

Carroll versus Galilei gravity

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ABSTRACT: We consider two distinct limits of General Relativity that in contrast to the standard non-relativistic limit can be taken at the level of the Einstein-Hilbert action instead of the equations of motion. One is a non-relativistic limit and leads to a so-called Galilei gravity theory, the other is an ultra-relativistic limit yielding a so-called Carroll gravity theory. We present both gravity theories in a first-order formalism and show that in both cases the equations of motion (i) lead to constraints on the geometry and (ii) are not sufficient to solve for all of the components of the connection fields in terms of the other fields. Using a second-order formalism we show that these independent components serve as Lagrange multipliers for the geometric constraints we found earlier. We point out a few noteworthy differences between Carroll and Galilei gravity and give some examples of matter couplings.

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1 Introduction

Einstein's classical theory of General Relativity is able to explain many experiments within certain distance scales. However, it is generally appreciated that there are issues both at small distances where the unification of General Relativity with quantum mechanics becomes relevant as well as at large distances where gravity may couple to as yet unseen dark matter and where we are facing the dark energy puzzle. A remarkable result of the quest for a theory of quantum gravity is the AdS/CFT correspondence [1–3] which states that a gravitational theory in a D -dimensional Anti-de Sitter (AdS) spacetime under certain conditions can be described by a relativistic Conformal Field Theory (CFT) that is defined at the boundary of that spacetime.

The AdS/CFT correspondence has been generalized to a non-relativistic correspondence where one considers gravitational background solutions in the bulk that preserve a number of non-relativistic symmetries such as the Schrödinger symmetries [4, 5] or Lifshitz symmetries [6]. There exists another approach, initiated in [7], where not only the boundary QFT is non-relativistic but also the String Theory. Non-relativistic strings came into the picture some time ago as a possibly solvable special sector of String Theory [8, 9]. In this

alternative approach one ends up with a non-relativistic vibrating string in the bulk [10]. When the curvature is small the non-relativistic string gives rise to a non-relativistic gravity theory in the bulk with a two-dimensional foliation, representing the time and the single spatial direction of the string. This gravity theory is a string-like version of a frame-independent formulation of Newton's theory of gravity, called Newton-Cartan (NC) gravity, which has a one dimensional foliation representing the absolute time.

In view of its role in the AdS/CFT correspondence, it is of interest to consider special limits of General Relativity, possibly with matter beyond the standard non-relativistic limit which gives rise to NC gravity.¹ Motivated by this we will consider in this paper two distinct limits of General Relativity with a one-dimensional foliation. The extension to a two-dimensional foliation can be done in a separate step and will not be considered in this paper.

The standard non-relativistic limit of General Relativity in four spacetime dimensions, leading to NC gravity, that is usually considered in the literature can only be defined at the level of the equations of motion.² This so-called NC limit leads to infinities when applied at the level of the Einstein-Hilbert (EH) action. A noteworthy feature of the resulting NC gravity theory is that it contains a central charge gauge field that couples to the current corresponding to the conservation of (massive) particles.

In this paper we will explore two different limits of General Relativity that, in contrast to the NC limit, can be defined at the level of the EH action. The first limit we will consider is an ultra-relativistic limit leading to a so-called Carroll gravity theory invariant under reparametrizations and the Carroll symmetries.³ These Carroll symmetries have recently occurred in studies of flat space holography [13]. The second limit that we will consider is a non-relativistic limit, the so-called Galilei limit, that differs from the NC limit in the sense that it does not involve a mass parameter and a central charge gauge field. The resulting Galilei gravity theory is invariant under reparametrizations and Galilei symmetries. Such symmetries, and extensions thereof, have occurred in a recent study on non-relativistic limits of string actions [14, 15].

In this paper we will present the limits of General Relativity leading to the Carroll and Galilei gravity theories using a first-order formulation where the spin-connection fields are considered to be independent variables. A noteworthy feature is that the equations of motion lead to constraints on the geometry. We next show that, in contrast to General Relativity, for both Carroll and Galilei gravity not all components of the spin-connection fields can be solved for by using the equations of motion. Instead, we find that, using a second-order formulation, the independent components of the spin-connection fields, occur as Lagrange multipliers that precisely reproduce the geometric constraints mentioned above.

The organization of this paper is as follows. In section 2 we review a few aspects of General Relativity that are relevant for the analysis in the next sections. In section 3 we

¹We will not consider in this paper the Newtonian limit, which is discussed in most text books, since that limit involves extra assumptions leading to a frame-dependent formulation.

²In three dimensions the non-relativistic limit has been considered at the level of the action by adding an extra term to the Einstein-Hilbert action [11].

³A different version of Carroll gravity has been studied in [12]. We will compare the two versions later in this paper.

symmetry	generators	gauge field	parameters	curvatures
spacetime translations	P_A	E_μ^A	η^A	$R_{\mu\nu}^A(P)$
Lorentz transformations	J_{AB}	Ω_μ^{AB}	Λ^{AB}	$R_{\mu\nu}^{AB}(J)$

Table 1. This table indicates the generators of the Poincaré algebra and the gauge fields, local parameters and curvatures that are associated to each of these generators.

explore Carroll gravity, both using a first-order as well as a second-order formulation. In section 4 we perform a similar analysis for Galilei gravity. In section 5 we discuss matter couplings for both Carroll and Galilei gravity. Finally, we give our conclusions in section 6.

2 General Relativity

Before taking limits we first summarize some relevant formulae of General Relativity including matter couplings which will be of use in the next sections. Our starting point is the D -dimensional Poincaré algebra of spacetime translations P_A and Lorentz transformations J_{AB} ($A = 0, 1, \dots, D - 1$)

$$[P_A, J_{BC}] = 2\eta_{A[C}P_{B]}, \tag{2.1}$$

$$[J_{AB}, J_{CD}] = 4\eta_{[A[D}J_{C]B]}, \tag{2.2}$$

where η_{AB} is the (mostly plus) Minkowski metric. To each generator of the Poincaré algebra we associate a gauge field, a local parameter parametrizing the corresponding symmetry and a curvature, see table 1. The gauge field E_μ^A is the Vielbein field while Ω_μ^{AB} is the spin-connection field.

According to the Poincaré algebra (2.1), (2.2) the gauge fields transform as follows:⁴

$$\delta E_\mu^A = \partial_\mu \eta^A + \Lambda^A_B E_\mu^B - \Omega_\mu^{AB} \eta_B, \tag{2.3}$$

$$\delta \Omega_\mu^{AB} = \partial_\mu \Lambda^{AB} + \Omega_\mu^A_C \Lambda^{BC} - \Omega_\mu^{BC} \Lambda^A_C. \tag{2.4}$$

These gauge fields transform as covariant vectors under general coordinate transformations with parameters ξ^μ . The curvatures indicated in table 1 transform covariantly under these transformations:

$$R_{\mu\nu}^A(P) = 2\partial_{[\mu} E_{\nu]}^A - 2\Omega_{[\mu}^A B E_{\nu]}^B, \tag{2.5}$$

$$R_{\mu\nu}^{AB}(J) = 2\partial_{[\mu} \Omega_{\nu]}^{AB} - 2\Omega_{[\mu}^{BC} \Omega_{\nu]}^A_C. \tag{2.6}$$

In arbitrary dimensions, it is not possible to write down a gauge-invariant action for the gauge fields [16]. Instead, we consider the following action which is invariant under general coordinate transformations and local Lorentz transformations:

$$S = -\frac{1}{16\pi G_N} \int E E_A^\mu E_B^\nu R_{\mu\nu}^{AB}(J) + S_{\text{matter}}. \tag{2.7}$$

⁴All parameters depend on the coordinates x^μ , even when not explicitly indicated.

Here $E = \det E_\mu^A$ and we have defined the inverse Vierbein E^μ_A

$$E_\mu^A E^\mu_B = \delta_B^A, \quad E_\mu^A E^\nu_A = \delta_\mu^\nu. \quad (2.8)$$

For generality we have included an arbitrary matter action S_{matter} . Note that we are using a first-order formulation where Ω_μ^{AB} is treated as an independent variable. The action (2.7) transforms under P -transformations as follows:

$$\delta_P S = -\frac{3}{8\pi G_N} \int E E_{[A}^\mu E_B^\nu E_C^\rho E_{D]}^\sigma R_{\mu\nu}{}^{AB}(J) R_{\rho\sigma}{}^C(P) \eta^D + \delta_P S_{\text{mat}}. \quad (2.9)$$

This shows that only for $D = 3$ the gravity kinetic term in the action (2.7) is invariant under both Lorentz and P -transformations. This is related to the fact that for $D = 3$ this kinetic term can be rewritten as a Chern-Simons gauge theory.

Varying the action (2.7) with respect to the independent gauge fields Ω_μ^{AB} and E^μ_A we obtain the following equations of motion:

$$R_{C[A}{}^C(P) E_{B]}^\mu + \frac{1}{2} E_C^\mu R_{AB}{}^C(P) = J_{AB}^\mu, \quad (2.10)$$

$$G_\mu{}^A = T_\mu{}^A, \quad (2.11)$$

where the Einstein tensor is defined by

$$G_\mu{}^A = R_{\mu C}{}^{AC}(J) - \frac{1}{2} E_\mu^A R_{CD}{}^{CD}(J) \quad (2.12)$$

and where we have defined the Lorentz transformation current J_{AB}^μ and the energy-momentum tensor $T_\mu{}^A$ as follows ($\kappa = 8\pi G_N$):

$$J_{AB}^\mu \equiv \frac{\kappa}{E} \frac{\delta S_{\text{mat}}}{\delta \Omega_\mu^{AB}}, \quad T_\mu{}^A = \frac{\kappa}{E} \frac{\delta S_{\text{mat}}}{\delta E_\mu^A}. \quad (2.13)$$

For $D > 2$ the equation of motion (2.10) can be rewritten as

$$R_{\mu\nu}{}^A(P) = 2J_{AB}^\rho E_\mu^A E_\nu^B E_\rho^C + \frac{4}{D-2} J_{AB}^\rho E_\rho^A E_{[\mu}^B E_{\nu]}^C. \quad (2.14)$$

By taking cyclic permutations, this equation can be further rewritten in terms of the Lorentz spin connection as follows:

$$\Omega_\mu^{AB} = -2E^{\rho[A} \partial_{[\mu} E_{\rho]}^{B]} + E_{\mu C} E^{\rho A} E^{\nu B} \partial_{[\rho} E_{\nu]}^C + X_\mu^{AB}, \quad (2.15)$$

with

$$X_\mu^{AB} = 2E_\mu^C \eta^{D[A} J_{CD}^{B]} - E_{\mu C} \eta^{AE} \eta^{BF} J_{EF}^C + \frac{4}{D-2} E_\mu^{[A} \eta^{B]D} J_{CD}^C. \quad (2.16)$$

The equations of motion (2.10) and (2.11) give relations between the curvatures and the currents. The curvatures satisfy the following Bianchi identities:

$$\mathcal{D}_{[\mu} R_{\nu\rho]}{}^A(P) + R_{[\mu\nu}{}^{AB}(J) E_{\rho]B} = 0, \quad (2.17)$$

$$\mathcal{D}_{[\mu} R_{\nu\rho]}{}^{AB}(J) = 0, \quad (2.18)$$

where \mathcal{D}_μ is the Lorentz-covariant derivative. By contraction these Bianchi identities imply that

$$2R_{[\mu\nu]} + \mathcal{D}_A R_{\mu\nu}{}^A(P) + 2\mathcal{D}_{[\mu} R_{\nu]A}{}^A(P) = 0, \quad (2.19)$$

$$2\mathcal{D}_A G_C{}^A - 2R_B{}^A R_{CA}{}^B(P) + R_{CD}{}^{AB}(J) R_{AB}{}^D(P) = 0, \quad (2.20)$$

with

$$R_\mu{}^A = R_{\mu\nu}{}^{AB}(J) E_B{}^\nu. \quad (2.21)$$

For the equations of motion to be consistent, these identities require the following on-shell relations among the currents:

$$T_{[AB]} = -\mathcal{D}_C J_{AB}^C + \frac{2}{D-2} J_{AB}^C J_{CD}^D, \quad (2.22)$$

$$\mathcal{D}_B T_A{}^B = 2J_{AB}^\mu T_\mu{}^B - R_{AB}{}^{CD}(J) J_{CD}^B + \frac{2}{D-2} (2T_A{}^B - T\delta_A^B) J_{BC}^C. \quad (2.23)$$

3 Carroll gravity

In this section we will consider Carroll gravity, i.e. the ultra-relativistic limit of General Relativity. The underlying algebra is a particular (ultra-relativistic) contraction of the Poincaré algebra which is called the Carroll algebra [17, 18]. This section consists of two subsections. In the first subsection we will review a few properties of the Carroll algebra while in the second one we will construct Carroll gravity. The addition of general matter couplings to Carroll gravity will be discussed in subsection 5.1.

3.1 The Carroll algebra

The Carroll algebra is obtained by a contraction of the Poincaré algebra. To define this contraction, we decompose the A -index into $A = \{0, a\}$ with $a = (1, \dots, D-1)$, and redefine the Poincaré generators according to

$$P_0 = \omega H, \quad (3.1)$$

$$J_{0a} = \omega G_a, \quad (3.2)$$

where H and G_a are the generators of time translations and boosts, respectively. The generators P_a of space translations and J_{ab} of spatial rotations are not redefined. Next, taking the limit $\omega \rightarrow \infty$ we obtain the following Carroll algebra:

$$\begin{aligned} [J_{ab}, P_c] &= 2\delta_{c[a} P_{b]}, & [J_{ab}, G_c] &= 2\delta_{c[a} G_{b]}, \\ [J_{ab}, J_{cd}] &= 4\delta_{[a[d} J_{c]b]}, & [P_a, G_b] &= \delta_{ab} H. \end{aligned} \quad (3.3)$$

To each generator of the Carroll algebra we associate a gauge field, a local parameter parametrizing the corresponding symmetry and a curvature, see table 2.

symmetry	generators	gauge field	parameters	curvatures
time translations	H	τ_μ	$\zeta(x^\nu)$	$R_{\mu\nu}(H)$
space translations	P_a	e_μ^a	$\zeta^a(x^\nu)$	$R_{\mu\nu}^a(P)$
boosts	G_a	ω_μ^a	$\lambda^a(x^\nu)$	$R_{\mu\nu}^a(G)$
spatial rotations	J_{ab}	ω_μ^{ab}	$\lambda^{ab}(x^\nu)$	$R_{\mu\nu}^{ab}(J)$

Table 2. This table indicates the generators of the Carroll algebra and the gauge fields, local parameters and curvatures that are associated to each of these generators.

The gauge field transformations according to the Carroll algebra are given by

$$\begin{aligned}
 \delta\tau_\mu &= \partial_\mu\zeta - \omega_\mu^a\zeta^a + e_\mu^a\lambda_a, \\
 \delta e_\mu^a &= (D_\mu\zeta)^a + \lambda^a{}_b e_\mu^b, \\
 \delta\omega_\mu^{ab} &= (D_\mu\lambda)^{ab}, \\
 \delta\omega_\mu^a &= (D_\mu\lambda)^a + \lambda^a{}_b\omega_\mu^b,
 \end{aligned}
 \tag{3.4}$$

where D_μ is the covariant derivative with respect to spatial rotations, e.g., $(D_\mu\zeta)^a = \partial_\mu\zeta^a - \omega_\mu^{ab}\zeta^b$. Like in the case of General Relativity, all gauge fields transform as covariant vectors under general coordinate transformations with parameter ξ^μ . In the following we will ignore the time and space translations but instead consider the general coordinate transformations.

By construction the curvatures

$$\begin{aligned}
 R_{\mu\nu}(H) &= 2\partial_{[\mu}\tau_{\nu]} - 2\omega_{[\mu}^a e_{\nu]}{}_a, \\
 R_{\mu\nu}^a(P) &= 2\partial_{[\mu}e_{\nu]}^a - 2\omega_{[\mu}^{ab} e_{\nu]}{}_b, \\
 R_{\mu\nu}^a(G) &= 2\partial_{[\mu}\omega_{\nu]}^a - 2\omega_{[\mu}^{ab}\omega_{\nu]}{}_b, \\
 R_{\mu\nu}^{ab}(J) &= 2\partial_{[\mu}\omega_{\nu]}^{ab} - 2\omega_{[\mu}^a{}_c\omega_{\nu]}{}^{cb},
 \end{aligned}
 \tag{3.5}$$

transform covariantly under the Carroll transformations (3.4). In particular, they transform under Carroll boosts and spatial rotations as follows:

$$\delta R_{\mu\nu}(H) = \lambda^a R_{\mu\nu}^a(P), \tag{3.6}$$

$$\delta R_{\mu\nu}^a(P) = \lambda^{ab} R_{\mu\nu}^b(P), \tag{3.7}$$

$$\delta R_{\mu\nu}^a(G) = \lambda^{ab} R_{\mu\nu}^b(G) - \lambda^b R_{\mu\nu}^{ab}(J), \tag{3.8}$$

$$\delta R_{\mu\nu}^{ab}(J) = \lambda^{bc} R_{\mu\nu}^{ac}(J) - \lambda^{ac} R_{\mu\nu}^{bc}(J). \tag{3.9}$$

Furthermore, they satisfy the following Bianchi identities:

$$\mathcal{D}_{[\mu}R_{\nu\rho]}(H) + R_{[\mu\nu}^a(G)e_{\rho]}^a = 0, \tag{3.10}$$

$$\mathcal{D}_{[\mu}R_{\nu\rho]}^a(P) + R_{[\mu\nu}^{ab}(J)e_{\rho]}^b = 0, \tag{3.11}$$

$$\mathcal{D}_{[\mu}R_{\nu\rho]}^a(G) = 0, \tag{3.12}$$

$$\mathcal{D}_{[\mu}R_{\nu\rho]}^{ab}(J) = 0, \tag{3.13}$$

where \mathcal{D}_μ is a Carroll-covariant derivative, i.e. it is covariant with respect to Carroll boosts and spatial rotations.

3.2 Carroll gravity

We will first derive an invariant action for Carroll gravity by taking the ultra-relativistic limit of the action of General Relativity (2.7). To define this limit, we redefine the gauge fields and symmetry parameters with the same parameter ω that occurs in the Carroll contraction defined by eqs. (3.1) and (3.2). Requiring that the generalized parameter ϵ and generalized gauge field A_μ defined by

$$\epsilon = \zeta H + \zeta^a P_a + \lambda^a G_a + \frac{1}{2} \lambda^{ab} J_{ab}, \quad (3.14)$$

$$A_\mu = \tau_\mu H + e_\mu^a P_a + \omega_\mu^a G_a + \frac{1}{2} \omega_\mu^{ab} J_{ab}, \quad (3.15)$$

are invariant under the redefinitions leads to the following redefinitions of the gauge fields and parameters:

$$E_\mu^0 = \omega^{-1} \tau_\mu, \quad \Omega_\mu^{0a} = \omega^{-1} \omega_\mu^a, \quad (3.16)$$

$$E_\mu^a = e_\mu^a, \quad \Omega_\mu^{ab} = \omega_\mu^{ab}, \quad (3.17)$$

$$\eta^0 = \omega^{-1} \zeta, \quad \Lambda^{0a} = \omega^{-1} \lambda^a, \quad (3.18)$$

$$\eta^a = \zeta^a, \quad \Lambda^{ab} = \lambda^{ab}. \quad (3.19)$$

One can show that performing these redefinitions in the relativistic transformation rules (2.3) and taking the limit $\omega \rightarrow \infty$ one recovers the Carroll transformations (3.4).

Performing the same ω -rescalings (3.16) and (3.17) in the relativistic action (2.7) we obtain

$$S_{\text{Car}} = -\frac{1}{16\pi G_N} \int \frac{e}{\omega} \left(2\tau^\mu e_a^\nu R(G)_{\mu\nu}{}^a + e_a^\mu e_b^\nu R(J)_{\mu\nu}{}^{ab} + O(\omega^{-2}) \right), \quad (3.20)$$

where $e = \det(\tau_\mu, e_\mu^a)$ is the ultra-relativistic determinant. We have defined here the projective inverses τ^μ and e^μ_a according to:

$$\begin{aligned} e_\mu^a e^\mu_b &= \delta_b^a, & \tau^\mu \tau_\mu &= 1, \\ \tau^\mu e_\mu^a &= 0, & \tau_\mu e^\mu_a &= 0, \\ e_\mu^a e^\nu_a &= \delta_\mu^\nu - \tau_\mu \tau^\nu. \end{aligned} \quad (3.21)$$

They transform under boosts and spatial rotations as follows:

$$\delta\tau^\mu = 0, \quad \delta e_a^\mu = -\lambda^a \tau^\mu + \lambda^{ab} e_b^\mu. \quad (3.22)$$

Rescaling $G_N \rightarrow \omega^{-1} G_C$ and taking the $\omega \rightarrow \infty$ limit in the action (3.20) we end up with the Carroll action⁵

$$S_{\text{Car}} = -\frac{1}{16\pi G_C} \int e \left(2\tau^\mu e_a^\nu R(G)_{\mu\nu}{}^a + e_a^\mu e_b^\nu R(J)_{\mu\nu}{}^{ab} \right). \quad (3.23)$$

⁵This limit shows similarities with the strong coupling limit considered in [19, 20], see also [21–23]. Note that both limits lead to a theory with a Carroll-invariant vacuum solution. This suggests that, although looking different at first sight, the result of the two limits might be the same up to field redefinitions. We thank Marc Henneaux and Max Niedermaier for a discussion on this point.

Using the variations (3.8), (3.9) and (3.22) it can easily be checked that this action is invariant under Carroll boosts and rotations. In $D = 3$, the Carroll algebra can be equipped with a non-degenerate, invariant bilinear form and as a consequence it is possible to write down a Chern-Simons action for the Carroll algebra. This Chern-Simons action is then equivalent to the one above.

The set of equations of motion obtained by varying τ^μ , e_a^μ , ω_μ^a and ω_μ^{ab} in the Carroll action (3.23) can be written for any $D > 2$ as follows:

$$R_{\mu\nu}(H) = 0, \quad (3.24)$$

$$R_{\mu\nu}{}^a(P) = 0, \quad (3.25)$$

$$R_{\mu a}{}^a(G) = 0, \quad (3.26)$$

$$R_{0b}{}^{ab}(J) = 0, \quad (3.27)$$

$$R_{ac}{}^{bc}(J) + R_{0a}{}^b(G) = 0, \quad (3.28)$$

where $R_{0b}{}^{ab}(J) = \tau^\mu e_b^\nu R_{\mu\nu}{}^{ab}(J)$ and we are using the same notation for the remaining projections of the curvatures. The equations (3.24)–(3.25) follow from manipulating the equations of motion corresponding to the spin-connections:

$$\frac{\kappa}{e} \frac{\delta S_{\text{Car}}}{\delta \omega_\mu^a} = \tau^\mu R_{ba}{}^b(P) + e_b^\mu R_{a0}{}^b(P) - e_a^\mu R_{b0}{}^b(P), \quad (3.29)$$

$$\frac{\kappa}{e} \frac{\delta S_{\text{Car}}}{\delta \omega_\mu^{ab}} = -\frac{1}{2} \tau^\mu R_{ab}(H) - R_{0[a}(H) e_{b]}^\mu - R_{c[a}{}^c(P) e_{b]}^\mu - \frac{1}{2} e_c^\mu R_{ab}{}^c(P). \quad (3.30)$$

The resulting equations (3.24) and (3.25) can be used to solve for the spin connections

$$\omega_\mu^a = \tau_\mu \tau^\nu e^{\rho a} \partial_{[\nu} \tau_{\rho]} + e^{\nu a} \partial_{[\mu} \tau_{\nu]} + S^{ab} e_\mu^b, \quad (3.31)$$

$$\omega_\mu^{ab} = -2e^{\rho[a} \partial_{[\mu} e_{\rho]}^b] + e_{\mu c} e^{\rho a} e^{\nu b} \partial_{[\rho} e_{\nu]}^c, \quad (3.32)$$

except for a symmetric component $S^{ab} = S^{(ab)} = e^{\mu(a} \omega_\mu^{b)}$ of the boost spin connection ω_μ^a which remains undetermined. Below we will give an interpretation for S^{ab} . The equation (3.25) can additionally be used to derive the constraint

$$K_{ab} = 0, \quad (3.33)$$

where we defined $K_{ab} = e_a^\mu e_b^\nu K_{\mu\nu}$ with $K_{\mu\nu}$ the extrinsic curvature given by the Lie derivative of $h_{\mu\nu} = e_\mu^a e_\nu^b \delta_{ab}$ along the vector field τ^μ

$$K_{\mu\nu} \equiv \frac{1}{2} \mathcal{L}_\tau(h_{\mu\nu}) = \frac{1}{2} (\tau^\rho \partial_\rho h_{\mu\nu} + h_{\mu\rho} \partial_\nu \tau^\rho + h_{\nu\rho} \partial_\mu \tau^\rho). \quad (3.34)$$

The fact that curvature constraints are not only used to solve for (part of) the spin-connections but also lead to constraints on the geometry has been encountered before in the construction of the so-called stringy Newton-Cartan gravity theory [24].

Let us stress that from equations (3.31) and (3.32) by themselves, it follows that the spin connections transform under Carroll boosts and rotations according to

$$\delta \omega_\mu^a = (D_\mu \lambda)^a + \lambda^a{}_b \omega_\mu^b + \tau_\mu \delta^{ab} K_{bc} \lambda^c, \quad (3.35)$$

$$\delta \omega_\mu^{ab} = D_\mu \lambda^{ab} - 2\lambda^{[a} \delta^{b]c} e_\mu^d K_{cd}. \quad (3.36)$$

Hence, it is only thanks to the constraint (3.33) imposed on the geometry that the transformation of the spin connections agrees with (3.4). In order to obtain (3.35) we used

$$\delta S^{ab} = \lambda^{ac} S^{bc} + \lambda^{bc} S^{ac} + e^{\mu(a} \partial_\mu \lambda^{b)} - \lambda^{(a} \omega_\mu^{b)} \tau^\mu - \lambda^c e^{\nu(a} \omega_\nu^{b)c}, \quad (3.37)$$

as can be directly deduced from (3.4) since $S^{ab} = e^{\mu(a} \omega_\mu^{b)}$.

The geometrical constraint (3.33) is closely related to the undetermined components S^{ab} of the boost spin connection. In order to see this, it is instructive to go to a second order formulation of Carroll gravity. Plugging the dependent expressions for the spin connections (3.31) and (3.32) into the Carroll action (3.23) we obtain

$$S_{\text{Car}} = -\frac{1}{16\pi G_C} \int e \left(2\tau^\mu e_a^\nu R(G)^a{}_{\mu\nu}|_{S^{ab}=0} + e_a^\mu e_b^\nu R(J)^{ab}{}_{\mu\nu} + 2K_{ab} S^{ab} - 2\delta^{ab} \delta_{cd} K_{ab} S^{cd} \right), \quad (3.38)$$

where we performed an integration by part on the S^{ab} dependent terms.⁶ From the expression (3.38) for the action it follows that the equation of motion for S^{ab} implies $K_{ab} = 0$. In other words, we conclude that the S^{ab} term is actually a Lagrange multiplier that enforces the constraint (3.33) which, previously in the first order formulation, was a consequence of the equations of motion for the spin connections.

Finally, Carroll gravity can be rewritten in a second order metric formulation in terms of the fields τ_μ , $h_{\mu\nu}$ and $S^{\mu\nu} = e_a^\mu e_b^\nu S^{ab}$. In order to do this we first trade the spin connections for a Christoffel connection. The spin connections can be related to a space-time connection by imposing a vielbein postulate

$$\partial_\mu \tau_\nu - \Gamma_{\mu\nu}^\rho \tau_\rho - \omega_\mu^a e_\nu^a \equiv 0, \quad (3.39)$$

$$\partial_\mu e_\nu^a - \Gamma_{\mu\nu}^\rho e_\rho^a - \omega_\mu^{ab} e_\nu^b \equiv 0. \quad (3.40)$$

The vielbein postulate implies the following relation between the space-time connection $\Gamma_{\mu\nu}^\rho$ and the spin connections

$$\Gamma_{\mu\nu}^\rho = \tau^\rho \partial_\mu \tau_\nu + e_a^\rho \partial_\mu e_\nu^a - \tau^\rho \omega_\mu^a e_\nu^b \delta_{ab} - e_a^\rho \omega_\mu^{ab} e_\nu^c \delta_{bc}. \quad (3.41)$$

A few remarks are in order here. By construction, the connection $\Gamma_{\mu\nu}^\rho$ would be Carroll invariant if the fields would transform as in (3.4). However, this is not the case at this stage since we have additional K_{ab} contributions in (3.35) and (3.36). Also, on general grounds it follows from the vielbein postulate that

$$K_{\mu\nu} = \left(h_{\mu\rho} \Gamma_{[\sigma\nu]}^\rho + h_{\nu\rho} \Gamma_{[\sigma\mu]}^\rho \right) \tau^\sigma, \quad (3.42)$$

where $\Gamma_{[\mu\nu]}^\rho$ represents the torsion. Hence, on a Carrollian geometry $K_{\mu\nu}$ is automatically vanishing whenever there is no spatial component to the torsion, namely whenever $e_\rho^a \Gamma_{[\mu\nu]}^\rho$ vanishes which is precisely the content of equation (3.25). The same constraint on the torsion also occurs in the context of the Carroll geometry of [25].

⁶This implies that we end up with a Lagrangian that is only Carroll invariant up to total derivative terms.

Now let us rewrite $\Gamma_{\mu\nu}^\rho$ in a metric formulation. Plugging (3.31) and (3.32) into (3.41) we obtain

$$\begin{aligned} \Gamma_{\mu\nu}^\rho &= \tau^\rho (\partial_{[\mu}\tau_{\nu]}) + \tau_\mu\tau^\sigma\partial_{[\nu}\tau_{\sigma]} + \tau_\nu\tau^\sigma\partial_{[\mu}\tau_{\sigma]} - h_{\mu\tau}h_{\nu\sigma}S^{\tau\sigma} \\ &\quad - h^{\rho\sigma}K_{\sigma\mu}\tau_\nu + \frac{1}{2}h^{\rho\sigma}(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}) . \end{aligned} \quad (3.43)$$

We then define a Riemann tensor with respect to the connection $\Gamma_{\mu\nu}^\rho$ in the usual way

$$R_{\mu\nu\rho}{}^\sigma = -\partial_\mu\Gamma_{\nu\rho}^\sigma + \partial_\nu\Gamma_{\mu\rho}^\sigma - \Gamma_{\mu\lambda}^\sigma\Gamma_{\nu\rho}^\lambda + \Gamma_{\nu\lambda}^\sigma\Gamma_{\mu\rho}^\lambda . \quad (3.44)$$

Finally, the Carroll invariant action in a second order metric formulation reads⁷

$$S_{\text{Car}} = \frac{1}{16\pi G_C} \int e h^{\mu\nu} \left(R_{\mu\nu} + \tau^\rho\tau_\sigma R_{\mu\rho\nu}{}^\sigma \right) , \quad (3.45)$$

with $\Gamma_{\mu\nu}^\rho$ given by (3.43) and where we defined the Ricci tensor as $R_{\mu\nu} = R_{\mu\sigma\nu}{}^\sigma$. Since we have seen that in the second order formulation the connection $\Gamma_{\mu\nu}^\rho$ is not Carroll invariant $\delta\Gamma_{\mu\nu}^\rho \neq 0$, it follows that the invariance of the action (3.45) is no longer manifest.

In the second order formulation, the equations of motion for $S^{\mu\nu}$ read

$$K_{\mu\nu} - h_{\mu\nu}K = 0 , \quad (3.46)$$

with $K = h^{\mu\nu}K_{\mu\nu}$ and for $D > 2$ this implies that $K_{\mu\nu} = 0$. We thus reproduce the constraint we initially obtained in the first order formalism. As we already learned from equation (3.38) $S^{\mu\nu}$ is hence to be seen as a Lagrange multiplier whose role is to impose this constraint on the geometry. Using that $K_{\mu\nu} = 0$ the remaining equations of motion obtained by varying τ^μ and $h^{\mu\nu}$,⁸ are

$$\left(\tau_\lambda h_\mu{}^\sigma - \frac{1}{2}\tau_\mu h_\lambda{}^\sigma \right) h^{\nu\rho} R_{\sigma\nu\rho}{}^\lambda = 0 , \quad (3.47)$$

$$R_{\mu\nu} - \frac{1}{2}h_{\mu\nu}\hat{R} = 0 , \quad (3.48)$$

with $h_\mu{}^\nu = h_{\mu\rho}h^{\nu\rho}$ and $\hat{R} = h^{\mu\nu}R_{\mu\nu} + \tau^\rho\tau_\sigma h^{\mu\nu}R_{\mu\rho\nu}{}^\sigma$. Note that with $K_{\mu\nu} = 0$ the terms $h_{\mu\nu}$, $R_{\mu\nu}$ and \hat{R} in equation (3.48) are all separately Carroll invariant. Moreover, in this case, the Ricci tensor becomes symmetric and since it satisfies $R_{\mu\nu}\tau^\nu = 0$ equation (3.48) leads to $\frac{1}{2}D(D-1)$ equations.

The Carroll theory we described in this section can be compared to the Carroll geometry developed in [12]. In [12] the extrinsic curvature $K_{\mu\nu}$ is not constrained to vanish but is kept arbitrary. Moreover, in [12] the Carroll symmetries are realised on the fields

⁷Alternatively, we can define a modified connection $\hat{\Gamma}_{\mu\nu}^\rho = \Gamma_{\mu\nu}^\rho - \tau^\rho\partial_{[\mu}\tau_{\nu]} + \tau^\rho\tau^\sigma(\tau_\mu\partial_{[\nu}\tau_{\sigma]} + \tau_\nu\partial_{[\mu}\tau_{\sigma]}) - \tau^\rho S_{\mu\nu}$ such that the action takes the simpler form $S_{\text{Car}} = \frac{1}{16\pi G_C} \int e h^{\mu\nu} \hat{R}_{\mu\nu}$ with $\hat{R}_{\mu\nu}$ the Ricci tensor relative to the shifted connection $\hat{\Gamma}_{\mu\nu}^\rho$.

⁸In varying $h^{\mu\nu}$ one should use that its variation is constrained due to $h^{\mu\nu}\tau_\nu = 0$. This implies that one should take care of projecting out the purely time-like components of the equation obtained by varying $h^{\mu\nu}$. E.g., upon varying $h^{\mu\nu}X_{\mu\nu}$, where $X_{\mu\nu}$ does not depend on $h^{\mu\nu}$, the correct equation of motion is $X_{\mu\nu} - \tau_\mu\tau_\nu\tau^\rho\tau^\sigma X_{\rho\sigma} = 0$.

$\tau_\mu, h_{\mu\nu}$ and a vector field $M^\mu = e_a^\mu M^a$. This is different from the present case where the additional field needed to realise the Carroll symmetries is a symmetric tensor S^{ab} . Furthermore, although when evaluated in the case $K_{\mu\nu} = 0$ the rotation spin connection (3.32) agrees precisely with the one obtained in [12], there exists no special choice of S^{ab} such that the boost spin connection (3.31) would match the one of [12]. The reason for this is that in the latter case the boost connection is by construction always of the form

$$\text{boost connection of [12]} : \quad \omega_\mu^a = \partial_\mu M^a - \omega_\mu^{ab} M^b. \quad (3.49)$$

In particular, $\tau^\mu \omega_\mu^a$ is then a function of M^a whereas in our case $\tau^\mu \omega_\mu^a$ is not a function of S^{ab} . Hence, there cannot be a choice of S^{ab} for which the connections would agree. For further comments, see the conclusions.

4 Galilei gravity

The kinematics of Galilei gravity can be obtained by gauging the Galilei algebra. In contrast to Newton-Cartan gravity, Galilei gravity has no mass parameter. In this section we will perform the same steps as for Carroll gravity thereby emphasizing the similarities as well as the differences. In the first subsection we will review a few properties of the Galilei algebra while in the second subsection we will construct Galilei gravity.

4.1 The Galilei algebra

The Galilei algebra is obtained by a contraction of the Poincaré algebra. To define this contraction, we decompose the A -index into $A = \{0, a\}$ with $a = (1, \dots, D - 1)$, and redefine the Poincaré generators according to

$$P_0 = \omega^{-1} H, \quad (4.1)$$

$$J_{0a} = \omega G_a, \quad (4.2)$$

where H and G_a are the generators of time translations and boosts, respectively. The generators P_a of space translations and J_{ab} of spatial rotations are not redefined. Next, taking the limit $\omega \rightarrow \infty$ we obtain the following Galilei algebra:

$$\begin{aligned} [J_{ab}, P_c] &= 2\delta_{c[a} P_{b]}, & [J_{ab}, G_c] &= 2\delta_{c[a} G_{b]}, \\ [J_{ab}, J_{cd}] &= 4\delta_{[a[d} J_{c]b]}, & [H, G_a] &= P_a. \end{aligned} \quad (4.3)$$

To each generator of the Galilei algebra we associate a gauge field, a local parameter parametrizing the corresponding symmetry and a curvature, for which we use the same notation as in the case of the Carroll algebra, see table 2.

The gauge field transformations according to the Galilei algebra are given by

$$\delta\tau_\mu = 0, \quad (4.4)$$

$$\delta e_\mu^a = \lambda^a \tau_\mu + \lambda^{ab} e_\mu^b, \quad (4.5)$$

$$\delta\omega_\mu^{ab} = (D_\mu \lambda)^{ab}, \quad (4.6)$$

$$\delta\omega_\mu^a = (D_\mu \lambda)^a + \lambda^a_b \omega_\mu^b. \quad (4.7)$$

Like in the Carroll case, all gauge fields transform as covariant vectors under general coordinate transformations with parameter ξ^μ . In the following we will ignore the time and space translations but instead consider the general coordinate transformations.

The curvatures that transform covariantly under the Galilei transformations (4.4)–(4.7) are given by

$$R_{\mu\nu}(H) = 2\partial_{[\mu}\tau_{\nu]}, \quad (4.8)$$

$$R_{\mu\nu}{}^a(P) = 2\partial_{[\mu}e_{\nu]}^a - 2\omega_{[\mu}^{ab}e_{\nu]}^b - 2\omega_{[\mu}^a\tau_{\nu]}. \quad (4.9)$$

$$R_{\mu\nu}{}^a(G) = 2\partial_{[\mu}\omega_{\nu]}^a - 2\omega_{[\mu}^{ab}\omega_{\nu]}^b, \quad (4.10)$$

$$R_{\mu\nu}{}^{ab}(J) = 2\partial_{[\mu}\omega_{\nu]}^{ab} - 2\omega_{[\mu}^{ac}\omega_{\nu]}^{cb}. \quad (4.11)$$

They transform under Galilean boosts and spatial rotations as follows:

$$\delta R_{\mu\nu}(H) = 0, \quad (4.12)$$

$$\delta R_{\mu\nu}{}^a(P) = \lambda^{ab}R_{\mu\nu}{}^b(P) + \lambda^a R_{\mu\nu}(H). \quad (4.13)$$

$$\delta R_{\mu\nu}{}^a(G) = \lambda^{ab}R_{\mu\nu}{}^b(G) - \lambda^b R_{\mu\nu}{}^{ab}(J), \quad (4.14)$$

$$\delta R_{\mu\nu}{}^{ab}(J) = \lambda^{bc}R_{\mu\nu}{}^{ac}(J) - \lambda^{ac}R_{\mu\nu}{}^{bc}(J) \quad (4.15)$$

and satisfy the following Bianchi identities:

$$\mathcal{D}_{[\mu}R_{\nu\rho]}(H) = 0, \quad (4.16)$$

$$\mathcal{D}_{[\mu}R_{\nu\rho]}{}^a(P) + R_{[\mu\nu}{}^a(G)\tau_{\rho]} + R_{[\mu\nu}{}^{ab}(J)e_{\rho]}^b = 0, \quad (4.17)$$

$$\mathcal{D}_{[\mu}R_{\nu\rho]}{}^a(G) = 0, \quad (4.18)$$

$$\mathcal{D}_{[\mu}R_{\nu\rho]}{}^{ab}(J) = 0, \quad (4.19)$$

where \mathcal{D}_μ is a Galilei-covariant derivative, i.e. it is covariant with respect to Galilei boosts and spatial rotations.

4.2 Galilei gravity

Like in the Carroll case an invariant action for Galilei gravity can be obtained by taking the non-relativistic limit of the action of General Relativity (2.7). To define this limit we redefine the gauge fields and symmetry parameters with the same parameter ω that occurs in the Carroll contraction defined by eqs. (4.1) and (4.2). Requiring that the generalized parameter ϵ and generalized gauge field A_μ defined by eqs. (3.14) and (3.15) are invariant under the redefinitions leads to the following redefinitions of the gauge fields and parameters:

$$E_\mu^0 = \omega\tau_\mu, \quad \Omega_\mu^{0a} = \omega^{-1}\omega_\mu^a, \quad (4.20)$$

$$E_\mu^a = e_\mu^a, \quad \Omega_\mu^{ab} = \omega_\mu^{ab}, \quad (4.21)$$

$$\eta^0 = \omega\zeta, \quad \Lambda^{0a} = \omega^{-1}\lambda^a, \quad (4.22)$$

$$\eta^a = \zeta^a, \quad \Lambda^{ab} = \lambda^{ab}. \quad (4.23)$$

Performing the same ω -rescalings (4.20) and (4.21) in the relativistic action (2.7), rescaling $G_N \rightarrow \omega G_G$ and taking the $\omega \rightarrow \infty$ limit we end up with the following Galilei action

$$S_{\text{Gal}} = -\frac{1}{2\kappa} \int e R_{\mu\nu}{}^{ab}(J) e_a^\mu e_b^\nu, \quad (4.24)$$

where $\kappa = 8\pi G_G$ and $e = \det(\tau_\mu, e_\mu^a)$ is the non-relativistic determinant. We have used here the same definition of the projective inverses τ^μ and e^μ_a like in the Carroll case, see eq. (3.21). These projective inverses transform under the Galilei boosts and spatial rotations as follows:

$$\delta\tau^\mu = -\lambda^a e_a^\mu, \quad \delta e_a^\mu = \lambda^{ab} e_b^\mu. \quad (4.25)$$

One may verify that the Galilei action (4.24) is not only Galilei invariant but it also has an accidental local scaling symmetry given by

$$\tau_\mu \rightarrow \lambda(x)^{-(D-3)} \tau_\mu, \quad (4.26)$$

$$e_\mu^a \rightarrow \lambda(x) e_\mu^a, \quad (4.27)$$

where $\lambda(x)$ is an arbitrary function. Hence, the full invariance of the Galilean gravity action is that of a Schrödinger algebra without central charge and with critical exponent $z = -(D-3)$.

For any $D > 2$ the equations of motion that follow from the variation of the Galilei action (4.24) are equivalent to a constraint on the geometry

$$R_{ab}(H) = e_a^\mu e_b^\nu \partial_{[\mu} \tau_{\nu]} = 0, \quad (4.28)$$

together with the following equations

$$R_{0a}(H) = \frac{D-3}{D-2} R_{ab}{}^b(P), \quad (4.29)$$

$$R_{ab}{}^c(P) = -\frac{2}{D-2} \delta_{[a}^c R_{b]d}{}^d(P), \quad (4.30)$$

$$R_{\mu b}{}^{ab}(J) = 0. \quad (4.31)$$

where the first two equations follow from manipulating the equations of motion with respect to the spin-connection. The constraint (4.28) means that this geometry has twistless torsion [26]. Clearly, we see from (4.29) that $D = 3$ is special, we will come back to this case below and first assume $D > 3$.

For $D > 3$ the equation of motion (4.29) and (4.30) can be used to solve for the spatial rotation spin connection $\omega_\mu{}^{ab}$ as

$$\omega_\mu{}^{ab} = \tau_\mu A^{ab} + e_{\mu c} \left(e^{\rho[a} e^{b]\nu} \partial_\rho e_\nu^c + e^{\rho[a} e^{c]\nu} \partial_\rho e_\nu^b - e^{\rho[b} e^{c]\nu} \partial_\rho e_\nu^a \right) + \frac{4}{D-3} e^{\rho[a} e_\mu^{b]} \tau^\nu \partial_{[\rho} \tau_{\nu]}, \quad (4.32)$$

except for A^{ab} which is an undetermined anti-symmetric tensor component of $\omega_\mu{}^{ab}$.

The constraint (4.28) is a restriction on the geometry which can be seen as the Galilean equivalent to the constraint (3.33) in the Carroll case. In the second order formulation the constraint (4.28) arises from the variation with respect to A^{ab} . Hence, we can interpret A^{ab} as a Lagrange multiplier. Indeed, in the case $D > 3$, plugging (4.32) into the action (4.24) to obtain it in a second order formulation leads to

$$S_{\text{Gal}} = -\frac{1}{2\kappa} \int e \left(R_{\mu\nu}{}^{ab}(J) e_a^\mu e_b^\nu \Big|_{A^{ab}=0} + A^{ab} R_{ab}(H) \right). \quad (4.33)$$

This makes manifest the fact that the variation with respect to A^{ab} of the second order action in equation (4.33) reproduces the constraint (4.28).

The field A^{ab} does not transform covariantly, as can be seen from (4.6). Since A^{ab} is undetermined we can make a redefinition

$$\bar{A}^{ab} = A^{ab} + \tau^\rho e^{\mu[a} \partial_\rho e_\mu^{b]}, \quad (4.34)$$

such that \bar{A}^{ab} transforms covariantly

$$\delta \bar{A}^{ab} = \lambda^{ac} \bar{A}^{cb} + \lambda^{bc} \bar{A}^{ac} - \lambda^c e_c^\mu \omega_\mu^{ab} - \lambda^c e_c^\mu e^{\nu[a} \partial_\mu e_\nu^{b]} - \lambda^{[a} e_c^{b]\nu} \tau^\mu \partial_\mu \tau_\nu. \quad (4.35)$$

The solution for ω_μ^{ab} given in equation (4.32) transforms according to

$$\delta \omega_\mu^{ab} = (D_\mu \lambda)^{ab} - \left(2\lambda^{[a} e^{b]\nu} e_c^\rho - e^{a\nu} e^{b\rho} \lambda^c \right) e_\mu^c \partial_{[\nu} \tau_{\rho]} + \frac{4}{D-3} e_\mu^{[a} e^{b]\nu} \lambda^c e_c^\rho \partial_{[\nu} \tau_{\rho]}. \quad (4.36)$$

Similar to the Carroll case, this transformation agrees with (4.6) only up to the geometrical constraint $e_a^\mu e_b^\nu \partial_{[\mu} \tau_{\nu]} = 0$ which we found in equation (4.28).

We will now rewrite the action (4.33) in a second order metric formulation in terms of τ_μ , $h_{\mu\nu}$ and $\bar{A}^{\mu\nu}$ following the same steps as we did in the Carroll case. This time however it will be necessary to use the redefined $\bar{A}^{\mu\nu} = e_a^\mu e_b^\nu \bar{A}^{ab}$ of equation (4.34) instead of $A^{\mu\nu} = e_a^\mu e_b^\nu A^{ab}$ in order to fully remove all vielbeins e_μ^a and obtain the theory in a metric formulation. Proceeding in a similar manner as before, namely trading the spin connections for a Christoffel connection $\Gamma_{\mu\nu}^\rho$ by imposing a vielbein postulate, we obtain⁹

$$S_{\text{Gal}} = \frac{1}{2\kappa} \int e h^{\mu\nu} R_{\mu\nu}, \quad (4.37)$$

with the same definitions for the Riemann and Ricci tensors we used before, see equation (3.44) and below equation (3.45). In this case the $\Gamma_{\mu\nu}^\rho$ connection that follows from the vielbein postulate and appears in equation (4.37) is given by

$$\Gamma_{\mu\nu}^\rho = \tau^\rho \partial_\mu \tau_\nu + e_a^\rho \partial_\mu e_\nu^a - e_a^\rho \omega_\mu^a \tau_\nu - e_a^\rho \omega_\mu^{ab} e_\nu^b, \quad (4.38)$$

$$\begin{aligned} &= \tau^\rho \partial_\mu \tau_\nu - e_a^\rho \omega_\mu^a \tau_\nu + \frac{2}{D-3} h^{\rho\sigma} \tau^\lambda (h_{\mu\nu} \partial_{[\lambda} \tau_{\sigma]} + h_{\mu\sigma} \partial_{[\nu} \tau_{\lambda]}) - \tau_\mu h_{\nu\sigma} \bar{A}^{\rho\sigma} - h^{\rho\sigma} K_{\sigma\mu} \tau_\nu \\ &+ \frac{1}{2} h^{\rho\sigma} (\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}) + \frac{1}{2} \tau_\mu h_\nu{}^\tau h^{\rho\sigma} \tau^\lambda (\partial_\sigma h_{\tau\lambda} - \partial_\tau h_{\sigma\lambda}). \end{aligned} \quad (4.39)$$

A few remarks are in order. First of all, note that due to the fact that $\Gamma_{\mu\nu}^\rho$ is obtained directly from a vielbein postulate, equation (4.38) being the result, the boost spin connection ω_μ^a naturally appears in $\Gamma_{\mu\nu}^\rho$. However, as expected all the terms containing ω_μ^a automatically cancel out in the action (4.37), leaving us with a second order formulation for Galilei gravity that depends only on τ_μ , $h_{\mu\nu}$ and $\bar{A}^{\mu\nu}$. Here, the use of $\bar{A}^{\mu\nu}$ over $A^{\mu\nu}$ is necessary since the difference between these two terms cannot be rewritten without using

⁹A related action occurs in [27] as the leading term in the non-relativistic expansion of an ADM formulation of the Einstein-Hilbert Lagrangian. This work does not mention, however, the occurrence of Galilean symmetries in this leading term.

the vielbein e_μ^a , see equation (4.34). In the second order formulation, the connection $\Gamma_{\mu\nu}^\rho$ is not Galilean invariant. This is due to the fact that the spin connection ω_μ^{ab} which appears in (4.38) transforms according to (4.36) instead of (4.6). As a direct consequence of this, the Lagrangian given in equation (4.37) is not an invariant. However, as we already observed in the Carroll case, the action is invariant.

The equations of motion obtained by varying the Galilean action (4.37) with respect to $\bar{A}^{\mu\nu}$ are

$$h_\mu{}^\rho h_\nu{}^\sigma (\partial_\rho \tau_\sigma - \partial_\sigma \tau_\rho) = 0. \tag{4.40}$$

As expected this is nothing else than the constraint (4.28). Using this geometric constraint, the remaining equations of motion obtained by varying the action with respect to τ^μ and $h^{\mu\nu}$, respectively, read

$$\tau_\mu R = 0, \tag{4.41}$$

$$h^{\rho\sigma} h_{\sigma(\mu} R_{\nu)\rho} - \frac{1}{2} h_{\mu\nu} R - \tau_{(\mu} h_{\nu)\tau} \tau^\rho h^{\sigma\lambda} R_{\rho\sigma\lambda}{}^\tau = 0, \tag{4.42}$$

with $R = h^{\mu\nu} R_{\mu\nu}$. Note that in this case $R_{\mu\nu}$ is not symmetric but both the Ricci and the Riemann tensors become invariants whenever the constraint (4.40) is satisfied.

The case $D = 3$ is special. In that case we may write $\omega_\mu^{ab} = \epsilon^{ab} \omega_\mu$ and it can be seen from the first order equations of motion (4.28)–(4.30) that the whole ω_μ remains undetermined. Hence, an interesting consequence is that there is intrinsically no second order formulation for $D = 3$. Also, in contrast to the $D > 3$ case, the equations of motion imply a stronger geometrical constraint, namely

$$R_{\mu\nu}(H) = \partial_{[\mu} \tau_{\nu]} = 0. \tag{4.43}$$

Using the identity $e\epsilon^{ab} e_a^\mu e_b^\nu = 2\epsilon^{\mu\nu\rho} \tau_\rho$, which is valid for $D = 3$, the Galilean action (4.24) can be rewritten as

$$S_{\text{Gal } 3\text{D}} = -\frac{1}{2\kappa} \int \epsilon^{\mu\nu\rho} \tau_\mu \partial_\nu \omega_\rho. \tag{4.44}$$

This form of the action makes manifest that its variation with respect to ω_μ precisely reproduces the constraint obtained in equation (4.43). Note that the Galilei algebra in $D = 3$ only allows for a degenerate invariant bilinear form. The above action corresponds to the Chern-Simons action for the Galilei algebra with this degenerate bilinear form. The degeneracy of the form explains why not all fields occur in the action.

5 Matter couplings

We generalize the discussion so far to include matter couplings. For this purpose, we consider the action

$$S_{\text{Tot}} = S_{\text{grav}} + S_{\text{mat}}, \tag{5.1}$$

where S_{grav} will be either Carroll or Galilei gravity and S_{mat} denotes a general matter action. We define the following currents

$$J^\mu{}_a = \frac{\kappa}{e} \frac{\delta S_{\text{mat}}}{\delta \omega_\mu^a}, \quad J^\mu{}_{ab} = \frac{\kappa}{e} \frac{\delta S_{\text{mat}}}{\delta \omega_\mu^{ab}}, \quad (5.2)$$

$$T_\mu = \frac{\kappa}{e} \frac{\delta S_{\text{mat}}}{\delta \tau^\mu}, \quad T_\mu{}^a = \frac{\kappa}{e} \frac{\delta S_{\text{mat}}}{\delta e_\mu^a}. \quad (5.3)$$

5.1 Matter coupled Carroll gravity

For any $D > 2$ the set of equations of motion obtained by varying the action (5.1) with respect to τ^μ , e_μ^a , ω_μ^a and ω_μ^{ab} can be written as follows:

$$R_{ab}(H) = 2J^0{}_{ab}, \quad (5.4)$$

$$R_{0b}(H) = J^0{}_b - \frac{1}{D-2} (J^0{}_b + 2J^a{}_{ab}), \quad (5.5)$$

$$R_{ab}{}^c(P) = \frac{2}{D-2} (J^0{}_{[a}\delta_{b]}^c + 2J^d{}_{d[a}\delta_{b]}^c) + 2J^c{}_{ab}, \quad (5.6)$$

$$R_{a0}{}^c(P) = \frac{1}{D-2} \delta_a^c J^b{}_b - J^c{}_a, \quad (5.7)$$

$$T_0 = -\frac{1}{2} R_{ab}{}^{ab}(J), \quad T_a = R_{ab}{}^b(G), \quad (5.8)$$

$$T_0{}^a = R_{0b}{}^{ab}(J), \quad (5.9)$$

$$T_a{}^b = R_{ac}{}^{bc}(J) - R_{a0}{}^b(G) + \delta_a^b R_{c0}{}^c(G) - \frac{1}{2} \delta_a^b R_{cd}{}^{cd}(J). \quad (5.10)$$

The equations (5.4)–(5.7) can be used to solve for the spin connections

$$\omega_{\mu a} = \tau_\mu \tau^\nu e_a^\rho \partial_{[\nu} \tau_{\rho]} + e_a^\nu \partial_{[\mu} \tau_{\nu]} + e_\mu^b S_{ab} - \left(\frac{(D-3)J^0{}_a + 2J^b{}_{ab}}{D-2} \right) \tau_\mu + J^0{}_{ab} e_\mu^b, \quad (5.11)$$

$$\begin{aligned} \omega_\mu{}^{ab} = & -2e^{\rho[a} \partial_{[\mu} e_{\rho]}^{b]} + e_{\mu c} e^{\rho a} e^{\nu b} \partial_{[\rho} e_{\nu]}^c - J^{[ab]} \tau_\mu - \frac{2}{D-2} (J^0{}_d + 2J^c{}_{cd}) \delta^{d[a} e_\mu^{b]} \\ & - 2J^{[a}{}_{cd} \delta^{b]d} e_\mu^c - \delta^{ac} \delta^{bd} J^f{}_{cd} e_{f\mu}. \end{aligned} \quad (5.12)$$

The same equations can also be used to derive the constraint

$$K_{ab} = J_{(ab)} - \frac{1}{D-2} \delta_{ab} J^c{}_c, \quad (5.13)$$

on the extrinsic curvature.

Like in the case of General Relativity discussed in section 2 the equations of motion (5.4)–(5.10) give relations between the curvatures and the currents. The Bianchi

identities (3.10)–(3.13) then lead to the following on-shell relations among the currents

$$T_{0a} = -\mathcal{D}_b J^b{}_a - \mathcal{D}_0 J^0{}_a + \frac{1}{D-2} \left(2J^b{}_a J^c{}_{bc} + J^0{}_a J^c{}_c - J^0{}_b J^b{}_a \right), \quad (5.14)$$

$$T_{[ab]} = -\mathcal{D}_0 J^0{}_{ab} - \mathcal{D}_c J^c{}_{ab} - \frac{1}{D-2} \left(J^0{}_c J^c{}_{ab} - 2J^c{}_{ab} J^f{}_{cf} - J^c{}_c J^0{}_{ab} \right), \quad (5.15)$$

$$\begin{aligned} \mathcal{D}_0 T_0 + \mathcal{D}_a T_0{}^a &= J^0{}_a T_0{}^a + J^a{}_b R_{ac}{}^{bc}(J) - J^a{}_{bc} R_{0a}{}^{bc}(J) \\ &+ \frac{2}{D-2} \left(J^a{}_a T_0 - J^0{}_a T_0{}^a - 2J^a{}_{ab} T_0{}^b \right), \end{aligned} \quad (5.16)$$

$$\begin{aligned} \mathcal{D}_0 T_c + \mathcal{D}_a T_c{}^a &= -J^\mu{}_c T_\mu + 2J^\mu{}_{cb} T_\mu{}^b + J^\mu{}_a R_{\mu c}{}^a(G) + J^\mu{}_{ab} R_{\mu c}{}^{ab}(J) \\ &+ \frac{2}{D-2} \left((J^0{}_c + 2J^a{}_{ac}) \frac{(T_a{}^a + T_0)}{2} - (J^0{}_b + 2J^a{}_{ab}) T_c{}^b + J^a{}_a T_c \right). \end{aligned} \quad (5.17)$$

5.2 Matter coupled Galilei gravity

The equations of motion that follow from the Galilei action with matter (5.1) for any $D > 2$ are given by

$$R_{ab}(H) = 2J^0{}_{ab}, \quad (5.18)$$

$$R_{0a}(H) = \frac{1}{D-2} \left(2J^b{}_{ab} + (D-3)R_{ab}{}^b(P) \right), \quad (5.19)$$

$$R_{ab}{}^c(P) = 2J^c{}_{ab} + \frac{2}{D-2} \delta_{[a}^c \left(2J^d{}_{b]d} - R_{b]d}{}^d(P) \right), \quad (5.20)$$

$$T_0 = -\frac{1}{2} R_{ab}{}^{ab}(J), \quad T_a = 0, \quad (5.21)$$

$$T_0{}^a = R_{0b}{}^{ab}(J), \quad (5.22)$$

$$T_{ab} = R_{acb}{}^c(J) - \frac{1}{2} \delta_{ab} R_{bc}{}^{bc}(J). \quad (5.23)$$

The fact that $T_a = 0$ is a direct consequence of the Galilei boost invariance of the action. Furthermore, the local scale invariance given by eqs. (4.26) and (4.27) implies

$$T_{aa} = (D-3)T_0. \quad (5.24)$$

For $D > 3$ the equations of motion (5.19) and (5.20) can be used to solve for the spatial rotation spin connection $\omega_\mu{}^{ab}$ as follows

$$\omega_\mu{}^{ab} = \tau_\mu A^{ab} + Z^{abc} e_{c\mu}, \quad (5.25)$$

$$\begin{aligned} Z_{abc} &= \left(\delta_{cd} e_{[a}^\mu e_{b]}^\nu + \delta_{bd} e_{[a}^\mu e_{c]}^\nu - \delta_{ad} e_{[b}^\mu e_{c]}^\nu \right) \partial_\mu e_\nu^d - J^d{}_{ab} \delta_{cd} + J^d{}_{bc} \delta_{ad} - J^d{}_{ac} \delta_{bd} \\ &+ \frac{4}{D-3} \left(e_{[a}^\mu \delta_{b]c} \tau^\nu \partial_{[\mu} \tau_{\nu]} - J^d{}_{a[b} \delta_{c]} \right), \end{aligned} \quad (5.26)$$

except for an anti-symmetric tensor component $A^{ab} = -A^{ba}$ of $\omega_\mu{}^{ab}$.

The case $D = 3$ is special. In this case we may write $J^\mu{}_{ab} = \epsilon_{ab} J^\mu$ and the equations (5.18) and (5.19) imply the constraint

$$\partial_{[\mu} \tau_{\nu]} = \epsilon_{ab} \left(J^0 e_\mu^a e_\nu^b + 2\tau_{[\mu} e_{\nu]}^a J^b \right). \quad (5.27)$$

The current J^μ automatically drops out from equation (5.20) which is solved by a fully undetermined spin connection $\omega_\mu^{ab} = \epsilon^{ab}\omega_\mu$.

Finally, like in General Relativity and Carroll gravity, the equations of motion (5.18)–(5.23) give relations between the curvatures and the currents. Using the Bianchi identities (4.16)–(4.19) the equations of motion imply the following additional on-shell relations between the currents:

$$T_{[ab]} = -\mathcal{D}_0 J^0{}_{ab} - \mathcal{D}_c J^c{}_{ab} - J^0{}_{ab} R_{0c}{}^c(P) + \frac{1}{D-2} J^c{}_{ab} \left(2J^d{}_{cd} - R_{cd}{}^d(P) \right), \quad (5.28)$$

$$\mathcal{D}_0 T_0 + \mathcal{D}_a T_0{}^a = -J^a{}_{bc} R_{0a}{}^{bc}(J) + T_0{}^a R_{ab}{}^b(P) + T_a{}^b R_{0b}{}^a(P) - T_0 R_{0a}{}^a(P) - \frac{2}{D-2} T_0{}^a \left(2J^b{}_{ba} + R_{ac}{}^c(P) \right), \quad (5.29)$$

$$\mathcal{D}_b T_a{}^b = J^\mu{}_{bc} R_{\mu a}{}^{bc}(J) + 2J^\mu{}_{ab} T_\mu{}^b - 2J^b{}_{ab} T_0 - \frac{2}{D-2} T_a{}^b \left(2J^c{}_{cb} + R_{bc}{}^c(P) \right). \quad (5.30)$$

5.3 Examples

In the previous section, we have left the matter action unspecified. In this section, we will consider specific examples of matter actions coupled to arbitrary Carrollian and Galilean backgrounds. In particular, we will consider actions for a real scalar field, a Dirac field and electromagnetism. The starting point in all cases will be the corresponding matter action coupled to a fixed relativistic background. After that, we will study the corresponding Carrollian and Galilean limits.

5.3.1 Spin 0

We first consider the action for a real Klein-Gordon field Φ , with mass M , minimally coupled to an arbitrary relativistic background

$$S_{\text{KG}} = -\frac{1}{2} \int d^D x \sqrt{-g} \left(g^{\mu\nu} \partial_\mu \Phi(x) \partial_\nu \Phi(x) + M^2 \Phi(x)^2 \right). \quad (5.31)$$

Focusing first on the Carrollian limit, we find that upon applying the rescalings (3.16), (3.17), along with $\Phi = \frac{1}{\sqrt{\omega}} \phi$ and $M = \omega m$, the $\omega \rightarrow \infty$ limit of (5.31) leads to the following Carroll action

$$S_{\text{KG}}^{\text{Car}} = \frac{1}{2} \int d^D x e \left(\tau^\mu \tau^\nu \partial_\mu \phi(x) \partial_\nu \phi(x) - m^2 \phi(x)^2 \right). \quad (5.32)$$

The equation of motion for ϕ is then given by

$$(\mathcal{D}_0^2 + m^2) \phi = 0, \quad (5.33)$$

where $\mathcal{D}_0^2 = \tau^\mu \partial_\mu (\tau^\nu \partial_\nu)$ is the second order Carroll-covariant time derivative. This equation of motion has appeared in a first order form in [28].

Another way of arguing that equation (5.33) is the correct equation of motion for a scalar field in an arbitrary Carroll background, is by considering the Carroll limit of a

relativistic particle in a relativistic curved background specified by the metric $g_{\mu\nu}$. The canonical action of such a particle is given by

$$S = \int d\tau \left[p_\mu \dot{x}^\mu - \frac{\lambda}{2} E \left(g^{\mu\nu} p_\mu p_\nu + M^2 \right) \right], \quad (5.34)$$

where $E = \det(E_\mu^A)$. The Carrollian limit is obtained by applying the rescalings (3.16), (3.17), along with $M = \omega m$ and by taking the limit $\omega \rightarrow \infty$. The dominant term is given by

$$S = \int d\tau \left[p_\mu \dot{x}^\mu - \frac{\lambda}{2} e \left(-\tau^\mu(t, \vec{x}) \tau^\nu(t, \vec{x}) p_\mu p_\nu + m^2 \right) \right], \quad (5.35)$$

where a factor of ω has been absorbed in λ . The equations of motion obtained by varying the coordinates and momenta are given by

$$\dot{x}^\mu = -e \lambda \tau^\mu \tau^\nu p_\nu, \quad \dot{p}_\mu = e \lambda (\partial_\mu \tau^\rho) \tau^\sigma p_\sigma p_\rho. \quad (5.36)$$

By varying with respect to the Lagrange multiplier λ , one obtains the mass-shell constraint for a Carroll particle

$$-\tau^\mu(t, \vec{x}) \tau^\nu(t, \vec{x}) p_\mu p_\nu + m^2 = 0. \quad (5.37)$$

Upon quantization, i.e. replacing $\tau^\mu p_\mu \rightarrow -i\mathcal{D}_0$, this mass-shell constraint indeed leads to the equation of motion (5.33) of a spin 0 field.

In the Galilean case we perform the same rescaling on the scalar field, $\Phi = \omega^{-\frac{1}{2}} \phi$, but we keep the mass M as it is. We thus obtain

$$S_{\text{KG}}^{\text{Gal}} = -\frac{1}{2} \int e \left(\delta^{ab} e_a^\mu e_b^\nu \partial_\mu \phi \partial_\nu \phi + M^2 \phi^2 \right). \quad (5.38)$$

The equation of motion for ϕ is given by

$$\left(\delta^{ab} \mathcal{D}_a \mathcal{D}_b + \frac{1}{D-2} R_{ab}{}^b(P) \mathcal{D}_a - M^2 \right) \phi = 0, \quad (5.39)$$

where $\mathcal{D}_a \mathcal{D}_b \phi = e_a^\mu (\partial_\mu \mathcal{D}_b \phi - \omega_\mu^{bc} \mathcal{D}_c \phi)$ is the second order Galilean-covariant spatial derivative. Written as such this result is valid for any $D \neq 2$. For $D > 3$ we have the additional relation $R_{ab}{}^b(P) \propto 2\partial_{[\mu} \tau_{\nu]} \tau^\mu e_a^\nu = R_{0a}(H)$.

5.3.2 Spin $\frac{1}{2}$

We now consider the coupling of a Dirac field to a curved background.¹⁰ The action is given by

$$S_{\text{Dirac}} = \int d^D x \sqrt{-g} \bar{\Psi} \gamma^\mu \left(\partial_\mu - \frac{1}{4} \Omega_\mu^{AB} \gamma_{AB} \right) \Psi, \quad (5.40)$$

where $\gamma^\mu = E^\mu_A \gamma^A$.

Using the rescalings (3.16) and (3.17) and taking the limit $\omega \rightarrow \infty$, one finds the following ‘Carroll-Dirac’ action

$$S_{\text{Carroll-Dirac}} = \int d^D x e \bar{\Psi} \gamma^0 \tau^\mu \left(\partial_\mu - \frac{1}{4} \omega_\mu^{ab} \gamma_{ab} \right) \Psi. \quad (5.41)$$

¹⁰The case of a spinning particle coupled to a curved background could also be studied.

As in the scalar field case, this action only contains a time-like derivative. Furthermore, it is interesting to note that it only contains the spin connection ω_μ^{ab} that does not contain any undetermined components.

The Galilean limit is obtained by applying the rescalings (4.20), (4.21) and $\Psi \rightarrow \frac{1}{\sqrt{\omega}}\Psi$ and taking the limit $\omega \rightarrow \infty$. This leads to the ‘Galilei-Dirac’ action

$$S_{\text{Galilei-Dirac}} = \int d^D x e \bar{\Psi} \gamma^a e^\mu{}_a \left(\partial_\mu - \frac{1}{4} \omega_\mu^{bc} \gamma_{bc} \right) \Psi. \quad (5.42)$$

Like for the scalar field, this action only contains a spatial derivative. It also contains the spin connection ω_μ^{ab} . It does however not contain the undetermined components of the latter, as these components lie along the τ_μ direction and are projected out of the above action since ω_μ^{bc} appears multiplied with $e^\mu{}_a$.

Unlike the Carroll case, in the Galilean case one could consider a different limit, with different components of the fermion scaling differently, that does lead to the appearance of a (undetermined) boost connection field in the action and fermions that transform under Galilean boosts. This other limit is basically the massless limit of the Newton-Cartan limit considered in [35], see eq. (2.6) of that paper.

5.3.3 Spin 1: electromagnetism

Starting from the action for Maxwell electromagnetism coupled to an arbitrary relativistic background

$$S_{\text{Maxwell}} = -\frac{1}{4} \int d^D x \sqrt{-g} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}, \quad (5.43)$$

with $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$, the Carrollian limit is obtained by applying the rescalings (3.16), (3.17) of the background fields, along with a rescaling $A_\mu \rightarrow \frac{1}{\sqrt{\omega}} A_\mu$ and by taking the limit $\omega \rightarrow \infty$. In this way, one obtains the following ‘Carroll-Maxwell’ action

$$S_{\text{Carroll-Maxwell}} = \frac{1}{2} \int d^D x e (\tau^\mu F_{\mu\nu}) (\tau^\rho F_{\rho\sigma}) h^{\nu\sigma}. \quad (5.44)$$

Similarly, the Galilean limit is obtained by taking the limit $\omega \rightarrow \infty$, after applying the rescalings (4.20), (4.21) and $A_\mu \rightarrow \frac{1}{\sqrt{\omega}} A_\mu$. This leads to the ‘Galilei-Maxwell’ action

$$S_{\text{Galilei-Maxwell}} = -\frac{1}{4} \int d^D x e h^{\mu\rho} h^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}. \quad (5.45)$$

One thus sees that the Carroll-Maxwell Lagrangian is the generalization of $\vec{E} \cdot \vec{E}$ to arbitrary Carroll backgrounds, where \vec{E} is the electric field.¹¹ Similarly, the Galilei-Maxwell Lagrangian is a suitable generalization of $\vec{B} \cdot \vec{B}$, with \vec{B} the magnetic field, to arbitrary Galilean backgrounds. While it may seem puzzling at first that only the electric field appears in the Carroll-Maxwell Lagrangian, this is consistent with the fact that the dynamics

¹¹When restricted to flat space-time, the Carroll-Maxwell action corresponds to the action of ‘Carrollian electromagnetism of the electric type’, considered in [29] and more recently in [30] in the context of flat space holography. In [29], ‘Carrollian electromagnetism of the magnetic type’ is also considered, whose Lagrangian is given by $\vec{B} \cdot \vec{B}$. This theory can, however, be obtained from Carrollian electromagnetism of the electric type, by interchanging $\vec{E} \rightarrow \vec{B}$ and $\vec{B} \rightarrow -\vec{E}$.

of Carroll particles and fields is trivial, in the sense that their equations of motion only involve time derivatives. As a consequence, minimal coupling to a vector potential will only involve the electric potential. Physically, since Carroll particles do not move, they will not induce a magnetic field nor will they be subjected to a Lorentz magnetic force. It therefore makes sense that the Carroll-Maxwell Lagrangian only involves the electric field, as that is the only field that will be relevant in coupling to Carroll particles and fields.

Similarly, actions for Galilei fields only involve spatial derivatives and minimal coupling to a vector potential will likewise only involve the spatial parts of the vector potential. The Galilei-Maxwell action then only contains the magnetic field, as that is the only contribution relevant for couplings to Galilei fields.

Note that the Galilei-Maxwell action above does not correspond to the action of what is known in the literature as Galilean electrodynamics [31] (for a review, see [32]; see [33, 34] for a discussion in the context of flat space holography), coupled to an arbitrary non-relativistic background. The latter contains contributions from both the electric and magnetic fields. While this action can not be obtained via the simple limit considered in this paper, it can be obtained by taking different limit procedures. In particular, it arises as a non-relativistic limit of an action that is a sum of the Maxwell action and the action for a real massless scalar field in an arbitrary relativistic background [35]. The action for Galilean electrodynamics in flat space-time has also been obtained via null reduction in [36]. As Galilean electrodynamics involves both electric and magnetic fields, it is the appropriate theory to consider when dealing with non-relativistic charged particles and fields, whose equations of motion involve both spatial and time derivatives. Examples of such fields have been studied in [35]. These examples involve massive fields and exhibit mass conservation. The appropriate non-relativistic background to couple such fields to is then a Newton-Cartan background, which we mentioned in the introduction. This Newton-Cartan background is an extension of a Galilean background, that apart from τ_μ and e_μ^a also involves an extra one-form m_μ , that plays the role of gauge field associated to the charge that expresses mass conservation.

6 Conclusions

In this paper we showed that there exist two consistent limits of the Einstein-Hilbert action describing General Relativity that lead to finite actions, upon making a redefinition of Newton's constant. This is in contrast to the Newton-Cartan limit, leading to Newton-Cartan gravity, that we defined in [37] and that can be taken at the level of the equations of motion only. The first, ultra-relativistic, limit leads to a so-called Carroll gravity action while the second limit is non-relativistic and leads to a so-called Galilei gravity action. We presented the actions both in first-order and second-order form. A noteworthy feature is that, unlike General Relativity, not all components of the spin connection fields can be solved for. We showed that the independent components occur as Lagrange multipliers in the action thereby imposing constraints on the geometry. The case of Carroll gravity is interesting in view of possible applications to flat space holography where the Carroll symmetries play an important role [13].

Here, we have considered Carrollian and Galilean limits of General Relativity at the level of the action. One could also consider these limits at the level of the equations of motion. However, this is not an unambiguous procedure. The relativistic equations of motion that one starts from can be written in different equivalent ways, that can however lead to different limits when $\omega \rightarrow \infty$. For instance, the limit taken directly in (2.10) (with $A = a$ and $B = 0$) is divergent in the Carroll case but not in Galilean one. On the other hand, the limit in the same equation of motion rewritten simply as $R_{\mu\nu}{}^A(P) = 0$ is divergent in the Galilean case and not in the Carroll one. It would be interesting to further investigate the possible limits of the equations of motion.

Given pure General Relativity, without additional fields, the Carroll and Galilei limits are the only consistent ones that can be taken at the level of the Einstein-Hilbert action.¹² Using an expansion of the fields in terms of the contraction parameter ω this limit picks out the leading term in an ω -expansion of the action. Introducing an additional vector field, a (non-relativistic) Newton-Cartan limit at the level of the equations of motion can be defined leading to the equations of motion of Newton-Cartan gravity. From the ω -expansion point of view, the vector field helps in cancelling the leading (divergent) term in an ω -expansion of the equations of motion with the effect that this new non-relativistic limit picks out the (finite) subleading term in an ω -expansion. It would be interesting to see whether, using the same vector field, also an ultra-relativistic limit can be defined that picks out the subleading term in the ω -expansion and whether the resulting ‘Carroll gravity’ theory is related to the one presented in [12].

After constructing the gravity actions, we also considered matter couplings and compared the results with the case of Newton-Cartan gravity. A characteristic feature of these matter couplings is that only time derivatives (Carroll limit) or spatial derivatives (Galilei limit) survive whereas in a Newton-Cartan limit both types of derivatives survive like in the case of the Schrödinger action. In the case of spin 0 Carroll matter, we showed that the results obtained are consistent with the point of view of a Carroll particle.

Besides taking the Carroll or Galilei limit of General Relativity, one could also consider taking these limits at the level of the effective actions that describe extended objects beyond particles. For instance, Carroll strings have been considered in [38]. Recently, a Galilean limit of a relativistic Green-Schwarz superstring action has been considered and the resulting non-relativistic so-called Galilean superstring, exhibiting kappa-symmetry, has been given [15]. One could also consider ‘stringy’ versions of the limits considered in this paper where, besides the time direction, one or more of the spatial directions, those in the direction of the world-volume of the extended object, play a special role.

It would be interesting to apply the Hamiltonian canonical quantisation procedure to Carroll and Galilei gravity and verify how many physical degrees of freedom exist in these models. This would enable one to find out whether the Lagrange multiplier fields do represent any kind of non-relativistic degree of freedom.

In a previous paper [39] we already discussed the extension of this work to include higher spins, i.e. fields describing particles with spin larger than 2. It would be interesting

¹²Using an ADM formulation one can additionally define the strong coupling limit of [19, 20].

to see whether the geometries discussed in [39] have applications to the non-relativistic higher-spins that have recently been discussed in the context of the fractional quantum Hall liquid [40] in the same way as Newton-Cartan geometry has found applications in Condensed Matter Theory, see, e.g., [41].

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