

CARLEMAN ESTIMATES AND ABSENCE OF EMBEDDED EIGENVALUES

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ABSTRACT. Let $L = -\Delta - W$ be a Schrödinger operator with a potential $W \in L^{\frac{n+1}{2}}(\mathbb{R}^n)$, $n \geq 2$. We prove that there is no positive eigenvalue. The main tool is an $L^p - L^{p'}$ Carleman type estimate, which implies that eigenfunctions to positive eigenvalues must be compactly supported. The Carleman estimate builds on delicate dispersive estimates established in [7]. We also consider extensions of the result to variable coefficient operators with long range and short range potentials and gradient potentials.

1. INTRODUCTION

Let $n \geq 2$. Suppose W is a potential in \mathbb{R}^n which decays at infinity. Then the Schrödinger operator

$$-\Delta_{\mathbb{R}^n} - W$$

has continuous spectrum $[0, \infty)$. In addition its spectrum may contain eigenvalues which could be positive, negative or zero. Positive eigenvalues in the continuous spectrum are undesirable. They are very unstable since they are destroyed even by weak interactions between the continuous spectrum and the eigenvalue (see [9]). Physically they correspond to trapped states in the continuous spectrum, and they are difficult to handle analytically. Moreover, excluding eigenvalues in the continuous spectrum is often a first step toward scattering. There is an extensive theory dealing with the absence of positive eigenvalues.

It is well known that under weak assumptions like

$$(1) \quad \lim_{|x| \rightarrow \infty} |x| |W(x)| = 0$$

there are no positive eigenvalues. The argument uses Carleman estimates in three steps as follows. Suppose that

$$-\Delta u - Wu = u$$

with $u \in L^2$, where the eigenvalue is normalized to 1 by scaling. Then one proves that:

- (1) The eigenfunction u decays faster than polynomially at infinity.
- (2) If u vanishes faster than polynomially at infinity then u has compact support.
- (3) If u has compact support then it must vanish.

These arguments work for many Schrödinger operators. However they do not cover Schrödinger operators for several particles (which are studied in [2] and [1]), neither do the standard arguments apply to the absence of bound states (i.e. L^2 solutions) in nonlinear optics modeled by problems of the type

$$-\Delta u = \omega u + a(x)|u|^\sigma u$$

with a bounded function a , because it is not clear how the assumption $u \in L^2(\mathbb{R}^n)$ is related to pointwise decay.

On the other hand the assumption (1) on pointwise decay is sharp: There is the famous Wigner-Von Neumann example of a positive eigenvalue and a potential decaying like $1/|x|$ but not better, see [12, 8].

Motivated by the above questions and by other potential applications one seeks to replace the pointwise bound (1) by an L^p bound. In terms of scaling any such bound must necessarily be weaker than (1) due to counterexamples by Jerison and Ionescu ([3]) with potentials concentrated close to $n-1$ dimensional planes. Jerison and Ionescu [3] have recently obtained absence of embedded eigenvalues for $W \in L^{n/2}$. In this paper we obtain the same result for a larger class of potentials which includes

$$(2) \quad W \in L^{\frac{n+1}{2}}.$$

We note that a higher index is better since it allows for potentials with less decay at infinity. Another way to look at this is that such a condition is mostly relevant for the low frequency part of W . The counterexample of Jerison and Ionescu ([3]) shows that this is the highest possible exponent.

Our method is robust enough so that it also allows us to add a long range potential, and also to replace the Laplacian with a (mildly) asymptotically flat second order elliptic operator. The latter generalization is more technical and less self-contained, so it is discussed only in the last section.

Thus we consider potentials which are the sum of weakly decaying long range potentials V and short range potentials W . We even include the eigenvalue $\lambda > 0$ into the long range potential and study the problem

$$(3) \quad (-\Delta - V)u = Wu.$$

To describe the long range potential we define the space $C_{\langle x \rangle}^2$ by

Definition 1. $C_{\langle x \rangle}^2$ is the space of C_{loc}^2 functions for which the following norm is finite:

$$\|f\|_{C_{\langle x \rangle}^2} := \max\{\sup_x |f(x)|, \sup_{\langle x \rangle} |Df|, \sup_{\langle x \rangle^2} |D^2 f|\}$$

Then we introduce the condition

Assumption A 1 (The long range potential). V belongs to $C_{\langle x \rangle}^2$ and satisfies

$$(4) \quad \liminf_{|x| \rightarrow \infty} V > 0, \quad \tau_0 := -\liminf_{|x| \rightarrow \infty} \frac{x \cdot \nabla V}{4V} < 1/2.$$

The bound from below on V corresponds to the condition $\lambda > 0$ while the last bound in (4) says that for large $|x|$ the function $|x|^2$ is strictly convex along the null Hamilton flow for $-\Delta - V$, and thus guarantees nontrapping outside a compact set.

To describe the short range potential we define the space

Definition 2. X is the space of $W_{loc}^{-\frac{1}{n+1}, \frac{2(n+1)}{n+3}}$ functions for which the following norm is finite:

$$\|W\|_X = \sup_{u \in C_0^\infty} \|Wu\|_{W^{-\frac{1}{n+1}, \frac{2(n+1)}{n+3}}} / \|u\|_{W^{\frac{1}{n+1}, \frac{2(n+1)}{n-1}}} \quad n \geq 3$$

$$\|W\|_X = \sup_{u \in C_0^\infty} \|Wu\|_{W^{-\frac{1}{3}+\epsilon, \frac{6}{3}}} / \|u\|_{W^{\frac{1}{3}-\epsilon, 6}} \quad n = 2, \epsilon > 0$$

For a domain $D \subset \mathbb{R}^n$ we denote

$$X(D) = \{1_D W; W \in X\}$$

Then we introduce

Assumption A 2 (The short range potential). W belongs to X_{loc} and can be decomposed as $W = W_1 + W_2$ where

$$(5) \quad \limsup_{j \rightarrow \infty} \|W_1\|_{X(\{|x|^{2j} \leq |x| \leq 2^{j+1}\})} < \delta$$

$$(6) \quad \limsup_{|x| \rightarrow \infty} |x| |W_2(x)| < \delta.$$

The W_2 component corresponds to the L^2 Carleman estimates. The class of allowed W_1 potentials includes $L^{\frac{n}{2}}$ and $L^{\frac{n+1}{2}}$ or even better¹ $l^{\frac{n+1}{2}}(L^{\frac{n}{2}})$ where the $l^{\frac{n+1}{2}}$ norm is taken with respect to a partition of \mathbb{R}^n into unit cubes.

Our main result is

Theorem 3. Assume that V and W satisfy Assumptions A1 and A2, let $\tau_1 > \tau_0$ and assume that δ is sufficiently small. Let $u \in H_{loc}^1(\mathbb{R}^n)$ satisfy (3) and $(1 + |x|^2)^{\tau_1 - \frac{1}{2}} u \in L^2$. Then $u \equiv 0$.

By comparison, the result of Jerison and Ionescu [3] applies to the case $V = 1$ and $W \in L^{\frac{n}{2}}$, $n \geq 3$. We note that the exponent $p = n/2$ is critical for weak unique continuation; for smaller exponents there are examples of compactly supported eigenfunctions, see [6].

The conditions (5) and (6) have a different scaling behavior. Nevertheless both are sharp, which can be seen by the Wigner-Von Neumann example and the non radial counter example of Jerison and Ionescu.

The proof uses Carleman estimates, following the same three steps indicated above. A combined L^2 - L^p Carleman inequality replaces the previous L^2 Carleman inequalities. Proving such inequalities is a highly nontrivial task and

¹ $l^{\frac{3}{2}} L^{1+}$ if $n = 2$.

relies on the bounds established in [7]. Conjugation of the operator $-\Delta - V$ with the weight of the Carleman inequality leads to a non-selfadjoint partial differential equation. A pseudo-convexity type condition is satisfied, but it degenerates for large x . This is related to the fact that the anti-selfadjoint part of the conjugated operator decays for large x in relevant coordinates.

Compared to earlier work and to the steps outlined above, we also consider a different family of weights in the Carleman estimates. Precisely, we begin with weights of the form $h(x) = e^{\tau\sqrt{|x|}}$ for part 2 of the argument, which we then flatten at infinity for part 1. This yields a more robust argument, and also better results in the variable coefficient case.

The paper is organized as follows. In the next section we state all the L^p Carleman estimates and show how they lead to the result on the absence of the embedded eigenvalues.

There are two main ingredients in the proof of the L^p Carleman estimates. The first is the L^2 Carleman estimates, which are proved in Section 3. The second is a dispersive estimate for second order operators which is obtained in Section 4 using an earlier result of the authors, namely Theorem 3 of [7]. This is of independent interest so we state it in more generality than needed here.

The L^p estimates are proved Section 5. The L^2 bounds obtained earlier are used to localize the L^p bounds to small spatial scales. Then we can rescale to a setting where the general dispersive estimates of Theorem 7 apply.

Finally, in the last two sections we discuss the extension of the results to second order elliptic operators with variable but asymptotically flat coefficients as well as unbounded gradient potentials. This goes along the same lines.

2. CARLEMAN ESTIMATES AND EMBEDDED EIGENVALUES

As explained above the proof depends on Carleman inequalities. In this section we explain the Carleman inequalities and their application whereas most of the proofs are postponed to the remaining sections.

Let $1 \leq p \leq \infty$ and $s \in \mathbb{R}$. We define the Sobolev space $W^{s,p}(\mathbb{R}^n)$ by the norm $\|f\|_{W^{s,p}} = \|(1 + |D|^2)^{s/2} f\|_{L^p}$ and $W^{s,p}(U)$ for open subsets U of \mathbb{R}^n through its norm which is the infimum of the norm of extensions.

Given a measurable function f and the Sobolev space $W^{s,q}$ we define the norm

$$\|f\|_{l^p W^{s,q}} = \left(\sum_{j=1}^{\infty} \|f\|_{W^{s,q}(\{2^{j-1} \leq |x| \leq 2^j\})}^p \right)^{1/p}$$

with the obvious modification for $p = \infty$.

Our Carleman estimates have the form

$$(7) \quad \begin{aligned} & \|e^{h(\ln(|x|))} v\|_{l^2 W^{\frac{1}{n+1}, \frac{2(n+1)}{n-1}}} + \|e^{h(\ln(|x|))} \rho v\|_{L^2} \lesssim \\ & \inf_{f_1 + f_2 = (-\Delta - V)v} \|e^{h(\ln(|x|))} \rho^{-1} f_1\|_{L^2} + \|e^{h(\ln(|x|))} f_2\|_{l^2 W^{-\frac{1}{n+1}, \frac{2(n+1)}{n+3}}} \end{aligned}$$

where ρ is given by

$$(8) \quad \rho = \left(\frac{h'(\ln(|x|))}{|x|^2} + \frac{h'(\ln(|x|))^2 h''_+(\ln(|x|))}{|x|^4} \right)^{\frac{1}{4}}$$

with h''_+ denoting the positive part of h'' . As a general rule, the function h is chosen to be

- (a) increasing, $h' \geq \tau_0$, with $h'(0)$ large.
- (b) slowly varying on the unit scale, $|h^{(j)}| \lesssim h'$ for $j = 2, 3, 4$.
- (c) strictly convex for as long as $h'(\ln(|x|)) \gtrsim |x|$.

More precise choices are made later on for convenience, but the estimates are in effect true for all functions h satisfying the above conditions.

The two terms in ρ have different origins. The second one simply measures the effect of the convexity of the function h . The first one, on the other hand, is due to the presence of the long range potential, which provides some extra pseudoconvexity for large $|x|$.

A simplifying assumption consistent with the choices of weights in this paper is to strengthen (c) to

- (c)' $h''(\ln(|x|)) \approx h'(\ln(|x|))$ for as long as $h'(\ln(|x|)) \gtrsim |x|$.

This allows us to simplify the expression of ρ to

$$(9) \quad \rho = \left(\frac{h'(\ln(|x|))}{|x|^2} \left(1 + \frac{h'(\ln(|x|))^2}{|x|^2} \right) \right)^{\frac{1}{4}}$$

Our Carleman estimates use weights which grow exponentially, but also allow for the possibility of leveling off the weight for large enough $|x|$.

Proposition 4. *Suppose that V satisfies Assumption A1. There is a universal constant ε_0 such that with*

$$(10) \quad h'_\varepsilon(t) = \tau_1 + (\tau e^{\frac{t}{2}} - \tau_1) \frac{\tau^2}{\tau^2 + \varepsilon e^t}$$

(7) holds with $h = h_\varepsilon$ for all $|\varepsilon| \leq \varepsilon_0$, v supported in $|x| > 1$ and satisfying $|x|^{\tau_1 - \frac{1}{2}} v \in L^2$, uniformly with respect to τ large enough.

The coefficient $\frac{1}{2}$ in the exponent is chosen somewhat arbitrarily. However, it must be smaller than 1 in order for stage (c) above to be reached. This is necessary if we are to be able to taper off the weight at infinity. We continue with a short discussion of the weight h_ε .

For small t it is uniformly convex in the sense that $h''_\varepsilon \approx h'_\varepsilon$. The first interesting threshold for it is t_0 defined by

$$e^{t_0} \approx \tau^2$$

This implies that $h'_\varepsilon(t_0) \approx e^{t_0}$. In the range $[0, t_0]$ the last factor in (10) is largely irrelevant, and h'_ε behaves like an exponential. In this region, the pseudoconvexity in the Carleman estimates is produced by the convexity of h .

After t_0 h_ϵ is still convex, roughly up to t_1 defined by

$$e^{t_1} \approx \epsilon^{-1} \tau^2$$

The region $t_1 + O(1)$ contains both the inflexion point t_1 and the maximum point for h'_ϵ . In between t_0 and t_1 the pseudo-convexity comes from the potential term, while the contribution from the convexity of h_ϵ is still positive but smaller.

Beyond $t_1 + O(1)$ the function $h'_\epsilon(t) - \tau_1$ decays in an exponential fashion. The last interesting threshold is t_2 where h'_ϵ approaches 1, given by

$$e^{t_2} \approx \epsilon^{-2} \tau^6$$

Between t_1 and t_2 there is still convexity coming from the potential V , which suffices in order to control the lack of convexity of h_ϵ . Finally, after t_2 the pseudoconvexity in the classical sense is lost, but there remains an Airy type gain to push the estimates through.

Proof of Theorem 3. Here we show that Proposition 4 implies Theorem 3.

STEP 1: We prove that u decays at infinity faster than $e^{-\tau\sqrt{|x|}}$. We choose R large enough so that (see Assumption A2)

$$(11) \quad \sup_{2^{j+1} > R} \|W_1\|_{X(\{|x|^{2j} \leq |x| \leq 2^{j+1}\})} < 2\delta$$

$$(12) \quad \sup_{|x| > R} |x| |W_2(x)| < 2\delta.$$

Choose $\phi \in C^\infty$ be identically 1 for $|x| \geq 2R$ and 0 for $|x| \leq R$. We set $v = \phi u$. Then

$$-\Delta v - Vv = Wv - (\Delta\phi)u - 2\nabla\phi \cdot \nabla u$$

For τ_1 as in Theorem 3 we have $|x|^{\tau_1 - \frac{1}{2}} v \in L^2$, therefore we can apply Proposition 4 with $\epsilon > 0$ to v to obtain

$$\begin{aligned} \|e^{h_\epsilon(\ln|x|)} v\|_{l^2 W^{\frac{1}{n+1}, \frac{2(n+1)}{n-1}}} + \|e^{h_\epsilon(\ln|x|)} \rho v\|_{L^2} &\lesssim \|e^{h_\epsilon(\ln|x|)} \rho^{-1} (|u| + |\nabla u|)\|_{L^2(B_{2R} \setminus B_R)} \\ &+ \|e^{h_\epsilon(\ln|x|)} W_1 v\|_{l^2 W^{-\frac{1}{n+1}, \frac{2(n+1)}{n+3}}} + \|e^{h_\epsilon(\ln|x|)} \rho^{-1} W_2 v\|_{L^2} \end{aligned}$$

By (11), (12) if δ is small enough then we can absorb the last two right hand side terms on the left to obtain

$$\|e^{h_\epsilon(\ln|x|)} v\|_{l^2 W^{\frac{1}{n+1}, \frac{2(n+1)}{n-1}}} + \|e^{h_\epsilon(\ln|x|)} \rho v\|_{L^2} \lesssim \|e^{h_\epsilon(\ln|x|)} \rho^{-1} (|u| + |\nabla u|)\|_{L^2(B_{2R} \setminus B_R)}$$

Then letting $\epsilon \rightarrow 0$ in the definition of h yields

$$(13) \quad \|e^{\tau\sqrt{|x|}} v\|_{l^2 W^{\frac{1}{n+1}, \frac{2(n+1)}{n-1}}} + \|e^{\tau\sqrt{|x|}} \rho v\|_{L^2} \lesssim \|e^{\tau\sqrt{|x|}} \rho^{-1} (|u| + |\nabla u|)\|_{L^2(B_{2R} \setminus B_R)}.$$

which shows that v and therefore u is rapidly decaying at infinity.

STEP 2: We prove that u vanishes outside a compact set. This is done using (13) (which can also be derived directly from Proposition 4 as above). From (13) we obtain

$$R^{-1}e^{-\tau\sqrt{2R}}\|e^{\tau\sqrt{|x|}}v\|_{L^2W^{\frac{1}{n+1},\frac{2(n+1)}{n-1}}} + \|e^{\tau\sqrt{|x|}}\rho v\|_{L^2} \lesssim \|(|u| + |\nabla u|)\|_{L^2(B_{2R}\setminus B_R)}.$$

Letting $\tau \rightarrow \infty$ shows that $v = 0$ outside B_{2R} . Then the same holds for u .

STEP 3: We prove that u is identically 0. Assume by contradiction that this is not the case, and choose r minimal so that u is supported in $\overline{B(0,r)}$. Our problem is scale invariant, so without any restriction in generality we can assume that $r > 1$. Take $x_0 \in \text{supp } u$ with $|x_0| = r$. The problem is also invariant with respect to translations so we can assume instead that $\text{supp } u \in \overline{B(x_0,r)}$ and $2x_0 \in \text{supp } u$.

To reach a contradiction we prove that there is $\alpha > 0$ so that u is supported in $B(0, 2r - \alpha)$. This follows as in STEP 2 provided we know that for every $\delta > 0$ we can find $\rho > 0$ such that

$$\|W_1v\|_{W^{-\frac{1}{n+1},\frac{n+1}{n+3}}} \leq \delta\|v\|_{W^{\frac{1}{n+1},\frac{n+1}{n-1}}}, \quad \text{supp } v \subset B(2x_0, \rho)$$

Then α is chosen so that

$$\{2r - \alpha < |x| < 2r\} \cap B(x_0, r) \subset B(2x_0, \rho)$$

Due to our choice of W this is a somewhat technical matter which is left for Proposition 14 in the appendix. This step can be approached alternatively by the unique continuation results of [7]. \square

3. THE L^2 CARLEMAN ESTIMATES

In this section we obtain the L^2 Carleman inequalities.

Proposition 5. *Suppose that V satisfies Assumption A1. Let h be as in (10) and ρ as in (8). Then for all u satisfying $|x|^{\tau_1 - \frac{1}{2}}u \in L^2$ we have*

$$(14) \quad \|e^{h(\ln|x|)}\rho u\|_{L^2} + \left\| \frac{|x|}{h'(\ln|x|) + |x|} e^{h(\ln|x|)}\rho \nabla u \right\|_{L^2} \lesssim \|e^{h(\ln|x|)}\rho^{-1}(\Delta + V)u\|_{L^2}.$$

uniformly with respect to τ sufficiently large and $0 < \varepsilon \leq \varepsilon_0$.

Proof. We use a conformal change of coordinates

$$t = \ln|x|, \quad y = x/|x| \in \mathbb{S}^{n-1}$$

Denote

$$\Delta u = g$$

and set

$$v(t, y) = e^{(n-2)t/2}u(e^ty), \quad f(t, y) = e^{(n+2)t/2}g(e^ty)$$

A routine computation shows that

$$|x|^{(n+2)/2}(\Delta + V)|x|^{(n+2)/2} = \frac{\partial^2}{\partial t^2} + \Delta_{\mathbb{S}^{n-1}} - ((n-2)/2)^2$$

therefore v solves the equation

$$(15) \quad Lv = f, \quad L = \partial_t^2 + \Delta_{\mathbb{S}^{n-1}} - ((n-2)/2)^2 + e^{2t}V$$

We also note that part of Assumption A1 in the new coordinates we get

$$-\liminf_{t \rightarrow \infty} \frac{V_t}{4V} = \tau_0 < \frac{1}{2}$$

By (4) we slightly readjust τ_0 and choose t_0 so that

$$(16) \quad -\frac{V_t}{4V} \leq \tau_0 < \frac{1}{2}, \quad t > t_0$$

For any exponential weight h we have

$$(17) \quad \int e^{2h(\ln|x|)} |u|^2 dx = \int_{\mathbb{R}} \int_{\mathbb{S}^n} e^{2h(t)+nt} |u(ty)|^2 dt dy = \|e^{h(t)} e^t v\|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})}^2,$$

$$(18) \quad \int e^{2h(\ln|x|)} |g|^2 dx = \int_{\mathbb{R}} \int_{\mathbb{S}^n} e^{2h(t)+nt} |g(ty)|^2 dt dy = \|e^{h(t)} e^{-t} f\|_{L^2(\mathbb{R} \times \mathbb{S}^{n-1})}^2.$$

Hence, in the new coordinates the bound (14) becomes

$$(19) \quad \|e^{h(t)} \rho_1 v\|_{L^2} + \|e^{h(t)} \frac{\rho_1}{e^t + h'(t)} \nabla v\|_{L^2} \lesssim \|e^{h(t)} \rho_1^{-1} f\|_{L^2},$$

where ∇v is the gradient of v with respect to y and t and, by (9),

$$\rho_1(t) = e^t \rho = h'(t) (e^{2t} + h'(t)^2)^{\frac{1}{4}}$$

To prove the above bound one would like to follow a standard strategy. This means conjugating the operator with respect to the exponential weight, and producing a commutator estimate for the self-adjoint and the skew-adjoint part of the conjugated operator. There are two small problems with this approach, both of which occur in the region where $h'(t)$ is small.

First we want to incorporate the weight ρ_1^{-1} on the right, which would require an additional conjugation. Where h' is small this cannot be treated as a small perturbation, so we really have to include ρ^{-1} in the exponential weight.

This leads to a second difficulty. After including ρ^{-1} in the exponential weight the commutator between the self-adjoint and the skew-adjoint part of the conjugated operator is no longer fully positive definite and we need a slightly modified argument.

To handle both issues we prove a slightly more general result and then we obtain (19) as a special case of it. Precisely, we consider an exponential weight ϕ as follows:

- (i) $\phi' \geq \tau_1 - \frac{1}{2}$, and $\phi'(0)$ is large.
- (ii) $1 + \phi'$ is slowly varying on the unit scale, i.e.

$$|\phi^{(j)}(t)| \lesssim 1 + \phi'(t) \quad j = 2, 3$$

- (iii) ϕ' can only have a limited exponential growth rate, $\phi'' \lesssim \frac{3}{4}(1 + \phi')$.

Together with (i) this yields the existence of a unique t_0 so that $\phi'(t_0) = e^{t_0}$. Our last assumption asks for uniform convexity up to t_0 :

(iv) $\phi''(t) \approx \phi'(t)$ for $0 \leq t \leq t_0 + C$ for some large parameter C .

We summarize the bound for the weight e^ϕ :

Lemma 6. *Consider a weight function ϕ satisfying the conditions (i)-(iv) above. Then for all v which are supported in $t > 0$ and with $e^{\phi(t)+t}v \in L^2$ we have*

$$(20) \quad \|e^{\phi(t)}(e^{2t} + \phi'(t)^2)^{\frac{1}{2}}v\|_{L^2} + \|e^{\phi(t)}\nabla v\|_{L^2} \lesssim \|e^{\phi(t)}(1 + \phi')^{-\frac{1}{2}}Lv\|_{L^2}.$$

Proof. First we conjugate with respect to the exponential weight. If we set $w = e^{\phi(t)}v$ then w solves the equation

$$L_\phi w = e^{\phi(t)}f, \quad L_\phi = e^{\phi(t)}Le^{-\phi(t)}$$

We decompose L_h into a selfadjoint and a skewadjoint part,

$$L_\phi^r = \partial_t^2 + \Delta - \left(\frac{n-2}{2}\right)^2 + e^{2t}V + \phi'^2, \quad L_\phi^i = -\phi'\partial_t - \partial_t\phi'$$

The bound to prove is

$$(21) \quad \|(e^{2t} + \phi'(t)^2)^{\frac{1}{2}}w\|_{L^2} + \|\nabla w\|_{L^2} \lesssim \|(1 + \phi')^{-\frac{1}{2}}L_\phi w\|_{L^2}.$$

The proof of this inequality is based on several integrations by parts. In a standard manner one verifies that the integrations by parts below are valid if $e^{\phi+t}v \in L^2$.

We multiply $P_\phi w$ by $-\frac{1}{2}w_t$ and integrate by parts to obtain

$$(22) \quad \int \phi'|w_t|^2 dy dt + \int \left(\frac{1}{4}\phi''' + \frac{1}{2}\phi'\phi''\right)|w|^2 dt dy + \int \frac{e^{2t}}{4}(2V + V_t)w^2 dy dt \\ = \frac{1}{2} \int w_t L_\phi w dy dt$$

This computation is essentially like taking the commutator of L_ϕ^r and L_ϕ^i . On the left we have mostly positive contributions, with the following qualifications:

- the first term can be negative where $\phi' < 0$
- the $\phi'\phi''$ term can also be negative, but only for $t > t_0 + C$ where it is controlled by the V term.
- the ϕ''' term is controlled either by the V term or by the $\phi'\phi''$ term.

To correct the first term in the region where ϕ' is negative we consider a cutoff function χ which equals δ in $\{\phi' > 2\}$ and which equals 1 in $\{\phi' < 1\}$. Here δ is a small universal parameter which we shall choose below. Since $\phi' + 1$ is slowly varying we can assume that χ has uniformly bounded derivatives.

Multiplying $P_\phi w$ by $\chi^2(t)w$ and integrating gives

$$(23) \quad \|\chi w_t\|_{L^2}^2 + \|\chi \nabla w\|_{L^2}^2 + \left(\frac{n-2}{2}\right)^2 \|\chi w\|_{L^2}^2 - \int \chi^2 (e^{2t}V + \phi^2) |w|^2 dy dt = \\ \int \frac{1}{2} (\partial_t^2 \chi^2) w^2 dy dt + \int w L_\phi w dy dt.$$

We multiply this by μ and add to the previous relation. This yields

$$(24) \quad \mu \|\chi \nabla w\|_{L^2}^2 + \int (\chi^2 \mu + \phi') |w_t|^2 dy dt + \int \left(\frac{1}{2} - \chi^2 \mu + \frac{V_t}{4V} \right) e^{2t} V w^2 dy dt \\ + \int \left(\frac{1}{2} \phi' \phi'' - \chi^2 \mu \phi'^2 \right) |w|^2 dy dt \\ = \int \left[-\phi''' / 4 - \left(\frac{n-2}{2} \right)^2 \mu \chi^2 + \partial_t^2 \chi^2 \mu \right] |w|^2 dy dt \\ + \int (\chi^2 \mu w + \frac{1}{2} w_t) L_\phi w dy dt.$$

To ensure that the left hand side is positive definite we recall that for large t

$$-\frac{V_t}{4V} \leq \tau_0 < \tau_1 \leq \frac{1}{2} + \phi'$$

Hence if we choose μ positive so that

$$\frac{1}{2} - \tau_1 < \mu < \frac{1}{2} - \tau_0$$

then the first three terms are positive definite.

For the fourth term we consider two possibilities. If $t < t_0 + C$ then $\chi = \delta$ while $\phi'' \approx \phi'$ so it yields a positive contribution. We choose the universal constant δ so that

$$\frac{1}{2} \phi' \phi'' - \chi \mu \phi'^2 \geq \frac{1}{4} \phi' \phi''$$

if $t \leq t_0 + C$. For larger t this fourth integrand may be negative but then it is controlled by the third. The first term on the right hand side is controlled by the left hand side and we obtain

$$\|\nabla w\|_{L^2}^2 + \|(1 + \phi')^{\frac{1}{2}} w_t\|_{L^2}^2 + \|(\phi'(t)^2 + e^{2t})^{\frac{1}{2}} w\|_{L^2}^2 \lesssim \int (\mu w + \frac{1}{2} w_t) L_\phi w dy dt$$

The proof is completed by an application of the Cauchy-Schwarz to the right hand side. □

Proof of Proposition 5, continued.

We obtain (19) from Lemma 6. For this we need to associate to each weight h a function ϕ satisfying (i)-(iv) with the property that

$$1 + \phi' \approx h', \quad (1 + \phi')^{-\frac{1}{4}} e^\phi \approx e^h (h'^2 + e^{2t})^{-\frac{1}{4}}$$

The natural choice for ϕ is

$$\phi(t) = h(t) - \frac{t}{2} + \frac{1}{4} \ln(1 + h'(t)) - \frac{1}{4} \ln(1 + e^{-t}h'(t))$$

Then

$$\phi' = h' - \frac{1}{2} + \frac{h''}{4(1+h')} + \frac{(h' - h'')e^{-t}}{4(1 + e^{-t}h')}$$

We verify the properties of ϕ . It is easy to see that $1 + \phi'$ is slowly varying. This implies that the last two terms in ϕ' are bounded and have bounded derivatives. Hence the properties (ii)-(iv) follow from the similar properties of h' .

It remains to check the bound $\phi' > \tau_1 - \frac{1}{2}$. This is clear when $h' \gg 1$ which corresponds to $\epsilon e^{\frac{t}{2}} \ll \tau^3$. For larger t we have

$$h'(t) = \tau_1 + \frac{1}{\epsilon} \tau^3 e^{-\frac{t}{2}} (1 + O(\tau^{-1}))$$

and

$$h''(t) = -\frac{1}{2\epsilon} \tau^3 e^{-\frac{t}{2}} (1 + O(\tau^{-1}))$$

Then

$$\phi'(t) > \tau_1 - \frac{1}{2} + \frac{1}{2\epsilon} \tau^3 e^{-\frac{t}{2}} (1 + O(\tau^{-1}))$$

so the desired bound is again verified. We note that what happens when h' is small is not so important anyway; in this region we can simply choose $\phi(t) = h(t) - \frac{t}{2}$. \square

4. A GENERAL DISPERSIVE ESTIMATE FOR SECOND ORDER OPERATORS

In this section we study the second order operator²

$$L_\mu = \partial_i a^{ij}(x) \partial_j + \mu^2 c(x) - i\mu(b_j(x) \partial^j + \partial_j b^j(x)),$$

in the unit ball $B \subset \mathbb{R}^n$, $n \geq 2$ with real coefficients a^{ij} and complex coefficients b^j and c . Here μ is sufficiently large and plays the role of a semiclassical parameter. Concerning the type and regularity of the coefficients we assume that

$$(REG) \quad \begin{cases} \text{the matrix } (a^{ij}(x)) \text{ is real, symmetric and positive definite} \\ \text{the functions } a^{ij}, b^i \text{ and } c \text{ are of class } C^2 \end{cases}$$

We define the symbol

$$l(x, \xi) = -\xi_i a^{ij}(x) \xi_j + c(x) + 2b_j \xi_j$$

The real part of l is a second degree polynomial in ξ with characteristic set

$$char_x \Re l(x, \xi) = \{\xi \in \mathbb{R}^n; \Re l(x, \xi) = 0\}$$

²We use the summation convention here and in the sequel.

The geometric assumption on the operator L is

$$(GEOM) \quad \begin{cases} \text{for each } x \text{ the characteristic set } \mathit{char}_x \Re l(x, \xi) \\ \text{is an ellipsoid of size } \approx 1. \end{cases}$$

Our third hypothesis is concerned with the size of the Poisson bracket of the real and imaginary part of L . We are interested in a principal normality type condition of the form

$$(25) \quad |\{\Re l(x, \xi), \Im l(x, \xi)\}| \lesssim \delta + |\Re l(x, \xi)| + |\Im l(x, \xi)|$$

where the relevant range for δ is $\mu^{-1} < \delta \ll 1$. This would suffice for our purposes if in addition we knew that all the coefficients of l are of class C^3 . In general for technical reasons we need to replace the inequality with a decomposition

$$(26) \quad \{\Re l, \Im l\}(x, \xi) = \delta q_0(x, \xi) + q_1^r(x, \xi) \Re l(x, \xi) + q_1^i(x, \xi) \Im l(x, \xi) + q_2(x, \xi)$$

Thus our last assumption has the form

$$(PN) \quad \begin{cases} \text{the Poisson bracket } \{\Re l, \Im l\} \text{ admits a representation (26) where} \\ |\partial_x^\alpha \partial_\xi^\beta q_i(x, \xi)| \leq c_{\alpha\beta} \quad |\alpha| \leq i \\ |q_0| \lesssim 1, \quad |q_1^r| + |q_1^i| \lesssim 1, \quad |q_2| \lesssim |l| \end{cases}$$

For L in the class of operators described above we are interested in constructing a parametrix T which has good $L^{p'} \rightarrow L^p$ and $L^2 \rightarrow L^p$ mapping properties, while the errors are always measured in L^2 . A dual form of this also allows us to estimate the L^p norm of a function u in terms of the L^2 norms of u and Lu .

In the context of the Carleman estimates such parametrices allow us to superimpose local $L^{p'} \rightarrow L^p$ bounds on top of the global $L^2 \rightarrow L^2$ estimates in order to obtain a global $L^{p'} \rightarrow L^p$ bound.

Such estimates are dispersive in nature and are strongly related to the spreading of singularities in the parametrix T . This in turn is determined by the nonvanishing curvatures of the characteristic set $\mathit{char}_x \Re l(x, \xi)$.

If L has constant coefficients and real symbol then the theorem below is nothing but a reformulation of the restriction theorem. If L has real symbol but variable coefficients then we are close to the spectral projection estimates of C. Sogge [10]. In the case when L has constant coefficients but complex symbol some bounds of this type were obtained in [4].

In the more general case considered here we rely on bounds and parametrix constructions in the author's earlier paper [7]. These apply to principally normal operators. The operator L_μ is principally normal on the unit spatial scale only if $\delta \approx \mu^{-1}$. Otherwise, we use a better spatial localization to the $(\delta\mu)^{-\frac{1}{2}}$ scale. On one hand L_μ is principally normal on this scale, while on

the other hand this localization is compatible with the L^2 estimates and this allows us to easily put the pieces back together.

All Sobolev norms in the theorem below are flattened at frequency μ instead of frequency 1 as usual. Hence we introduce the notation

$$W_\mu^{s,p} = \{u \in S'; (\mu^2 + D^2)^{\frac{s}{2}}u \in L^p\}$$

with the corresponding norm.

We note that the operator L is elliptic at frequencies larger than μ so all the estimates are trivial in that case. All the interesting action takes place at frequency $\lesssim \mu$, where we can identify all Sobolev norms with L^p norms.

Theorem 7. *Suppose that the operator L_μ satisfies the conditions (REG), (GEOM) and (PN) for some $\delta > \mu^{-1}$. Let $\phi \in C(B_2(0))$ have compact support. Then*

A) *There exists an operator T such that*

$$(27) \quad \begin{aligned} & \|Tf\|_{W_\mu^{\frac{1}{n+1}, \frac{2(n+1)}{n-1}}} + (\delta\mu)^{1/4} \mu^{-1/2} \|Tf\|_{H_\mu^1} \\ & \lesssim \inf_{f=f_1+f_2} (\delta\mu)^{-1/4} \mu^{-1/2} \|f_1\|_{L^2} + \|f_2\|_{W_\mu^{-\frac{1}{n+1}, \frac{2(n+1)}{n+3}}} \end{aligned}$$

and

$$(28) \quad \begin{aligned} & (\delta\mu)^{-1/4} \mu^{-1/2} \|LT\phi f - \phi f\|_{L^2} \lesssim \\ & \inf_{f=f_1+f_2} (\delta\mu)^{-1/4} \mu^{-1/2} \|f_1\|_{L^2} + \|f_2\|_{W_\mu^{-\frac{1}{n+1}, \frac{2(n+1)}{n+3}}} \end{aligned}$$

B) *For all functions u in $B_2(0)$ we have*

$$(29) \quad \begin{aligned} & \|\phi u\|_{W_\mu^{\frac{1}{n+1}, \frac{2(n+1)}{n-1}}} \lesssim (\delta\mu)^{1/4} \mu^{1/2} \|u\|_{L^2} \\ & + \inf_{Lu=f_1+f_2} (\delta\mu)^{-1/4} \mu^{-1/2} \|f_1\|_{L^2} + \|f_2\|_{W_\mu^{-\frac{1}{n+1}, \frac{2(n+1)}{n+3}}} \end{aligned}$$

C) *Suppose that in addition the problem is pseudoconvex in the sense that*

$$(30) \quad q_0(x, \xi) \approx \delta \gg \mu^{-1} \quad x \in B_2(0), \quad \mu \gg 1$$

Then for all functions u with compact support in $B_2(0)$ we have

$$(31) \quad \begin{aligned} & \|u\|_{W_\mu^{\frac{1}{n+1}, \frac{2(n+1)}{n-1}}} + (\delta\mu)^{1/4} \mu^{1/2} \|u\|_{L^2} \\ & \lesssim \inf_{Lu=f_1+f_2+f_3} (\delta\mu)^{-1/4} \mu^{-1/2} \|f_1\|_{L^2} + \|f_2\|_{W_\mu^{-\frac{1}{n+1}, \frac{2(n+1)}{n+3}}} \end{aligned}$$

The difficult part of this theorem is the existence of the rough parametrix in Part A. This existence will be derived from Theorem 3 in [7]. The arguments repeat partially those of Section 3, 7 and 8 of [7].

Proof. Part A. (i) Localization. We first reduce the problem to the case when $\delta = \mu^{-1}$. This is done by localization to a small spatial scale and then by rescaling. The appropriate spatial scale is $r = (\mu\delta)^{-\frac{1}{2}}$. We cover the support

of ϕ with balls B_j of radius r and choose a subordinate partition of unity of the form

$$\sum \phi_j^2 = 1$$

Suppose that within B_j there exists a parametrix T_j satisfying the desired estimates. Then we set

$$T = \sum_{j=1}^N \phi_j T_j \phi_j.$$

The bound (27) for T follows directly by square summing the similar bounds for T_j . For (28) we compute

$$I - LT = \sum_{j=1}^N \phi_j (I - LT_j) \phi_j + \sum_{j=1}^N [L, \phi_j] T_j \phi_j$$

For the first term we use (28) for T_j while for the second we estimate the commutators using (27) for T_j .

In order to obtain the localized parametrices T_j we rescale B_j to the unit scale. Then the problem reduces to the original one but with $\delta = \mu^{-1}$.

(ii) The elliptic high frequency parametrix.

For each x the zero set of \mathfrak{R} is an ellipse contained in a ball of radius $B_{R\mu}(0)$ with $R \sim 1$. Let $\psi \in C^\infty(\mathbb{R}^n)$ be a nonnegative radial radially decreasing function supported in $B_2(0)$ and identically 1 in $B_1(0)$. Let ϕ be as in the statement of the theorem. We fix a nonnegative function $\phi_0 \in C^\infty(B_2(0))$, identically 1 on the support of ϕ . We define T_{high} by its Weyl symbol

$$\phi_0(x) l_\mu^{-1}(x, \xi) (1 - \psi(\xi/\mu R)) \phi_0(x).$$

Then the following L^2 bounds are immediate:

$$\begin{aligned} \|T_{high} f\|_{H_\mu^1} &\lesssim \|f\|_{H_\mu^{-1}} \\ \|(1 - LT_{high})(1 - \psi(D/(2\mu R)))\phi f\|_{L^2} &\lesssim \|f\|_{H_\mu^{-1}} \end{aligned}$$

This estimates are the elliptic versions of the parametrix bounds. By Sobolev embeddings they imply bounds of the type of Theorem 4.

(iii) The low frequency parametrix. We first mollify the coefficients of L_μ on a scale $\mu^{-1/2}$ and note that this does not affect the hypothesis of the Theorem. We also modify its symbol for large ξ and extend it to \mathbb{R}^{2n} so that it is of size μ^2 and so that it satisfies

$$|\partial_x^\alpha \partial_\xi^\beta \tilde{l}_\mu(x, \xi)| \lesssim \begin{cases} \mu^{2-|\beta|} & \text{if } |\alpha| \leq 2 \\ \mu^{1+|\alpha|/2-|\beta|} & \text{if } |\alpha| \geq 3 \end{cases}$$

By Theorem 3 of [7] there exists a parametrix T_{low} for \tilde{l}_μ satisfying

$$\begin{aligned} (32) \quad \mu^{\frac{1}{n+1}} \|T_{low} f\|_{L^{\frac{2(n+1)}{n+3}}} &+ \mu^{1/2} \|T_{low} f\|_{L^2} \\ &\lesssim \inf_{f=f_1+f_2} \mu^{-1/2} \|f_1\|_{L^2} + \mu^{-\frac{1}{n+1}} \|f_2\|_{L^{\frac{2(n+1)}{n-1}}} \end{aligned}$$

and the error estimate

$$(33) \quad \begin{aligned} & \mu^{-1/2} \|(1 - \tilde{T}_\mu^w(x, D)T_{low})\psi(D/(2\mu R))\phi f\|_{L^2} \\ & \lesssim \inf_{f=f_1+f_2} \mu^{-1/2} \|f_1\|_{L^2} + \mu^{-\frac{1}{n+1}} \|f_2\|_{L^{\frac{2(n+1)}{n-1}}} \end{aligned}$$

(iv) The complete parametrix In the final step we combine the low and high frequency parametrices. We set

$$T = T_{high}(1 - \psi(D/2\mu R)\phi_0 + \phi_0\psi(D/4\mu R)T_{low}\psi(D/2\mu R)\phi_0$$

The estimate (27) follows easily from the similar bounds for T_{high} and T_{low} . It remains to consider the error estimate. We have

$$\begin{aligned} (I - LT)\phi f &= (I - LT_{high})(1 - \psi(D/2\mu R)\phi f \\ &+ \phi_0\psi(D/4\mu R)(I - \tilde{L}_\mu^w T_{low})\psi(D/2\mu R)\phi f \\ &+ [\tilde{L}_\mu^w, \phi_0\psi(D/4\mu R)]T_{low}\psi(D/2\mu R)\phi f \\ &+ (L - \tilde{L}_\mu^w)\phi_0\psi(D/4\mu R)T_{low}\psi(D/2\mu R)\phi f \end{aligned}$$

For the first two terms we use the error estimates for T_{high} , respectively T_{low} . In the third term the commutator has size μ in L^2 so we can use the L^2 bound for T_{high} . The operator

$$(L - \tilde{L}_\mu^w)\phi_0\psi(D/4\mu R)$$

also has size μ in L^2 since the original coefficients differ from the mollified ones by μ^{-1} . This complete the proof of the inequality (28).

Part B. We prove (29) by duality as in Section 3 of [7]. Let $g \in W_\mu^{-\frac{1}{n+1}, \frac{2(n+1)}{n+3}}$. We decompose ϕg as

$$\phi g = h + L^*\bar{T}\phi g$$

where \bar{T} is the operator of Theorem 7 constructed for the formal adjoint operator L^* . By part A of the theorem we have

$$\begin{aligned} & (\delta\mu)^{-1/4}\mu^{-1/2}\|h\|_{L^2} + (\delta\mu)^{1/4}\mu^{1/2}\|\bar{T}\phi g\|_{L^2} + \|\bar{T}\phi g\|_{W_\mu^{\frac{1}{n+1}, \frac{2(n+1)}{n-1}}} \\ & \lesssim \|g\|_{W_\mu^{-\frac{1}{n+1}, \frac{2(n+1)}{n+3}}} \end{aligned}$$

Therefore we can write

$$\begin{aligned} |\langle \phi u, g \rangle| &= |\langle u, \phi g \rangle| \\ &\leq |\langle u, h \rangle| + |\langle u, L^*\bar{T}\phi g \rangle| \\ &= |\langle u, h \rangle| + |\langle Lu, \bar{T}\phi g \rangle| \\ &\lesssim \left((\delta\mu)^{1/4}\mu^{1/2}\|u\|_{L^2} + \inf_{Lu=f_1+f_2} (\delta\mu)^{-1/4}\mu^{-1/2}\|f_1\|_{L^2} + \|f_2\|_{W_\mu^{-\frac{1}{n+1}, \frac{2(n+1)}{n+3}}} \right) \\ &\quad \times \|g\|_{W_\mu^{-\frac{1}{n+1}, \frac{2(n+1)}{n+3}}} \end{aligned}$$

This implies the estimate (29).

Part C. We begin with an L^2 estimate. The principal symbol of

$$\bar{L}_\mu = L_\mu(\mu^2 + |D|^2)^{-1/2}$$

is

$$\bar{l}_\mu(x, \xi) = \left(-a^{ij}(x)\xi_i\xi_j + \mu^2 W(x) + 2\mu g^j \xi_j \right) (\mu^2 + |\xi|^2)^{-1/2}.$$

A short calculation shows that

$$\delta\mu - \{\Re \bar{l}_\mu(x, \xi), \Im \bar{l}_\mu(x, \xi)\} \lesssim |\bar{l}_\mu(x, \xi)|$$

and hence, by Corollary II.14 of [11], we obtain the bound

$$\delta\mu \|w\|_{L^2} \lesssim \|\bar{L}_\mu w\|_{L^2} + \|w\|_{L^2}$$

If $\delta\mu \gg 1$ then the norm of u on the right hand side can be hidden on the left hand side. Applying this to $w = (\mu^2 + |D|^2)^{1/2}v$ we obtain

$$(34) \quad \delta\mu \|v\|_{H_\mu^1}^2 \lesssim \|L_\mu v\|_{L^2}^2$$

For u as in the theorem we write

$$u = v + TL_\mu u$$

The bounds for the second term come from part A. On the other hand,

$$L_\mu v = (1 - L_\mu T)L_\mu u$$

for which we can use the error estimate (28) to obtain

$$(\delta\mu)^{-1/4} \mu^{-1/2} \|Lv\|_{L^2} \lesssim \inf_{L_\mu u = f_1 + f_2} (\delta\mu)^{-1/4} \mu^{-1/2} \|f_1\|_{L^2} + \|f_2\|_{W_\mu^{-\frac{1}{n+1}, \frac{2(n+1)}{n+3}}}$$

Then we successively apply (34) and (29) to v , concluding the proof. \square

5. THE L^p CARLEMAN INEQUALITY

In this section we prove Proposition 4. We first conjugate with respect to the exponential weight. If we set $w = e^{h(\ln|x|)}v$ then we can rewrite (7) in the form

$$\|w\|_{l^2 W^{\frac{1}{n+1}, \frac{2(n+1)}{n-1}}} + \|\rho w\|_{L^2} \lesssim \inf_{L_h w = f_1 + f_2} \|\rho^{-1} f_1\|_{L^2} + \|f_2\|_{l^2 W^{-\frac{1}{n+1}, \frac{2(n+1)}{n+3}}}$$

where

$$L_h = \Delta + Vw + h'(\ln|x|)^2 |x|^{-2} - h'(\ln|x|) \left[\nabla \frac{x}{|x|^2} + \frac{x}{|x|^2} \nabla \right]$$

We want to apply Theorem 7 on dyadic annuli

$$A_j = \{x | 2^{j-1} < |x| < 2^{j+1}\}$$

The rescaling $y = 2^{-j}x$ transforms this set to A_0 and the operator L_h to

$$L_h^j = \Delta + 2^{2j} \tilde{V} + h'(\ln(2^j|y|))^2 |y|^{-2} - h'(\ln(2^j|y|)) \left[\nabla \frac{y}{|y|^2} + \frac{y}{|y|^2} \nabla \right]$$

We verify that we can apply Theorem 7 to L_h^j . Since h' varies slowly on the unit scale we can take the corresponding value for μ to be

$$\mu_j = \sqrt{2^{2j} + h'(j \ln 2)^2}$$

The coefficients b and c are given by

$$c = \mu_j^{-2}(2^{2j}V + h'(\ln(2^j|y|))^2/|y|^2), \quad b_j = -\frac{h'(\ln(2^j|y|))}{\mu_j} \frac{y_j}{|y|^2}$$

and are clearly of class C^2 and size $O(1)$. We have

$$\Re l_h^j(x, \xi) = -\xi^2 + c, \quad \Im l_h^j(x, \xi) = 2b \cdot \xi$$

Their Poisson bracket has the form

$$\begin{aligned} \{-|\xi|^2 + c, b \cdot \xi\} &= \frac{h'(t)}{\mu|y|^2}(-|\xi|^2 + c) + 2y \cdot \xi \left(\frac{1}{|y|^4} - \frac{h''(t)}{h'(t)|y|^3} \right) b \cdot \xi \\ &\quad - \frac{2^{2j}h'(t)}{|y|^2\mu_j^3} y \cdot \nabla V - \frac{2h'(t)^2h''(t)}{|y|^4\mu_j^3}, \quad t = \ln(2^j|y|) \end{aligned}$$

Then we can apply Theorem 7 with δ comparable to the size of the third term. For our choice of h we have $|h''| \lesssim h'$ and also

$$h''(t) < 0 \implies h'(t) \ll e^t$$

Hence we can choose

$$\delta_j = \mu_j^{-3} (2^{2j}h'(j \ln 2) + h'(j \ln 2)^2h''_+(j \ln 2))$$

Let $\phi \in C_0^\infty(\mathbb{R})$ be a nonnegative function supported in $[-1, 1]$ with

$$\sum_{j=-\infty}^{\infty} \phi^2(t - j) = 1$$

and let $\phi_j(x) = \phi(\ln|x| - j)$. After rescaling, part A of Theorem 7 yields a parametrix T_j for L_h in A_j with the property that

$$\begin{aligned} &\|T_j g\|_{W^{\frac{1}{n+1}, \frac{2(n+1)}{n-1}}} + \|\rho T_j g\|_{L^2} + \left\| \rho \frac{|x|}{h'(\ln|x|) + |x|} \nabla(T_j g) \right\|_{L^2} \\ &+ \|\rho^{-1}(L_h T_j - 1)\phi_j g\|_{L^2} \lesssim \inf_{g=g_1+g_2} \|\rho^{-1}g_1\|_{L^2(A_j)} + \|g_2\|_{W^{-\frac{1}{n+1}, \frac{2(n+1)}{n+3}}(A_j)}. \end{aligned}$$

We define a parametrix for L_h by

$$T = \sum_{j=0}^{\infty} \phi_j T_j \phi_j$$

Summing up the bounds on T_j we obtain a bound for T ,

$$\|Tg\|_{l^2 W^{\frac{1}{n+1}, \frac{2(n+1)}{n-1}}} + \|\rho Tg\|_{L^2} \lesssim \inf_{g=g_1+g_2} \|\rho^{-1}g_1\|_{L^2} + \|g_2\|_{l^2 W^{-\frac{1}{n+1}, \frac{2(n+1)}{n+3}}}.$$

We also compute the error

$$1 - L_h T = \sum_{j=0}^{\infty} \phi_j (1 - L_h T_j) \phi_j - \sum_{j=0}^{\infty} [L_h, \phi_j] T_j \phi_j$$

Since

$$[L_h, \phi_j] = O(|x|^{-1}) \nabla + O(h'(\ln |x|) |x|^{-2})$$

and

$$|x|^{-1} \lesssim \rho^2 \frac{|x|}{h'(\ln |x|) + |x|}, \quad h'(\ln |x|) |x|^{-2} \lesssim \rho^2$$

we can bound the error by

$$\|\rho^{-1}(1 - LT)g\|_{L^2} \lesssim \inf_{g=g_1+g_2} \|\rho^{-1}g_1\|_{L^2} + \|g_2\|_{l^2 W^{-\frac{1}{n+1}, \frac{2(n+1)}{n+3}}}.$$

Now, after the construction of the parametrix the assertion of Proposition 4 follows exactly as the corresponding part of Theorem 7. We repeat the argument. Split w into

$$w = v + TLw$$

Then the second term satisfies the desired bounds while for the first we know that

$$\|\rho^{-1}Lv\|_{L^2} = \|\rho^{-1}(LT - 1)Lw\|_{L^2} \lesssim \inf_{Lw=g_1+g_2} \|\rho^{-1}g_1\|_{L^2} + \|g_2\|_{l^2 W^{-\frac{1}{n+1}, \frac{2(n+1)}{n+3}}}.$$

Lemma 5 allows us to also estimate

$$\|\rho v\|_{L^2}$$

On the other hand by Theorem 7, B rescaled and applied to v in A_j we get

$$\|\phi_j v\|_{W^{\frac{1}{n+1}, \frac{2(n+1)}{n-1}}} \lesssim \|\rho v\|_{L^2(A_j)} + \|\rho^{-1}Lv\|_{L^2(A_j)}$$

and after summation in j ,

$$\|v\|_{l^2 W^{\frac{1}{n+1}, \frac{2(n+1)}{n-1}}} \lesssim \|\rho v\|_{L^2} + \|\rho^{-1}Lv\|_{L^2}$$

thereby concluding the proof.

6. EQUATIONS WITH GRADIENT POTENTIALS

In this section we discuss the corresponding results which are obtained when short range gradient potentials are added. Thus we consider equations of the form

$$(35) \quad (-\Delta - V)u = Wu + Z^l \nabla u + \nabla Z^r u$$

with V and W as before. The gradient potential $Z = (Z^l, Z^r)$ is subject to the following conditions:

Assumption A 3 (The short range gradient potential). *The gradient potential $Z \in l^\infty(L^n)$ satisfies*

$$(36) \quad \limsup_{j \rightarrow \infty} \|Z\|_{L^n(\{|x|^{2j} \leq |x| \leq 2^{j+1}\})} \leq \delta$$

In addition for some $R \gg \|V\|_{L^\infty}$ the low frequency part $S_{<R}Z$ of Z satisfies the conditions in Assumption A2.

The L^n assumption is natural due to scaling. The low frequency condition is also natural, since on the characteristic set of $-\Delta - V$ the frequency has size $O(1)$, and at frequency one there is no difference between the potential and the gradient potential. Under these conditions we have

Theorem 8. *Assume that V , W and Z satisfy Assumptions A1, A2 respectively A3. Let $\tau_1 > \tau_0$ and assume that δ is sufficiently small. Let $u \in H_{loc}^1(\mathbb{R}^n)$ satisfy (3) and $(1 + |x|^2)^{\tau_1 - \frac{1}{2}}u \in L^2$. Then $u \equiv 0$.*

By scaling we obtain the following result on the absence of embedded eigenvalues:

Corollary 9. *Assume that V , W and Z satisfy Assumptions A1, A2 respectively A3 with $\delta = 0$. Then there are no embedded eigenvalues for the operator*

$$-\Delta - W - Z^l \nabla - \nabla Z^r$$

The problem of introducing gradient potentials has long been considered in the context of the unique continuation and the strong unique continuation problems for the same operators as here. There the key breakthrough came in Wolff's work [13] who proved that $Z \in L^n$ suffices for the unique continuation property. He also obtained the same result for strong unique continuation but only in low dimension. Later his ideas were used by the authors in [5] to complete the picture for strong unique continuation in high dimension, working with gradient potentials $Z \in l^1 L^n$. This latter paper is more relevant to the present context as it provides Carleman estimates in largely the same format as here.

Ideally, one would like to include matching gradient estimates to our L^p Carleman inequalities. This would solve the problem but unfortunately cannot work. Wolff's contribution was to show that by osculating the weight one can considerably improve the bounds for the gradient term in the equation. Thus the choice of weights ultimately depends both on the gradient potentials and on the solution u . In our context this argument is needed only at spatial scales where the frequency of the conjugated operator is larger than one. Elsewhere the gradient does not contribute much to the problem. Thus we are led to consider perturbed weights

$$(37) \quad \psi_{\epsilon, \tau}(x) = h_\epsilon(\ln |x|) + k(x)$$

where k is not spherically symmetric but is small in an appropriate sense. The assumptions on k are summarized in what follows:

$$(38) \quad \begin{cases} \text{supp } k \subset \{|x| \leq \tau^2\} \\ |x|^\alpha |\nabla^\alpha k(x)| \ll h'(\ln |x|) \quad \alpha = 1, 2, 3 \end{cases}$$

Part of the Carleman estimates below describes what happens in elliptic regions of the conjugated operator L_ϕ . To select (part of) this elliptic region we introduce a pseudodifferential operator $\chi_{>R}$ which selects the region

$$E = \{|x| \gtrsim \tau^2, |\xi| \gtrsim R\}$$

Here both the truncation in x and in ξ are done on the dyadic scale, while R is chosen sufficiently large so that E is away from the characteristic set of P_ϕ . Then the Carleman estimates are as follows:

Theorem 10. *Assume that the long range potential V satisfies A1. Let Z satisfy A3 with*

$$\|Z\|_{L^\infty L^n} + \|S_{<R}Z\|_{L^\infty X + \langle x \rangle^{-1} L^\infty} \leq 1$$

Then for each $0 < \epsilon \leq \epsilon_0$, τ large enough, $\tau_1 > \tau_0$ and v which satisfies $(1 + |x|^2)^{\tau_1 - \frac{1}{2}} v \in L^2$ there is a weight perturbation k satisfying (37),(38) so that the following estimate holds with constants independent of $0 < \epsilon \leq \epsilon_0$, $\tau > \tau_0$:

$$(39) \quad \begin{aligned} & \|e^{\psi_{\epsilon,\tau}(x)} v\|_{L^2 W^{\frac{1}{n+1}, \frac{2(n+1)}{n-1}}} + \|\chi_{>R} e^{\psi_{\epsilon,\tau}(x)} v\|_{H^1} + \|e^{\psi_{\epsilon,\tau}(x)} \rho v\|_{L^2} \\ & + \|e^{\psi_{\epsilon,\tau}(x)} Z^l \nabla v\|_{L^2 W^{-\frac{1}{n+1}, \frac{2(n+1)}{n+3}} + \chi_{>R} H^{-1}} + \|e^{\psi_{\epsilon,\tau}(x)} \nabla Z^r v\|_{L^2 W^{-\frac{1}{n+1}, \frac{2(n+1)}{n+3}} + \chi_{>R} H^{-1}} \\ & \lesssim \inf_{f_1 + f_2 = (-\Delta - V)v} \|e^{\psi_{\epsilon,\tau}(x)} \rho^{-1} f_1\|_{L^2} + \|e^{\psi_{\epsilon,\tau}(x)} f_2\|_{L^2 W^{-\frac{1}{n+1}, \frac{2(n+1)}{n+3}} + \chi_{>R} H^{-1}} \end{aligned}$$

The key feature of the theorem is that the weight $\psi_{\epsilon,\tau}(x)$ depends both on the potential Z and on the solution v itself. Once this result is established, it leads as before to the conclusion that solutions to (35) must be compactly supported. Then (a variation of) Wolff's weak unique continuation result [13] takes over and implies that v must be identically 0. We also refer the reader to [5], where the estimates are formulated in a way similar to this paper, and where both left and right gradient potentials are considered.

The proof requires the following steps:

(i) Conjugate the equation with respect to the exponential weight and set $w = e^{\psi_{\epsilon,\tau}(x)} v$. This eliminates the exponential weight from the equation and replaces the operator $-\Delta - V$ by its conjugated operator $L_{\psi_{\epsilon,\tau}}$.

(ii) prove the L^2 estimate for v uniformly for all weights $\phi_{\epsilon,\tau}$ with k satisfying (38). This is done exactly as in Section 3. The size of the perturbation k is so that its effect is negligible in this computation.

(iii) prove the estimate (7), again uniformly with respect to all choices for k . This repeats the arguments in Section 5 with no change.

(iv) Add in the H^{-1} and H^1 norms, thus proving (39) for $Z = 0$. This is done in an elliptic fashion, by constructing an elliptic parametrix for L_ϕ away from its characteristic set. For this the norms involving ρ are used only to estimate errors, while the L^p norms are all used via L^2 norms and Sobolev embedding. The standard pseudodifferential calculus can be applied since the coefficients of P_ϕ are smooth on the dyadic scale in x .

(v) observe that the L^2 estimates allow localization on the dyadic spatial scale. Thus we separate the estimate into two regions, $\{|x| < \tau^2\}$ and $\{|x| > \tau^2/2\}$.

(iv) show that within the first region it is possible to choose the weight k so that the estimate with Z included holds. This is the part that uses Wolff's osculation lemma, and it is explained in detail in [5]. Our case here is somewhat simpler than in [5] since in the region $\{|x| < \tau^2\}$ we have uniform convexity of the weight, $h'' \approx h'$. Also the L^p bound here is stronger than in [5], which only makes things better.

(v) prove the estimate in each dyadic component of the second region $\{|x| > \tau^2/2\}$ with Z included. This starts from the estimates without Z and uses only elliptic bounds. We outline the argument. Since we use dual norms on the left and on the right of (39), it suffices to do it for the $Z\nabla$ term. The bounds for ∇Z will work out similarly but in dual spaces.

We split Z into a low and a high frequency part,

$$Z = Z_{<R} + Z_{>R}$$

and the gradient also,

$$\nabla = \nabla_{<R/2} + \nabla_{>R/2}$$

Using the $L^{\frac{n+1}{2}}$ bound on $Z_{<R/2}$ we can directly estimate the contribution of $Z_{<R}\nabla_{<R/2}$ which is located at low frequency.

The contribution of $Z_{>R}\nabla_{<R/2}$ lies at high frequency, so it suffices to bound it in H^{-1} . We can actually bound it in L^2 ,

$$\|Z_{>R}\nabla_{<R/2}w\|_{L^2} \lesssim \|Z\|_{L^n} \|w\|_{W^{\frac{1}{n+1}, \frac{2(n+1)}{n-1}}}$$

For $Z\nabla_{>R/2}$ we can use the H^1 bound to write

$$\|Z\nabla_{>R/2}w\|_{L^{\frac{2n}{n+2}}} \leq \|Z\|_{L^n} \|\chi_{>R/2}w\|_{H^1}$$

and conclude by Sobolev embeddings.

7. ASYMPTOTICALLY FLAT METRICS

In this section we describe how the results on the absence of embedded eigenvalues extend to variable coefficient asymptotically flat metrics. We replace the Laplacian with a second order elliptic selfadjoint operator

$$L = -\partial_j a^{jk} \partial_k + i(b^j \partial_j + \partial_j b^j) + c$$

where the coefficients a, b, c are real. We assume that P is flat at infinity in the sense that (see Definition 1) :

$$(40) \quad \begin{aligned} & a^{jk}, b^j, c \in C_{\langle x \rangle}^2 \subset L^\infty \\ & \limsup_{|x| \rightarrow \infty} |x| |\nabla a^{ij}| \leq \delta_0, \\ & \limsup_{|x| \rightarrow \infty} |b(x)| + |x| |\nabla b(x)| \leq \delta_1, \quad \limsup_{|x| \rightarrow \infty} |c(x)| + |x| |\nabla c(x)| \leq \delta_1^2 \end{aligned}$$

We also slightly strenghten the assumption A3 to make it stable with respect to changes of variable:

Assumption A 4 (The short range gradient potential). *The gradient potential $Z \in l^\infty(L^n)$ satisfies*

$$(41) \quad \limsup_{j \rightarrow \infty} \|Z\|_{L^n(\{|x|^{2j} \leq |x| \leq 2^{j+1}\})} \leq \delta$$

In addition $\langle D \rangle^{-N} Z$ satisfies the conditions in Assumption A2 for some N sufficiently large.

Then we have

Theorem 11. *Assume that W, V and Z satisfy Assumptions A1, A2 and A4 with small enough δ , that $\tau_1 > \tau_0$ and that the coefficients of P satisfy (40) with δ_0 and δ_1 sufficiently small. If $u \in H_{loc}^1(\mathbb{R}^n)$ solves*

$$(42) \quad Lu + Vu = Wu + Z^l \nabla u + \nabla Z^r u$$

and $(1 + |x|^2)^{\tau_1 - \frac{1}{2}} u \in L^2$ then $u \equiv 0$.

The assumption of Theorem 11 are not scale invariant. For the following straightforward consequence we rescale the operator.

Corollary 12. *Assume that the coefficients of the operator P satisfy (40) with δ_0 sufficiently small. Let W, Z be as in Assumptions A2, A4 with $\delta = 0$. Then there exists $C > 0$ so that $P + W$ has no eigenvalues $\lambda > C\delta_1$.*

The proof follows the same outline as in the constant coefficient case. We describe the steps in what follows, and discuss the necessary modifications.

First one needs to augment (40) to gain also the relation

$$(43) \quad \limsup_{|x| \rightarrow \infty} |a(x) - I_n| \lesssim \delta_0$$

This is achieved using a change of coordinates somewhat similar to the one introduced in [5]. Due to (40), within each spatial dyadic region this can be achieved with a linear change of coordinates. But from one dyadic region to the next these linear maps differ by $O(\delta)$. Hence gluing them together yields a nonlinear function χ which achieves (43) and has the regularity

$$|\partial^\alpha \chi(x)| \lesssim \delta_1 |x|^{1-|\alpha|} \quad |\alpha| \geq 2$$

It is easy to verify that such a change of coordinates does not affect δ_1 by more than a fixed factor.

If χ were linear then the Assumption A1 on V would rest unchanged. As it is, we have to modify τ_1 by $O(\delta_0)$, which is suitably small.

Finally, the operator L is still L^2 selfadjoint in the new coordinates but with respect to the measure given by the Jacobian J of the change of coordinates. This implies that JL is selfadjoint with respect to the Lebesgue measure. This requires replacing V and W by JV and JW , which has no significant effect on our assumptions.

Once (43) is gained the Carleman estimates (7) in Proposition 4 remain valid with essentially no change. The only minor modification that is needed is concerned with what happens within a compact set, where we have no control over the geometry of the coefficients a^{ij} in the principal part. But this can be easily addressed by adding some additional convexity to the exponential weight within this compact set. Precisely, a weight of the form

$$h(t) = \tau e^{\lambda t}$$

would suffice for bounded t provided λ is large enough.

The L^2 Carleman estimates are established using integration by parts, and do not require any bounds on the second derivatives of a^{ij} .

The L^p Carleman estimates are derived from the L^2 ones exactly as in Section 5. For comparison purposes, we recall that the L^p estimates proved in [7] and [5] only require bounds on the first derivatives of a^{ij} . This is because the spatial localization which is allowed by the Carleman estimates is on a scale on which one is allowed to freeze the a^{ij} modulo negligible errors. The same applies here for $|x| \lesssim \tau^2$ (which corresponds to $e^t \lesssim \tau^2$). However, beyond this threshold the rescaled skewadjoint part becomes very small and the problem is close to the spectral projection estimates respectively the Strichartz estimates for wave equations with C^2 coefficients. The spatial localization scale is $h'(\ln(|x|))^{-\frac{1}{2}}|x|$ while the frequency, instead of decaying, remains $O(1)$ due to the long range potential V . Hence the difference between P and its frozen coefficient version is $O(h'(\ln(|x|))^{-\frac{1}{2}})$, which is more than the constant ρ^2 in the L^2 estimates. This is why we need also bounds on the second derivatives of a^{ij} , as required by pTheorem 7.

Finally, the gradient potential can be added in as explained in the previous section.

8. APPENDIX

We consider a dyadic partition of unity in \mathbb{R}^n ,

$$1 = \sum_{j=-\infty}^{\infty} \chi_j(x)$$

where $\chi_j(x) = \chi_0(2^{-j}x)$ is supported in $|x| \approx 2^j$. We also consider bump functions $\tilde{\chi}_j(x) = \tilde{\chi}_j(2^{-j}x)$ with slightly larger support, which equal 1 within the support of χ_j such that $\tilde{\chi}_j\tilde{\chi}_l = 0$ if $|j - l| \geq 2$.

Lemma 13. *Let $1 < p < \infty$ and $-\frac{n}{p'} < s < \frac{n}{p}$. Then*

$$\|u\|_{W^{s,p}}^p \approx \sum \|\chi_j u\|_{W^{s,p}}^p$$

Proof. Let (u_j) be a sequence in $W^{s,p}$. Arguing by duality it suffices to prove the bound

$$\left\| \sum_{j=-\infty}^{\infty} \chi_j u_j \right\|_{W^{s,p}}^p \lesssim \sum \|u_j\|_{W^{s,p}}^p$$

With $\langle D \rangle = (1 + |D|^2)^{1/2}$ we have

$$\left\| \sum_{j=-\infty}^{\infty} \chi_j u_j \right\|_{W^{s,p}} = \|\langle D \rangle^s \sum \chi_j u_j\|_{L^p}$$

We write

$$\langle D \rangle^s \sum_{j=-\infty}^{\infty} \chi_j u_j = \sum_{j=-\infty}^{\infty} \tilde{\chi}_j \langle D \rangle^s \chi_j u_j + \sum_{j=-\infty}^{\infty} (1 - \tilde{\chi}_j) \langle D \rangle^s \chi_j u_j$$

The terms in the first sum have almost disjoint supports and are easy to estimate. It remains to consider the second sum. We use bounds on the kernel of $\langle D \rangle^{-s}$ and its derivatives to estimate

$$|(1 - \tilde{\chi}_j) \langle D \rangle^s \chi_j u_j(x)| \lesssim \|u_j\|_{L^p} 2^{(s+\frac{n}{p'})j} (2^j + |x|)^{-n-s}.$$

Then we conclude using

$$\left\| \sum_{j=-\infty}^{\infty} a_j 2^{(s+\frac{n}{p'})j} (2^j + |x|)^{-n-s} \right\|_{L^p}^p \approx \sum_{j=-\infty}^{\infty} |a_j|^p \quad s > -\frac{n}{p'}$$

□

This is the main ingredient in the proof of

Proposition 14. *Let $\delta > 0$. Suppose that $W \in X$ (see Definition 2). Then we have*

$$\lim_{\alpha \rightarrow 0} \|W\|_{X(B(0,\alpha))} = 0$$

Proof. We assume that $n \geq 3$, the case $n = 2$ is similar. The result follows from the estimate

$$(44) \quad \|W\|_X \approx \|\chi_j W\|_{l^{\frac{n+1}{2}}(X)}$$

For one direction we write

$$\begin{aligned}
|\langle Wu, v \rangle| &= \left| \sum \langle \chi_j W \tilde{\chi}_j u, \tilde{\chi}_j v \rangle \right| \\
&\lesssim \sum \| \chi_j W \|_X \| \tilde{\chi}_j u \|_{W^{\frac{1}{n+1}, \frac{2(n+1)}{n-1}}} \| \tilde{\chi}_j v \|_{W^{\frac{1}{n+1}, \frac{2(n+1)}{n-1}}} \\
&\lesssim \| \chi_j W \|_{l^{\frac{n+1}{2}}} \| \tilde{\chi}_j u \|_{l^{\frac{2(n+1)}{n-1}} W^{\frac{1}{n+1}, \frac{2(n+1)}{n-1}}} \| \tilde{\chi}_j v \|_{l^{\frac{2(n+1)}{n-1}} W^{\frac{1}{n+1}, \frac{2(n+1)}{n-1}}} \\
&\lesssim \| \chi_j W \|_{l^{\frac{n+1}{2}}} \| u \|_{W^{\frac{1}{n+1}, \frac{2(n+1)}{n-1}}} \| v \|_{W^{\frac{1}{n+1}, \frac{2(n+1)}{n-1}}}
\end{aligned}$$

For the other, we consider separately sums with j even and with j odd:

$$\begin{aligned}
\sum_{j \text{ even}} \langle \chi_j W u_j, v_j \rangle &= \sum_{j \text{ even}} \langle \chi_j W \tilde{\chi}_j u_j, \tilde{\chi}_j v_j \rangle \\
&= \langle W \sum_{j \text{ even}} \tilde{\chi}_j u_j, \sum_{j \text{ even}} \tilde{\chi}_j v_j \rangle \\
&\lesssim \| W \|_X \sum_{j \text{ even}} \| \tilde{\chi}_j u_j \|_{W^{\frac{1}{n+1}, \frac{2(n+1)}{n-1}}} \| \sum_{j \text{ even}} \tilde{\chi}_j v_j \|_{W^{\frac{1}{n+1}, \frac{2(n+1)}{n-1}}} \\
&\lesssim \| W \|_X \| u_j \|_{l^{\frac{2(n+1)}{n-1}} W^{\frac{1}{n+1}, \frac{2(n+1)}{n-1}}} \| u_j \|_{l^{\frac{2(n+1)}{n-1}} W^{\frac{1}{n+1}, \frac{2(n+1)}{n-1}}}
\end{aligned}$$

□

REFERENCES

- [1] Richard Froese and Ira Herbst. Exponential bounds and absence of positive eigenvalues for N -body Schrödinger operators. *Comm. Math. Phys.*, 87(3):429–447, 1982/83.
- [2] Richard Froese, Ira Herbst, Maria Hoffmann-Ostenhof, and Thomas Hoffmann-Ostenhof. On the absence of positive eigenvalues for one-body Schrödinger operators. *J. Analyse Math.*, 41:272–284, 1982.
- [3] A. D. Ionescu and D. Jerison. On the absence of positive eigenvalues of Schrödinger operators with rough potentials. *Geom. Funct. Anal.*, 13(5):1029–1081, 2003.
- [4] C. E. Kenig, A. Ruiz, and C. D. Sogge. Uniform Sobolev inequalities and unique continuation for second order constant coefficient differential operators. *Duke Math. J.*, 55(2):329–347, 1987.
- [5] Herbert Koch and Daniel Tataru. Carleman estimates and unique continuation for second-order elliptic equations with nonsmooth coefficients. *Comm. Pure Appl. Math.*, 54(3):339–360, 2001.
- [6] Herbert Koch and Daniel Tataru. Sharp counterexamples in unique continuation for second order elliptic equations. *J. Reine Angew. Math.*, 542:133–146, 2002.
- [7] Herbert Koch and Daniel Tataru. Dispersive estimates for principally normal pseudo-differential operators. *Comm. Pure Appl. Math.*, 58(2):217–284, 2005.
- [8] Michael Reed and Barry Simon. *Methods of modern mathematical physics. IV. Analysis of operators*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1978.
- [9] A. Soffer and M.I. Weinstein. Time dependent resonance theory. *Geom. Funct. Anal.*, 8(6):1086–1128, 1998.
- [10] Christopher D. Sogge. *Fourier integrals in classical analysis*, volume 105 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1993.
- [11] Daniel Tataru. On the Fefferman-Phong inequality and related problems. *Comm. Partial Differential Equations*, 27(11-12):2101–2138, 2002.

- [12] J. von Neumann and E.P. Wigner. Über merkwürdige diskrete Eigenwerte. *Z. Phys.*, 30:465–467, 1929.
- [13] Thomas H. Wolff. Recent work on sharp estimates in second order elliptic unique continuation problems. In *Fourier analysis and partial differential equations (Miraflores de la Sierra, 1992)*, pages 99–128. CRC, Boca Raton, FL, 1995.

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