

# Carleson measures and uniformly discrete sequences in strongly pseudoconvex domains

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## ABSTRACT

We characterize, using the Bergman kernel, Carleson measures of Bergman spaces in strongly pseudoconvex bounded domains in  $\mathbf{C}^n$ , generalizing to this setting theorems proved by Duren and Weir for the unit ball. We also show that uniformly discrete (with respect to the Kobayashi distance) sequences give examples of Carleson measures, and we compute the speed of escape to the boundary of uniformly discrete sequences in strongly pseudoconvex domains, generalizing results obtained in the unit ball by Jevtić, Massaneda and Thomas, by Duren and Weir, and by MacCluer.

## 1. Introduction

In his celebrated solution of the corona problem in the disc, Carleson [2] introduced an important class of measures to study the structure of the Hardy spaces of the unit disc  $\Delta \subset \mathbf{C}$ . Let  $A$  be a Banach space of holomorphic functions on a domain  $D \subset \mathbf{C}^n$ , and assume that  $A$  is contained in  $L^p(D)$  for some  $p > 0$ . A finite positive Borel measure  $\mu$  on  $D$  is a *Carleson measure* of  $A$  if there exists a constant  $C > 0$  such that

$$\forall f \in A \quad \int_D |f|^p d\mu \leq C \|f\|_A^p.$$

Carleson studied Carleson measures of the Hardy spaces  $H^p(\Delta)$ , showing that a finite positive Borel measure  $\mu$  is a Carleson measure of  $H^p(\Delta)$  if and only if there exists a constant  $C > 0$  such that  $\mu(S_{\theta_0, h}) \leq Ch$  for all sets

$$S_{\theta_0, h} = \{re^{i\theta} \in \Delta \mid 1 - h \leq r < 1, |\theta - \theta_0| \leq h\}$$

(see [5]); in particular the set of Carleson measures of  $H^p(\Delta)$  does not depend on  $p$ .

In 1975, Hastings [10] (see also Oleinik [22] and Oleinik and Pavlov [23]) proved a similar characterization for the Carleson measures of the Bergman spaces  $A^p(\Delta)$ : a finite positive Borel measure  $\mu$  is a Carleson measure of  $A^p(\Delta)$  if and only if there exists a constant  $C > 0$  such that  $\mu(S_{\theta_0, h}) \leq Ch^2$  for all  $\theta \in [0, 2\pi]$  and  $h \in (0, 1)$ . As a consequence, again, the set of Carleson measures of  $A^p(\Delta)$  does not depend on  $p$ .

The sets  $S_{\theta_0, h}$  clearly are not invariant under automorphisms of the disc, whereas one would like to characterize Carleson measures for Bergman spaces in an invariant way, related to the intrinsic (hyperbolic) geometry of the disc and not to the extrinsic Euclidean geometry. This was done in 1983 by Luecking [18]; indeed he proved (see also [6, Theorem 14, p. 62]) that a finite positive Borel measure  $\mu$  is a Carleson measure of  $A^p(\Delta)$  if and only if for some (and hence all)  $0 < r < 1$  there is a constant  $C_r > 0$  such that  $\mu(B_\Delta(z_0, r)) \leq C_r \nu(B_\Delta(z_0, r))$  for all  $z_0 \in \Delta$ , where  $\nu$  is the Lebesgue (area) measure and  $B_\Delta(z_0, r) \subset \Delta$

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is the Poincaré disc

$$B_{\Delta}(z_0, r) = \left\{ z \in \Delta \left| \left| \frac{z - z_0}{1 - \bar{z}_0 z} \right| < r \right. \right\},$$

of centre  $z_0 \in \Delta$  and pseudohyperbolic radius  $r$ .

The first characterization for Carleson measures of the Bergman spaces of the unit ball  $B^n \subset \mathbf{C}^n$  has been given by Cima and Wogen [4], using again sets defined in terms of the Euclidean geometry of  $B^n$ , and thus not invariant under automorphisms. Again, as essentially noticed by Luecking [18] and explicitly stated by Duren and Weir [8], it is possible to give a characterization for the Carleson measures of Bergman spaces of  $B^n$  by using the balls for the Bergman (or Kobayashi, or pseudohyperbolic) distance: a finite positive Borel measure  $\mu$  is a Carleson measure of  $A^p(B^n)$  if and only if, for some (and hence all)  $0 < r < 1$ , there is a constant  $C_r > 0$  such that  $\mu(B_{B^n}(z_0, r)) \leq C_r \nu(B_{B^n}(z_0, r))$  for all  $z_0 \in B^n$ , where  $\nu$  is the Lebesgue  $2n$ -dimensional measure and  $B_{B^n}(z_0, r) \subset B^n$  is the ball for the Bergman distance of centre  $z_0 \in B^n$  and radius  $(1/2) \log(1+r)/(1-r) \in \mathbf{R}^+$  (that is, of radius  $r$  in the pseudohyperbolic distance; recall that the pseudohyperbolic distance  $\rho$  is related to the Bergman or Kobayashi distance  $k_{B^n}$  by the formula  $\rho = \tanh(k_{B^n})$ ).

In 1995, Cima and Mercer [3] characterized Carleson measures of the Bergman spaces of strongly pseudoconvex domains; in particular, they proved that in this case too the class of Carleson measures of  $A^p(D)$  does not depend on  $p$ . Their characterization is stated again in terms of the extrinsic Euclidean geometry of the domain, but the proof uses in an essential way the intrinsic geometry of strongly convex domains, as well as the construction of particular *ad hoc* functions.

A particularly important Bergman space is, of course,  $A^2(D)$ , where the Bergman kernel lives. This suggests the question of whether it is possible to characterize Carleson measures using the Bergman kernel. This has been done by Duren and Weir [8] for the unit ball; our first main result is the generalization of their characterization to strongly pseudoconvex domains.

Let  $K : D \times D \rightarrow \mathbf{C}$  be the Bergman kernel of a strongly pseudoconvex domain  $D \subset \subset \mathbf{C}^n$ . For any finite positive Borel measure  $\mu$  on  $D$ , the *Berezin transform* of  $\mu$  is the function  $B\mu : D \rightarrow \mathbf{R}$  given by

$$B\mu(z) = \int_D \frac{|K(\zeta, z)|^2}{K(z, z)} d\mu(\zeta).$$

Then (see Theorem 3.4 for a more complete statement) we obtain the following theorem.

**THEOREM 1.1.** *Let  $\mu$  be a finite positive Borel measure on a strongly pseudoconvex bounded domain  $D \subset \subset \mathbf{C}^n$ . Then  $\mu$  is a Carleson measure of  $A^p(D)$  if and only if its Berezin transform  $B\mu$  is bounded.*

Previous proofs of characterizations of Carleson measures in the ball heavily relied on the homogeneity of the ball under its automorphisms group, and on the explicit expression of the automorphisms; but this approach cannot be used in our setting, because strongly pseudoconvex domains that are not biholomorphic to the ball admit very few automorphisms. Our proof depends instead on a detailed understanding of the intrinsic Kobayashi geometry of strongly pseudoconvex domains, and on Fefferman's estimates on the Bergman kernel.

A natural question is how to construct explicit examples of Carleson measures in strongly pseudoconvex domains. As in the unit disc and in the unit ball, an important family of examples is provided by uniformly discrete sequences. Let  $(X, d)$  be a metric space; then a sequence  $\{x_j\} \subset X$  of points in  $X$  is *uniformly discrete* if there exists  $\delta > 0$  such that  $d(x_j, x_k) \geq \delta$  for all  $j \neq k$ . Then we are able to prove the following generalization of results proved by Duren

and Weir [8], Jevtić, Massaneda and Thomas [13], and Massaneda [21] in the unit ball (see Theorem 4.2 for a more complete statement).

**THEOREM 1.2.** *Let  $D \subset \subset \mathbf{C}^n$  be a strongly pseudoconvex bounded domain, and let  $\Gamma = \{z_j\}$  be a sequence in  $D$ . Then  $\Gamma$  is a finite union of uniformly discrete (with respect to the Kobayashi distance) sequences if and only if  $\sum_{z_j \in \Gamma} d(z_j, \partial D)^{n+1} \delta_{z_j}$  is a Carleson measure of  $A^p(D)$ , where  $\delta_{z_j}$  is the Dirac measure at  $z_j$  and  $d(\cdot, \partial D)$  is the Euclidean distance from the boundary.*

Finally, Duren, Schuster and Vukotić [7], Duren and Weir [8], and MacCluer [20] also studied the speed of escape to the boundary of uniformly discrete (with respect to the pseudohyperbolic or Bergman distances of the ball) sequences in the unit ball. We are able to generalize their results to strongly pseudoconvex domains (see Theorem 4.6, and Proposition 4.4 for a similar result not requiring pseudoconvexity).

**THEOREM 1.3.** *Let  $D \subset \mathbf{C}^n$  be a strongly pseudoconvex bounded domain. Let  $\Gamma = \{z_j\} \subset D$  be a uniformly discrete (with respect to the Kobayashi distance) sequence with  $d(z_j, \partial D) < 1$  for all  $j \in \mathbf{N}^*$ . Then*

$$\sum_{z_j \in \Gamma} d(z_j, \partial D)^n h\left(-\frac{1}{\log d(z_j, \partial D)}\right) < +\infty, \tag{1.1}$$

for all increasing functions  $h : \mathbf{R}^+ \rightarrow \mathbf{R}^+$  such that

$$\sum_{m=1}^{+\infty} h\left(\frac{1}{m}\right) < +\infty.$$

## 2. The intrinsic geometry of strongly pseudoconvex domains

In this section, we shall prove a number of estimates on the intrinsic geometry of strongly pseudoconvex domains, as described by the Kobayashi distance. In particular, we shall study the boundary behaviour of Kobayashi balls, and we shall prove a sort of submean property for nonnegative plurisubharmonic functions in Kobayashi balls.

Let us briefly recall the definition and the main properties of the Kobayashi distance; we refer to [1, 12, 14] for details and much more. Let  $k_\Delta$  denote the Poincaré distance on the unit disc  $\Delta \subset \mathbf{C}$ . If  $X$  is a complex manifold, the *Lempert function*  $\delta_X : X \times X \rightarrow \mathbf{R}^+$  of  $X$  is defined by

$$\delta_X(z, w) = \inf\{k_\Delta(\zeta, \eta) \mid \text{there exists a holomorphic } \phi : \Delta \rightarrow X \text{ with } \phi(\zeta) = z \text{ and } \phi(\eta) = w\},$$

for all  $z, w \in X$ . The *Kobayashi pseudodistance*  $k_X : X \times X \rightarrow \mathbf{R}^+$  of  $X$  is the largest pseudodistance on  $X$  bounded above by  $\delta_X$ . We say that  $X$  is (*Kobayashi*) *hyperbolic* if  $k_X$  is a true distance, and in that case it is known that the metric topology induced by  $k_X$  coincides with the manifold topology of  $X$  (see, for example, [1, Proposition 2.3.10]). For instance, all bounded domains are hyperbolic (see, for example, [1, Theorem 2.3.14]).

The main property of the Kobayashi (pseudo)distance is that it is contracted by holomorphic maps: if  $f : X \rightarrow Y$  is a holomorphic map, then

$$\forall z, w \in X \quad k_Y(f(z), f(w)) \leq k_X(z, w).$$

In particular, the Kobayashi distance is invariant under biholomorphisms, and decreases under inclusions: if  $D_1 \subset D_2 \subset \subset \mathbf{C}^n$  are two bounded domains, then we have  $k_{D_2}(z, w) \leq k_{D_1}(z, w)$  for all  $z, w \in D_1$ .

It is easy to see that the Kobayashi distance of the unit disc coincides with the Poincaré distance. Furthermore, the Kobayashi distance of the unit ball  $B^n \subset \mathbf{C}^n$  coincides with the Bergman distance (see, for example, [1, Corollary 2.3.6]).

If  $X$  is a hyperbolic manifold,  $z_0 \in X$ , and  $r \in (0, 1)$ , then we shall denote by  $B_X(z_0, r)$  the Kobayashi ball of centre  $z_0$  and radius  $(1/2) \log(1+r)/(1-r)$ :

$$B_X(z_0, r) = \{z \in X \mid \tanh k_X(z_0, z) < r\}.$$

Note that  $\rho_X = \tanh k_X$  is still a distance on  $X$ , because  $\tanh$  is a strictly convex function on  $\mathbf{R}^+$ . In particular,  $\rho_{B^n}$  is the pseudohyperbolic distance of  $B^n$ .

The Kobayashi distance of bounded strongly pseudoconvex domains enjoys several important properties. First of all, it is complete (see, for example, [1, Corollary 2.3.53]), and hence closed Kobayashi balls are compact. Furthermore, we can very precisely describe the boundary behaviour of the Kobayashi distance: if  $D \subset \subset \mathbf{C}^n$  is a strongly pseudoconvex bounded domain and  $z_0 \in D$ , then there exist  $c_0, C_0 > 0$  such that

$$\forall z \in D \quad c_0 - \frac{1}{2} \log d(z, \partial D) \leq k_D(z_0, z) \leq C_0 - \frac{1}{2} \log d(z, \partial D), \tag{2.1}$$

where  $d(\cdot, \partial D)$  denotes the Euclidean distance from the boundary of  $D$  (see [1, Theorems 2.3.51 and 2.3.52]).

Let us finally recall a couple of facts on Kobayashi balls of  $B^n$ ; for proofs see [1, Section 2.2.2; 8; 25, Section 2.2.7]. The ball  $B_{B^n}(z_0, r)$  is given by

$$B_{B^n}(z_0, r) = \left\{ z \in B^n \mid \frac{(1 - \|z_0\|^2)(1 - \|z\|^2)}{|1 - \langle z, z_0 \rangle|^2} > 1 - r^2 \right\}. \tag{2.2}$$

Geometrically, it is an ellipsoid of the (Euclidean) centre

$$c = \frac{1 - r^2}{1 - r^2 \|z_0\|^2} z_0,$$

its intersection with the complex line  $\mathbf{C}z_0$  is a Euclidean disc of radius

$$r \frac{1 - \|z_0\|^2}{1 - r^2 \|z_0\|^2},$$

and its intersection with the affine subspace through  $z_0$  orthogonal to  $\mathbf{C}z_0$  is a Euclidean ball of the larger radius

$$r \sqrt{\frac{1 - \|z_0\|^2}{1 - r^2 \|z_0\|^2}}.$$

Let  $\nu$  denote the Lebesgue volume measure of  $\mathbf{R}^{2n}$ , normalized so that  $\nu(B^n) = 1$ . Then the volume of a Kobayashi ball  $B_{B^n}(z_0, r)$  is given by (see [8])

$$\nu(B_{B^n}(z_0, r)) = r^{2n} \left( \frac{1 - \|z_0\|^2}{1 - r^2 \|z_0\|^2} \right)^{n+1}. \tag{2.3}$$

A similar estimate holds for the volume of Kobayashi balls in strongly pseudoconvex bounded domains.

LEMMA 2.1. *Let  $D \subset \subset \mathbf{C}^n$  be a strongly pseudoconvex bounded domain. Then there exist  $c_1 > 0$  and, for each  $r \in (0, 1)$ , a  $C_{1,r} > 0$  depending on  $r$  such that*

$$c_1 r^{2n} d(z_0, \partial D)^{n+1} \leq \nu(B_D(z_0, r)) \leq C_{1,r} d(z_0, \partial D)^{n+1},$$

for every  $z_0 \in D$  and  $r \in (0, 1)$ .

*Proof.* Since  $D$  is bounded and with smooth boundary, there is  $\delta > 0$  so that any Euclidean ball internally tangent to  $\partial D$  with radius  $\delta$  is completely contained in  $D$ . Take  $z_0 \in D$  with  $d(z_0, \partial D) < \delta$ , and choose  $x \in \partial D$  so that  $d(z_0, \partial D) = \|x - z_0\|$ . Then the Euclidean ball  $B$  tangent to  $\partial D$  in  $x$  and with radius  $\delta$  is contained in  $D$  and contains  $z_0$ , and  $d(z_0, \partial D) = d(z_0, \partial B)$ . Since the Kobayashi distance decreases under inclusions,  $B_B(z_0, r) \subset B_D(z_0, r)$ . Hence, (2.3) yields (assuming without loss of generality that  $B$  is centred at the origin)

$$\begin{aligned} \nu(B_D(z_0, r)) &\geq \nu(B_B(z_0, r)) = \delta^{2n} \nu(B_{B^n}(z_0/\delta, r)) \\ &= \delta^{2n} r^{2n} \left( \frac{1 - \|z_0/\delta\|^2}{1 - r^2 \|z_0/\delta\|^2} \right)^{n+1} = \delta^{2n} r^{2n} \left( \frac{\delta^2 - \|z_0\|^2}{\delta^2 - r^2 \|z_0\|^2} \right)^{n+1} \\ &\geq \delta^{2n} r^{2n} \left( \frac{\delta(\delta - \|z_0\|)}{\delta^2} \right)^{n+1} = \delta^{n-1} r^{2n} d(z_0, \partial D)^{n+1}, \end{aligned}$$

and we are done in this case.

Assume now  $d(z_0, \partial D) \geq \delta$ , and let  $B \subset D$  be the Euclidean ball of centre  $z_0$  and radius  $d(z_0, \partial D)$ . Then  $B_B(z_0, r)$  is a Euclidean ball of radius  $rd(z_0, \partial D)$ , and so

$$\nu(B_D(z_0, r)) \geq \nu(B_B(z_0, r)) = r^{2n} d(z_0, \partial D)^{2n} \geq \delta^{n-1} r^{2n} d(z_0, \partial D)^{n+1},$$

and we have obtained the lower estimate in this case too.

For the upper estimate, results of [16, 17] show that there exists a  $\delta_1 > 0$  such that if  $d(z_0, \partial D) < \delta_1$ , then there exist  $a_1(r), a_2(r) > 0$  so that  $B_D(z_0, r)$  is contained in a (possibly rotated) polydisc  $P$  of centre  $z_0$  and polyradius

$$(a_1(r)d(z_0, \partial D), a_2(r)\sqrt{d(z_0, \partial D)}, \dots, a_2(r)\sqrt{d(z_0, \partial D)}).$$

Thus if  $d(z_0, \partial D) < \delta_1$ , then we have

$$\nu(B_D(z_0, r)) \leq \nu(P) \leq ca_1(r)^2 a_2(r)^{2(n-1)} d(z_0, \partial D)^{n+1},$$

for a suitable constant  $c > 0$  independent of  $r$ .

Finally, if  $d(z_0, \partial D) \geq \delta_1$ , then we have

$$\nu(B_D(z_0, r)) \leq \frac{\nu(D)}{\delta_1^{n+1}} d(z_0, \partial D)^{n+1},$$

and we are done. □

REMARK 1. If  $D$  is strongly convex, then, using Euclidean balls externally tangent to  $D$ , it is possible to show that one can take

$$C_{1,r} = C_1 r^{2n} (1 - r^2)^{-(n+1)},$$

where  $C_1 > 0$  is a constant independent of  $r$ .

The next lemma contains an estimate on the Euclidean size of Kobayashi balls.

LEMMA 2.2. *Let  $D \subset\subset \mathbf{C}^n$  be a strongly pseudoconvex bounded domain. Then there is  $C_2 > 0$  such that, for every  $z_0 \in D$  and  $r \in (0, 1)$ , one has*

$$\forall z \in B_D(z_0, r) \quad \frac{C_2}{1-r} d(z_0, \partial D) \geq d(z, \partial D) \geq \frac{1-r}{C_2} d(z_0, \partial D).$$

*Proof.* Let us fix  $w_0 \in D$ . Then (2.1) yields  $c_0, C_0 > 0$  such that

$$\begin{aligned} c_0 - \frac{1}{2} \log d(z, \partial D) &\leq k_D(w_0, z) \leq k_D(z_0, z) + k_D(z_0, w_0) \\ &\leq \frac{1}{2} \log \frac{1+r}{1-r} + C_0 - \frac{1}{2} \log d(z_0, \partial D), \end{aligned}$$

for all  $z \in B_D(z_0, r)$ , and hence

$$e^{2(c_0 - C_0)} d(z_0, \partial D) \leq \frac{2}{1-r} d(z, \partial D).$$

The left-hand inequality is obtained in the same way, reversing the roles of  $z_0$  and  $z$ . □

REMARK 2. Using again externally tangent Euclidean balls, it is possible to show that if  $D$  is strongly convex, then one can take  $C_2 = 4$ .

The two previous lemmas (together with Corollary 2.7 below) give the main properties needed in Luecking’s approach [18] to the characterization of Carleson measures. However, to get the characterization involving the Berezin transform, we shall need precise information on the behaviour of the Bergman kernel in Kobayashi balls, which we shall deduce from Fefferman’s estimates using another estimate on the shape of Kobayashi balls. In the unit ball the latter estimate has the following form.

LEMMA 2.3. *Let  $B^n \subset \mathbf{C}^n$  be the unit ball in  $\mathbf{C}^n$ , and take  $z_0 \in B^n$  and  $r \in (0, 1)$ . Then*

$$\forall z \in B_{B^n}(z_0, r) \quad 1 - \|z_0\|^2 > \frac{1-r^2}{4} (\|z - z_0\|^2 + |\langle z - z_0, z_0 \rangle|).$$

*Proof.* First of all, it is easy to check that  $z \in B_{B^n}(z_0, r)$  if and only if

$$\frac{(1 - \|z_0\|^2)(1 - \|z\|^2)}{|1 - \langle z, z_0 \rangle|^2} > 1 - r^2;$$

therefore, it suffices to prove that

$$\frac{|1 - \langle z, z_0 \rangle|^2}{1 - \|z\|^2} \geq \frac{1}{4} (\|z - z_0\|^2 + |\langle z - z_0, z_0 \rangle|). \tag{2.4}$$

Let us write  $z_0 = \lambda z + w$ , with  $\lambda \in \mathbf{C}$  and  $w \perp z$ . Then  $\langle z, z_0 \rangle = \bar{\lambda} \|z\|^2$  and  $\|z - z_0\|^2 = |1 - \lambda|^2 \|z\|^2 + \|w\|^2$ ; moreover,  $|\lambda|^2 \|z\|^2 + \|w\|^2 = \|z_0\|^2 < 1$ , that is,  $\|w\|^2 < 1 - |\lambda|^2 \|z\|^2$ . Now, a computation shows that  $\|z\|^4 |\lambda - 1|^2 \geq 0$  is equivalent to

$$|1 - \bar{\lambda} \|z\|^2|^2 \geq (1 - \|z\|^2)(|1 - \lambda|^2 \|z\|^2 + 1 - |\lambda|^2 \|z\|^2);$$

therefore,

$$\begin{aligned} \frac{|1 - \langle z, z_0 \rangle|^2}{1 - \|z\|^2} &= \frac{|1 - \bar{\lambda} \|z\|^2|^2}{1 - \|z\|^2} \\ &\geq |1 - \lambda|^2 \|z\|^2 + 1 - |\lambda|^2 \|z\|^2 \\ &> |1 - \lambda|^2 \|z\|^2 + \|w\|^2 = \|z - z_0\|^2. \end{aligned} \tag{2.5}$$

Write  $z = \mu z_0 + w$ , with  $\mu \in \mathbf{C}$  and  $w \perp z_0$ . Then  $\langle z, z_0 \rangle = \mu \|z_0\|^2$  and  $|\langle z - z_0, z_0 \rangle| = |1 - \mu| \|z_0\|^2$ ; moreover,  $|\mu|^2 \|z_0\|^2 + \|w\|^2 = \|z\|^2 < 1$ , implying  $|\mu| \|z_0\| < 1$ . Now,

$$\frac{|1 - \langle z, z_0 \rangle|^2}{1 - \|z\|^2} = \frac{|1 - \mu \|z_0\|^2|^2}{1 - |\mu|^2 \|z_0\|^2 - \|w\|^2} \geq \frac{|1 - \mu \|z_0\|^2|^2}{1 - |\mu|^2 \|z_0\|^2} = \frac{|1 - \mu \|z_0\|^2|}{1 - |\mu| \|z_0\|} \cdot \frac{|1 - \mu \|z_0\|^2|}{1 + |\mu| \|z_0\|}. \tag{2.6}$$

By assumption we have  $2 \geq |\mu| \|z_0\| (1 + \|z_0\|)$ . A computation shows that this is equivalent to

$$|1 - |\mu| \|z_0\|^2|^2 \geq (1 - |\mu| \|z_0\|)^2;$$

therefore,

$$\frac{|1 - \mu \|z_0\|^2|}{1 - |\mu| \|z_0\|} \geq \frac{|1 - |\mu| \|z_0\|^2|}{1 - |\mu| \|z_0\|} \geq 1. \tag{2.7}$$

Next,  $1 + \|z_0\|^2 \geq 2\|z_0\| \geq 2|\mu| \|z_0\|^2 \geq 2\|z_0\|^2 \operatorname{Re} \mu$ ; multiplying this by  $1 - \|z_0\|^2$ , we end up with

$$|1 - \mu \|z_0\|^2|^2 \geq |1 - \mu|^2 \|z_0\|^4 = |\langle z - z_0, z_0 \rangle|^2. \tag{2.8}$$

Since  $1 + |\mu| \|z_0\| < 2$ , putting together (2.6), (2.7), and (2.8), we get

$$\frac{|1 - \langle z, z_0 \rangle|^2}{1 - \|z\|^2} > \frac{1}{2} |\langle z - z_0, z_0 \rangle|. \tag{2.9}$$

Putting together (2.5) and (2.9), we get (2.4), and thus the assertion.  $\square$

A defining function for a smooth domain  $D \subset \subset \mathbf{C}^n$  is a smooth function  $\psi : \mathbf{C}^n \rightarrow \mathbf{R}$  such that  $D = \{\psi > 0\}$  and the gradient  $\nabla \psi$  is never zero on  $\partial D$ . If  $D$  is strongly pseudoconvex, then one can find a defining function strictly plurisubharmonic in a neighbourhood of  $\partial D$ .

LEMMA 2.4. *Let  $D \subset \subset \mathbf{C}^n$  be a strongly pseudoconvex bounded domain, and  $\psi : \mathbf{C}^n \rightarrow \mathbf{R}$  be a defining function for  $D$ . Then, for every  $r \in (0, 1)$ , there exists  $c_{2,r} > 0$  depending on  $r$  such that, for every  $z_0 \in D$ , one has*

$$\forall z \in B_D(z_0, r) \quad d(z_0, \partial D) \geq c_{2,r} (\|z - z_0\|^2 + |\partial \psi_{z_0}(z - z_0)|).$$

*Proof.* Fix  $r \in (0, 1)$ . Since  $D$  is bounded, it suffices to prove the statement for  $z_0$  close enough to  $\partial D$ .

By Narasimhan’s lemma [15, Lemma 3.2.3] we can cover  $\partial D$  with open sets  $U_1, \dots, U_l \subset \mathbf{C}^n$  so that, for each  $j = 1, \dots, l$ , there is a biholomorphism  $\Phi_j : U_j \rightarrow \Phi_j(U_j) \subset \mathbf{C}^n$  such that  $\Phi_j(U_j \cap D)$  is strongly convex. Furthermore, we can assume that each  $\Phi_j$  is defined in a slightly larger neighbourhood, and hence find  $c > 0$  such that  $d(z_0, \partial D) \geq cd(\Phi_j(z_0), \partial \Phi_j(U_j \cap D))$  for all  $z_0 \in U_j \cap D$  close enough to  $\partial D$  and all  $j = 1, \dots, l$ . For the same reason, and recalling Lemma 2.2,  $|\partial \psi_{z_0}(z - z_0)|$  and  $|\partial(\psi \circ \Phi_j^{-1})_{\Phi_j(z_0)}(\Phi_j(z) - \Phi_j(z_0))|$  are comparable as soon as  $z_0 \in U_j \cap D$  is close enough to  $\partial D$  and  $z \in B_D(z_0, r)$ . Finally, by the localization property of the Kobayashi distance (see [1, Theorem 2.3.65]) if  $z_0 \in U_j \cap D$  is close enough to the boundary of  $D$ , then we can find  $r_1 \in (0, 1)$  depending only on  $r$  such that  $B_D(z_0, r) \subseteq B_{U_j \cap D}(z_0, r_1)$ . The upshot of these remarks is that it suffices to prove the statement when  $D$  is strongly convex.

So, assume that  $D$  is strongly convex, and let  $\delta > 0$  be such that if  $d(z_0, \partial D) < \delta$ , then there exists a unique  $x = x(z_0) \in \partial D$  so that  $d(z_0, \partial D) = \|x - z_0\|$ ; again it suffices to prove the statement for  $d(z_0, \partial D) < \delta$ .

Take then  $z_0 \in D$  with  $d(z_0, \partial D) < \delta$ ; since  $D$  is strongly convex, there exists a Euclidean ball  $B$  containing  $D$  and tangent to  $D$  in  $x = x(z_0)$ ; in particular,  $d(z_0, \partial D) = d(z_0, \partial B)$ . Let  $R = R(z_0) > 0$  be the radius of  $B$ ; up to a translation, we can assume that  $B$  is centred at the origin.

Let  $z \in B_D(z_0, r)$ . Since  $B_D(z_0, r) \subseteq B_B(z_0, r)$ , Lemma 2.3 implies

$$2Rd(z_0, \partial D) \geq R^2 - \|z_0\|^2 > \frac{1-r^2}{4}(\|z-z_0\|^2 + |\langle z-z_0, z_0 \rangle|).$$

Now,  $|\langle z-z_0, z_0 \rangle|$  is (a uniform multiple of) the distance of  $z$  from the complex hyperplane  $\pi$  passing through  $z_0$  and parallel to the complex hyperplane tangent to  $\partial B$  at  $x$ . But the latter coincides with the complex hyperplane tangent to  $\partial D$  at  $x$ , which is exactly given by  $\partial\psi_x(z-x) = 0$ . Therefore,  $\pi$  has equation  $\partial\psi_{z_0}(z-z_0) = 0$ . It follows that the difference between  $|\langle z-z_0, z_0 \rangle|$  and  $|\partial\psi_{z_0}(z-z_0)|$  is (uniformly in  $z$  and  $z_0$ ) of the order of  $\|z-z_0\| \cdot \|z_0-x\| = d(z_0, \partial D)\|z-z_0\|$ ; so we get

$$(2R + C\|z-z_0\|(1-r^2))d(z_0, \partial D) \geq \frac{1-r^2}{4}(\|z-z_0\|^2 + |\partial\psi_{z_0}(z-z_0)|),$$

for a suitable constant  $C > 0$ . Since  $D$  is bounded,  $R$  too is bounded as a function of  $z_0$ , and the assertion follows. □

REMARK 3. If  $D$  is strongly convex, then the proof shows that we can take  $c_{2,r} = c_2(1-r^2)$  for a suitable  $c_2 > 0$  independent of  $r$ .

We now prove a covering lemma for  $D$ .

LEMMA 2.5. *Let  $D \subset\subset \mathbf{C}^n$  be a strongly pseudoconvex bounded domain. Then, for every  $r \in (0, 1)$ , there exist  $m \in \mathbf{N}$  and a sequence  $\{z_k\} \subset D$  of points such that  $D = \bigcup_{k=0}^\infty B_D(z_k, r)$  and no point of  $D$  belongs to more than  $m$  of the balls  $B_D(z_k, R)$ , where  $R = (1/2)(1+r)$ .*

*Proof.* Let  $\{B_j\}_{j \in \mathbf{N}}$  be a sequence of Kobayashi balls of radius  $r/3$  covering  $D$ . We can extract a subsequence  $\{\Delta_k = B_D(z_k, r/3)\}_{k \in \mathbf{N}}$  of disjoint balls in the following way: set  $\Delta_1 = B_1$ . Suppose that we have already chosen  $\Delta_1, \dots, \Delta_l$ . We define  $\Delta_{l+1}$  as the first ball in the sequence  $\{B_j\}$  which is disjoint from  $\Delta_1 \cup \dots \cup \Delta_l$ . In particular, by construction every  $B_j$  must intersect at least one  $\Delta_k$ .

We now claim that  $\{B_D(z_k, r)\}_{k \in \mathbf{N}}$  is a covering of  $D$ . Indeed, let  $z \in D$ . Since  $\{B_j\}_{j \in \mathbf{N}}$  is a covering of  $D$ , there is  $j_0 \in \mathbf{N}$  so that  $z \in B_{j_0}$ . As remarked above, we get  $k_0 \in \mathbf{N}$  so that  $B_{j_0} \cap \Delta_{k_0} \neq \emptyset$ . Take  $w \in B_{j_0} \cap \Delta_{k_0}$ . Then

$$\rho_D(z, z_{k_0}) \leq \rho_D(z, w) + \rho_D(w, z_{k_0}) \leq \frac{2}{3}r,$$

and  $z \in B_D(z_{k_0}, r)$ .

To conclude the proof, we have to show that there is  $m = m_r \in \mathbf{N}$  so that each point  $z \in D$  belongs to at most  $m$  of the balls  $B_D(z_k, R)$ . Put  $r_1 = (1/3) \min\{r, 1-r\}$  and  $R_1 = (1/6)(5+r)$ . Since  $z \in B_D(z_k, R)$  is equivalent to  $z_k \in B_D(z, R)$ , we have that  $z \in B_D(z_k, R)$  implies  $B_D(z_k, r_1) \subset B_D(z, R_1)$ . Furthermore, Lemmas 2.1 and 2.2 yield

$$\nu(B_D(z_k, r_1)) \geq c_1 r_1^{2n} d(z_k, \partial D)^{n+1} \geq \frac{c_1}{C_2^{n+1}} (1-R)^{n+1} r_1^{2n} d(z, \partial D)^{n+1},$$

when  $z_k \in B_D(z, R)$ . Therefore, since the balls  $B_D(z_k, r_1)$  are pairwise disjoint, using again Lemma 2.1, we get

$$\text{card}\{k \in \mathbf{N} \mid z \in B_D(z_k, R)\} \leq \frac{\nu(B_D(z, R_1))}{\nu(B_D(z_k, r_1))} \leq \frac{C_2^{n+1} C_{1,R_1}}{c_1} \frac{1}{r_1^{2n} (1-R)^{n+1}},$$

and we are done. □



Our last aim for this section is a sort of submean property in Kobayashi balls for nonnegative plurisubharmonic functions. Let us first prove it in a Euclidean ball.

LEMMA 2.6. *Let  $B \subset \subset \mathbf{C}^n$  be a Euclidean ball of radius  $R > 0$ . Then*

$$\forall z_0 \in B \forall r \in (0, 1) \quad \chi(z_0) \leq \frac{4^{n+1}}{R^{n-1}} \frac{1}{r^{2n} d(z_0, \partial B)^{n+1}} \int_{B_B(z_0, r)} \chi \, d\nu,$$

for all nonnegative plurisubharmonic functions  $\chi : B \rightarrow \mathbf{R}^+$ .

*Proof.* Without loss of generality, we can assume that  $B$  is centred at the origin. Let  $\phi_{z_0/R} : B^n \rightarrow B^n$  be the usual involutive automorphism of  $B^n$  sending the origin in  $z_0/R$  (see [25, Section 2.2]), and let  $\Phi_{z_0} : B^n \rightarrow B$  be given by  $\Phi_{z_0} = R\phi_{z_0/R}$ ; in particular,  $\Phi_{z_0}$  is a biholomorphism with  $\Phi_{z_0}(O) = z_0$ , and thus  $\Phi_{z_0}(B_{B^n}(O, r)) = B_B(z_0, r)$ . Furthermore (see [25, Theorem 2.2.6]),

$$|\text{Jac}_{\mathbf{R}} \Phi_{z_0}(z)| = R^{2n} \left( \frac{R^2 - \|z_0\|^2}{|R - \langle z, z_0 \rangle|^2} \right)^{n+1} \geq \frac{R^{n-1}}{4^{n+1}} d(z_0, \partial B)^{n+1},$$

where  $\text{Jac}_{\mathbf{R}} \Phi_{z_0}$  denotes the (real) Jacobian determinant of  $\Phi_{z_0}$ . It follows that

$$\int_{B(z_0, r)} \chi \, d\nu = \int_{B_{B^n}(O, r)} (\chi \circ \Phi_{z_0}) |\text{Jac}_{\mathbf{R}} \Phi_{z_0}| \, d\nu \geq \frac{R^{n-1}}{4^{n+1}} d(z_0, \partial B)^{n+1} \int_{B_{B^n}(O, r)} (\chi \circ \Phi_{z_0}) \, d\nu.$$

Using [25, 1.4.3 and 1.4.7.(1)], we obtain

$$\int_{B_{B^n}(O, r)} (\chi \circ \Phi_{z_0}) \, d\nu = 2n \int_{\partial B^n} d\sigma(x) \frac{1}{2\pi} \int_0^r \int_0^{2\pi} \chi \circ \Phi_{z_0}(te^{i\theta}x) t^{2n-1} \, dt \, d\theta,$$

where  $\sigma$  is the area measure on  $\partial B^n$  normalized so that  $\sigma(\partial B^n) = 1$ . Now,  $\zeta \mapsto \chi \circ \Phi_{z_0}(\zeta x)$  is subharmonic on  $r\Delta = \{|\zeta| < r\} \subset \mathbf{C}$  for any  $x \in \partial B^n$ , since  $\Phi_{z_0}$  is holomorphic and  $\chi$  is plurisubharmonic. Therefore [11, Theorem 1.6.3], yields

$$\frac{1}{2\pi} \int_0^r \int_0^{2\pi} \chi \circ \Phi_{z_0}(te^{i\theta}x) t^{2n-1} \, dt \, d\theta \geq \chi(z_0) \int_0^r t^{2n-1} \, dt = \frac{1}{2n} r^{2n} \chi(z_0).$$

So

$$\int_{B_{B^n}(O, r)} (\chi \circ \Phi_{z_0}) \, d\nu \geq r^{2n} \chi(z_0),$$

and the assertion follows. □

Then we have the following corollary.

COROLLARY 2.7. *Let  $D \subset \subset \mathbf{C}^n$  be a strongly pseudoconvex bounded domain, and  $r \in (0, 1)$ . Then there exists a  $C_{3,r} > 0$  depending on  $r$  such that*

$$\forall z_0 \in D \quad \chi(z_0) \leq \frac{C_{3,r}}{\nu(B_D(z_0, r))} \int_{B_D(z_0, r)} \chi \, d\nu,$$

for all nonnegative plurisubharmonic functions  $\chi : D \rightarrow \mathbf{R}^+$ .

*Proof.* Since  $D$  has smooth boundary, there exists a radius  $\rho > 0$  such that, for every  $x \in \partial D$ , the Euclidean ball  $B_x(\rho)$  internally tangent to  $\partial D$  at  $x$  is contained in  $D$ .

Let  $z_0 \in D$ . If  $d(z_0, \partial D) \leq \rho$ , then let  $x \in \partial D$  be such that  $d(z_0, \partial D) = \|z_0 - x\|$ ; in particular,  $z_0$  belongs to the ball  $B = B_x(\rho) \subset D$ . If  $d(z_0, \partial D) > \rho$ , then let  $B \subset D$  be the

Euclidean ball of centre  $z_0$  and radius  $d(z_0, \partial D)$ . In both cases we have  $d(z_0, \partial D) = d(z_0, \partial B)$ ; moreover, the decreasing property of the Kobayashi distance yields  $B_D(z_0, r) \supseteq B_B(z_0, r)$  for all  $r \in (0, 1)$ .

Let  $\chi$  be a nonnegative plurisubharmonic function. Then Lemmas 2.1 and 2.6 imply

$$\begin{aligned} \int_{B_D(z_0, r)} \chi \, d\nu &\geq \int_{B_B(z_0, r)} \chi \, d\nu \geq \frac{\rho^{n-1}}{4^{n+1}} \frac{r^{2n} d(z_0, \partial D)^{n+1}}{\nu(B_D(z_0, r))} \nu(B_D(z_0, r)) \chi(z_0) \\ &\geq \frac{\rho^{n-1}}{4^{n+1} C_{1,r}} \nu(B_D(z_0, r)) \chi(z_0), \end{aligned}$$

and we are done. □

REMARK 4. If  $D$  is strongly convex, then one can take  $C_{3,r} = C_3(1 - r^2)^{-(n+1)}$ , where  $C_3 > 0$  is independent of  $r$ .

In a similar way we get another useful estimate.

COROLLARY 2.8. *Let  $D \subset\subset \mathbf{C}^n$  be a strongly pseudoconvex bounded domain. Given  $r \in (0, 1)$ , set  $R = (1/2)(1 + r) \in (0, 1)$ . Then there exists a  $K_r > 0$  depending on  $r$  such that*

$$\forall z_0 \in D \, \forall z \in B_D(z_0, r) \quad \chi(z) \leq \frac{K_r}{\nu(B_D(z_0, r))} \int_{B_D(z_0, R)} \chi \, d\nu,$$

for every nonnegative plurisubharmonic function  $\chi : D \rightarrow \mathbf{R}^+$ .

*Proof.* Let  $r_1 = (1/2)(1 - r)$ ; by the triangle inequality,  $z \in B_D(z_0, r)$  yields  $B_D(z, r_1) \subset B_D(z_0, R)$ . Corollary 2.7 then implies

$$\begin{aligned} \chi(z) &\leq \frac{C_{3,r_1}}{\nu(B_D(z, r_1))} \int_{B_D(z, r_1)} \chi \, d\nu \leq \frac{C_{3,r_1}}{\nu(B_D(z, r_1))} \int_{B_D(z_0, R)} \chi \, d\nu \\ &= C_{3,r_1} \frac{\nu(B_D(z_0, r))}{\nu(B_D(z, r_1))} \cdot \frac{1}{\nu(B_D(z_0, r))} \int_{B_D(z_0, R)} \chi \, d\nu, \end{aligned}$$

for all  $z \in B_D(z_0, r)$ . Now Lemmas 2.1 and 2.2 yield

$$\frac{\nu(B_D(z_0, r))}{\nu(B_D(z, r_1))} \leq \frac{C_{1,r} C_2^{n+1}}{c_1 (1 - r)^{n+1} r^{2n}},$$

and so

$$\chi(z) \leq \frac{C_{3,r_1} C_{1,r} C_2^{n+1}}{c_1 r_1^{2n} (1 - r)^{n+1}} \frac{1}{\nu(B_D(z_0, r))} \int_{B_D(z_0, R)} \chi \, d\nu. \quad \square$$

### 3. Carleson measures

Let  $D \subset\subset \mathbf{C}^n$  be a strongly pseudoconvex bounded domain in  $\mathbf{C}^n$ . Given  $0 < p < +\infty$ , the Bergman space  $A^p(D)$  of  $D$  is the Banach space of holomorphic  $L^p$ -functions on  $D$ , that is,  $A^p(D) = L^p(D) \cap \mathcal{O}(D)$ , endowed with the  $L^p$ -norm

$$\|f\|_p^p = \int_D |f(z)|^p \, d\nu,$$

where  $\nu$  is the Lebesgue measure normalized so that  $\nu(B^n) = 1$ .

A finite positive Borel measure  $\mu$  on  $D$  is said to be a *Carleson measure* of  $A^p(D)$  if there exists  $C_p > 0$  such that

$$\forall f \in A^p(D) \quad \int_D |f(z)|^p d\mu \leq C_p \|f\|_p^p.$$

As explained in Section 1, our aim is to give a characterization of Carleson measures involving the Bergman kernel of  $D$ .

Let  $K : D \times D \rightarrow \mathbf{C}$  be the *Bergman kernel* of  $D$  (see, for example, [15, Section 1.4]); it has the reproducing property

$$\forall f \in A^2(D) \quad \forall z \in D \quad f(z) = \int_D K(z, \zeta) f(\zeta) d\nu.$$

Since  $K(\cdot, \zeta) = \overline{K(\zeta, \cdot)} \in A^2(D)$ , in particular we have

$$K(z, z) = \int_D |K(z, \zeta)|^2 d\nu(\zeta) = \|K(z, \cdot)\|_2^2.$$

For each  $z_0 \in D$  let  $k_{z_0} \in A^2(D)$  be the *normalized Bergman kernel* given by

$$k_{z_0}(z) = \frac{K(z, z_0)}{\|K(\cdot, z_0)\|_2} = \frac{K(z, z_0)}{\sqrt{K(z_0, z_0)}};$$

clearly,  $\|k_{z_0}\|_2 = 1$ . The *Berezin transform*  $B\mu$  of a finite measure  $\mu$  on  $D$  is the function given by

$$B\mu(z) = \int_D |k_z(\zeta)|^2 d\mu(\zeta),$$

for all  $z \in D$ .

We recall the following estimate from above on the Bergman kernel.

LEMMA 3.1. *Let  $D \subset\subset \mathbf{C}^n$  be a strongly pseudoconvex bounded domain. Then there exists  $C_4 > 0$  such that*

$$\forall z_0 \in D \quad |K(z_0, z_0)| \leq \frac{C_4}{d(z_0, \partial D)^{n+1}}.$$

*Proof.* The proof follows immediately from [11, Theorem 3.5.1] or from [24, p. 186]. □

Our next result is an estimate from below on the Bergman kernel, valid close enough to the boundary.

LEMMA 3.2. *Let  $D \subset\subset \mathbf{C}^n$  be a strongly pseudoconvex bounded domain. Then, for every  $r \in (0, 1)$ , there exist  $c_{5,r} > 0$  and  $\delta_r > 0$  such that if  $z_0 \in D$  satisfies  $d(z_0, \partial D) < \delta_r$ , then*

$$\forall z \in B_D(z_0, r) \quad |K(z, z_0)| \geq \frac{c_{5,r}}{d(z_0, \partial D)^{n+1}}.$$

*Proof.* Let  $\psi$  be a defining function for  $D$ ; in particular, there are  $C_6, c_6 > 0$  such that

$$c_6 d(z, \partial D) \leq |\psi(z)| \leq C_6 d(z, \partial D) \tag{3.1}$$

in a neighbourhood of  $\bar{D}$ .

The main Theorem 2 in [9] implies that there is  $\eta > 0$  so that

$$|K(z, z_0)| \geq \frac{c}{(\psi(z) + \psi(z_0) + \rho(z, z_0))^{n+1}},$$

for a suitable constant  $c > 0$  as soon as  $d(z_0, \partial D) + d(z, \partial D) + \|z - z_0\| < \eta$ , where

$$\rho(z, z_0) = \|z - z_0\|^2 + |\partial\psi_{z_0}(z - z_0)|.$$

Put

$$\delta_r = \frac{\eta}{3} \min \left\{ 1, \frac{1-r}{C_2}, \frac{c_{2,r}\eta}{3} \right\};$$

then Lemmas 2.2 and 2.4 imply that  $d(z_0, \partial D) < \delta_r$  yields  $d(z_0, \partial D) + d(z, \partial D) + \|z - z_0\| < \eta$  for all  $z \in B_D(z_0, r)$ . Using Lemmas 2.2 and 2.4 and (3.1), we then get that if  $d(z_0, \partial D) < \delta_r$ , then

$$\begin{aligned} |K(z, z_0)| &\geq \frac{c}{\psi(z_0)^{n+1}} \left( \frac{1}{1 + (\psi(z)/\psi(z_0)) + (\rho(z, z_0)/\psi(z_0))} \right)^{n+1} \\ &\geq \frac{c(1-r)^{n+1}}{C_6^{n+1}d(z_0, \partial D)^{n+1}} \left( \frac{c_6}{c_6 + C_2C_6 + 1/c_{2,r}} \right)^{n+1}, \end{aligned}$$

for all  $z \in B_D(z_0, r)$ , and we are done. □

REMARK 5. When  $D$  is strongly convex, we can take  $c_{5,r} = c_5(1 - r^2)^{n+1}$  with  $c_5 > 0$  independent of  $r$ .

As a corollary we get a crucial estimate from below for the normalized Bergman kernel.

COROLLARY 3.3. *Let  $D \subset\subset \mathbf{C}^n$  be a strongly pseudoconvex bounded domain. Then, for every  $r \in (0, 1)$ , there exist  $c_{7,r} > 0$  and  $\delta_r > 0$  such that if  $z_0 \in D$  satisfies  $d(z_0, \partial D) < \delta_r$ , then*

$$\forall z \in B_D(z_0, r) \quad |k_{z_0}(z)|^2 \geq \frac{c_{7,r}}{d(z_0, \partial D)^{n+1}}.$$

*Proof.* The proof follows from Lemmas 3.1 and 3.2, with  $c_{7,r} = c_{5,r}^2/C_4$ . □

REMARK 6. When  $D$  is strongly convex we can take  $c_{7,r} = c_7(1 - r^2)^{2(n+1)}$  with  $c_7 > 0$  independent of  $r$ .

Now we can finally prove the promised characterization for the Carleson measures of  $A^p(D)$ . We recall that  $\nu$  denotes the Lebesgue volume measure of  $\mathbf{R}^{2n}$ , normalized so that  $\nu(B^n) = 1$ ; see (2.3).

THEOREM 3.4. *Let  $\mu$  be a finite positive Borel measure on a strongly pseudoconvex bounded domain  $D \subset\subset \mathbf{C}^n$ . Then the following statements are equivalent:*

- (i)  $\mu$  is a Carleson measure of  $A^p(D)$  for some (and hence all)  $p \in (0, +\infty)$ ;
- (ii) the Berezin transform of  $\mu$  is a bounded function;
- (iii) for every  $r \in (0, 1)$  there exists  $C_r > 0$  such that  $\mu(B_D(z_0, r)) \leq C_r \nu(B_D(z_0, r))$  for all  $z_0 \in D$ ;
- (iv) for some  $r \in (0, 1)$  there exists  $C_r > 0$  such that  $\mu(B_D(z_0, r)) \leq C_r \nu(B_D(z_0, r))$  for all  $z_0 \in D$ .

*Proof.* (i)  $\implies$  (ii): By [3] we can assume that  $\mu$  is a Carleson measure of  $A^2(D)$ . Then

$$B\mu(z_0) = \int_D |k_{z_0}(z)|^2 d\mu(z) \leq C \|k_{z_0}\|_2^2 = C,$$

for a suitable  $C > 0$ , and  $B\mu$  is bounded.

(ii)  $\implies$  (iii): Fix  $r \in (0, 1)$ . If  $d(z_0, \partial D) \geq \delta_r$ , where  $\delta_r > 0$  is given by Lemma 3.2, then using Lemma 2.1 we get

$$\mu(B_D(z_0, r)) \leq \mu(D) \leq \frac{\mu(D)}{c_1 r^{2n} \delta_r^{n+1}} \nu(B_D(z_0, r)),$$

as desired.

Assume now  $d(z_0, \partial D) < \delta_r$ . Since the Berezin transform is bounded, there exists  $C_8 > 0$  independent of  $z_0$  and  $r$  such that

$$\int_{B_D(z_0, r)} |k_{z_0}(z)|^2 d\mu(z) \leq B\mu(z_0) \leq C_8.$$

Hence, Corollary 3.3 yields

$$\frac{c_{7,r}}{d(z_0, \partial D)^{n+1}} \mu(B_D(z_0, r)) \leq C_8.$$

Recalling Lemma 2.1, we get

$$\mu(B_D(z_0, r)) \leq \frac{C_8}{c_{7,r}} d(z_0, \partial D)^{n+1} \leq \frac{C_8}{c_1 c_{7,r} r^{2n}} \nu(B_D(z_0, r)),$$

and we are done in this case too.

(iii)  $\implies$  (iv): The proof is obvious.

(iv)  $\implies$  (i): The proof follows from Lemmas 2.1, 2.2, Corollary 2.7, and [18]; for the sake of completeness we give here a slightly different proof. Let  $\{z_k\}$  be the sequence given by Lemma 2.5. Clearly

$$\int_D |f(z)|^p d\mu(z) \leq \sum_{k=1}^{\infty} \int_{B_D(z_k, r)} |f(z)|^p d\mu(z),$$

for all  $f \in A^p(D)$ . Since  $|f|^p$  is plurisubharmonic and nonnegative, Corollary 2.8 and (iv) yield

$$\begin{aligned} \int_{B_D(z_k, r)} |f(z)|^p d\mu(z) &\leq \frac{K_r}{\nu(B_D(z_k, r))} \int_{B_D(z_k, r)} d\mu(z) \int_{B_D(z_k, R)} |f(\zeta)|^p d\nu(\zeta) \\ &= K_r \frac{\mu(B_D(z_k, r))}{\nu(B_D(z_k, r))} \int_{B_D(z_k, R)} |f(\zeta)|^p d\nu(\zeta) \\ &\leq K_r C_r \int_{B_D(z_k, R)} |f(\zeta)|^p d\nu(\zeta), \end{aligned}$$

where  $R = (1/2)(1 + r)$ . Hence,

$$\int_D |f(z)|^p d\mu(z) \leq K_r C_r \sum_{k=1}^{\infty} \int_{B_D(z_k, R)} |f(\zeta)|^p d\nu(\zeta) \leq K_r C_r m \|f\|_p^p, \tag{3.2}$$

where  $m$  is given by Lemma 2.5, and so  $\mu$  is a Carleson measure of  $A^p(D)$ . □

REMARK 7. Note that (3.2) says that if  $\mu$  is a Carleson measure, then we can find  $C > 0$  so that

$$\int_D |f(z)|^p d\mu(z) \leq C \|f\|_p^p,$$

for all  $f \in A^p(D)$  and all  $p \in (0, +\infty)$ ; in other words, the constant  $C$  is independent of  $p$ .

4. Uniformly discrete sequences

Let  $(X, d)$  be a metric space. A sequence  $\Gamma = \{x_j\} \subset X$  of points in  $X$  is *uniformly discrete* if there exists  $\delta > 0$  such that  $d(x_j, x_k) \geq \delta$  for all  $j \neq k$ . In this case  $\inf_{j \neq k} d(x_j, x_k)$  is the *separation constant* of  $\Gamma$ .

Furthermore, given  $x_0 \in X$ ,  $r > 0$ , and a subset  $\Gamma \subset X$ , we shall denote by  $N(x_0, r, \Gamma)$  the number of points of  $\Gamma$  contained in the ball of centre  $x_0$  and radius  $r$ . We can use  $N(x_0, r, \Gamma)$  to detect finite unions of uniformly discrete sequences.

LEMMA 4.1. *Let  $X$  be a metric space, and  $\Gamma = \{x_n\}_{n \in \mathbf{N}} \subset X$  be a sequence in  $X$ . If there are  $N \geq 1$  and  $r > 0$  such that  $N(x, r, \Gamma) \leq N$  for all  $x \in X$ , then  $\Gamma$  is the union of at most  $N$  uniformly discrete sequences.*

*Proof.* We shall define  $N$  disjoint uniformly discrete sequences  $\Gamma_0, \dots, \Gamma_{N-1} \subset \Gamma$  so that  $\Gamma = \Gamma_0 \cup \dots \cup \Gamma_{N-1}$ . To do so, we start with  $\Gamma_0 = \dots = \Gamma_{N-1} = \emptyset$  and, arguing by induction on  $n$ , we shall put each  $x_n$  in a specified  $\Gamma_j$ . As a matter of notation, we shall denote by  $B(x, r)$  the metric ball of centre  $x$  and radius  $r$ , and if  $x_i \in \Gamma_j$ , then we write  $m(x_i) = j$ .

Put  $x_0 \in \Gamma_0$ . Assume that we have already defined  $m(x_i)$  for  $i \leq n$ , and consider  $x_{n+1}$ . By assumption,  $\Gamma \cap B(x_{n+1}, r)$  contains at most  $N$  points, one of which is  $x_{n+1}$ . Hence,  $\{x_0, \dots, x_n\} \cap B(x_{n+1}, r)$  contains at most  $N - 1$  points, and we can define

$$m(x_{n+1}) = \min\{i \in \{0, \dots, N - 1\} \mid i \neq m(x_j) \text{ for all } 0 \leq j \leq n \text{ such that } x_j \in B(x_{n+1}, r)\}.$$

In this way  $d(x_{n+1}, x_j) \geq r$  for all  $x_j \in \Gamma_{m(x_{n+1})}$  with  $j < n + 1$ .

It now follows easily that  $\Gamma_0, \dots, \Gamma_{N-1}$  are uniformly discrete sequences with separation constant at least  $r$ , because by construction if  $x_h, x_k \in \Gamma_j$  with  $h > k$ , we have  $d(x_h, x_k) \geq r$ . □

We are now able to prove that uniformly discrete sequences give examples of Carleson measures.

THEOREM 4.2. *Let  $D \subset \subset \mathbf{C}^n$  be a strongly pseudoconvex bounded domain, considered as a metric space with the distance  $\rho_D = \tanh k_D$ . Let  $\Gamma = \{z_j\}_{j \in \mathbf{N}}$  be a sequence in  $D$ . Then the following statements are equivalent:*

- (i)  $\Gamma$  is a finite union of uniformly discrete sequences;
- (ii)  $\sup_{z_0 \in D} N(z_0, r, \Gamma) < +\infty$  for some  $r \in (0, 1)$ ;
- (iii)  $\sup_{z_0 \in D} N(z_0, r, \Gamma) < +\infty$  for all  $r \in (0, 1)$ ;
- (iv) there exists  $p \in (0, +\infty)$  such that  $\sum_{z_j \in \Gamma} d(z_j, \partial D)^{n+1} \delta_{z_j}$  is a Carleson measure of  $A^p(D)$ , where  $\delta_{z_j}$  is the Dirac measure in  $z_j$ ;
- (v) there exists  $C_9 > 0$  such that, for all  $p \in (0, +\infty)$ , we have

$$\forall f \in A^p(D) \quad \sum_{z_j \in \Gamma} d(z_j, \partial D)^{n+1} |f(z_j)|^p \leq C_9 \|f\|_p^p;$$

in particular,  $\sum_{z_j \in \Gamma} d(z_j, \partial D)^{n+1} \delta_{z_j}$  is a Carleson measure of  $A^p(D)$  for all  $p \in (0, +\infty)$ .

*Proof.* (v)  $\implies$  (iv): The proof is obvious.

(iv)  $\implies$  (iii): Fix  $r \in (0, 1)$ , and let  $\delta_r > 0$  be given by Lemma 3.2. By Lemma 2.2, if  $d(z_0, \partial D) \geq \delta_r$ , then  $z \in B_D(z_0, r)$  implies  $d(z, \partial D) \geq (1/C_2)(1 - r)\delta_r$ . Using (iv), it is easy

to see that only a finite number of  $z_j \in \Gamma$  can have  $d(z_j, \partial D) \geq (1/C_2)(1-r)\delta_r$ ; therefore, to get (iii) it suffices to prove that the supremum is finite when  $d(z_0, \partial D) < \delta_r$ .

Given  $z_0 \in D$  with  $d(z_0, \partial D) < \delta_r$ , Corollary 3.3 and Lemma 2.2 yield

$$\forall z \in B_D(z_0, r) \quad d(z, \partial D)^{n+1} |k_{z_0}(z)|^2 \geq \frac{c_{7,r}}{C_2^{n+1}} (1-r)^{n+1}.$$

By [3] we can assume  $p = 2$ ; hence,

$$\begin{aligned} N(z_0, r, \Gamma) &\leq \frac{C_2^{n+1}}{c_{7,r}(1-r)^{n+1}} \sum_{z \in B_D(z_0, r) \cap \Gamma} d(z, \partial D)^{n+1} |k_{z_0}(z)|^2 \\ &\leq \frac{C_2^{n+1} C}{c_{7,r}(1-r)^{n+1}} \|k_{z_0}\|_2^2 = \frac{C_2^{n+1} C}{c_{7,r}(1-r)^{n+1}}, \end{aligned}$$

for a suitable  $C > 0$ , as desired.

(iii)  $\implies$  (ii): The proof is obvious.

(ii)  $\implies$  (i): The proof is from Lemma 4.1.

(i)  $\implies$  (v): Clearly it suffices to prove the assertion when  $\Gamma$  is a single uniformly discrete sequence. Let  $\delta > 0$  be the separation constant of  $\Gamma$ , and put  $r = \delta/2$ . By the triangle inequality, the Kobayashi balls  $B_D(z_j, r)$  are pairwise disjoint. Hence,

$$\int_D |f(z)|^p d\nu \geq \sum_{z_j \in \Gamma} \int_{B_D(z_j, r)} |f(z)|^p d\nu.$$

Now,  $|f|^p$  is plurisubharmonic and nonnegative; hence, Corollary 2.7 and Lemma 2.1 yield

$$\int_{B_D(z_j, r)} |f(z)|^p d\nu \geq \frac{1}{C_{3,r}} \nu(B_D(z_j, r)) |f(z_j)|^p \geq \frac{c_1}{C_{3,r}} r^{2n} d(z_j, \partial D)^{n+1} |f(z_j)|^p,$$

for all  $z_j \in \Gamma$ . Setting  $C_9 = C_{3,r}/c_1 r^{2n}$ , the assertion follows. □

Now we would like to study how fast a uniformly discrete sequence can escape to the boundary. A first result in this vein is an immediate corollary of the previous theorem.

**COROLLARY 4.3.** *Let  $\Gamma = \{z_j\} \subset D$  be a uniformly discrete sequence in a strongly pseudoconvex bounded domain  $D \subset \mathbf{C}^n$ . Then*

$$\sum_{z_j \in \Gamma} d(z_j, \partial D)^{n+1} < +\infty.$$

*Proof.* It suffices to take  $f \equiv 1$  in Theorem 4.2(v). □

The next result gives a worse estimate, but one which is valid in any hyperbolic domain with finite Euclidean volume.

**PROPOSITION 4.4.** *Let  $D \subset \mathbf{C}^n$  be a hyperbolic domain with finite Euclidean volume, endowed with the distance  $\rho_D = \tanh k_D$ . Let  $\Gamma = \{z_j\} \subset D$  be a uniformly discrete sequence with  $d(z_j, \partial D) < 1$  for all  $z_j \in \Gamma$ . Then*

$$\sum_{z_j \in \Gamma} d(z_j, \partial D)^{2n} h \left( -\frac{1}{\log d(z_j, \partial D)} \right) < +\infty,$$

for any increasing function  $h : \mathbf{R}^+ \rightarrow \mathbf{R}^+$  such that

$$\sum_{m=1}^{+\infty} h\left(\frac{1}{m}\right) < +\infty.$$

*Proof.* For  $m \in \mathbf{N}$  put

$$\Omega_m = \left\{ z \in D \mid m < \log \frac{1}{d(z, \partial D)} \leq m + 1 \right\} = \{z \in D \mid e^{-(m+1)} \leq d(z, \partial D) < e^{-m}\}.$$

Set  $\Gamma_m = \Gamma \cap \Omega_m$ ; by assumption,  $\Gamma = \bigcup_{m=0}^{+\infty} \Gamma_m$ . Let  $\delta > 0$  be the separation constant of  $\Gamma$ , and put  $r = \delta/2$ . The Kobayashi balls  $B_D(z_j, r)$  centred in points of  $\Gamma$  are pairwise disjoint; in particular,

$$\sum_{z_j \in \Gamma_m} \nu(B_D(z_j, r)) \leq \nu(D). \tag{4.1}$$

If  $z \in D$ , then the euclidean ball  $B$  of centre  $z$  and radius  $d(z, \partial D)$  is contained in  $D$ . But then  $B_D(z, r)$  contains the Kobayashi ball  $B_B(z, r)$  of  $B$  centred in  $z$  and of radius  $r$ , which is a Euclidean ball of radius  $rd(z, \partial D)$ . Thus,

$$\nu(B_D(z_j, r)) \geq r^{2n} d(z_j, \partial D)^{2n}.$$

Let  $N_m$  be the cardinality of  $\Gamma_m$ . Then (4.1) yields

$$N_m \leq \nu(D) r^{-2n} e^{2n(m+1)} = C_9 e^{2nm},$$

for a suitable constant  $C_9 > 0$  independent of  $m$ . Therefore,

$$\begin{aligned} \sum_{z_j \in \Gamma} d(z_j, \partial D)^{2n} h\left(-\frac{1}{\log d(z_j, \partial D)}\right) &= \sum_{m=0}^{+\infty} \sum_{z_j \in \Gamma_m} d(z_j, \partial D)^{2n} h\left(-\frac{1}{\log d(z_j, \partial D)}\right) \\ &\leq \sum_{z_j \in \Gamma_0} d(z_j, \partial D)^{2n} h\left(-\frac{1}{\log d(z_j, \partial D)}\right) \\ &\quad + \sum_{m=1}^{+\infty} N_m e^{-2mn} h\left(\frac{1}{m}\right) \\ &\leq C_{10} + C_9 \sum_{m=1}^{+\infty} h\left(\frac{1}{m}\right) < +\infty. \quad \square \end{aligned}$$

REMARK 8. If  $D \subset \subset \mathbf{C}^n$  is bounded and  $\Gamma = \{z_j\} \subset D$  is a uniformly discrete (with respect to any distance inducing the natural topology of  $D$ ) sequence, then  $d(z_j, \partial D) < 1$  for all but a finite number of elements of  $\Gamma$ .

To get the sharp estimate valid in strongly pseudoconvex bounded domains, we replace the Euclidean measure by the Eisenman–Kobayashi invariant measure (but see Remark 9).

For simplicity, let us recall the definition of the Eisenman–Kobayashi measure in domains of  $\mathbf{C}^n$  only. Let  $D \subset \mathbf{C}^n$  be a domain; the *Eisenman–Kobayashi density*  $K_D : D \rightarrow \mathbf{R}^+$  is given by

$$K_D(z_0) = \inf\{|\text{Jac}_{\mathbf{R}} f(O)|^{-1} | f : B^n \rightarrow D \text{ holomorphic, } f(O) = z_0, df_O \text{ invertible}\}.$$

It is not difficult to prove that  $K_D$  is upper semicontinuous (see, for example, [1, Proposition 2.3.37]); then the *Eisenman–Kobayashi measure* of  $D$  is just  $\tilde{\kappa}_D = K_D \nu$ . When  $D = B^n$ , the Eisenman–Kobayashi measure coincides with the Bergman (or hyperbolic) volume (see, for example, [1, Proposition 2.3.36]).



The next lemma contains an estimate on the Eisenman–Kobayashi measure of Kobayashi balls.

LEMMA 4.5. *Let  $D \subset \subset \mathbf{C}^n$  be a strongly pseudoconvex bounded domain. Then there exist  $0 < c_{11}, C_{11}$  such that*

$$c_{11}r^{2n}(1-r)^{n+1} \leq \tilde{\kappa}_D(B_D(z_0, r)) \leq \frac{C_{11}}{d(z_0, \partial D)^n(1-r)^n},$$

for all  $z_0 \in D$  and  $r \in (0, 1)$ .

*Proof.* The estimates in [19] show that there are two constants  $c_{12}, C_{12} > 0$  such that

$$\forall z \in D \quad \frac{c_{12}}{d(z, \partial D)^{n+1}} \leq K_D(z) \leq \frac{C_{12}}{d(z, \partial D)^{n+1}}.$$

In particular, Lemmas 2.1 and 2.2 yield

$$\begin{aligned} \tilde{\kappa}_D(B_D(z_0, r)) &= \int_{B_D(z_0, r)} K_D(z) \, d\nu \geq c_{12} \int_{B_D(z_0, r)} \frac{1}{d(z, \partial D)^{n+1}} \, d\nu \\ &\geq \frac{c_{12}(1-r)^{n+1}}{C_2^{n+1}d(z_0, \partial D)^{n+1}} \nu(B_D(z_0, r)) \geq \frac{c_{12}c_1}{C_2^{n+1}} r^{2n}(1-r)^{n+1}. \end{aligned}$$

For the upper estimate, note that since  $\partial D$  is smooth and bounded, there is an  $\epsilon > 0$  such that, setting

$$U_\epsilon = D \setminus K_\epsilon = \{z \in D \mid d(z, \partial D) < \epsilon\},$$

there exists a smooth diffeomorphism  $\Psi : \bar{U}_\epsilon \rightarrow [0, \epsilon] \times \partial D$  with

$$\Psi^{-1}(\{t\} \times \partial D) = \{z \in D \mid d(z, \partial D) = t\},$$

for all  $t \in (0, \epsilon]$ .

Now Lemma 2.2 implies

$$B_D(z_0, r) \subset \{z \in D \mid d(z, \partial D) \geq l(r)\},$$

where  $l(r) = (1/C_2)(1-r)d(z_0, \partial D)$ . Thus, denoting by  $\sigma$  the usual  $(2n-1)$ -measure on  $\partial D$ , and by  $\lambda$  the product measure on  $[0, \epsilon] \times \partial D$ , and using the compactness of  $\bar{U}_\epsilon$ , we have

$$\begin{aligned} \tilde{\kappa}_D(B_D(z_0, r)) &= \int_{B_D(z_0, r)} K_D(z) \, d\nu \leq C_{12} \int_{B_D(z_0, r)} \frac{1}{d(z, \partial D)^{n+1}} \, d\nu \\ &\leq C_{12} \int_{K_\epsilon} \frac{1}{d(z, \partial D)^{n+1}} \, d\nu + C_{12} \int_{B_D(z_0, r) \cap U_\epsilon} \frac{1}{d(z, \partial D)^{n+1}} \, d\nu \\ &\leq C_{13} + C_{14} \int_{\Psi(B_D(z_0, r) \cap U_\epsilon)} \frac{1}{t^{n+1}} \, d\lambda \leq C_{13} + C_{14} \int_{[l(r), \epsilon] \times \partial D} \frac{1}{t^{n+1}} \, d\lambda \\ &\leq C_{13} + C_{14} \sigma(\partial D) \left( \frac{1}{l(r)^n} - \frac{1}{\epsilon^n} \right) \leq \frac{C_{15}}{l(r)^n} = \frac{C_2^n C_{15}}{d(z_0, \partial D)^n(1-r)^n}, \end{aligned}$$

for suitable constants  $C_{13}, C_{14}, C_{15} > 0$  independent of  $z_0$  and  $r$ . □

REMARK 9. We stated this lemma in terms of the Eisenman–Kobayashi measure just to keep with the invariant approach of this paper; but for the purpose of the next theorem any measure providing the same estimates would work. For instance, we might use the measure  $\mu = d(\cdot, \partial D)^{-(n+1)}\nu$ .

We are now able to prove the promised sharp estimate for strongly pseudoconvex domains.

**THEOREM 4.6.** *Let  $D \subset\subset \mathbf{C}^n$  be a strongly pseudoconvex bounded domain, endowed with the distance  $\rho_D = \tanh k_D$ . Let  $\Gamma = \{z_j\} \subset D$  be a uniformly discrete sequence with  $d(z_j, \partial D) < 1$  for all  $z_j \in \Gamma$ . Then*

$$\sum_{z_j \in \Gamma} d(z_j, \partial D)^n h \left( -\frac{1}{\log d(z_j, \partial D)} \right) < +\infty,$$

for any increasing function  $h : \mathbf{R}^+ \rightarrow \mathbf{R}^+$  such that

$$\sum_{m=1}^{+\infty} h \left( \frac{1}{m} \right) < +\infty.$$

*Proof.* Fix  $z_0 \in D$  and, for  $m \in \mathbf{N}$ , set

$$D_m = \left\{ z \in D \mid \frac{m}{2} \leq k_D(z_0, z) < \frac{m+1}{2} \right\},$$

and let  $\Gamma_m = D_m \cap \Gamma$ . Let  $\delta > 0$  be the separation constant of  $\Gamma$  (with respect to  $k_D$ ), and let  $r = \tanh(\delta/2)$ . Clearly, if  $z_j \in \Gamma_m$  and  $z \in B_D(z_j, r)$ , then  $k_D(z_0, z) < (1/2)(m+1+\delta)$ . Since, as usual, the Kobayashi balls  $B_D(z_j, r)$  are pairwise disjoint, using Lemma 4.5, we get

$$c_{11}r^{2n}(1-r)^{n+1}N_m \leq \sum_{z_j \in \Gamma_m} \tilde{\kappa}_D(B_D(z_j, r)) \leq \tilde{\kappa}_D(B_D(z_0, R_m)) \leq \frac{C_{11}}{d(z_0, \partial D)^n(1-R_m)^n},$$

where  $R_m = \tanh(m+1+\delta)/2$ ; hence, there is  $C_{16} > 0$  (depending on  $z_0$  and  $\delta$  but not on  $m$ ) such that

$$N_m \leq C_{16}e^{mn}.$$

Now, the estimates (2.1) on the boundary behaviour of the Kobayashi distance yield  $C_{17}, C_{18} > 0$  such that

$$d(z, \partial D) \leq C_{17} \exp(-2k_D(z_0, z)),$$

and

$$-\frac{1}{\log d(z, \partial D)} \leq \frac{1}{2k_D(z_0, z) - C_{18}},$$

as soon as  $2k_D(z_0, z) > C_{18}$ . In particular, if  $z_j \in \Gamma_m$ , then we have

$$d(z_j, \partial D) \leq C_{17}e^{-m} \quad \text{and} \quad -\frac{1}{\log d(z, \partial D)} \leq \frac{1}{m - C_{18}},$$

as soon as  $m > C_{18}$ . Then

$$\begin{aligned} \sum_{z_j \in \Gamma} d(z_j, \partial D)^n h \left( -\frac{1}{\log d(z_j, \partial D)} \right) &= \sum_{m=0}^{+\infty} \sum_{z_j \in \Gamma_m} d(z_j, \partial D)^n h \left( -\frac{1}{\log d(z_j, \partial D)} \right) \\ &\leq C_{19} + C_{17}^n \sum_{m=m_0}^{+\infty} N_m e^{-nm} h \left( \frac{1}{m - C_{18}} \right) \\ &\leq C_{19} + C_{17}^n C_{16} \sum_{m=0}^{+\infty} h \left( \frac{1}{m} \right) < +\infty, \end{aligned}$$

where  $m_0$  is the smallest integer greater than  $C_{18}$ . □

In particular, taking  $h(x) = e^{-1/x}$ , we recover Corollary 4.3.

REMARK 10. The statement of Theorem 4.6 is sharp. In fact, MacCluer [20] has constructed, for any increasing function  $h : \mathbf{R}^+ \rightarrow \mathbf{R}^+$  with  $\lim_{t \rightarrow 0^+} h(t) = 0$  and  $\sum_{m=1}^{\infty} h(1/m) = +\infty$ , a uniformly discrete sequence  $\Gamma = \{z_j\}$  in  $B^n$  such that

$$\sum_{z_j \in \Gamma} d(z_j, \partial B^n)^n h\left(-\frac{1}{\log d(z_j, \partial B^n)}\right) = +\infty.$$

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