptyset$). Now $A^*\neq\emptyset$ for $A\in\mathscr{A}$ since X is compact, hence $|X|\geqslant|\mathscr{A}| = \varkappa^\omega$.

It follows from Lemma 5.4 that $|X|^\omega = |X|$.

To see that $|X|\geqslant 2^{2^\omega}$ observe that if N is a countable relatively discrete subset of X then $\overline{N} = \beta N$, which is homeomorphic to $\beta\omega$, and recall that $|\beta\omega| = 2^{2^\omega}$, [C, 2.4], [CN, 7.4] or [GJ, 9.3]. ■

7.4. Remarks (a) The proof of 7.2 resembles the proof of the theorem of Hajnal and Juhász that if X is regular and $\mathrm{cf}[s(X)] = \omega$, then X has a relatively discrete subset of cardinality $s(X)$ (i.e. sup = max), see [HJ$_1$] or [Ju$_1$, 3.3].

(b) We did not use the full force of X being an F-space, nor of X being compact for the first part of the theorem. What we proved then is the following.

7.5. THEOREM. *Let X be an infinite countably compact space with the property that*

(*) *for all countably relatively discrete $D\subseteq X$, and for all $A\subseteq D$ it is true that $\overline{A}\cap(D-A)^- = \emptyset$.*

Then $|X|^\omega = |X|$. ■

It is known that every F-space satisfies (*), [GJ, 14N.5], but even for compact X(*) does not imply that X is an F-space, [vD$_2$].

8. Character. The main result of this section is Theorem 8.3. We begin with an easy observation which undoubtedly has been made before.

8.1. Proposition. *Let X be a compact space, and let $x \in X$. If $\mathrm{cf}[\chi(x, X)] = \omega$, then there is a nontrivial sequence in X that converges to x.*

Proof. There is a sequence $\langle \mathcal{U}_n \rangle_n$ of open families in X with

(1) $|\mathcal{U}_n| < \chi(x, X)$ for $n \in \omega$,

(2) $\mathcal{U}_n \subseteq \mathcal{U}_{n+1}$ for $n \in \omega$, and $\bigcup_n \mathcal{U}_n$ is a local base at x.

Lemma 4.2 tells that $|\bigcap \mathcal{U}_n| \geqslant \omega$ for all $n \in \omega$. Hence there is a sequence $\langle s_n \rangle_n$ of distinct points with $s_n \in \bigcap \mathcal{U}_n$ for $n \in \omega$. This sequence converges to x by (2). ∎

8.2. Corollary. *If X is a compact F-space, then $\mathrm{cf}[\chi(x, X)] \neq \omega$ for all $x \in X$.*

Proof. A compact F-space has no nontrivial convergent sequences, e.g. by 6.2a. ∎

8.3. Theorem. *If X is a compact F-space, then $\mathrm{cf}[\chi(X)] \neq \omega$.*

Proof. Let $\varkappa$ be any cardinal with $\mathrm{cf}\varkappa = \omega$ and $\varkappa \leqslant \chi(X)$. We will show that $\chi(X) > \varkappa$ by finding $p \in X$ with $\chi(p, X) > \varkappa$; because of 8.2 it suffices to find p with $\chi(p, X) \geqslant \varkappa$.

Let $\langle \varkappa_n \rangle_n$ be a strictly increasing sequence of cardinals with $\varkappa = \sup_n \varkappa_n$. For $n \in \omega$ pick $p_n \in X$ with $\chi(p_n, X) > \varkappa_n$.

If $\chi(p_n, X) \geqslant \varkappa$ for some n we are done. If not, we may assume without loss of generality that the p_n's are distinct. Hence we may even assume that $\langle p_n \rangle_n$ is relatively discrete (for every infinite subset of X has an infinite relatively discrete subset). Since X is regular we can find a disjoint open sequence $\langle P_n \rangle_n$ in X with $p_n \in P_n$ for $n \in \omega$.

Let p be any cluster point of $\langle x_n \rangle_n$, and let $\mathcal{U}$ be an open neighborhood base for p. We claim that $|\mathcal{U}| \geqslant \varkappa$, for suppose not. Without loss of generality $|\mathcal{U}| < \varkappa_0$. Because of Lemma 4.2 we can choose

$$q_n \in (P_n \cap \bigcap \{U \in \mathcal{U}: p_n \in U\}) - \{p\}$$

for $n \in \omega$. Since $\langle P_n \rangle_n$ is a disjoint open family; we have

$$\{p_n: n \in \omega\}^- \cap \{q_n: n \in \omega\} = \{p_n: n \in \omega\} \cap \{q_n: n \in \omega\}^- = \emptyset .$$

Since $p \in \{p_n: n \in \omega\}^-$, and since X is an F-space, it follows that $p \notin \{q_n: n \in \omega\}^-$.

Hence there is $U \in \mathcal{U}$ with $U \cap \{q_n: n \in \omega\} = \emptyset$. But U must contain some p_n, and hence some q_n, since p is a cluster point of $\langle p_n \rangle_n$. ∎

More information about character follows from the following result of Pospíšil, [Po] (see e.g. [C, 2.7], [CN, 7.15] or [Ku_1, 2.8] for a recent reference).

8.4. Theorem. *There is $p \in \beta\omega$ such that $\chi(p, \beta\omega) = 2^\omega$.* ∎

8.5. Corollary. *If X is an infinite compact F-space, then there is a $p \in X$ such that $\chi(p, X) \geqslant 2^\omega$. Hence $\chi(X) \geqslant 2^\omega$.* ∎

8.6. Remark. In 8.2, 8.3 and 8.4 it suffices to assume that X is a compact space satisfying (∗) of 7.5.

9. Hereditary Lindelöf degree.

9.1. Theorem. *If X is a compact F-space, then $\mathrm{cf}[hL(X)] \neq \omega$.*

We need the following easy observation.

9.2. Lemma. *Let X be any space.*

(a) *$hL(X) = \sup\{L(Y): Y$ is an open subspace of $X\}$.*

(b) *If $\mathcal{A}$ is a cover of X (not necessarily consisting of open sets), then*

$$hL(X) \leqslant \sum_{A \in \mathcal{A}} hL(A) . \blacksquare$$

The following lemma gives half of the proof of Theorem 9.1.

9.3. Lemma. *Let $\langle \varkappa_n \rangle_n$ be a strictly increasing sequence of infinite cardinals, and let $\varkappa = \sup_n \varkappa_n$. Let X be a compact F-space.*

If there is a disjoint clopen sequence $\langle K_n \rangle_n$ in X such that $hL(X_n) > \varkappa_n$ for $n \in \omega$, then $hL(X) > \varkappa$.

Proof. Because of 9.2a and 4.2 we can choose a closed $F_n \subseteq K_n$ with $\chi(F_n, X_n) \geqslant \varkappa_n$ for $n \in \omega$. (We have $\geqslant$, not $>$ here because we do not want to get involved in the sup = max problem; see [Ju_1, Ch. 3] about this sort of problems.) Define

$$F = (\bigcup_n F_n)^- .$$

We claim that $\chi(F, X) > \varkappa$; in view of 4.2 this will prove that $hL(X) > \varkappa$. We combine the proofs of 8.1 and 8.3.

Suppose $\chi(F, X) \leqslant \varkappa$. Then there is a sequence $\langle \mathcal{U}_n \rangle_n$ of open families such that

(1) $|\mathcal{U}_n| < \varkappa_n$, for $n \in \omega$;

(2) $\mathcal{U}_n \subseteq \mathcal{U}_{n+1}$ for $n \in \omega$, and $\bigcup_n \mathcal{U}_n$ is a local base at F.

Since $|\mathcal{U}_n \cup \{K_n\}| < \chi(F_n, X)$ for $n \in \omega$, we can pick

$$q_n \in ((\bigcap \mathcal{U}_n) \cap K_n) - F_n$$

for $n \in \omega$, by 4.2. Since $\langle K_n \rangle_n$ is a disjoint open sequence we have

$$F \cap \{q_n: n \in \omega\} = (\bigcup_n F_n) \cap \{q_n: n \in \omega\}^- = \emptyset ,$$

hence $F \cap \{q_n: n \in \omega\}^- = \emptyset$. It follows from (2) that there are $k \in \omega$ and $U \in \mathcal{U}_k$ such that $U \cap \{q_n: n \in \omega\} = \emptyset$. But then $(\bigcap \mathcal{U}_k) \cap \{q_n: n \in \omega\} = \emptyset$, which contradicts the choice of our q_n's. ∎

9.4. Proof of Theorem 9.1. For convenience we assume that X is zero-dimensional. Evidently we may assume that $|X| \geqslant \omega$, then also $\chi(X) \geqslant \omega$.

Since $\chi(X) \leqslant hL(X)$, by 4.2, and since we know already from 8.3 that $\mathrm{cf}[\chi(X)] \neq \omega$, we may assume that $\chi(X) < hL(X)$.

Let $\varkappa$ be any cardinal with $\chi(X) < \varkappa \leqslant hL(X)$ and $\mathrm{cf}\varkappa = \omega$, and let $\langle \varkappa_n \rangle_n$ be a strictly increasing sequence of cardinals with $\sup_n \varkappa_n = \varkappa$ and $\varkappa_0 > \chi(X)$ (so $\varkappa_0 \geq \omega$). We will construct a disjoint clopen sequence $\langle K_n \rangle_n$ in X with $hL(X_n) > \varkappa_n$ for $n \in \omega$. This will show that $hL(X) > \varkappa$ because of 9.3.

Since X is compact, it follows from 9.2b that there is a $p \in X$ such that

(1) if U is any neighborhood of p, then $hL(U) = hL(X)$.

Let $n < m$, and suppose we have constructed K_n already for $n < m$, satisfying the additional condition that $p \notin K_n$ for $n < m$. Then $U = X - \bigcup_{n<m} K_n$ is a clopen set containing p. Since $\chi(p, U) = \chi(p, X) < \varkappa_0 \leq \varkappa_m$, and since X is zero-dimensional, there is a clopen family $\mathscr{K}$ in X with $\bigcup \mathscr{K} = U - \{p\}$ and $|\mathscr{K}| < \varkappa_m$.

Since $hL(\{p\}) = 1$, it follows from (1) and 9.2b that there is a $K_m \in \mathscr{K}$ with $hL(K_m) > \varkappa_m$.

This completes the construction of the K_n's. ■

9.5. Remark. In a similar way one can prove that if X is an infinite compact F-space, then

(a) $\mathrm{cf}[w(X)] \neq \omega$;

(b) if $s(X) > \chi(X)$, then $s(X)^\omega = s(X)$;

(c) if $hd(X) > \chi(X)$, then $\mathrm{cf}[hd(X)] \neq \omega$.

There is a better result than (a) available, see § 12, and I do not think (b) and (c) are interesting enough to warrant inclusion of the proof.

9.6. Remark. If X is an infinite compact F-space then $hL(X) \geq 2^\omega$. This follows from the facts thats X has a subspace homeomorphic to $\beta\omega$ and that $\beta\omega$ has a relatively discrete subset of cardinality 2^ω. Alternatively, use Corollary 8.5 and the fact that $hL(X) \geq \chi(X)$.

10. Spread.

10.1. Theorem. *If X is a compact F-space, then $s(X)$ is not a strong limit with countable cofinality. In fact, if λ is any strong limit with $\mathrm{cf}\lambda = \omega$ and $\lambda \leq s(X)$, then X has a relatively discrete subset of cardinality λ^ω.*

Proof. X has a relatively discrete subset D with $|D| = \lambda$: if $s(X) > \lambda$ this is clear, and if $s(X) = \lambda$ it follows from the theorem of Hajnal and Juhász quoted in 7.4. By 5.2 there is an almost disjoint $\mathscr{A} \subseteq [D]^\omega$ with $|\mathscr{A}| = \lambda^\omega$. For $A \subseteq X$ put $A^* = \bar{A} - A$. As in 7.2 each A^* is nonempty. In 7.2 we also showed that

(1) $$A^* \cap B^* = \emptyset \quad \text{for distinct } A, B \in \mathscr{A}.$$

We would like to show that each A^* is open in $\bigcup_{B \in \mathscr{A}} B^*$, for then it would follow from (1) that X has a discrete subset of cardinality $|\mathscr{A}| = \lambda$. Unfortunately this need not be true, see Example 13.10. We will use the fact that λ is a strong limit cardinal to overcome this annoying fact: we simply construct a $\mathscr{B} \subseteq \mathscr{A}$ with $|\mathscr{B}| = \lambda^\omega$ such that each A^* with $A \in \mathscr{B}$ is open in $\bigcup_{B \in \mathscr{B}} B^*$.

Since $\mathrm{cf}\lambda = \omega$, there is a sequence $\langle D_n \rangle_n$ of subsets of D with $\bigcup_n D_n = D$, such that $|D_n| < \lambda$ for $n \in \omega$. Since λ is a strong limit, we also have $|\bar{D}_n| < \lambda$ for $n \in \omega$. (Recall that $|Y| \leq \exp^2 d(Y)$ for every Hausdorff space Y, see e.g. [Ju$_1$, 2.4] for the easy proof.) Hence the set

$$E = \bigcup_n \bar{D}_n$$

is an F_σ including D with $|E| = \lambda$. We now define $\mathscr{B} \subseteq \mathscr{A}$ by

$$\mathscr{B} = \{A \in \mathscr{A}:\ A^* \cap E = \emptyset\}.$$

Then $|\mathscr{B} - \mathscr{A}| \leq |E| = \lambda$ by (1), hence $|\mathscr{B}| = \lambda^\omega$ since $|\mathscr{A}| = \lambda^\omega > \lambda$. (Recall that $\mathrm{cf}\lambda = \omega$.)

Let $A \in \mathscr{B}$ be arbitrary. We have to show that A^* is open in $\bigcup_{B \in \mathscr{B}} \mathscr{B}^*$, or, equivalently, that

$$A^* \cap (\bigcup \{\mathscr{B}^*:\ B \in \mathscr{B} - \{A\}\})^- = \emptyset.$$

If $B \in \mathscr{B} - \{A\}$, then $B^* \subseteq (D - A)^-$ since $|A \cap B| < \omega$ (and $B \subseteq D$). So it suffices to show that

$$\bar{A} \cap (D - A)^- = \emptyset.$$

Let $F = E \cap (D - A)^-$, then F and A are two F_σ-subsets of the F-space X. Clearly $\bar{F} \cap A = (D - A)^- \cap A = \emptyset$ since D is relatively discrete. But also $F \cap \bar{A} = \emptyset$ since $F \subseteq E$ and $(\bar{A} - A) \cap E = \emptyset$ and since $F \cap A = \emptyset$. So $\bar{A} \cap \bar{F} = \emptyset$, therefore $\bar{A} \cap (D - A)^- = \emptyset$ since $\bar{F} = (D - A)^-$. ■

11. Density and cellularity in special compact F-spaces. There are no restrictions on the density and cellularity of a compact F-space in general, see Example 14.1. We are interested in compact spaces in which nonempty G_δ-subsets have non-empty interior (or in Boolean algebras B with the property that if $\langle a_n \rangle_n$ is a sequence in B with $a_n \geq a_{n+1} > 0$ for $n \in \omega$, then there is a $b \in B - \{0\}$ with $a_n \geq b$ for $n \in \omega$). We have the following easy result.

11.1. Theorem. *If X is an infinite compact F-space in which every nonempty G_δ has nonempty interior, then $d(X)$ is not a strong limit with countable cofinality.*

Proof. Let λ be a strong limit with countable cofinality, and assume $\lambda \leq d(X)$. Let D be any subset of X with $|D| = \lambda$. In the proof of 10.1 we showed that there is an F_σ-subset E in X with $|E| = \lambda$ and $E \supseteq D$. Since $|X| = \lambda^\omega > \lambda$ by Theorem 7.1, we see that $X - E$ is a nonempty G_δ, so $\mathrm{Int}(X - E) \neq \emptyset$. Hence D is not dense. ■

11.2. Corollary. *If X is a (noncompact) σ-compact locally compact space, then $d(\beta X - X)$ is not a strong limit with countable cofinality.*

This corollary shows that 11.1 is not vacuous, Theorem 17.1 also shows that the next result belongs to this section.

11.3. Theorem. *If X is a (noncompact) σ-compact locally compact space, then $c(\beta X - X)$ is not a strong limit with countable cofinality. In fact, for every strong* *limit λ with countable cofinality, if $c(\beta X-X)\geq\lambda$, then $\beta X-X$ has a disjoint open family with cardinality λ^ω.*

Proof. We give the proof for zero-dimensional X. Let B be the clopen algebra of X, and let $\mathscr{I}$ be the ideal of compact members in B. Then $c(\beta X-X) = c(B/\mathscr{I})$, see Section 17. Hence

(*) for all $\varkappa$, there is a disjoint open family in $\beta X-X$ of cardinality $\varkappa$ iff there is a $\mathscr{U}\subseteq B-\mathscr{I}$ with $|\mathscr{U}| = \varkappa$, such that $U\cap V\in\mathscr{I}$ for any two distinct $U, V\in\mathscr{U}$.

Define

$$\gamma = \min\{c(X-A):\ A\in\mathscr{I}\}.$$

Since $\beta X-X = \beta(X-A)-(X-A)$ for all $A\in\mathscr{I}$, we may assume without loss of generality that $\gamma = c(X)$.

Claim. $c(\beta X-X)\leq\exp c(X)$.

This is due to Ginsburg and Woods, and does not depend on X being zero-dimensional, [GW]. For completeness sake we give the argument. There is an increasing sequence $\langle I_n\rangle_n$ in $\mathscr{I}$ with $\bigcup_n I_n = X$. Suppose there is $\mathscr{U}$ as in (*) with $|\mathscr{U}|>\exp c(X)$. For $n\in\omega$ define $P_n = \{\{U, V\}\in[\mathscr{U}]^2:\ U\cap V\subseteq I_n\}$. Then $[\mathscr{U}]^2 = \bigcup_n P_n$, hence from the partition relation $(2^\varkappa)^+\to(\varkappa^+)^2_\varkappa$, [E], or [CN, 8.10] or [Ju$_1$, A4.4], we find $\mathscr{V}\subseteq\mathscr{U}$ with $|\mathscr{V}|>c(X)$ and $n\in\omega$ such that $[\mathscr{V}]^2\subseteq I_n$. But then $\{V-I_n:\ V\in\mathscr{V}\}$ is a disjoint open family in X of cardinality $>c(X)$.

It follows that $\lambda\geq\gamma$ since λ is a strong limit. Let $\langle\lambda_n\rangle_n$ be a strictly increasing sequence of cardinals with $\sup_n\lambda_n = \lambda$. Since X is σ-compact and zero-dimensional, we can find a countable disjoint $\mathscr{D}\subseteq\mathscr{I}-\{\emptyset\}$ with $\bigcup\mathscr{D} = X$ such that for all $n<\omega$ there are only finitely many $D\in\mathscr{D}$ with $|D|\leq n$. Then for all finite $\mathscr{F}\subseteq\mathscr{D}$ we have

$$c(\bigcup(\mathscr{D}-\mathscr{F})) = \sup\{c(D):\ D\in\mathscr{D}-\mathscr{F}\}.$$

Hence we can find a disjoint sequence $\langle D_n\rangle_n$ in $\mathscr{D}$ with $c(D_n)>\lambda_n$ for $n\in\omega$. For each $n\in\omega$ we can choose a disjoint family $K_n\subseteq\mathscr{I}-\{\emptyset\}$ with $|K_n| = \lambda_n$ and $\bigcup K_n\subseteq D_n$. It should be clear now how to construct $\mathscr{U}$ as in (*) with $|\mathscr{U}| = \lambda^\omega$ from the K_n's, using Lemma 5.1. ■

11.4. Remarks. (a) I do not know if $c(X)$ cannot be a strong limit with countable cofinality if X is a compact F-space in which nonempty G_δ-subsets have nonempty interior.

(b) The equivalence in (*) is a weak version of a similar characterization of Comfort and Gordon which holds for all X, [CG, 3.3]. The construction of $\mathscr{U}$ resembles the argument of [CG, 3.1].

(c) For the construction of $\mathscr{U}$ we only used the fact that $\mathrm{cf}\,\gamma = \omega$. So because of 5.4 the following theorem is a corollary to the proof of 11.2, at least for zero-dimensional X (but it holds as stated).

11.5. Theorem. *If X is a (noncompact) σ-compact locally compact space, and $\gamma = \min\{c(X-K):\ K\subseteq X$ compact$\}$, then $\gamma^\omega\leq c(\beta X-X)\leq 2^\gamma$.* ■

11.6. Corollary. *If K is an infinite compact space, and $X = \omega\times K$, then $c(K)^\omega\leq c(\beta X-X)\leq\exp c(K)$.* ■

11.7. Remark. In answer to my question Fleissner has shown that it is consistent that there be a compact K with $c(K) = 2^\omega$ but if $X = \omega\times K$ then $c(\beta X-X) = 2^{2^\omega}$, [F1]. In Example 14.4 we will see that there also is a K with $c(K) = 2^\omega$ for which $c(\beta X-X) = 2^\omega$. So 11.5 is in a certain sense best possible.

12. Weight.

12.1. Theorem. *If X is an infinite compact F-space then $w(X)^\omega = w(X)$.*

This was proved for extremally disconnected X by Pierce, [P], and for basically disconnected X by Comfort and Hager, [CH$_1$]. I proved Theorem 12.1 by modifying their proof. After being informed of 12.1, Wis Comfort and Donald Monk have pointed out that S. Koppelberg has proved 12.1 for zero-dimensional X (in the Boolean algebraic translation), also by modifying the proof of Comfort and Hager, [K$_2$]. The assumption of zero-dimensional reduces the technical complexity somewhat, but the complete proof of 12.1 does not require new ideas, hence it will not be included; the major difficulty is the following theorem of Efimov, [E], see also [CH$_2$], which is almost trivial in case the X_n's are zero-dimensional.

12.2. Theorem. *If Σ denotes the topological sum of a sequence $\langle X_n\rangle_n$ of infinite compact spaces, then $w(\beta\Sigma) = \prod_n w(X_n)$.* ■

12.3. Remark. Pierce, [P] and Efimov, [E], show that Theorem 12.1 is best possible for extremally disconnected spaces in the sense that for every $\varkappa$ with $\varkappa^\omega = \varkappa$ there is a compact extremally disconnected space X with $w(X) = \varkappa$. (As pointed out in [CH$_2$, p. 378], the argument Efimov offers in [E] as a proof of 12.1 for extremally disconnected spaces is wrong.)

13. Relations between cardinal functions on extremally disconnected compacta.

13.1. Theorem. *If X is an infinite compact extremally disconnected space, then*

$$s(X) = hL(X) = \chi(X) = w(X) \quad and \quad |X| = \exp w(X).$$

Proof. Balcar and Franěk have recently shown that every infinite complete Boolean algebra B has an independent subset of cardinality $|B|$, [BF]. So if σ denotes $w(X)$, then X admits a continuous map onto ${}^\sigma 2$, the product of σ copies of 2. The rest is routine.

Since X can be mapped onto ${}^\sigma 2$ we have $|X|\geq{}^\sigma 2$, and since $|Y|\leq\exp w(Y)$ for every T_0-space Y, [Ju$_1$, 2.2], we have $|X| = 2^\sigma$ as noted in [BF].

Since X can be mapped onto ${}^\sigma 2$ and $s({}^\sigma 2) = \sigma$, we have $s(X)\geq\sigma$. Hence $s(X) = hL(X) = w(X)$ since trivially $s(Y)\leq hL(Y)\leq w(Y)$ for every Y.

Finally, since X can be mapped onto ${}^\sigma 2$, there is a $p\in X$ with $\chi(p, X)\geq\sigma$, this is a result of Pospíšil, [Po]; see [C, 2.6] or [CN, 7.13] for a recent reference. ■

In view of this proof we point out that Example 14.7 shows that there exists a compact basically disconnected space X such that $\chi(p, X) < \chi(X)$ for all $p \in X$.

14. Examples.

A. Sharpness of results. Our results put restrictions on the values taken on by certain cardinal functions within the class of compact F-spaces or within the (narrower) class of spaces of the form $\beta X - X$, with X σ-compact and locally compact. We show that our results are best possible, at least under GCH, in the sense that there are no further restrictions; our results about cardinality are best possible even without GCH.

We will show that our results are best possible not only within the class of compact F-spaces, but also within the narrower class of compact basically disconnected spaces and also within the narrower class of spaces of the form $\beta X - X$, with X σ-compact and locally compact. (These classes are disjoint.) For the second class we also have to consider c and d, because of 11.1 and 11.3. We first point out that analogues of 11.1 and 11.3 do not hold for compact F-spaces in general.

14.1. Example. *For every $\varkappa \geqslant \omega$ there is a compact extremally disconnected space X with $c(X) = d(X) = \varkappa$.*

Proof. $X = \beta\varkappa$ will do. ∎

We next consider compact basically disconnected spaces.

14.2. Example. *For every $\varkappa \geqslant \omega$ with $\varkappa^{\omega} = \varkappa$ there is a compact basically disconnected space X with*

(a) $|X| = \varkappa \cdot 2^{2^{\omega}}$;

(b) $\chi(X) = hL(X) = s(X) = w(X) = \varkappa$;

(c) *there is a point $p \in X$ with $\chi(p, X) = \varkappa$.*

Proof. We construct X as the Stone space of a suitable Boolean algebra B, so we calculate cardinal functions of B, rather than X. We will use 4.1 without explicit reference.

Let B be the Boolean algebra of countable and cocountable subsets of $\varkappa$. Clearly B is countably complete and $|B| = \varkappa^{\omega} = \varkappa$.

Let $\mathscr{F}$ be the filter of cocountable subsets of $\varkappa$. Then $\mathscr{F}$ is an ultrafilter in B. If $\mathscr{G} \subseteq \mathscr{F}$ has $|\mathscr{G}| < \varkappa$, then $|\varkappa - \bigcap \mathscr{G}| \leqslant |\mathscr{G}| \cdot \omega = \varkappa$, hence $\mathscr{G}$ does not generate $\mathscr{F}$. Since $|\mathscr{B}| = \varkappa$, it follows that $\chi(\mathscr{F}, B) = \varkappa$. Therefore $hL(B) \geqslant \chi(B) \geqslant \varkappa$, hence, using the fact that $|B| = \varkappa$ again, we see that $hL(B) = \chi(B) = \varkappa$.

From 5.2 we see that $c(B/[\varkappa]^{<\omega}) \geqslant \varkappa^{\omega} = \varkappa$, hence $s(B) = hc(B) \geqslant \varkappa$. Since $s(B) \leqslant |B| = \varkappa$, it follows that $s(B) = \varkappa$. ∎

Before we proceed to our next example we give a topological description of Example 14.2; we do so because we will use the fact that Example 14.2 can be embedded in our next example.

If $\varkappa$ is any cardinal $> \omega$, let $\mathscr{F}_{\varkappa}$ be the filter of cocountable subsets of $\varkappa$, let $\lambda\varkappa$ be the space with underlying set $\varkappa \cup \{\infty\}$ (where $\infty \notin \varkappa$), which has

$$\{U \subseteq \varkappa \cup \{\infty\}: \text{ if } \infty \in \varkappa \text{ then } \varkappa \cap U \in \mathscr{F}_{\varkappa}\}$$

as a topology; $\lambda\varkappa$ is the "one-point Lindelöfization" of $\varkappa$. Note that $\lambda\varkappa$ is a P-space ($\equiv$ every G_{δ} is open), hence $\lambda\varkappa$ is basically disconnected, hence $\beta\lambda\varkappa$ is basically disconnected, [GJ, 6M.1]. It is easy to see that if $\varkappa = \varkappa^{\omega}$ then $\beta\lambda\varkappa$ is Example 14.2, cf. § 17.

14.3. Lemma. *If $b\lambda\varkappa$ is any compactification of $\lambda\varkappa$, then the following are equivalent:*

(a) $b\lambda\varkappa = \beta\lambda\varkappa$;

(b) *$b\lambda\varkappa$ is basically disconnected;*

(c) *$b\lambda\varkappa$ is an F-space;*

(d) *condition (∗) of 7.5 holds for $b\lambda\varkappa$.*

Proof. It suffices to prove that (d) implies (a). Since $\lambda\varkappa$ is strongly zero-dimensional, it suffices to prove that each clopen subset of $\lambda\varkappa$ has clopen closure in $b\lambda\varkappa$. Let $U \subseteq \lambda\varkappa$ be clopen. It suffices to show that U and $\lambda\varkappa - U$ have disjoint closures, so we may assume $\infty \notin U$. Then $\infty \notin \overline{U}$ (closure in $b\lambda\varkappa$), hence there is an open V in $b\lambda\varkappa$ with $\infty \in V$ and $\overline{V} \cap U = \emptyset$. Now $W = \lambda\varkappa - V$ is countable, and $U \subseteq W$, hence $\overline{U} \cap (W - U)^{-} = \emptyset$ by (∗). Since

$$(\lambda\varkappa - U)^{-} = (\lambda\varkappa \cap V)^{-} \cup \big(\lambda\varkappa - (V \cup U)^{-}\big) \subseteq V^{-} \cup (W - U)^{-}$$

it follows that $(\lambda\varkappa - U)^{-} \cap \overline{U} = \emptyset$, as required. ∎

14.4. Example. *For every $\varkappa > \omega$ with $\varkappa^{\omega} = \varkappa$ there is a σ-compact locally compact space X such that*

(a) $|\beta X - X| = \varkappa \cdot 2^{2^{\omega}}$;

(b) *$\varphi(\beta X - X) = \varkappa$ for $\varphi \in \{c, \chi, d, hL, s, w\}$;*

(c) *there is a $p \in \beta X - X$ with $\chi(p, \beta X - X) = \varkappa$.*

Proof. Let $\infty \notin \varkappa$, and let $\alpha\varkappa$ be the one-point compactification of $\alpha\varkappa$. Then our example is $X = \omega \times \alpha\varkappa$. Note that $c(\alpha\varkappa) = w(\alpha\varkappa) = \varkappa$.

First we note that $c(\beta X - X) \geqslant \varkappa^{\omega} = \varkappa$ by 11.5, and $w(\beta X - X) \leqslant \varkappa^{\omega} = \varkappa$ by 12.2. Since for every space Y we have

$$c(Y) \leqslant hc(Y) = s(Y) \leqslant hL(Y) \leqslant w(Y) \quad \text{and} \quad c(Y) \leqslant d(Y) \leqslant w(Y)$$

we conclude that $\varphi(\beta X - X) = \varkappa$ for $\varphi \in \{c, d, hL, s, w\}$.

Next we calculate $|\beta X - X|$. Since $c(Y) \leqslant |Y|$ for every space Y, we see from 7.1 that $|\beta X - X| \geqslant K \times 2^{2^{\omega}}$. In order to prove the reverse inequality we recall that $|Y| \leqslant \exp^2 d(Y)$ for every Hausdorff space Y, [Ju$_1$, 2.4], hence it suffices to prove the following

Claim 1. *For every $x \in \beta X - X$ there is a countable $A \subseteq X$ with $x \in \overline{A}$.*

Indeed, if $x \in (\omega \times \{\infty\})^{-}$, there is nothing to prove. If not, then x has a closed neighborhood U in βX which misses $w \times \{\infty\}$, and then $A = U \cap X$ is as required. For clearly $x \in \overline{A}$, and since A is a σ-compact subspace of X all points of which are isolated, it must be countable.

Since we know that $\beta\lambda\varkappa$ is Example 14.2, and since we know that $\chi(\beta X - X) \leqslant \varkappa$ since $w(\beta X - X) = \varkappa$, we complete the proof of (b) and prove (c) by proving the following

CLAIM 2. *$\beta\lambda\varkappa$ can be embedded into $\beta X - X$.*

Since $\beta X - X$ is an F-space, by 17.1, it suffices to show that $\lambda\varkappa$ can be embedded into $\beta X - X$, because of 14.3. Since $\lambda\varkappa$ and $\alpha\varkappa$ have the same underlying set, and $\lambda\varkappa$ is obtained from $\alpha\varkappa$ by letting the G_δ-subsets of $\alpha\varkappa$ be a base for $\lambda\varkappa$, this follows from our next lemma.

14.5. LEMMA. *Let K be any compact space. Let K_π be K, retopologized by using the G_δ-sets as a base. Then K_π can be embedded into $\beta(\omega \times K) - (\omega \times K)$.*

Proof. Denote $\omega \times K$ by P. Let $\pi\colon P \to \omega$ be the projection, let $\beta\pi\colon \beta P \to \beta\omega$ be its Stone extension. For each $y \in K$ let H_y be the horizontal line $\omega \times \{y\}$.

For each $y \in K$ the restriction $\pi \restriction H_y$ is a homeomorphism of H_y onto ω; since $\beta\omega$ is the biggest compactification of ω it follows that the restriction $\beta\pi|\overline{H}_y$ is a homeomorphism of $\overline{H}_y$ onto $\beta\omega$.

So if we pick any $p \in \beta\omega - \omega$, then we can define a function $f\colon K_\pi \to \beta X - X$ by assigning to each $y \in K_\pi$ the unique point of $\overline{H}_y \cap (\beta\pi)^{\leftarrow}\{p\}$.

CLAIM 1. *f is an injection.*

Indeed, if y and z are distinct points of K_π, then H_y and H_z are disjoint closed subsets of P, hence $\overline{H}_y \cap \overline{H}_z = \emptyset$ since P is normal.

CLAIM 2. *f is open.*

Let $y \in K_\pi$, and let U be a neighborhood of y in K_π. There is a decreasing sequence $\langle W_n\colon n \in \omega\rangle$ of open sets in K_π with $y \in \bigcap_n W_n \subseteq U$. Let $W = \bigcup_n \{n\} \times W_n$. Then $W^e = (\beta P - P) - (P - W)^-$ is an open set in $\beta P - P$. If z is any point of $K_\pi - U$, then $H_z - W$ is finite, hence $f(z) \in (X - W)^-$, hence $f(z) \notin W^e$. It follows that $W^e \subseteq f^{\to} U$. Since H_y and $P - W$ are disjoint closed sets in P, their closures in βP are disjoint. Consequently $f(y) \in W^e$.

CLAIM 3. *f is continuous.*

Let $y \in K_\pi$ and let W be a neighborhood of $f(y)$ in $f^{\to} K_\pi$. Then

$$F = [(f^{\to} K_\pi) - W]^-$$

is a subset of $(\beta\pi)^{\leftarrow}\{p\}$ (since $f^{\to} K_\pi \subseteq (\beta\pi)^{\leftarrow}\{p\}$) which is closed but does not contain $f(y)$. Since $f(y)$ is the only point of $\overline{H}_y \cap (\beta\pi)^{\leftarrow}\{p\}$, it follows that F and $\overline{H}_y$ are disjoint closed subsets of βP. Let U be an open neighborhood of $\overline{H}_y$ in βP whose closure misses F. For each $n \in \omega$ the set

$$U_n = \{z \in K\colon \langle n, z\rangle \in U\}$$

is open in K and contains y. Hence $\bigcap_n U_n$ is a neighborhood of y in K_π. If $z \in \bigcap_n U_n$ then $H_z \subseteq U$ hence $\overline{H}_z \cap F \subseteq \overline{U} \cap F = \emptyset$. It follows that $f^{\to} \bigcap_n U_n \subseteq W$. ■

14.6. Remark. In 14.4 we found a copy C of $(\alpha\varkappa)_\pi = \lambda\varkappa$ in $\beta X - X$ such that $\overline{C} = \beta C$. This is a happy coincidence. If, for example, one lets K be the closed unit interval in 14.5, then $w(\beta(\omega \times K)) = 2^\omega$ by 12.2, but K_π is discrete, so $w(\beta K_\pi) = 2^{2^\omega}$, hence βK_π cannot be embedded into $\beta(\omega \times K) - (\omega \times K)$.

Our next example gives information about character, not supplied by Example 14.2.

14.7. EXAMPLE. *For every cardinal $\varkappa$ with $\mathrm{cf}\varkappa > \omega$ there is a basically disconnected space X containing a point p with $\chi(p, X) = \mathrm{cf}\varkappa$.*

If $\varkappa$ is a singular strong limit, then X can be choosen to satisfy

$$|X| = \chi(X) = \varkappa, \quad \text{but} \quad \chi(x, X) < \varkappa \quad \text{for all } x \in X.$$

Proof. Let X be the Stone space of the subalgebra

$$\{A \subseteq \varkappa\colon \exists \alpha < \varkappa [A \cap (\alpha, \varkappa) = \emptyset \text{ or } (\alpha, \varkappa) \subseteq A]\}$$

of $\mathscr{P}(\varkappa)$. We omit the straightforward verification. ■

B. Additional examples. We first show that in our results the condition that X be a compact F-space cannot be weakened to the condition that X be a compact space without nontrivial convergent sequences, with exception of 8.2 of course. The example is due to Hodel, [Ho, 4.4].

14.8. EXAMPLE. *If $\varkappa$ is any uncountable strong limit cardinal with countable cofinality, then there is a compact space X such that*

$$\varphi(X) = \varkappa \quad \text{for} \quad \varphi \in \{|\ |, c, \chi, d, hL, s, w\}$$

yet X has no nontrivial convergent sequences, and $\chi(x, X) < \varkappa$ for all $x \in X$.

Proof. Let $\langle D_n\rangle_n$ be a decomposition of $\varkappa$ with $|D_n| < \varkappa$ for $n \in \omega$. Let $f\colon \varkappa \to \omega$ be the function

$$f = \bigcup_n (D_n \times \{n\})$$

and let $\beta f\colon \beta\varkappa \to \beta\omega$ be its Stone extension. Let X be the quotient space obtained from $\beta\varkappa$ by collapsing every $(\beta f)^{\leftarrow}\{p\}$, with $p \in \beta\omega - \omega$, to a point. A straightforward consideration of cases shows that X is Hausdorff. Let $q\colon \beta\varkappa \to X$ be the quotient map, we may assume that $q \restriction f^{\leftarrow}\omega$ is the identity. (One can visualize X as $\bigcup_n \overline{D}_n$, compactified by pasting on a copy of $\beta\omega - \omega$.)

$\varkappa$ is a dense set of isolated points of X, hence $c(X) = \varkappa$.

$\overline{D}_n = \beta D_n$, hence $|D_n| = \exp^2 |D_n| < \lambda$ for all $n \in \omega$. It follows that

$$|X| \leqslant \omega \cdot \varkappa + |\beta\omega - \omega| = \varkappa + 2^{2^\omega} = \varkappa.$$

Since $c(Y) \leqslant \varphi(Y) \leqslant |Y|$ for all φ considered if Y is just any compact space, if follows that $\varphi(X) = \varkappa$ for all φ considered.

We show that X has no nontrivial convergent sequences by showing that

(*) every infinite closed subset of X has cardinality $\geqslant 2^{2^\omega}$.

There obviously is a mapping $g\colon \varphi \to \beta\omega$ such that $f = g \circ q$. Then $g \restriction g^{\leftarrow}(\beta\omega - \omega)$ is injective, and so is $g \restriction f^{\leftarrow}\omega$, as noted above. Since every infinite closed subset of $\beta\varkappa$ or of $\beta\omega$ has cardinality $\geqslant 2^{2^\omega}$, by the second half of 7.1, it now is easy to verify (*).

Since X has no nontrivial convergent sequences, and $\chi(X) = \varkappa$, it follows from 8.1 that $\chi(x, X) < \varkappa$ for all $x \in X$. (This also can be easily verified directly.) ■

Our next example was promised in the introduction. It is due to Gillman and Henriksen, [GH, 2.8].

14.9. EXAMPLE. *There is an infinite connected compact F-space.*

Proof. If H is the half-line $[0, \infty)$, then $X = \beta H - H$ is such an example. X is clearly compact and infinite. It is connected since $X = \bigcap_n [n, \infty)^-$. It is an F-space because of 17.1. [One can make X arbitrarily large by multiplying H with a big continuum.) ■

Our final example was promised in the proof of 10.1.

14.10. EXAMPLE. *There is a compact F-space X with a discrete subset D and an almost disjoint collection $\mathscr{A}$ of subsets of D such that $\bar{N} - N$ is not open in $\bigcup\{\bar{A} - A \colon A \in \mathscr{A}\}$ for some $N \in \mathscr{A}$.*

Proof. As before, let $\lambda\omega_1 = \omega_1 \cup \{\infty\}$ be the one-point Lindelöfization of ω_1, then $\beta\lambda\omega_1$ is basically disconnected. Let N be a countable discrete space and assume βN and $\beta\lambda\omega_1$ are disjoint. Pick any $p \in \beta N - N$, and let X be the quotient space obtained from topological sum $\beta N + \beta\lambda\omega_1$ by identifying the points ∞ and p.

One can easily verify that X is a compact F-space using the fact that ∞ is a P-point of $\beta\lambda\omega_1$ (i.e. every G_δ-subset of $\beta\lambda\omega_1$ that contains ∞ is a neighborhood of ∞ in $\beta\lambda\omega_1$).

Let $\mathscr{D}$ be a disjoint family of cardinality ω_1 consisting of countable subsets of ω_1. We claim that $D = N \cup \omega_1$ and $\mathscr{A} = \mathscr{D} \cup \{N\}$ are as required. Clearly D is discrete, and $\{p, \infty\} \in \bar{N} - N$. Also each neighborhood of ∞ in $\beta\lambda\omega$ must intersect a member of $\mathscr{D}$ since $|\bigcup \mathscr{D}| = \omega_1$, hence $\{p, \infty\} \in (\bigcup\{\bar{A} - A \colon A \in \mathscr{A}\})^-$. As $(\bar{N} - N) \cap (\bar{A} - A) = \emptyset$ for $A \in \mathscr{D}$ (indeed, $p \notin \mathrm{Cl}_{\beta\lambda\omega} A$ for $A \in \mathscr{D}$, it follows that $\bar{N} - N$ is not open in $\bigcup\{\bar{A} - A \colon A \in \mathscr{A}\}$. ■

15. Cardinality of closed subsets of extremally disconnected compacta. Since $\beta\omega$ has no closed subspaces of cardinality $\varkappa$ if $\omega \leqslant \varkappa \leqslant 2^\omega = \omega^\omega$, one may wonder if $\beta\sigma$ has no closed subspaces of cardinality $\varkappa$ if $\sigma \leqslant \varkappa \leqslant \sigma^\omega$ if σ is an uncountable cardinal with $\mathrm{cf}\sigma = \omega$ (possibly a strong limit). This turns out not to be the case.

We will use results from § 14 in the proofs below.

15.1. THEOREM. *If $\varkappa$ and σ are cardinals with $2^{2^\omega} \leqslant \varkappa \leqslant 2^\sigma$ and $\varkappa^\omega = \varkappa$, then $\beta\sigma$ has a closed subset of cardinality $\varkappa$.*

It is a result of Balcar and Simon, [BS], and of Kunen, [Ku$_2$], and of Shelah, [S], that if $\omega_1 \leqslant \mu \leqslant 2^\sigma$, then $\beta\sigma$ has a subspace homeomorphic to the one-point-Lindelöfization $\lambda\mu$ of μ. Now $|\beta\lambda\mu| = 2^{2^\omega} \cdot \mu^\omega$ by 14.2 and $\mathrm{Cl}_{\beta\sigma}\lambda\mu = \beta\lambda\mu$ by 14.3. ■

15.2. COROLLARY (GCH). *If $\varkappa$ and σ are infinite cardinals then $\beta\sigma$ has a closed subset of cardinality $\varkappa$ if and only if $2^{2^\omega} \leqslant \varkappa \leqslant 2^{2^\sigma}$ and $\varkappa^\omega = \varkappa$.*

Proof. Recall that $|\beta\sigma| = 2^{2^\sigma}$, [C, 2.4], [CN, 7.4] or [GJ, 9.2], and note that there are no cardinals between 2^σ and 2^{2^σ} because of GCH so that Theorem 5.1 applies. ■

In fact we can prove something better.

15.3. THEOREM (GCH). *If X is a compact extremally disconnected space, then the following are equivalent for an infinite cardinal $\varkappa$:*

(a) $2^{2^\omega} \leqslant \varkappa \leqslant |X|$ *and* $\varkappa^\omega = \varkappa$.

(b) *X has a closed subspace of cardinality $\varkappa$.*

Proof. (b) → (a): Theorem 7.1.

(a) → (b): As in the proof of Theorem 13.1, if σ denotes $w(X)$, then X admits a continuous map onto ${}^\sigma 2$. Hence some closed subspace K of X admits an irreducible map f onto ${}^\sigma 2$ (i.e. $f^{\to} F \neq {}^\sigma 2$ for every proper closed subset F of K), hence $c(K) = \omega$. Since K is an F-space, it follows that K is extremally disconnected, hence K is the absolute of ${}^\sigma 2$ (i.e. the clopen algebra of K is the completion of the free algebra on σ generators). But K has a subspace homeomorphic to $\lambda\sigma$, [BS], [Ku$_2$], [S], hence has a subspace homeomorphic to $\lambda\mu$ for every μ with $\omega_1 \leqslant \mu \leqslant \sigma$. Now $\mathrm{Cl}_X \lambda\mu = \beta\lambda\mu$ by 14.3, hence $|\mathrm{Cl}_X \lambda\mu| = 2^{2^\omega} \cdot \mu^\omega$ by 14.2.

Consequently X has a closed subspace of cardinality $\varkappa$ for every $\varkappa$ with $2^{2\omega} \leqslant \varkappa \leqslant \sigma^\omega$ (actually $\sigma^\omega = \sigma$ but we do not need this), and also of cardinality $|X|$ of course. Since $|X| \leqslant \exp(w(X))$, cf. [Ju$_1$, 2.2], and we assume GCH, this completes the proof. ■

15.4. Remark. The reader can easily formulate analogous results for the other cardinal functions considered.

16. Questions.

16.1. QUESTION. If X is an infinite compact F-space and $\varphi \in \{\chi, hL\}$, then is $\varphi(X)^\omega = \varphi(X)$?

16.2. QUESTION. If X is an infinite compact F-space, is $s(X)^\omega = s(X)$? Is at least $\mathrm{cf}(s(X)) \neq \omega$?

Since we know that the answer is in the affirmative under GCH, a negative answer can be at most a consistency result. However, things are not this simple. Since for every infinite cardinal λ one has

$$\lambda^\omega = \lambda \cdot \sup\{\varkappa^\omega \colon \varkappa \leqslant \lambda \text{ and } \mathrm{cf}\varkappa = \omega\},$$

a negative answer to 16.1 would imply, because of 8.3, 8.5, and 9.6, that

(*) there is a $\varkappa > 2^\omega$ such that $\mathrm{cf}\varkappa = \omega$ and $\varkappa^\omega > \varkappa^+$.

Now Dodd and Jensen, quoted in [M], have shown that CON(ZFC+(*)) implies CON(ZFC+∃measurable), hence one cannot prove CON(ZFC+(*)) from CON(ZFC) alone (but Magidor has proved it from CON(ZFC+∃supercompact), [M]). At any rate, the most elegant way to settle 16.1 and the first half of 16.2 negatively, modulo (*), is to answer the following affirmatively without using additional axioms.

16.3. QUESTION. Does there exist for every $\varkappa > 2^\omega$ with $\mathrm{cf}\varkappa \neq \omega$ a compact F-space with $\chi(X) = \varkappa$? with $\mathrm{hL}(X) = \varkappa$? with $s(X) = \varkappa$? (It would be sufficient to consider $\varkappa$ of the form λ^+ with $\mathrm{cf}\lambda = \omega$.)

In view of Theorem 13.1 we ask

16.4. QUESTION. Is $\chi(X) = hL(X) = s(X) = w(X)$ if X is an infinite compact F-space? (This really is 3 questions.)

An affirmative answer would imply an affirmative question to 16.1 and 16.2, because of 12.1.

Of course Example 14.7 and its construction lead to the following questions.

16.5. QUESTION. Does there exist for every cardinal $\varkappa$ with $\mathrm{cf}\varkappa > \omega$ a compact F-space X containing a point p with $\chi(p, X) = \varkappa$? (YES if YES to the next question.)

16.6. QUESTION. Does there exist for every cardinal $\varkappa$ with $\mathrm{cf}\varkappa > \omega$ a countably complete filter $\mathscr{F}$ on some set such that $\mathscr{F}$ has a set of $\varkappa$ generators, but does not have a set of fewer than $\varkappa$ generators?

(A natural candidate would be the cocountable filter $\mathscr{F}_\varkappa$ on $\varkappa$, which is not λ-generated for $\lambda < \varkappa$. But if $\varkappa \geqslant 2^\omega$ then $\mathscr{F}_\varkappa$ is $\varkappa$-generated iff $\varkappa = \varkappa^\omega$, so one would have to assume (*). It also is known that $\mathscr{F}_\varkappa$ is $\varkappa$-generated for all $\varkappa$ with $\mathrm{cf}\varkappa > \omega$, in particular for such $\varkappa$ with $\varkappa < 2^\omega$, if $0^\#$ does not exist, [Pr, Prop. 4].)

Finally, Theorems 11.2 and 11.3, and Theorem 15.3 suggest the following two questions.

16.7. QUESTION. If X is noncompact, σ-compact and locally compact, and $\varphi \in \{c, d\}$, is $\varphi(\beta X - X)^\omega = \varphi(\beta X - X)$? Is at least $\mathrm{cf}(\varphi(\beta X - X)) > \omega$?

16.8. QUESTION. If σ and $\varkappa$ are cardinals with $2^{2^\omega} \cdot 2^\sigma < \varkappa < 2^{2^\sigma}$ and $\varkappa^\omega = \varkappa$ does $\beta\sigma$ have a closed subspace of cardinality $\varkappa$?

17. Appendix: some special spaces and Boolean algebras. In this appendix we collect some facts we needed in §§ 11, 14.

Let X be any zero-dimensional space, and let B be its clopen algebra. If X is not compact, then X is not the Stone space of B. However, it is easy to describe the Stone space of B: Let ζX be the largest zero-dimensional compactification of X, [B]. ζX has the property that $\overline{U}$ is open in X for all $U \in B$, so the function $U \to \overline{U}$ is an isomorphism from B to the clopen algebra of ζX. In other words, ζX is the Stone space of B. The Boolean algebraic reader can consider this to be the definition of ζX: note that then X is embedded in ζX in a natural way. (In §§ 11, 14 we used ζX since we dealt with strongly zero-dimensional X, and $\beta X = \zeta X$ for such X.)

If X is zero-dimensional and locally compact, let $\mathscr{I}$ be the ideal of compact open subsets of X. Then $\bigcup \mathscr{I} = X$, hence the Stone space of $B/\mathscr{I}$ is $\zeta X - X$. Hence the following known result shows that Theorem 11.1 is not vacuous.

17.1. THEOREM. *Let X be a noncompact σ-compact locally compact zero-dimensional space. (Then $\zeta X = \beta X$.)*

(a) *$\beta X - X$ is an F-space.*

(b) *Every nonempty G_δ in $\beta X - X$ has nonempty interior.*

Proof. We do not need the fact that $\zeta X = \beta X$. Let B and $\mathscr{I}$ be as above, note that $\mathscr{I}$ is countably generated. We prove (a), the proof of (b) is similar. We show that $B/\mathscr{I}$ is a WCC algebra. We have to show that if $\langle F_n \rangle_n$ and $\langle G_n \rangle_n$ are sequences in B such that $\langle F_n/\mathscr{I} \rangle_n$ and $\langle F_n/\mathscr{I} \rangle_n$ are disjointed, then the latter sequences can be separated in $B/\mathscr{I}$.

Since $\mathscr{I}$ is countably generated, we can find a sequence $\langle I_n \rangle_n$ in $\mathscr{I}$ such that

$$\langle I_n \rangle_n \text{ generates } \mathscr{I}, \text{ and } F_i \cap G_j \subseteq I_n \subseteq I_{n+1} \text{ if } i, j \leqslant n < \omega.$$

Put

$$S = \bigcup_n (F_n - I_n).$$

Then S is clopen in X (since $\bigcup_n I_n = X$ and $I_n \subseteq I_{n+1}$ for $n \in \omega$), and $F_n - S \in \mathscr{I}$ and $G_n \cap S \in \mathscr{I}$ for $n \in \omega$, so that $S/\mathscr{I}$ separates $\langle F_n/\mathscr{I} \rangle_n$ from $\langle G_n/\mathscr{I} \rangle_n$, as required. ■

17.2. Remark. Theorem 17.1(a), even without the condition that X be zero-dimensional, is due to Gillman and Henriksen, [GH, 2.7]; our proof illustrates the fact that the additional assumption of zero-dimensionality often reduces the technical complexity of proofs without affecting the basic ideas. (A very elegant proof for the general case of 17.1(a) can be found in [N], but that argument does not yield 17.1(b).)

Theorem 17.1(b), without the condition that X be zero-dimensional, is due to Fine and Gillman, [FG, 3.1].

17.3. Remark. Theorem 17.1 also holds if X is an (infinite) discrete space.

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Accepté par la Rédaction le 25. 2. 1980